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$SU(2)$ ACTION-ANGLE VARIABLES

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Abstract

Operator angle-action variables are studied in the frame of the $SU(2)$ algebra, and their eigenstates and coherent states are discussed. The quantum mechanical addition of action-angle variables is shown to lead to a novel non commutative Hopf algebra. The group contraction is used to make the connection with the harmonic oscillator.

1 Introduction

Action-angle variables in quantum mechanics one known to lack, in the operator level, some of properties of their classical analogues [1,2]. Especially the exponential phase operators for the harmonic oscillator, occuring in the polar decomposition of the bosonic creation and annihilation operators (an operator analogon of the polar decomposition of a complex number), lack the unitary and satisfy the weaker condition of one side-unitary or isometry operator. Based on the mathematical fact that, unlike in finite dimensional Hilbert spaces as the Fock space of harmonic oscillator, in finite spaces an isometry is equivalent to a unitary operator, we have in recent works, suggested a group theoretical construction of a unitary phase operator by introducing action-angle variables for the $SU(2)$ algebra and going over to their oscillator counterparts via the Inönü-Wigner method of group contraction [3-6]. In this report we will briefly review and then expand this work with respect to two aspects: first, a set of coherent states will be introduced along the lines of the displacement operator creating the usual coherent states from the vacuum state and second, we will show that addition of spins in terms of their action-angles (polar) operators, unlike the usual addition in terms of the step (cartesian) operators, involves a genuine no commutative, no co-commutative Hopf algebra structure and relates interestingly the phase operators subject to the subject of quantum groups.

2 Action-angle Variables and States

Let us start with the $SU(2)$ action-angle operators

$$J_- = e^{i\Phi}\sqrt{J_+J_-} = \sqrt{J_-J_+}e^{i\Phi} \tag{1}$$

$$J_+ = e^{-i\Phi}\sqrt{J_-J_+} = \sqrt{J_+J_-}e^{-i\Phi} \tag{2}$$

where

$$J_+ = \sum_{m=0}^{2j} \sqrt{m(2j-m+1)} |J; m+1 >< J; m| \qquad , \qquad J_- = J_+^+ \tag{3}$$

$$J_3 = \sum_{m=0}^{2j} (m-j) |J; m >< J; m| \tag{4}$$

and

$$e^{i\Phi} = \sum_{\ell=0}^{2j} |J; \ell >< J; \ell+1| \, , \tag{5}$$

mod$(2j+1)$, and $hh^+ = h^+h = \mathbf{1}$ with $h \equiv e^{i\Phi}$, $h^+ \equiv e^{-i\Phi}$ the unitary angle operator. Then from the fact that h, generates the cyclic group Z_{2j+1} acting as a cyclic permutation in the weight space of the algebra we can construct phase states

$$|\Phi; k >= F|J; k >= \frac{1}{\sqrt{2j+1}} \sum_{m=0}^{2j} \omega^{kn} |J; n > \tag{6}$$

through the finite Fourier transform $FF^+ = F^+F = 1$, which maps action eigenstates to angle eigenstates and conjugates the respective variables, where $\omega = \exp i(2\pi/2j+1)$. Indeed, if $g := \omega^{J_3+j1}$ then $FgF^+ = h$, $FhF^+ = g^{-1}$ and $g(h)$ acts as step operator in the angle (action) state basis, i.e,

$$h|J; n >= |J; n+1 > \quad , \quad h|\Phi; m >= \omega^m |\Phi; m > \tag{7}$$

while

$$g^{-1}|\Phi; n >= |\Phi; n+1 > \quad , \quad g|J; m >= \omega^m |J; m > \tag{8}$$

mod$(2j+1)$ and $h^{2j+1} = g^{2j+1} = \mathbf{1}$, (notice that the state $|J; n>$ and $|\Phi; m>$ where denoted as $|n>$ and $|\varphi_m>$ respectively, in Refs. 3-6). The noncommutativity between the action and the angle variables is best expressed by the formula

$$\omega g h = h g \tag{9}$$

which resembles the exponential form of the Heisenberg canonical commutation relations (CR) as were originally written by Weyl with the association that here the action operator J_3 is a finite version of the position operator and the angle operator stands for the momentum operator. By virtue of this analogy we may interpret eqs. (7-8) as the translations along the two different directions of the phase space of our problem, which due to the module condition is a lattice torus, parametrized by the discrete action and angle values. Also eq. (9), exhibits the unusual noncommutative character of two succesive translations along different directions. Moreover, the effect of group contraction which is discussed below, is to increase the density of the lattice points until the continous limit $j \to \infty$. Furthermore this association to position and momentum suggests that we should look for the "number states" $|N;m>$, $m=0,1,...,2j$ in our finite system. Indeed by diagonalizing the finite Fourier transform $F|N;m>= i^m|N;m>$, we find the number states $|N;m>$, related e.g. with the orthonormal action states as:

$$|N;k>=\sum_{m=0}^{2j}|J;m><J;m|N;k> , \tag{10}$$

with expansion coefficients given in terms of the Hermite polynomial, H_k with discrete argument,

$$<J;m|N;k>=\sum_{p=-\infty}^{\infty} e^{-\frac{\pi}{2j+1}(p(2j+1)+m)^2} H_k\Big(\sqrt{\frac{2\pi}{2j+1}}(p(2j+1)+m)\Big) \tag{11}$$

This situation is akin to that of the harmonic oscillator number states which are similarly eigenstates of the usual Fourier transform operator which conjugates position and momentum operators, a fact that stems from the property of the oscillator eigenstates $\exp(-\frac{1}{2}x^2)H_k(x)$, to be their own Fourier transforms. Especially the vacum or lowest number state is,

$$|N;0>=\sum_{m=0}^{2j}\omega^{\frac{1}{2}m^2}\theta_3(im|i(2j+1))|J;m> \tag{12}$$

where θ_3 is the theta-Jacobi function [7]:

$$\theta_3(z|\tau)=\sum_{s=-\infty}^{\infty} e^{zi2\pi s+\tau i\pi s^2} \tag{13}$$

Having the action $|J;m>$, the angle $|\Phi;n>$ and the number states $|N;k>$ as were given above, we can further built, as have been outlined in Ref. 4, the quantum theory of action-angle variables by introducing the corresponding coherent states acting on the vacum $|N;0>$, with a displacement operator. Such an operator is furnished by the unitary traceless elements $J_{m_1,m_2} := \omega^{m_1m_2/2}g^{m_1}h^{m_2}$, where $J^{+}_{m_1,m_2} = J_{-m_1,-m_2} = J_{2j+1-m_1,2j+1-m_2}$, with (m_1,m_2) pairs belonging to the square index-lattice $0\le m_1,m_2\le 2j$ with boundary conditions and the $(0,0)$ pair excluded.

The following interesting properties of these operators suggest them as the Glauber displacement operator of our case; first they constitute an orthonormal set of $(2j+1)^2-1$ elements obeying the relation

$$<J_{\vec{m}},J_{\vec{n}}>:= Tr\ J_{\vec{m}}J_{\vec{n}}=(2j+1)\delta_{\vec{m}+\vec{n},\vec{0}} , \tag{14}$$

where e.g. $J_{\vec{m}}=J_{m_1m_2}$, and further,

$$J_{\vec{m}}J_{\vec{n}}=\omega^{-\frac{1}{2}\vec{m}\times\vec{n}}J_{\vec{m}+\vec{n}} \tag{15}$$

and

$$J_{\vec{n}}J_{\vec{m}}=\omega^{\vec{m}\times\vec{n}}J_{\vec{m}}J_{\vec{n}} \tag{16}$$

and finally

$$[J_{\vec{m}},J_{\vec{n}}]=-2i\sin\Big[\frac{\pi}{2j+1}\vec{m}\times\vec{n}\Big]J_{\vec{m}+\vec{n}} \tag{17}$$

mod$(2j+1)$, while $\vec{m} \times \vec{n} = m_1 n_2 - m_2 n_1$. With the aid of these operators we now introduce coherent states $|\vec{\ell}>$, for the action-angle system by acting on the vacum:

$$|\vec{\ell}>:= J_{\vec{\ell}}|N;0>=\omega^{\frac{1}{2}\ell_1\ell_2}\sum_{m=0}^{2j}\omega^{\ell_1 m+\frac{1}{2}m^2}\theta_3(im|i(2j+1))|J;m+\ell_2> \tag{18}$$

These are now coherent states defined on the lattice phase space which is the appropriate phase space of the quantum action-angle variables. They involve the Jacobi theta functions which are also appearing in the case of the ordinary coherent states when, looking for a complete subset out of the over complete set of coherent states we lattice the phase space. Elsewhere, the normalization and minimum uncertainty properties of the states will be studied in detail.

3 Quantum Angles Addition

Let us now turn to the case where there are several action-angle degrees of freedom and search for the way we combine them quantum mechanically. The similar problem for the "cartesian" generators J_i, with $[J_i, J_j] = 2i\epsilon_{ijk}J_k$ is the fundamental theme of addition of spins and customanily is solved by tensoring the generators,

$$\Delta J_i := J_i \otimes \mathbf{1} + \mathbf{1} \otimes J_i \tag{19}$$

which again satisfy the commutation relations, $[\Delta J_i, \Delta J_j] = 2i\epsilon_{ijk}\Delta J_k$. In our case, for the "polar" generators $g = \omega^{(J_3+j1)}$ and $h = \omega^{F(J_3+j1)F^+}$ with $\omega gh = hg$ we must find an appropriate tensoring (coproduct in the jargon of Hopf algebras), which provides such Δg and Δh that $\omega\Delta g = \Delta h$. Two such coproducts we have found,

$$\Delta g = g \otimes g \quad , \quad \Delta h = h \otimes \mathbf{1} + g \otimes h \tag{20}$$

and

$$\Delta g = g \otimes g \quad , \quad \Delta h = h \otimes g + g^{-1} \otimes h \tag{21}$$

which both have the remarkable property of not been the same under permutation of their components involved in the tensor products. This is distingly different to the usual addition of spins, where there is no sence of order in the tensoring the spins. Technically speaking we have here a natural case of no co-commutativity unlike in eq. (19), where the product is co-commutative [8-11]. We end here this discussion, as we intent to expand it elsewhere, by saying that it is also possible to show the Hopf and quasi triangular Hopf algebra structure of the above tensoring and then to find the R-matrix and to verify the Yang-Baxter equation.

4 Contraction to the Oscillator

Before we came to conclusions let us mention that as was shown in Ref. 3 via the group contraction that the $SU(2)$ action-angle variables can be contructed to those of the oscillator and the dynamical aspects of this procces could be exemplified by studing the Jaynes-Cummings model. We illustrate now this idea be contracting the $SU(2)$ generators to the oscillator generators in the Bargmann analytic realization. In the space of analytic polynomials of degree $2j$ the $SU(2)$ algebra is realized as,

$$J_+ = -z^2 \frac{d}{dz} + z2_j \qquad J_- = \frac{d}{dz} \qquad J_3 = z\frac{d}{dz} - j \tag{22}$$

where z is the complex label of the spin coherent states, and geometrically stands for the projective coordinate of the coset sphere $SU(2)/U(1) \sim S^1$. Transforming now the generators like $J_\pm \rightarrow J_\pm/\sqrt{2j}$ and $J_3 \rightarrow J_3 + j\mathbf{1}$ we find in the large j limit, the oscillator generators in their Bargmann form as follows:

$$\frac{J_+}{\sqrt{2j}} = -\frac{(\sqrt{2j}z)^2}{2j}\frac{d}{d(\sqrt{2j}z)} + \sqrt{2j}z \approx \alpha = a^+ \tag{23}$$

$$\frac{J_-}{\sqrt{2j}} = \frac{d}{d(\sqrt{2j}z)} \approx \frac{d}{d\alpha} = a \tag{24}$$

and

$$J_3 + j = \sqrt{2j}z\frac{d}{d(\sqrt{2j}z)} \approx \alpha\frac{d}{d\alpha} = N \tag{25}$$

where $\sqrt{2j}z \approx \alpha$ is the complex variable of the Glauber coherent states which is now becoming the coordinate of the tangent phase plane of the harmonic oscillator. One can further show that the overlap, the completeness relation and all other notions of the spin coherent states can be contracted to their respective oscillator counterparts. Moreover in Ref. 5 has been shown how a q-deformed oscillator with q deformation parameter to be root of unity can be employed to define action-angles variables in a finite Fock Hilbert space and a number of their properties have been worked out. In such an approach we have shown [5], that the contraction method is substituted by the limit procedure of undeforming the q-oscillator to the usual ocillators.

5 Conclusion

In conclusion, we have shown that the quantization of action-angle classical variables can be developed in the framework of the $SU(2)$ algebra in a manner which allows for the classical properties of these variables to find well defined operator analogues. Interesting relations to the quantum groups and Hopf algebras are naturally emerge from the present method of angle quantization which will be pursued further, together with the introduction of the Wigner function for the action-angles variables and the star and Moyal product defined between functions of the phase space of our problem.

References

[1] P.A.M. Dirac, Proc. R. Soc. Lond. A, **114**, 243 (1927).

[2] P. Carruthers and M.M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).

[3] D. Ellinas, J. Math. Phys. **32**, 135 (1991).

[4] D. Ellinas, J. Mod. Optics. **38**, 2393 (1991).

[5] D. Ellinas, Phys. Rev. A. **45**, 3358 (1992).

[6] M. Chaichian and D. Ellinas, J. Phys. A **23**, L291 (1990).

[7] D. Mumford, *Tata Lectures on Theta* (Birkhauser, Boston, 1984).

[8] M.E. Sweedler, *Hopf algebras*, (W.A. Benjamin, Inc. New York 1969).

[9] V.G. Drienfeld, Sov. Math. Pokl. **32**, 254 (1985).

[10] M. Jimbo, Lett. Math. Phys. **10**, 63 (1985); Commun. Math. Phys. **102**, 537 (1986).

[11] N. Yu. Reshetikhin, L.A. Takhtajan and L.D. Faddeev, Leningrad Math. J. **1**, 193 (1990).