q-HARMONIC OSCILLATORS, q-COHERENT STATES AND THE q-SYMPLECTON† ‡

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Abstract

The recently introduced notion of a quantum group is discussed conceptually and then related to deformed harmonic oscillators ("q-harmonic oscillators"). Two developments in applying q-harmonic oscillators are reviewed: q-coherent states and the q-symplecton.

1 Introduction

It is not unfamiliar in physics that a new theory appears in the form of a 'deformation' of a previous 'classical' theory; thus, for example, quantum mechanics can be considered to be a deformation of classical mechanics (which is recovered in the limit that the 'deformation parameter' $\hbar \to 0$), and Einsteinian relativity to be a deformation of Newtonian relativity (which is recovered when the 'deformation parameter' $c \to \infty$). Recently this notion of deformation has been applied [1,2] to symmetry itself, leading to the concept of a 'quantum group' as a deformation of a classical (Lie) group with a deformation parameter denoted by q. This new development has had numerous important applications in both physics and mathematics [3,4]. Since harmonic oscillators have played a fundamental—and pervasive!—rôle in the applications of symmetry in quantum physics, it is not surprising that the concepts of quantum groups, and

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deformations, are important here also, and hence relevant to the present conference. Accordingly, it is our purpose to discuss here deformed harmonic oscillators ("q-harmonic oscillators"), deformed coherent states ("q-coherent states") and the deformed algebraic structure (based on harmonic oscillators) called the "q-symplecton".

We will begin by discussing, in conceptual and motivational terms, the simplest of quantum groups—$SU_q(2)$, the q-deformed quantal rotation group—to set the stage for introducing deformed harmonic oscillators, and then the remaining topics mentioned above.

2 The Quantum Group $SU_q(2)$

The commutation relations for the three generators $\{J_+^q, J_-^q, J_z^q\}$ defining the quantum group $SU_q(2)$ are given by:

$$[J_z^q, J_\pm^q] = \pm J_\pm^q, \tag{2.1}$$

$$[J_+^q, J_-^q] = \frac{q^{J_z^q} - q^{-J_z^q}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad q \in \mathbb{R}^+. \tag{2.2}$$

These defining relations for $SU_q(2)$ differ from those of ordinary angular momentum ($SU(2)$) in two ways:

(a) the commutator in (2.2) is not $2J_z$ as usual, but an infinite series (for generic q) involving *all odd powers*: $(J_z^q)^1, (J_z^q)^3, \ldots$. Each such power is a linearly independent operator in the enveloping algebra; accordingly, the Lie algebra of $SU_q(2)$ is ***not*** of finite dimension.

(b) For $q \to 1$, the right hand side of (2.2) becomes $2J_z$. Thus we recover in the limit the usual commutation relations for angular momentum.

The differences noted in (a) and (b) are expressed by saying that the quantum group $SU_q(2)$ is a *deformation* of the enveloping algebra of $SU(2)$.

The deformation parameter q occurs in $SU_q(2)$ in a characteristic way, as *q-integers* denoted by $[n]_q$ such that:

$$\begin{aligned} [n]_q &= \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \\ &= q^{\frac{(n-1)}{2}} + q^{\frac{(n-3)}{2}} + \ldots q^{-\frac{(n-1)}{2}}, \quad n \in \mathbb{Z}. \end{aligned} \tag{2.3}$$

These q-integers, $[n]_q$ obey the rule: $[-n]_q = (-1)[n]_q$, with $[0]_q = 0$ and $[1]_q = 1$. Note that $[n]_q = [n]_{q^{-1}}$, so that the defining relations (2.1) and (2.2) are invariant to $q \leftrightarrow q^{-1}$.

The quantum group concept involves much more than just deforming the commutation relations of the classical group generators. Actually an interesting new algebraic structure is also imposed, that of a ***Hopf algebra*** [5]. Let us first define this new structure and then discuss its meaning.

Consider an associative algebra A, with a unit element, $\mathbf{1}$, over a field say, $\mathbb{C}$. Then the algebra involves the operations:

$$\textit{multiplication:} \quad m : A \otimes A \to A, \quad \text{and,} \tag{2.4}$$

$$\textit{unit:} \quad \mathbf{1} : \mathbb{C} \to A, \tag{2.5}$$

subject to the familiar axioms of associativity and the compatibility of addition and multiplication.

We can extend this algebra to become a Hopf algebra if we can "reverse the arrows" in (2.4) and (2.5) above, that is, if we can define two new operations:

$$\textit{co-multiplication:} \quad \Delta : A \to A \otimes A, \quad \text{and,} \tag{2.6}$$
$$\textit{co-unit:} \quad \epsilon : A \to \mathbb{C}. \tag{2.7}$$

Since for a quantum group the algebra A is a group algebra, it is reasonable to require that one have a third new operation:

$$\gamma : \quad A \to A, \tag{2.8}$$

called "antipode", (the analog to the inverse in the group).

These three new operations must satisfy the requirement that Δ and ϵ are homomorphisms of the algebra A and that γ is an anti-homomorphism. In addition, the operations must satisfy the compatibility axioms:

$$\text{Associativity of co-multiplication: } (id \otimes \Delta)\Delta(a) = (\Delta \otimes id)\Delta(a), \quad a \in A \tag{2.9}$$
$$\text{Antipode axiom: } m(id \otimes \gamma)\Delta(a) = m(\gamma \otimes id)\Delta(a) = \epsilon(a)1, \tag{2.10}$$
$$\text{Co-unit axiom: } (\epsilon \otimes id)\Delta(a) = (id \otimes \epsilon)\Delta(a) = a. \tag{2.11}$$

• For a physicist, the introduction of such complicated and heavy algebraic machinery "out of the blue" is very disconcerting. Certainly it requires motivation. The obvious question is: "why a Hopf algebra"? Let us try to answer this.

Physicists are already very familiar with the algebraic approach to symmetry in quantum mechanics; what is needed is a physical reason for "reversing the arrows". What this really means, in effect, is that all one needs is a simple motivating physical example.

Here is that example. Consider angular momentum: there is a natural, classical, concept for adding angular momenta, which is taken over in quantum mechanics. Consider $\mathbf{J}_{\text{total}}$ as the total angular momentum operator which is to be the sum of two independent constituent angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$. Writing the total angular momentum operator $\mathbf{J}_{\text{total}}$ as an action on the two constituent state vectors we have:

$$\mathbf{J}_{\text{total}}|\psi\rangle_{\text{total}} = \mathbf{J}_1|\varphi\rangle_1 \otimes \mathbf{1}|\chi\rangle_2 + \mathbf{1}|\varphi\rangle_1 \otimes \mathbf{J}_2|\chi\rangle_2, \tag{2.12}$$

where we have been careful to use a precise notation for the tensor product $\otimes$ of the two independent systems.

Writing this same result in an abstract formal manner, we discover that what we have really done by "adding angular momentum" is to define a *co-multiplication*:

$$\Delta(J) = \mathbf{J} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{J}, \tag{2.13}$$

where $\mathbf{J}$ denotes a generic angular momentum (defined as obeying the commutation relations).

In other words: *The vector addition of angular momenta defines a commutative co-product in a Hopf algebra.* One sees accordingly that a (commutative) Hopf algebra structure is not only very natural in quantum physics, but actually implicit, and in fact essential—unfamiliar only because unrecognized. The remaining Hopf algebra axioms are required to make the structures compatible and well-defined, and in a sense analogous to group concepts.

What we wish to emphasize is that the deformation of the algebraic structure in a quantum group is only part of the basic concept—requiring the additional Hopf algebra structure, which is natural to quantum mechanics, provides an important constraint on the freedom to deform the commutation relations.

One can now understand intuitively from our example the fundamental significance of quantum groups for physics: one now has the new possibility of defining a *non-commutative* co-multiplication, as actually occurs for the quantum group $SU_q(2)$. This means that:

(i) the fundamental commutation relations are changed ("deformed"); that is, one has *kinematic symmetry breaking*. (Recall that Hamiltonian perturbation theory is dynamical and leaves commutation relations (which are kinematical) *invariant*);

(ii) the "addition of q-angular momentum" depends on the *order* of addition.

There is one other feature of the commutation relations for $SU_q(2)$ that deserves comment: the relations (2.1) and (2.2) single out J^q_z and thus appear to break the rotational symmetry. For generic values of q this seeming result is incorrect: the degeneracy structure of q-group irreps is in fact *preserved*, a consequence of the Rosso-Lusztig theorem. (We take this opportunity to note that ref. [6] is misleading on this particular point.)

For completeness, since we have emphasized the importance of the complete Hopf algebra structure, let us give explicitly the remaining Hopf algebra operations for the quantum group $SU_q(2)$:

$$\Delta(J^q_z) = J^q_z \otimes 1 + 1 \otimes J^q_z, \tag{2.14}$$

$$\Delta(J^q_\pm) = J^q_\pm \otimes q^{\frac{J^q_z}{4}} + q^{-\frac{J^q_z}{4}} \otimes J^q_\pm, \tag{2.15}$$

$$\epsilon(1) = 1, \quad \epsilon(J^q_\pm) = \epsilon(J^q_z) = 0, \tag{2.16}$$

$$\gamma(J^q_\pm) = -q^{\mp\frac{1}{2}} J^q_\pm, \quad \gamma(J^q_z) = -J^q_z. \tag{2.17}$$

3 q-Boson operators

In order to understand the meaning of the deformed commutation relations (2.1) and (2.2) it is natural to look for representations of the operators $J^q_\pm, J^q_z$ as finite-dimensional matrices. For the usual angular momentum group, there is a standard way to do this: one uses the *Jordan-Schwinger map* [7], which maps the 2×2 matrices $\{J_\pm, J_z\}$ of the fundamental irrep into *boson operators*.

Let us recall how this works. One begins with a realization of the operators $J_\pm, J_z$ in terms of a pair of commuting boson creation operators (a_1, a_2) and annihilation operators, $(\overline{a}_1, \overline{a}_2)$, and defines the Jordan-Schwinger map:

$$J_+ \to a_1\overline{a}_2, \quad J_- \to a_2\overline{a}_1, \quad J_z \to \tfrac{1}{2}(a_1\overline{a}_1 - a_2\overline{a}_2). \tag{3.1a, b, c}$$

This map preserves the angular momentum commutation relations (that is, the Jordan-Schwinger map is a homomorphism) and from this map one can explicitly construct *all* unitary irreps of $SU(2)$.

Is there a q-analog to the Jordan-Schwinger map? There is indeed! (Refs. [8,9,10]). *The basic idea is to construct q-analogs to the boson operators.* To do so introduce the *q-creation operator* a^q, its Hermitian conjugate the *q-destruction operator* $\overline{a}^q$, and the *q-boson vacuum ket vector* $|0\rangle$ defined by the equation

$$\overline{a}^q|0\rangle = 0. \tag{3.2}$$

Instead of the Heisenberg relation, $[\overline{a}, a] = 1$, let us postulate the algebraic relation:

$$\overline{a}^q a^q - q^{\frac{1}{2}} a^q \overline{a}^q = q^{-\frac{N^q}{2}}, \tag{3.3}$$

where N^q is the Hermitian *number operator* satisfying

$$[N^q, a^q] = a^q, \quad [N^q, \overline{a}^q] = -\overline{a}^q, \quad \text{with } N^q|0\rangle \equiv 0. \tag{3.4a,b,c}$$

This algebra is a deformation of the Heisenberg-Weyl algebra, which is recovered in the limit $q \to 1$. (Note that the q-number operator N^q is now no longer the operator $a\overline{a}$ as in the Heisenberg case.)

Orthonormal ket vectors corresponding to states of n q-quanta are given by:

$$|n\rangle_q \equiv ([n]_q!)^{-\frac{1}{2}}(a^q)^n|0\rangle, \tag{3.5}$$

$$\text{with :} \quad N^q|n\rangle_q = n|n\rangle_q. \tag{3.6}$$

It is now easy to define a q-analog for the algebra of the generators of the quantum group $SU_q(2)$. In the language of q-boson operators, one defines a pair of *mutually commuting q-bosons* a_i^q for $i = 1, 2$. That is, for each, i, a_i^q and $\overline{a}_i^q$ obey equations (3.3), (3.4) and, in addition, the relations:

$$\text{for } i \neq j: \quad [a_i^q, a_j^q] = [\overline{a}_i^q, \overline{a}_j^q] = [a_i^q, \overline{a}_j^q] = 0. \tag{3.7}$$

The generators $\{J_+^q, J_-^q, J_z^q\}$ of $SU_q(2)$ are then realized by

$$J_+^q \to a_1^q \overline{a}_2^q, \quad J_-^q \to a_2^q \overline{a}_1^q, \quad J_z^q \to \tfrac{1}{2}(N_1^q - N_2^q). \tag{3.8a,b,c}$$

The construction of all unitary irreps of the quantum group $SU_q(2)$—for generic q—is now straightforward [6] but will be omitted.

Remarks: (1) We have emphasized in Section 2 that the Hopf algebra structure—more particularly co-multiplication—is an important constraint on possible deformations. Let us note that the deformation of q-bosons given by eq. (3.3) does allow a (non-commutative) co-product to be defined. However, as shown by Prof. T. Palev (private communication), a complete Hopf algebra structure is not possible.

(2) The deformation given in eq. (3.3) can be put into many differently appearing, but equivalent, forms. For example, if we define $A^q = a^q q^{\frac{1}{4}N^q}$ and $\overline{A}^q = q^{\frac{1}{4}N^q}\overline{a}^q$, then eq. (3.3) becomes:

$$\overline{A}^q A^q = q A^q \overline{A}^q + 1, \tag{3.9}$$

a form often found in the literature.

4 The q-Harmonic Oscillator

We have motivated the introduction of q-deformed bosons as a way to implement the concept of a quantum group.

Let us now examine the q-harmonic oscillator on its own merits. From the q-boson operators $a, \bar{a}$ we can define q-momentum (P) and q-position (Q) operators in the same way as for boson operators. That is, we define:

$$P \equiv i\sqrt{\frac{m\hbar\omega}{2}}(a^q - \bar{a}^q), \tag{4.1}$$

$$Q \equiv \sqrt{\frac{\hbar}{2m\omega}}(a^q + \bar{a}^q). \tag{4.2}$$

The commutator $[P, Q]$ is then (using (3.3)):

$$i[P, Q] = \hbar[\bar{a}^q, a^q] = \hbar([N+1]_q - [N]_q). \tag{4.3}$$

The eigenvalues ($N \to n$) of the right hand side are therefore

$$\hbar([n+1]_q - [n]_q) = \hbar\frac{\cosh(\frac{1}{4}(2n+1)\log q)}{\cosh(\frac{1}{4}\log q)}. \tag{4.4}$$

One sees that the Heisenberg uncertainty in the q-harmonic oscillator is minimal (and independent of q) *only* in the limit $q \to 1$; the uncertainty *increases* with n for $q \neq 1$.

The q-harmonic oscillator Hamiltonian is defined from P, Q according to

$$\begin{aligned}\mathcal{H} &\equiv \frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2, \\ &= \frac{\hbar\omega}{2}(\bar{a}^q a^q + a^q \bar{a}^q).\end{aligned} \tag{4.5}$$

From (3.3) we find

$$\mathcal{H} = \frac{\hbar\omega}{2}([N+1]_q + [N]_q), \tag{4.6}$$

showing that the eigenvalues of $\mathcal{H}$ are

$$E(n) = \frac{\hbar\omega}{2}([n+1]_q + [n]_q). \tag{4.7}$$

The normalized eigenstates $|n\rangle$ are:

$$|n\rangle = ([n]!)^{-\frac{1}{2}}(a^q)^n|0\rangle. \tag{4.8}$$

The energy spectrum for the q-harmonic oscillator is uniformly spaced only for $q = 1$, the undeformed case. For q large, one sees that the spectrum becomes exponential: $E(n) \sim \hbar\omega q^{\frac{n}{2}}(1 + \mathcal{O}(\frac{1}{q}))$.

5 Coherent States

It is natural to ask, once one has defined q-deformed bosons, whether or not coherent states exist for this new harmonic oscillator structure. The answer is yes [11], as one might expect. Let us review this structure briefly here.

There are two key characteristics of the (usual) coherent states, as identified by Klauder and Skagerstam [12]:

(a) *continuity* of the coherent state $|z\rangle$ as a function of z.
and (b) *the resolution of unity:*

$$1 = \int |z\rangle\langle z| \; d\mu(z), \tag{5.1}$$

where the integration takes place with respect to a positive measure $d\mu(z)$.

The best known examples of coherent states, which certainly satisfy these two characteristics, are the canonical coherent states generated by the (usual) creation and annihilation operators a and $\bar{a}$. These canonical coherent states are defined by [8]

$$\begin{aligned} |z\rangle &\equiv e^{-|z|^2/2} e^{za} |0\rangle \\ &= e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \, |n\rangle, \end{aligned} \tag{5.2}$$

where $|n\rangle$ denotes the orthonormal vectors generated by the creation operator a.

We can immediately write down q-coherent states $|z\rangle_q$ by replacing the boson operator of (5.2) by its q-boson analog, and replacing the exponential in (5.2) by the *q-exponential function* $\exp_q$:

$$\begin{aligned} |z\rangle_q &= \left(\exp_q(|z|^2)\right)^{-\frac{1}{2}} \exp_q(z a^q)|0\rangle_q \\ &= \left(\exp_q\left(|z|^2\right)\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_q!}} |n\rangle_q. \end{aligned} \tag{5.3}$$

These states satisfy:

$$\bar{a}_q |z\rangle_q = z|z\rangle_q, \tag{5.4}$$

showing that the q-coherent state $|z\rangle_q$ is an eigenstate of the annihilation operator $\bar{a}^q$ with eigenvalue z and, since $z = {}_q\langle z|\bar{a}^q|z\rangle_q$ (assuming the states $|z\rangle_q$ are normalized), the label z is the *mean* of $\bar{a}^q$ in the state $|z\rangle_q$. The definition (5.3) is not a unique q-extension of (5.2), for we could have chosen any one of the family e_q^λ of exponential functions in [13]; this would introduce explicit q-factors in equations such as (5.4). We outline below how the particular q-harmonic oscillator model of Section 4 (above) leads naturally to these q-coherent states. (The states (5.3) were first considered in Ref. [8] and subsequently also in Refs. [14-17]. In fact, as with many q-analogs of classical and quantum concepts, some q-generalizations were obtained before the appearance of quantum groups [18]).

Let us now consider the two characteristic properties of coherent states, continuity and completeness. (a) The continuity properties of $|z\rangle_q$, as a function of z, follow immediately from the continuity of the deformed exponential function, $\exp_q$ in (5.3).

(b) The resolution of unity within the Hilbert space, in terms of the states $|z\rangle_q$, has been considered by Gray and Nelson [15] and also Bracken et al [17]. The q-analog of Euler's formula for $\Gamma(x)$ is required, and is expressed in terms of the q-integration defined in [13]:

$$\int_0^{\zeta} \exp_q(-x)x^n d_q x = [n]_q! \tag{5.5}$$

where ζ is the largest zero of $\exp_q(x)$ (note that, unlike e^x, $\exp_q(x)$ alternates in sign as $x \to -\infty$). A natural restriction is $|z|^2 < \zeta$ and then, with the help of (3.5), the resolution of unity can be derived [17],

$$1 = \int |z\rangle_q \, {}_q\langle z| \, d\mu(z) \tag{5.6}$$

where the measure $d\mu(z)$ is given by

$$d\mu(z) = \tfrac{1}{2\pi} \exp_q(|z|^2) \exp_q(-|z|^2) d_q |z|^2 d\theta, \tag{5.7}$$

where $\theta = \arg(z)$. It follows from (5.6) that an arbitrary state can be expanded in terms of the states $|z\rangle_q$. (In fact, q-coherent states are overcomplete, for an arbitrary q-coherent state is non-orthogonal to $|z\rangle_q$, for any z.)

Coherent states arise naturally within the framework of the harmonic oscillator of Section 4, by defining boson operators from position and momentum operators, Q, P, putting dimensional factors to unity:

$$a^q = \tfrac{1}{\sqrt{2}}(Q - iP), \quad \overline{a}^q = \tfrac{1}{\sqrt{2}}(Q + iP). \tag{5.8}$$

Conversely, we can use these formulas to define momentum and position operators and so, given q-boson operators, these formulas also provide convenient q-analog definitions of q-momentum and position operators [8].

Alternatively, one can define a q-harmonic oscillator by starting with Schrödinger's equation and replacing the derivative by a finite difference operator which provides an alternative form for the deformation. We use the following q-derivative,

$$\nabla_q f(x) = \frac{f(xq) - f(x)}{x(q-1)}, \tag{5.9}$$

and the q-harmonic oscillator states are now determined by the equation

$$\tfrac{1}{2}(-\nabla_q^2 + qx^2)\psi(x) = E\psi(x). \tag{5.10}$$

Effectively, we have chosen q-momentum and q-position operators Q_q, P_q satisfying

$$qQ_qP_q - P_qQ_q = i, \tag{5.11}$$

with the realization $Q_q = x, \;\; P_q = i\nabla_q$. (This is yet another realization different from (3.9) for the deformation.)

Solutions of the difference equation (5.10) have been given by several authors [19,20], and involve q-extensions of the Hermite polynomials. The ground state ψ_0 is given by

$$\psi_0(x) = \sum_{n=0}^{\infty} \frac{(-)^n q^{-\frac{n^2}{2}} x^{2n}}{[2n]_q!!}, \tag{5.12}$$

where $[2n]_q!! = [2n]_q[2n-2]_q \dots [2]_q$. Upon using the identity $[2n]_q = [2]_q[n]_{q^2}$ we can identify the function (5.12) as one of the family of q-exponential functions given by Exton [13].

The eigenstates ψ_n of the deformed Schrödinger equation (5.10) are labelled by an integer n, and the energy levels are $E_n = \frac{1}{2}[2n+1]_q$. (For comparison, note that in the model defined in Section 4, the energy levels are different: $E_n = \frac{1}{2}([n+1]_q + [n]_q) = \frac{1}{2}[2n+1]_{q^{1/2}}$). The eigenstates of (5.10), ψ_n, take the form

$$\psi_n(x) = H_n^q(x)\psi_0\left(xq^{-\frac{n}{2}}\right), \tag{5.13}$$

where ψ_0 is given by (5.12) and $H_n^q(x)$ denotes a q-extension of the classical Hermite polynomial, with the explicit formula:

$$H_n^q(x) = \sum_{r=0}^{n} \frac{C_r x^r q^{-\frac{(2n+1)r}{4}}}{[r]_q!}, \tag{5.14}$$

where the coefficients C_r are given (for even or odd r) by

$$C_{2m} = (-)^m q^{(2n+1)m/2}[2n]_q[2n-4]_q \dots [2n-4m+4]_q \tag{5.15a}$$

$$C_{2m+1} = (-)^m q^{(2n+1)m/2}[2n-2]_q[2n-6]_q \dots [2n-4m+2]_q. \tag{5.15b}$$

From the explicit eigenstates one can identify q-boson operators which step between the eigenstates $\psi_n(x)$, from which one can form the q-coherent states of this model of the q-harmonic oscillator [20].

6 The q-Symplecton

The idea behind the symplecton construction has a close relationship to harmonic oscillators. In the Jordan-Schwinger realization of angular momentum one obtains uniformly all unitary irreps in terms of *two* independent harmonic oscillators. This naturally suggests the question: can one do better and realize all irreps *uniformly* in terms of *one* harmonic oscillator? The answer is (of course) yes—this is the symplecton realization [7,21], which uses the creation operator (a) as the spin-$\frac{1}{2}$ "up" state and the destruction operator $(\bar{a})$ as the "down" state. This implies that there is no longer a vacuum ket $|0\rangle$ annihilated by $\bar{a}$. Instead we define a *formal ket* $|\rangle$ *and seek to interpret both* $a|\rangle$ *and* $\bar{a}|\rangle$ *as non-vanishing vectors.*

Operators in this symplecton calculus will be defined as polynomials over $(a, \bar{a})$ with complex numbers as scalars. State vectors will be defined as operators multiplied on the right by the basic formal ket, i.e.,

$$|\nu\rangle \equiv \mathcal{O}_\nu|\rangle, \tag{6.1}$$

where $|\nu\rangle$ is a vector and $\mathcal{O}_\nu$ the operator creating this vector. The action of the generators on state vectors will be defined as commutation on the relevant operator $\mathcal{O}_\nu$, that is,

$$J_i(|\nu\rangle) \equiv [J_i, \mathcal{O}_\nu]|\rangle. \tag{6.2}$$

To be completely explicit we are considering (for the undeformed symplecton) a single boson operator a and its conjugate $\bar{a}$ obeying:

$$[\bar{a}, a] = 1, \tag{6.3}$$

all other commutators zero. The generators of $SU(2)$ are realized by:

$$J_+ \rightarrow -\tfrac{1}{2}a^2, \text{ (note the sign!) } \quad J_- \rightarrow \tfrac{1}{2}\bar{a}^2, \quad J_0 \rightarrow \tfrac{1}{4}(a\bar{a} + \bar{a}a). \tag{6.4}$$

It is easily verified that this realization obeys the desired commutation relations:

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0. \tag{6.5}$$

Note that the action of these generators on symplecton state vectors, verifying the commutation relations, succeeds precisely because of the Jacobi identity. Using commutation under the generators, the labels J and M can be assigned to define characteristic polynomials $\mathcal{P}_J^M$. The angular momentum irrep eigenvectors are then given by the set of vectors $\mathcal{P}_J^M|\rangle$.

The *adjoint polynomial* $(\mathcal{P}_J^M)^{\text{adj}}$ is defined by:

$$(\mathcal{P})^{\text{adj}} = (-1)^{J-M}\mathcal{P}_j^{-M}, \tag{6.6}$$

with $\bar{a}$ taken to be adjoint to a. The adjoint (dual space) vector to $\mathcal{P}_j^M|\rangle$ is defined as $\langle|(\mathcal{P}_j^M)^{\text{adj}}$.

The crucial problem in this (undeformed) symplecton construction is the proper definition of an inner product for the Hilbert space of the irreps. Omitting details [7], the answer is obtained from the *multiplication law for symplecton eigen-polynomials.*

THEOREM [21]: *Let $\mathcal{P}_a^\alpha$ and $\mathcal{P}_b^\beta$ be normalized eigen-polynomials of the generators J_i. Then these polynomials obey the product law:*

$$\mathcal{P}_a^\alpha \mathcal{P}_b^\beta = \sum_{c=|a-b|}^{a+b} \langle c|a|b\rangle \; C_{\beta\alpha}^{bac} \; \mathcal{P}_c^{\alpha+\beta}, \tag{6.7a}$$

where

$$\langle c|a|b\rangle = (2c+1)^{-\frac{1}{2}} \cdot \Delta(abc), \tag{6.7b}$$

$$\Delta(abc) \equiv \left[\frac{(a+b+c+1)!}{(a+b-c)!(a-b+c)!(-a+b+c)!}\right]^{\frac{1}{2}}, \tag{6.7c}$$

and $C_{\beta\alpha}^{bac}$ is the usual Wigner-Clebsch-Gordan coefficient for $SU(2)$.

Using this theorem it is now easy to understand the inner product $\langle\mu|\nu\rangle$: one applies the product law to the polynomials $\mathcal{O}_{\mu}^{\text{adj}}$ and $\mathcal{O}_{\nu}$ and then *projects onto* the $J = 0$ part. The Wigner-Clebsch-Gordan coefficient (for $J = 0$) quite literally defines here a metric!

Remark: It is clear also that one can extend this structure by adjoining additional symplectons. That is, one considers a symplecton having n "internal" states: $a_1, a_2, \ldots, a_n$ and their conjugates $\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n$. Just as the adjunction of a boson with n "internal" states suffices to realize $SU(n)$, so does an n state symplecton suffice to realize the structure $Sp(2n)$.

An important consequence of the symplecton construction is the definition of a new invariant angular momentum function: *the triangle coefficient* $\Delta(abc)$, eq. (6.7c). This triangle function, $\Delta(abc)$, has gratifyingly simple properties. It is a function defined symmetrically on three "lengths" or "sides" a, b, c, which (from the properties of the factorial function) vanishes unless the triangle conditions (that the sum of any two sides equals or exceeds the third side) are fulfilled. The symplecton realization of angular momentum yields the triangle rule of vector addition in a particularly graphic way.

The triangle function is clearly a rotationally *invariant* function defined on three angular momenta; as such, it fits very nicely into the series of invariant functions defined on $3n$ angular momenta: $(6j)$ [Racah coefficient] and $(9j)$ [Fano coefficient]. The Wigner coefficients are often called "$(3j)$ symbols", but in view of the fact—emphasized by Wigner—that these coefficients are coordinate frame dependent (i.e., involve magnetic quantum numbers) one might consider the triangle function as the more appropriate to designate as the $(3j)$ symbol.

The triangle function obeys the following transformation law, Ref. [21]:

$$\Delta(acf)\Delta(bdf) = (2f+1)\sum \Delta(abe)\Delta(cde)W(abcd; ef). \tag{6.8}$$

It is quite remarkable that the Racah function appears here as a *tetrahedral* function coupling four *triangles* by pairs.

Having reviewed now the symplecton construction it is time to return to our main theme: can one define a deformed symplecton ("q-symplecton") using a single deformed harmonic oscillator? The answer (of course) is *yes*, but there are some surprises [22]. We will develop the deformed structure using finite q-transformations, which provides further insights into the deformation process [23]. (The infinitesimal approach—which obtains the q-generators $\{J_i^q\}$ using a single q-boson, the q-boson analogs to eqs. (6.4)—was developed earlier in ref. [24].)

Let a_q and $\overline{a}_q$ be q-boson creation and annihilation operators obeying:

$$\overline{a}_q a_q - q^{\frac{1}{2}} a_q \overline{a}_q = 1. \tag{6.9}$$

This q-commutation relation is invariant under the transformation of q-spaces [23]:

$$(a, \overline{a})' = (a, \overline{a}) \begin{pmatrix} x & u \\ v & y \end{pmatrix}, \tag{6.10}$$

where:

$$ux = q^{\frac{1}{2}} xu, \quad vx = q^{\frac{1}{2}} xv, \quad yv = q^{\frac{1}{2}} vy, \tag{6.11a, b, c}$$

$$yu = q^{\frac{1}{2}} uy, \quad uv = vu, \tag{6.11d, e}$$

$$xy - q^{-\frac{1}{2}} vu = yx - q^{\frac{1}{2}} vu = 1. \tag{6.11f}$$

The *adjoint* to $(a, \overline{a})$ is: $\left(q^{\frac{1}{4}}\overline{a}, -q^{-\frac{1}{4}}a\right)$ and obeys:

$$\left(q^{\frac{1}{4}}\overline{a}, -q^{-\frac{1}{4}}a\right)' = \left(q^{\frac{1}{4}}\overline{a}, -q^{-\frac{1}{4}}a\right)\begin{pmatrix} x^* & u^* \\ v^* & y^* \end{pmatrix}, \tag{6.12}$$

with:

$$x^* = y, \quad u^* = -q^{-\frac{1}{2}}v, \quad v^* = -q^{\frac{1}{2}}u, \quad y^* = x. \tag{6.13a, b, c, d}$$

Let us denote the q-symplecton eigenpolynomials by: $\mathcal{Q}_j^m$. Then $\mathcal{Q}_j^m$ is a polynomial of order $j+m$ in a and $j-m$ in $\overline{a}$ and defined to transform as:

$$\mathcal{Q}_j^m(a', \overline{a}') = \sum_n d^j_{nm}(x,u,v,y)\mathcal{Q}_j^n(a, \overline{a}). \tag{6.14}$$

Here $d^j_{n,m}(x,u,v,y)$ is the *q-rotation matrix* which obeys:

$$d^{j'}_{m'k'}(x,u,v,y) \;\; d^{j''}_{m''k''}(x,u,v,y) = \sum_j {}_qC^{j'j''j}_{m'm''m} \times {}_qC^{j'j''j}_{k'k''k} \;\; d^j_{mk}(x,u,v,y), \tag{6.15}$$

where ${}_qC^{\cdots}_{\cdots}$ are q-WCG coefficients. It follows that the set $\{\mathcal{Q}_j^m,\ m=-j,-j+1,\ldots,j\}$ is an irreducible tensor of rank j. Moreover $\mathcal{Q}_j^m$ is a q-symmetric function:

$$\begin{aligned}\left(\begin{bmatrix} 2j \\ j+m \end{bmatrix}\right)^{\frac{1}{2}} \mathcal{Q}_j^m = {} & q^{-\frac{(j+m)(j-m)}{4}} a^{j+m}\overline{a}^{j-m} \\ & + q^{-\frac{(j+m)(j-m)}{4}+\frac{1}{2}} a^{j+m-1}\overline{a}a\overline{a}^{j-m-1} + \ldots \\ & + q^{-\frac{(j+m)(j-m)}{4}+\frac{\ell}{2}} f(a,\overline{a}) + \ldots + q^{\frac{(j-m)(j+m)}{4}} \overline{a}^{j-m}a^{j+m}. \end{aligned} \tag{6.16}$$

Here ℓ is the least number of transpositions needed to put $f(a,\overline{a})$ in normal-ordered form.

$$\text{Example:} \quad [4]^{\frac{1}{2}}\mathcal{Q}_2^1 = q^{-\frac{3}{4}}a^3\overline{a} + q^{-\frac{1}{4}}a^2\overline{a}a + q^{\frac{1}{4}}a\overline{a}a^2 + q^{\frac{3}{4}}\overline{a}a^3. \tag{6.17}$$

As is clear from our review (of the usual symplectons), the major task is to prove a product law for the deformed q-eigenpolynomials, $\mathcal{Q}_j^m$.

THEOREM [23]: *Let $\mathcal{Q}_{j'}^{m'}$ and $\mathcal{Q}_{j''}^{m''}$ be normalized q-eigenpolynomials. Then:*

$$\mathcal{Q}_{j'}^{m'}(a,\overline{a})\mathcal{Q}_{j''}^{m''}(a,\overline{a}) = \sum_j N(j'j''j) \cdot {}_qC^{j'j''j}_{m'm''m} \cdot \mathcal{Q}_j^m(a,\overline{a}), \tag{6.18}$$

where: ${}_qC^{j'j''j}_{m'm''m}$ is the q-Wigner-Clebsch-Gordan coefficient, and $N(j'j''j)$ is a scalar function of q dependent only on j', j'', j.

$N(j'j''j)$ obeys the recursion relation:

$$
\begin{aligned}
([2j''][2j+1])^{\frac{1}{2}} N(j'j''j) = {} & ([j'-j''+j+1]_q[j'+j''-j]_q)^{\frac{1}{2}} \\
& \times N\left(j', j''-\tfrac{1}{2}, j+\tfrac{1}{2}\right) N\left(j+\tfrac{1}{2}, \tfrac{1}{2}, j\right) \\
& + ([j'+j''+j+1]_q[-j'+j''+j]_q)^{\frac{1}{2}} \\
& \times N\left(j', j''-\tfrac{1}{2}, j-\tfrac{1}{2}\right).
\end{aligned} \tag{6.19}
$$

The determination of the coefficient $N(j'j''j)$ is very difficult. It helps to see a few special cases. We find:

$$N(j\ 0\ j) = 1, \tag{6.20}$$

$$N(j_1, j_2, j_1+j_2) = 1 \tag{6.21}$$

$$N\left(j, \tfrac{1}{2}, j-\tfrac{1}{2}\right) = \frac{-q^{-\frac{1}{4}}F(2j)}{([2j][2j+1])^{\frac{1}{2}}}, \tag{6.22}$$

$$\text{with:}\quad F(n) \equiv [1]+[2]+\ldots+[n], \quad F(0) \equiv 0. \tag{6.23}$$

We remark that the appearance of the function $F(n)$ is characteristic of relations involving the q-symplecton [23].

One can prove the further property, at this stage, that the function $N(j', j'', j)$ is symmetric in the first two indices. One of the surprising properties [23] is that the (q-rotationally invariant) function $N(j', j'', j)$ is ***not*** symmetric under $q \to q^{-1}$.

These results show that $N(j', j'', j)$ is not the proper q-analog to the triangle function $\Delta(a, b, c)$, despite the fact that the q-symplecton product law seemingly appears to define $N(j', j'', j)$ in the proper form. It has been shown in Ref. [22], that the proper way to proceed is via the definition:

$$\Delta_q(abc) \equiv (-1)^{a+b+c} N(abc) q^{\frac{(a+b-c)}{4}} \sqrt{\frac{F(2c)![2a+1]![2b+1]!}{F(2a)!F(2b)![2c]!}}. \tag{6.24}$$

This q-triangle coefficient has the desired symmetry. As shown in Ref. [22], $\Delta_q(j_1j_2j_3)$ *is totally symmetric in its arguments* j_1, j_2, j_3—precisely the same property possessed by the (undeformed) triangle coefficient $\Delta(j_1j_2j_3)$ in (6.7c).

Moreover, it is now possible [22] to obtain the proper q-analog of (6.8):

$$\Delta_q(acf)\Delta(bdf) = [2f+1] \sum_e \Delta_q(abe)\Delta_q(cde)W_q(abcd; ef). \tag{6.25}$$

Let us conclude by citing the product law for q-eigenpolynomials in the proper form now to show the desired q-analog structure [22]:

$$\mathcal{Q}_a^\alpha \mathcal{Q}_b^\beta = \sum_{c=|a-b|}^{a+b} [2c+1]^{-\frac{1}{2}} \Delta_q(abc)(ba\beta\alpha|c\alpha+\beta)_{\frac{1}{q}} \mathcal{Q}_c^{\alpha+\beta}. \tag{6.26}$$

Note the surprising appearance of the q-WCG coefficient involving q^{-1} as the proper form to show the analogy.

Space is lacking for more than this brief survey of the q-symplecton and the associated subtleties of q-analysis. More detail can be found in [22], and related discussions—from the aspect of Weyl-ordered boson polynomials—is given in [25] and [26].

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