N93-27322

PHASE OF THE QUANTUM HARMONIC OSCILLATOR WITH APPLICATIONS TO OPTICAL POLARIZATION

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Abstract

The phase of the quantum harmonic oscillator, the temporal distribution of a particle in a square-well potential, and a quantum theory of angles are derived from a general theory of complementarity. Schwinger's harmonic oscillator model of angular momenta [1] is modified for the case of photons. Angular distributions for systems of identical and distinguishable particles are discussed. Unitary and antiunitary time reversal operators are then presented and applied to optical polarization states in birefringent media.

1 General Theory of Complementarity

The fact that linear momentum is the generator of translations in space, leads to the Fourier transform relations between the momentum and spatial representations of Schrodinger's wave-mechanics [2]. Similarly, since energy generates translations in time, there are Fourier transform relations between the energy and temporal representations [3]. For the case of the harmonic oscillator, the energy eigenspectrum is proportional to the integers $n = 0, 1, 2...$ (recall $\hat{H} = \hbar\omega(\hat{n} + 1/2)$, where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the photon number operator) and this spectrum is aperiodic (i.e. not periodic). Therefore the temporal distribution of the oscillator will be continuous and periodic. Indeed, the simplest way (that I have found) to describe the phase ($\Phi = \omega t$) of the quantum harmonic oscillator is to form the wavefunction

$$\psi(\Phi) = \sum_{n=0}^{\infty} \psi_n e^{-in\Phi} \tag{1}$$

which is the Fourier series of the n-space wavefunction (or number-ket expansion coefficients) $\psi_n \equiv \langle n|\psi\rangle$, where $\hat{n}|n\rangle = n|n\rangle$. The probability density for finding Φ on any 2π interval (the period of $\psi(\Phi)$) is then simply $|\psi(\Phi)|^2/2\pi$. The wavefunction approach circumvents complications associated with the equally correct perspective [4] that this phase distribution corresponds to the realizable measurement of the Susskind-Glogower (SG) [5] phase operator.

Suppose we wish to study the temporal behavior of a particle in a one dimensional box (the "phase of the infinite square well"). We do not have to start all over, we can simply take the Fourier (series) transform of the discrete energy wavefunction, which underlies the discrete energy eigenspecta:

$$E_i = \frac{\hbar^2\pi^2}{2mL^2}(i)^2 \quad (i = 1, 2, 3...) \tag{2}$$

where L is the length of the box and m the mass of the particle. In other words, labeling the energy eigenstates, $\{|E_n\rangle\}$, according to the value of $n \equiv (i)^2$, we'd use the $\psi_n \equiv \langle E_n|\psi\rangle$ as the Fourier series coefficients in $\psi(\Phi) = \sum_n \psi_n e^{-in\Phi}$, where $\Phi = t(\hbar\pi^2/2mL^2)$. The temporal distribution is therefore like that of a harmonic oscillator for which $\psi_2 \equiv 0 \equiv \psi_3$, $\psi_5 \equiv 0 \equiv \psi_6 \equiv \psi_7 \equiv \psi_8$, etc. For a well of finite depth, the bound state eigenenergies will be perturbed from being proportional to the squares of integers, but they will still be discrete and we would still sum over the $\langle E_i|\psi\rangle$ with each one weighted by $e^{-iE_it/\hbar}$, to form $\psi(t)$ which is quasi-periodic (it can't be *exactly* periodic since the E_i are no longer integer multiples of each other, however the difference between $\psi(t)$ and $\psi(t+T)$ can be made as small as we wish by making T large enough — hence the term "quasi-periodic"). The unbound states for this problem, however, have a continuous distribution in energy and for these we would form the aperiodic

$$\psi(t) = \int dE\, \psi(E)\, e^{-iEt/\hbar} \tag{3}$$

where $\psi(E) \equiv \langle E|\psi\rangle$. Notice that the unbound states exhibit an aperiodic temporal distribution, i.e. they can be "here today and gone tomorrow" as they zip past the potential well, whereas the bound states are trapped into quasi-periodic oscillations.

From the general theory of complementarity we can also obtain a quantum theory of angles. The z component of angular momentum, $\hat{J}_z$, is (by definition) the generator of translations in the angle about the z axis, which shall be denoted as ϕ. It is well known that $\hat{J}_z$ has discrete eigenvalues given by $m\hbar$ where $m \in \{-j, -j+1, ...j-1, j\}$ and j is the label of the discrete eigenvalues of the simultaneously measurable $\hat{J}^2 \equiv \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$ which are $j(j+1)\hbar^2$. For states in which each value of m is uniquely represented (the degenerate case will be discussed in the next section), such as a particle of spin s (i.e. $j = s =$ a fixed number), we can form the angle representation

$$\psi(\phi) = \sum_m \psi_m e^{-im\phi} \tag{4}$$

where $\psi_m \equiv \langle j, m|\psi\rangle$ and the angular distribution is $p(\phi) = |\psi(\phi)|^2/2\pi$. Since $\psi(\Phi)$ is periodic its transform ψ_m must be discrete, i.e. the quantization of angular momentum (projected onto an axis) is a simple and immediate consequence of the periodicity of the angle (about that axis).

2 Harmonic Oscillator Models of Angular Momenta

In 1952, Schwinger [1] demonstrated a connection between the algebra of two uncoupled harmonic oscillators and the algebra of angular momenta. The key points of Schwinger's model are as follows:

$$\hat{J}_- \equiv \hbar\hat{a}_d^\dagger\hat{a}_u \;\; and \;\; \hat{J}_z \equiv \frac{\hbar}{2}(\hat{n}_u - \hat{n}_d), \tag{5}$$

where $\hat{a}_u$ and $\hat{a}_d$ denote the annihilation operators for the "up type" and "down type" oscillators. From this we obtain the fundamental commutation relations of angular momentum:

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z \;\; and \;\; [\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm, \tag{6}$$

where $\hat{J}_+ \equiv (\hat{J}_-)^\dagger$ and $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$, so $[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$ etc. Since the quanta of these oscillators behave like spin 1/2 objects (as seen from eq.(5)), yet only totaly symmetrical states are constructed by this method, these quanta are not believed to correspond to actual particles and the connection is merely within the mathematics [2].

We put some physics into this connection by considering a rotation of a single frequency electromagnetic wave about the z axis (along which the $\bar{k}$ vector lies) which leads to the well known result that a right handed circularly polarized photon is an eigenstate of $\hat{J}_z/\hbar$ with eigenvalue $m = +1$. Similarly, a left handed photon is associated with $m = -1$ and since we need only consider transverse components of the vector potential, the photon is said to be a particle of "spin 1 with m=0 missing" [6]. Since the photon is a boson which resembles a spin 1/2 object in the sense that its spin space is two dimensional, it seems reasonable to attempt to reconstruct the algebra of angular momenta from these physically significant photonic primitives. Indeed, taking

$$\hat{J}_- \equiv 2\hbar \hat{a}_l^\dagger \hat{a}_r \quad and \quad \hat{J}_z \equiv \hbar(\hat{n}_r - \hat{n}_l), \tag{7}$$

where $\hat{a}_r$ and $\hat{a}_l$ are the annihilation operators for the right and left circularly polarized modes of a single frequency, z propagating, electromagnetic wave, we obtain

$$[\hat{J}_+, \hat{J}_-] = 4\hbar \hat{J}_z \quad and \quad [\hat{J}_z, \hat{J}_\pm] = \pm 2\hbar \hat{J}_\pm. \tag{8}$$

where as before $\hat{J}_+ = (\hat{J}_+)^\dagger$ and $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$, so $[\hat{J}_x, \hat{J}_y] = 2i\hbar\hat{J}_z$ etc. This is the same group, however $\hat{J}_-$ now lowers m by 2 (rather than by 1) which is exactly what we want for photons. Notice that a differential phase shift between these circularly polarized modes (or between the up and down oscillators for the case of fermions and ordinary, i.e. "non-photonic", bosons) is equivalent to a rotation about the z axis:

$$\hat{R}_z(\phi)|\psi\rangle = \sum_{n_r,n_l} \psi_{n_r,n_l} e^{-i(n_r-n_l)\phi}|n_r, n_l\rangle. \tag{9}$$

We can relable our two-mode number states according to the values of $j \equiv n_r + n_l$ (or $j = (n_u + n_d)/2$) and $m \equiv n_r - n_l$ (or $m = (n_u - n_d)/2$). To obtain the angle representation for the case of identical particles (e.g. all these states are photons, or they are all electrons, etc.) we should allow for quantum interference of all these states (i.e. we should add amplitudes rather than probabilities) and therefore simply use

$$\psi_m \equiv \sum_j \langle j, m|\psi\rangle \tag{10}$$

in eq.(4) for these cases. Since the ψ_m defined in eq.(10) are no longer normalized (for m states degenerate in j) we must renormalize:

$$\psi(\phi) \rightarrow \psi(\phi)/\sqrt{c} \quad where \quad c \equiv \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}|\psi(\phi)|^2. \tag{11}$$

For bosons or photons the minimal non-zero value of $|m|$ is one therefore the period of $\psi(\phi)$ is at most 2π, and since $\Delta m_{min} = 1$ the period of $p(\phi)$ is at most 2π. For fermions m can be 1/2 so the period of $\psi(\phi)$ can be 4π. This indicates the rotational Berry's phase "for fermions"[7], which we now see to be more correctly stated as being "for fermions which have non-zero overlap with $m = \pm 1/2$ states." Since Δm_{min} for fermions is still one, $p(\phi)$ is still mod 2π, indicating that observation of the "mod 4π Berry's phase" requires interference of this state with another one, as is well known.

Notice that if we allowed for **particles** comprised of mixtures of integer and half-integer spin we could have $\Delta m_{min} = 1/2$ so that $p(\phi)$ (not just $\psi(\phi)$) would be periodic mod 4π! Since no interference with another state is required, the existence of proposed particles of this type would radically alter our conceptualization of space (each point like a Mobius strip?). Alternatively, we might argue that it is physicaly reasonable to require that $p(\phi)$ have at most a period of 2π and therefore we would have a theoretical explanation of the "... empirical fact that a mixed symmetry does not occur" [2]. If however, we had a **system** comprised of a fermion and a boson (e.g. an electron and a photon) then since these distinguishable particles do not interfere, the angular distribution should be (at most) mod 2π.

For the case of distinguishable particles we should add probabilities (rather than amplitudes), i.e. rather than the procedure defined by eq.s (4), (10), and (11), we should do the following. For each distinguishable particle we should form an angular wavefunction, *then* square its magnitude and divide by 2π to form each different particle's individual angular distribution, *then* add these individual distributions to form the angular distribution of the entire system. When these distinguishable particles have distinct values of spin (such as a system comprised of a spin 1/2 electron and a spin 1 photon for example) this proceedure is as follows. For each fixed value of $j = s$ we form

$$\psi^{(j)}(\phi) \equiv \sum_m \psi_{j,m} e^{-im\phi} \quad and \quad p^{(j)}(\phi) \equiv |\psi^{(j)}(\phi)|^2/2\pi \tag{12}$$

from which we obtain the system's angular distribution: $p(\phi) = \sum_j p^{(j)}(\phi)$. This procedure corresponds to the measurement of $\hat{Z} \equiv (\hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z)^{-1/2} \hat{J}_-$ where the leading term obviates the $\sqrt{(j+m)(j-m+1)}$ factor from $\hat{J}_-$ so that the lowering of $\hat{Z}$ is "pure": $\hat{Z}|j,m\rangle = |j,m-1\rangle$.

3 Unitary and Antiunitary Time Reversal

Although in the literature to date [2] it has been argued that a time reversal operator must be antiunitary (so that kinetic energy, for example, remains non-negative) it is more appropriate for our purposes to define a *unitary* time reversal operator since we are mainly interested in *relative* (rather than absolute) time and relative time (e.g differential phase) is complementary to the *relative energy* (e.g. photon number difference) which can be negative.

For a quantum mechanical operation to conserve probability the corresponding operator must either be unitary or antiunitary [8] (or some combination thereof). In either case it is reasonable to require that a time reversal operator, $\hat{T}$, should satisfy [2]

$$\hat{U}(t)\hat{T} = \hat{T}\hat{U}(-t), \tag{13}$$

where $\hat{U}(t)$ denotes (unitary) time evolution of an amount t. Equivalently, we could require $\hat{U}(t)\hat{T}\hat{U}(t)\hat{T} = \hat{I}$ and we are neglecting (as we did in eq.(13)) any overall phase which might be aquired in getting back to the "same" state.

Any antiunitary operator can be expressed as a product of a "complex conjugator" (of c-numbers) and a unitary operator [8]. Thus the unitary time reversal operator, $\hat{T}_u$, is simply the unitary part of the antiunitary time reversal operator

$$\hat{T}_a = \hat{C}\hat{T}_u, \tag{14}$$

where $\hat{C}$ denotes complex conjugation. It has previously been demonstrated that the auxiliary (a) mode (associated with the realizable measurement of the SG operator) must be "time reversed" with respect to the original system (s) mode [9]. Therefore, $\hat{T}_u$, should permute these modes so that $\hat{T}_a$ acting on a two-mode state, $|\psi\rangle \equiv \sum_{n_s}\sum_{n_a}\psi_{n_s,n_a}|n_s\rangle_s|n_a\rangle_a$, yields

$$\hat{T}_a|\psi\rangle = \sum_{n_s}\sum_{n_a}\psi^*_{n_s,n_a}|n_a\rangle_s|n_s\rangle_a = \sum_{n_s}\sum_{n_a}\psi^*_{n_a,n_s}|n_s\rangle_s|n_a\rangle_a. \tag{15}$$

Subsequent time evolution (i.e. *absolute* phase shift) of this state results in

$$\hat{U}(t)\hat{T}_a|\psi\rangle = \sum_{n_s}\sum_{n_a}\psi^*_{n_a,n_s}e^{-i(n_s+n_a+1)\omega t}|n_s\rangle_s|n_a\rangle_a, \tag{16}$$

where $\hat{U}(t) = e^{-i(\hat{n}_s+\hat{n}_a+1)\omega t}$. If instead, we first propagated the initial state $|\psi\rangle$ "backwards" in time, and then time reversed we'd obtain $\hat{T}_a\hat{U}(-t)|\psi\rangle =$

$$\hat{T}_a\sum_{n_s}\sum_{n_a}\psi_{n_s,n_a}e^{+i(n_s+n_a+1)\omega t}|n_s\rangle_s|n_a\rangle_a = \sum_{n_s}\sum_{n_a}\psi^*_{n_a,n_s}e^{-i(n_s+n_a+1)\omega t}|n_s\rangle_s|n_a\rangle_a, \tag{17}$$

which is the same state as in eq.(16) and therefore the requirement of eq.(13) is satisfied.

For unitary time reversal, we simply omit the complex conjugation of the expansion coefficients and we find that in order to satisfy eq.(13) we must consider a *differential* (rather than absolute) phase shift $\hat{U}_d(t) \equiv e^{-i(\hat{n}_s-\hat{n}_a)\omega t}$. Explicitly, we have

$$\hat{U}_d(t)\hat{T}_u|\psi\rangle = \hat{U}_d(t)\sum_{n_s}\sum_{n_a}\psi_{n_a,n_s}|n_s\rangle_s|n_a\rangle_a = \sum_{n_s}\sum_{n_a}\psi_{n_a,n_s}e^{-i(n_s-n_a)\omega t}|n_s\rangle_s|n_a\rangle_a, \tag{18}$$

which is equivalent to $\hat{T}_u\hat{U}_d(-t)|\psi\rangle =$

$$\hat{T}_u\sum_{n_s}\sum_{n_a}\psi_{n_s,n_a}e^{+i(n_s-n_a)\omega t}|n_s\rangle_s|n_a\rangle_a = \sum_{n_s}\sum_{n_a}\psi_{n_a,n_s}e^{-i(n_s-n_a)\omega t}|n_s\rangle_s|n_a\rangle_a. \tag{19}$$

Thus the "time" to be associated with unitary time reversal is the difference time, translations in which are generated by the energy difference $\hbar\omega(\hat{n}_s - \hat{n}_a)$.

We have already demonstrated that the differential phase between the two oscillators of our angular momenta model is equivalent to the angle ϕ. Therefore $\hat{T}_u$ corresponds to angle inversion ($\phi \to -\phi$) when we take the s and a modes to be the right and left circularly polarized electromagnetic modes (or the up and down oscillators), i.e. under $\hat{T}_u$ we have:

$$\psi_{n_r,n_l} \to \psi_{n_l,n_r} \quad or \quad \psi_{j,m} \to \psi_{j,-m} \quad so \quad \psi(\phi) \to \psi(-\phi) \tag{20}$$

(in the antiunitary case, we'd have $\psi(\phi) \to \psi^*(\phi)$ under $\hat{T}_a$). A $\hat{T}_u$ eigenstate ($\psi(\phi) = \psi(-\phi)$) will therefore have an angular distribution symmetrically centered about $\phi = 0$, so that any vector associated with this state can only be along the x axis. Indeed, from $\psi_{n_r,n_l} = \psi_{n_l,n_r}$ we can show $\langle(\hat{a}_r)^p\rangle = \langle(\hat{a}_l)^p\rangle \;\; \forall p \in \{0,1,2,...\}$ and from the $p = 1$ result we have $\langle\hat{E}_y\rangle = 0$. The $\hat{T}_u$ eigenstates (in the cicularly polarized basis) therefore correspond to polarization which is *linear* in terms of the polarization "signal" (i.e. the $\langle\hat{E}\rangle$) so that they resemble (and include) the case of putting one linear polarized mode in the vacuum state, but they can achieve this with a reduction in polarization "noise" (e.g. $\Delta^2 E_x$ or $\Delta^2 E_y$). These states therefore provide a foundation for the study of quantum limits on the performance of devices which utilize circularly birefringent media (e.g. Faraday rotators, optical isolators, etc.).

As a simple example, compare these two $\hat{T}_u$ eigenstates: one an x polarized coherent state (with the y polarization unexcited), $|\alpha\rangle_r|\alpha\rangle_l = |\sqrt{2}\alpha\rangle_x|0\rangle_y$; and the other $(|\alpha\rangle_r|0\rangle_l + |0\rangle_r|\alpha\rangle_l)/\sqrt{2} + (1-\sqrt{2})e^{-|\alpha|^2/2}|0\rangle_r|0\rangle_l$, which I'll refer to as the pseudo-coherent state. Both states yield similar polarization "signals" $\langle\hat{E}_x\rangle \simeq -2\alpha\sin(\omega t)$ *and* $\langle\hat{E}_y\rangle = 0$, yet, the polarization "noise" of the pseudo-coherent state ($\Delta^2 E_x = 1/2$) is 3 dB below the shot noise limit of the coherent state ($\Delta^2 E_x = 1$), where we assume $|\alpha|^2 >> 1$ (else the pseudo-coherent and coherent states both tend towards the vacuum).

We can also use the phase representation to describe the measurement of the differential phase shift of two linearly polarized modes which is germane to optical polarization states propagating through linearly birefringent media. The sense in which $\psi(\phi)$ would describe the polarization state for the linear mode set would be different however since we lose the connection with the angular measurement as the energy eigenstates in the linear basis are not eigenstates of angular momenta. Nonetheless, the mode exchange eigenstates in this basis correspond to an expected value of the electric field operator that resembles circular polarization and these states provide a foundation for the study of quantum limits on the performance of quarter-wave plates, etc.

References

[1] J. Schwinger, U.S. Atomic Energy Report No. NYO-3071 (U.S. GPO, Washington, D.C., 1952).

[2] J. J. Sakurai, *Modern Quantum Mechanics* (Benjamin/Cummings, Menlo Park, CA, 1985).

[3] S. R. Shepard, Ph.D. Thesis Proposal (Massachusetts Institute of Technology, Cambridge, MA, 1990).

[4] S. R. Shepard and J. H. Shapiro, OSA '88 Technical Digest (Optical Society of America, Washington, D.C., 1988).

[5] L. Susskind and J. Glogower, Physics **1**, 49 (1964).

[6] V. B. Berestetskii, et. al., *Quantum Electrodynamics* (Permagon Press, Elmsford, NY, 1982).

[7] S. A. Werner, et. al., Phys. Rev. Lett. **35**, 1053 (1975).

[8] K. Gottfried, *Quantum Mechanics* (W. A. Benjamin, New York, NY, 1966).

[9] S. R. Shepard, Workshop on Squeezed States and Uncertainty Relations (NASA CP-3135, Washington, D.C., 1991).