TWO PHOTON ANNIHILATION OPERATORS AND SQUEEZED VACUUM

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Abstract

We introduce inverses of the harmonic oscillator creation and annihilaton operators by their actions on the number states. Three of the two photon annihilation operators, viz., $\hat{a}^{+-1}\hat{a}$, $\hat{a}\hat{a}^{+-1}$ and $\hat{a}^2$, have normalizable right eigenstates with non vanishing eigenvalues. We discuss the eigenvalue equation of these operators and obtain their normalized eigenstates. We find that the Fock state representation, in each case separates into two sets of states, one involving only the even number states while the other involving only the odd number states. We show that the even set of eigenstates of the operator $\hat{a}^{+-1}\hat{a}$ is the customary squeezed vacuum $\hat{S}(\sigma)|0\rangle$.

1 Introduction

In quantum optics several different representations of the harmonic oscillator states have been discussed such as number states, coherent states [1], squeezed states [2-4], squeezed number states [5], near number states [6], and photon added coherent states [7]. The basic operators are the boson annihilation and creation operators $\hat{a}$ and $\hat{a}^+$, satisfying the usual commutation relation $[\hat{a},\hat{a}^+] = 1$. These operators are defined in terms of their actions on number states as

$$\hat{a}|n\rangle = n^{1/2}|n - 1\rangle, \tag{1.1}$$

$$\hat{a}^{+}|n\rangle = (n + 1)^{1/2}|n + 1\rangle. \tag{1.2}$$

One may introduce the generalized inverses [8] of $\hat{a}$ and $\hat{a}^{+}$:

$$\hat{a}^{-1}|n\rangle = (n + 1)^{-1/2}|n + 1\rangle, \tag{1.3}$$

$$\hat{a}^{+-1}|n\rangle = (1 - \delta_{n,0})\,(n)^{-1/2}|n - 1\rangle, \tag{1.4}$$

The operator $\hat{a}^{-1}$ behaves as a creation operator whereas $\hat{a}^{+-1}$ behaves as an annihilation operator. Further $\hat{a}^{-1}$ is the right inverse of $\hat{a}$ and $\hat{a}^{+-1}$ is the left inverse of $\hat{a}^{+}$, i.e.,

$$\hat{a}\hat{a}^{-1} = \hat{a}^{+-1}\hat{a}^{+} = I. \tag{1.5}$$

On the other hand $\hat{a}^{-1}\hat{a}$ and $\hat{a}^{+}\hat{a}^{+-1}$ give

$$\hat{a}^{-1}\hat{a} = \hat{a}^{+}\hat{a}^{+-1} = I - |0\rangle\langle 0|, \tag{1.6}$$

where $|0\rangle\langle 0|$ is the projection operator on the vacuum.

Five of the operators exhibiting two photon processes, viz., $\hat{a}^{+2}$, $\hat{a}^{-2}$, $\hat{a}^{+-2}$, $\hat{a}^{+}\hat{a}^{-1}$ and $\hat{a}^{-1}\hat{a}^{+}$ do not have any normalizable right eigenstate with non-zero eigenvalue. We can solve the eigenvalue problem for the remaining three, viz., $\hat{a}^{+-1}\hat{a}$, $\hat{a}\hat{a}^{+-1}$ and $\hat{a}^{2}$. These three are the two photon annihilation operators (TAO). The matrix representation of these TAOs may readily be obtained by noting their actions on the number states $|n\rangle$. Using Eqs.(1.1)-(1.4) we obtain, for $n \geq 2$

$$\hat{a}^{+-1}\hat{a}|n\rangle = [n/(n - 1)]^{1/2}|n-2\rangle, \tag{1.7}$$

$$\hat{a}\hat{a}^{+-1}|n\rangle = [(n - 1)/n]^{1/2}|n-2\rangle, \tag{1.8}$$

$$\hat{a}^{2}|n\rangle = [n(n - 1)]^{1/2}|n-2\rangle, \tag{1.9}$$

whereas their action on $|n\rangle$ with $n = 0$ or 1 gives zero. We, therefore, find the following matrix elements for these operators:

$$\langle m|\hat{a}^{+-1}\hat{a}|n\rangle = [n/(n-1)]^{1/2}\,\delta_{m,n-2}, \tag{1.10}$$

$$\langle m|\hat{a}\hat{a}^{+-1}|n\rangle = [(n-1)/n]^{1/2}\,\delta_{m,n-2}\,, \tag{1.11}$$

$$\langle m|\hat{a}^{2}|n\rangle = [n(n-1)]^{1/2}\,\delta_{m,n-2}\,. \tag{1.12}$$

We now consider the eigenvalue problem for these TAOs in detail.

2 Eigenstates of $\hat{a}^{+-1}\hat{a}$

We write an eigenvalue equation for the operator $\hat{a}^{+-1}\hat{a}$ as:

$$\hat{a}^{+-1}\hat{a}|\lambda,1\rangle = \lambda|\lambda,1\rangle, \tag{2.1}$$

where $|\lambda,1\rangle$ is a right eigenstate of the first TAO $\hat{a}^{+-1}\hat{a}$ with eigenvalue λ and obtain a solution for suitable complex number λ. Expressing $|\lambda,1\rangle$ in the form

$$|\lambda,1\rangle = \sum_{n=0}^{\infty} C_n|n\rangle \tag{2.2}$$

we obtain the following recurrence relation for C_n :

$$C_n = \lambda\,[(n-1)/n]^{1/2}\,C_{n-2}\,. \tag{2.3}$$

From this recurrence relation it is observed that the eigenstates $|\lambda,1\rangle$ separate into two sets of states involving either even number states or odd number states as follows:

$$|\lambda,+1\rangle = N_{+}\sum_{n=0}^{\infty}\frac{[(2n)!]^{1/2}}{2^{n}n!}\,\lambda^{n}|2n\rangle \tag{2.4}$$

and

$$|\lambda,-1\rangle = N_- \sum_{n=0}^{\infty} \frac{2^n\, n!}{[(2n+1)!]^{1/2}} \lambda^n |2n+1\rangle. \tag{2.5}$$

Here N_+ and N_- are the normalization constants given by

$$N_+ = (1 - |\lambda|^2)^{1/4}, \qquad |\lambda| < 1, \tag{2.6}$$

$$N_- = \left[\frac{|\lambda|(1-|\lambda|^2)^{1/2}}{\sin^{-1}|\lambda|}\right]^{1/2}, \qquad |\lambda| < 1. \tag{2.7}$$

Both of the states $|\lambda,+1\rangle$ and $|\lambda,-1\rangle$ correspond to the same eigenvalue λ and hence any linear combination of these states is the general eigenstate of the TAO $\hat{a}^{+-1}\hat{a}$.

3 Eigenstates of $\hat{a}\hat{a}^{+-1}$

We write the eigenvalue equation for this operator as

$$\hat{a}\hat{a}^{+-1}|\lambda,2\rangle = \lambda|\lambda,2\rangle, \tag{3.1}$$

where $|\lambda,2\rangle$ is the right eigenstate of the second operator $\hat{a}\hat{a}^{+-1}$ with an eigenvalue λ. Proceeding in a manner strictly analogous to that followed in Sec.2, we find that these eigenstates also separate into two sets, one involving even number states and the other involving odd number states

$$|\lambda,+2\rangle = M_+ \sum_{n=0}^{\infty} \frac{2^n n!}{[(2n)!]^{1/2}} \lambda^n |2n\rangle \tag{3.2}$$

and

$$|\lambda,-2\rangle = M_- \sum_{n=0}^{\infty} \frac{[(2n+1)!]^{1/2}}{2^n n!} \lambda^n |2n+1\rangle, \tag{3.3}$$

where M_+ and M_- are the normalization constants given by

$$M_+ = \left[\frac{(1-|\lambda|^2)^{3/2}}{(1-|\lambda|^2)^{1/2} + |\lambda| \sin^{-1} |\lambda|}\right]^{1/2}, \qquad |\lambda| < 1, \tag{3.4}$$

$$M_- = [1 - |\lambda|^2]^{3/4}, \qquad |\lambda| < 1. \tag{3.5}$$

A general eigenstate of the TAO $(\hat{a}\hat{a}^{+-1})$ is a linear combination of the states $|\lambda, +2\rangle$ and $|\lambda, -2\rangle$.

4 Eigenstates of $\hat{a}^2$

Coherent states are the right eigenstates of the annihilation operator $\hat{a}$, and so that of $\hat{a}^2$ also. These states [9] separate neatly into the even and odd parts both being the eigenstates of $\hat{a}^2$, as in the case of the other TAOs. Hence the normalized eigenstates of $\hat{a}^2$ with eigenvalue λ can be expressed in the form

$$|\lambda, +3\rangle = (\cosh |\lambda|)^{-1/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{[(2n)!]^{1/2}} |2n\rangle \tag{4.1}$$

and

$$|\lambda, -3\rangle = \left[\frac{\sinh |\lambda|}{|\lambda|}\right]^{-1/2} \sum_{n=0}^{\infty} \frac{\lambda^n}{[(2n+1)!]^{1/2}} |2n+1\rangle. \tag{4.2}$$

Any linear superposition of $|\lambda, +3\rangle$ and $|\lambda, -3\rangle$ states is an eigenstate of $\hat{a}^2$. Of course, a particular linear combination happens to be the coherent state $|(\alpha)^{1/2}\rangle$. Further there is no restriction on the value of $|\lambda|$, whereas in the earlier cases $|\lambda|$ was restricted to be less than 1.

5 Squeezed Vacuum as an Eigenstate of $\hat{a}^{+-1}\hat{a}$

It is interesting to note that the state $|\lambda, +1\rangle$ [Eq. (2.4)] is essentially the squeezed vacuum discussed in literature [3, 10-12].

The squeezed vacuum is generated by the action of the squeeze operator $\hat{S}(\sigma)$ on vacuum

$$\hat{S}(\sigma)|0\rangle = \exp\left[1/2\left(\sigma\hat{a}^{+2} - \sigma^{*}\hat{a}^{2}\right)\right]|0\rangle. \qquad (5.1)$$

Using the normal ordered form of the operator $S(\sigma)$ we find the number state representation of the squeezed vacuum as

$$\hat{S}(\sigma)|0\rangle = (\cosh r)^{-1/2} \times \sum_{n=0}^{\infty} \frac{[(2n)!]^{1/2}}{2^{n}n!} (e^{i\theta} \tanh r)^{n}|2n\rangle, \qquad (5.2)$$

where the squeeze parameter $\sigma = r\, e^{i\theta}$. Comparing Eqs.(2.4) and (5.2) we find that

$$|\lambda,+1\rangle = \hat{S}(\sigma)|0\rangle, \qquad (5.3)$$

where the eigenvalue λ is related to the squeeze parameter σ by

$$\lambda = e^{i\theta} \tanh r. \qquad (5.4)$$

Hence we conclude that the squeezed vacuum is an eigenstate of our TAO $\hat{a}^{+-1}\hat{a}$. In a similar manner we can show that the squeezed first number state $\hat{S}(\sigma)|n=1\rangle$ is an eigenstate $|\lambda,-2\rangle$ of the operator $\hat{a}\hat{a}^{+-1}$.

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