SQUEEZED STATES OF DAMPED OSCILLATOR CHAIN

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Abstract

The Caldirola-Kanai model of one-dimensional damped oscillator is extended to the chain of coupled parametric oscillators with damping. The correlated and squeezed states for the chain of coupled parametric oscillators with damping are constructed. Based on the concept of the integrals of motion it is demonstrated how squeezing phenomenon arises due to parametric excitation.

1 Introduction

A number of papers devoted to finding exact solutions of the Schrödinger equation with explicitly time-dependent quadratic Hamiltonians were published over the past quarter of a century. Several different but equivalent approaches are usually exploited for this purpose. In this paper we would like to present the model of quantum chain of coupled parametric oscillators with damping extending the model of known one-dimensional damped oscillator, and to demonstrate how squeezing phenomenon arrives in the chain based on the concept of time-dependent quantum integrals of motion. This method was elaborated in [1, 2, 3], while a detailed review was given in [4]. The exact formulae for propagators, wave functions, coherent states, density matrices, Wigner function, transition amplitudes and probabilities were given in [2, 3, 4] in the most convenient and explicit forms for quite general quadratic systems and numerous special cases.

Here we will apply developed approach to the model describing oscillator chain of coupled parametric oscillators with damping. This model is the partial case of general problem of multi-dimensional parametric oscillator, but the dynamics of these systems may be investigated in the explicit form due to the possibility of using usual normal mode transformation in spite of the coefficients being time-dependent. It is necessary to note that the problem of different kinds of quantum closed chains was discussed in recent papers [5–12] and the problem of of unclosed chain in [13, 14].

2 Integrals of Motion

Let us consider a quantum chain consisting of N coupled harmonic parametric oscillators with damping. All oscillators vibrate with frequency $\omega_0(t)$ which depends on time and interacts linearly with neighbors. The interaction constant $\omega(t)$ depends on time too. When the distance between neighbors approaches zero, and number N tends to infinity, the chain turns into the parametric string.

The Hamiltonian of this system depends on time and has the form

$$\hat{H} = \frac{1}{2}\sum_{n=1}^{N}\left(\frac{p_n^2 e^{-2\Gamma(t)}}{m} + m\Omega^2(t)e^{2\Gamma(t)}(q_n - q_{n+1})^2 + m\Omega_0^2(t)e^{2\Gamma(t)}q_n^2\right), \tag{1}$$

where q_n is an operator of a shift from the equilibrium point of a n-th oscillator, p_n is a momentum operator of the oscillator, m is the mass of each oscillator, and Γ is a damping coefficient depending on time.

In this model, damping is described in the framework of a phenomenological Hamiltonian first suggested for one-dimensional quantum oscillators with damping by Caldirola [15] and Kanai [16]. In this model, the mass of the oscillator increases exponentially. That dependence models the interaction of the oscillator with external degrees of freedom. Hamiltonian (1) is an extension of the Caldirola-Kanai Hamiltonian to the case of quantum chain of coupled parametric oscillators with damping.

The equations of motion corresponding to Hamiltonian (1) are

$$p_n = \dot{q}_n e^{2\Gamma(t)}, \qquad \ddot{q}_n = \Omega^2(t)\left(q_{n+1} + q_{n-1} - 2q_n\right) - \Omega_0^2(t)q_n - 2\dot{\Gamma}(t)\dot{q}_n. \tag{2}$$

We take into consideration the closed chain, so we have the condition

$$q_{1+N} = q_1.$$

A property of this model is that the time-dependence of the coefficients does not prevent from the application of the usual normal-mode reduction formulae. So, let us introduce new variables

$$x_s = \sqrt{\frac{2}{N}}\sum_{m=1}^{N}q_m \cos\left(2\pi sm/N\right), \tag{3}$$

$$y_s = \sqrt{\frac{2}{N}}\sum_{m=1}^{N}q_m \sin\left(2\pi sm/N\right), \tag{4}$$

$$x_N = \frac{1}{\sqrt{N}}\sum_{m=1}^{N}q_m. \tag{5}$$

For simplicity, we consider the chain consisting of an odd number of oscillators, so the number s changes from 1 to $p = (N - 1)/2$. The normal mode transformation (3)-(5) reduces the system of N coupled oscillators (2) to a set of N free oscillators vibrating independently according to the equations

$$\ddot{x}_s - 2\dot{\Gamma}(t)\dot{x}_s + \Omega_s^2(t)x_s = 0, \tag{6}$$

$$\ddot{y}_s - 2\dot{\Gamma}(t)\dot{y}_s + \Omega_s^2(t)y_s = 0, \tag{7}$$

$$\ddot{x}_N - 2\dot{\Gamma}(t)\dot{x}_N + \Omega_0^2(t)x_N = 0, \tag{8}$$

where the frequencies are given by the relation

$$\Omega_s^2(t) = 4\Omega^2(t)\sin^2\left(\pi s/N\right) + \Omega_0^2(t). \tag{9}$$

One can see that equations (6)-(8) are the trajectory equations of classical damped oscillators with frequencies (9).

Following the usual procedure (see,for example, [4]) "annihilation" operators for variable-frequency chain with damping can be constructed

$$\hat{A}_s(t) = \frac{i}{\sqrt{N}}\sum\nolimits_{m=1}^{N} \left(\frac{l_s \epsilon_s p_m}{\hbar} - \frac{\dot{\epsilon}_s e^{2\Gamma(t)} q_m}{l_s \Omega_s(0)} \right) \cos\left(2\pi sm/N\right), \qquad (10)$$

$$\hat{B}_s(t) = \frac{i}{\sqrt{N}}\sum\nolimits_{m=1}^{N} \left(\frac{l_s \epsilon_s p_m}{\hbar} - \frac{\dot{\epsilon}_s e^{2\Gamma(t)} q_m}{l_s \Omega_s(0)} \right) \sin\left(2\pi sm/N\right), \qquad (11)$$

$$\hat{A}_N(t) = \frac{i}{\sqrt{2N}}\sum\nolimits_{m=1}^{N} \left(\frac{l_0 \epsilon_0 p_m}{\hbar} - \frac{\dot{\epsilon}_0 e^{2\Gamma(t)} q_m}{l_0 \Omega_0(0)} \right), \qquad (12)$$

where

$$l_s = \left(\frac{\hbar}{m\Omega_s(0)} \right)^{1/2}, \qquad l_0 = \left(\frac{\hbar}{m\Omega_0(0)} \right)^{1/2}.$$

The complex functions $\epsilon_s(t)$ and $\epsilon_0(t)$ are the solutions of the equations of motion of classical parametric oscillators with damping

$$\ddot{\epsilon}_s(t) + 2\dot{\Gamma}(t)\dot{\epsilon}_s(t) + \Omega_s^2(t)\epsilon_s(t) = 0, \qquad \ddot{\epsilon}_0(t) + 2\dot{\Gamma}(t)\dot{\epsilon}_0(t) + \Omega_0^2(t)\epsilon_0(t) = 0. \qquad (13)$$

"Annihilation" operators and their Hermitian conjugate operators satisfy the boson commutation relations

$$\left[\hat{A}_s(t), \hat{A}_i^\dagger(t)\right] = \left[\hat{B}_s(t), \hat{B}_i^\dagger(t)\right] = \delta_{si}, \qquad \left[\hat{A}_N(t), \hat{A}_N^\dagger(t)\right] = 1,$$

and

$$\left[\hat{A}_s(t), \hat{B}_i^\dagger(t)\right] = \left[\hat{A}_s(t), \hat{A}_N^\dagger(t)\right] = \left[\hat{A}_s(t), \hat{B}_i(t)\right] = \left[\hat{A}_s(t), \hat{A}_N(t)\right] = \left[\hat{B}_s(t), \hat{A}_N(t)\right] = 0,$$

if the functions $\epsilon_s(t)$, $\epsilon_0(t)$ satisfy the additional conditions

$$e^{2\Gamma(t)} \left(\dot{\epsilon}_s(t)\epsilon_s^*(t) - \dot{\epsilon}_s^*(t)\epsilon_s(t) \right) = 2i\Omega_s(0), \qquad (14)$$

$$e^{2\Gamma(t)} \left(\dot{\epsilon}_o(t)\epsilon_o^*(t) - \dot{\epsilon}_o^*(t)\epsilon_o(t) \right) = 2i\Omega_0(0). \qquad (15)$$

One can check that the full derivatives of operators (10)-(12) and their Hermitian conjugates are equal to zero, so they are the linear integrals of the motion of the quantum parametric chain with damping.

3   Squeezed Correlated and Fock States

The ground state of the parametric chain with damping can be constructed with the help of the integrals of motion (10)-(12) using the relations

$$\hat{A}_s(t)\psi_{\vec{0}}(\vec{q}, t) = \hat{B}_s(t)\psi_{\vec{0}}(\vec{q}, t) = \hat{A}_N(t)\psi_{\vec{0}}(\vec{q}, t) = 0, \qquad (16)$$

where $\vec{q} = (q_1, ..., q_N)$. So the normalized ground state satisfying the Schrödinger equations with Hamiltonian (1) has the following wave function in coordinate representation

$$\psi_{\vec{0}}(\vec{q}, t) = \pi^{-N/4}(l_0\epsilon_0)^{-1/2}\prod_{s=1}^{p}(l_s\epsilon_s)^{-1}$$

$$\exp\left\{\sum_{m,m'=1}^{N} q_m q_{m'}\left(\frac{i\dot{\epsilon}_0 e^{2\Gamma(t)}}{2\epsilon_0 N\Omega_0(0)l_0^2} + \sum_{s=1}^{p}\frac{i\dot{\epsilon}_s e^{2\Gamma(t)}}{\epsilon_s N\Omega_s(0)l_s^2}\cos\left(2\pi s(m - m')/N\right)\right)\right\}. \quad (17)$$

Constructing with the help of integrals of motion (10)-(12) the displacement operator

$$D(\vec{\alpha}) = \prod_{s=1}^{p}\exp\left(\alpha_s \hat{A}_s^\dagger - \alpha_s^* \hat{A}_s + \beta_s \hat{B}_s^\dagger - \beta_s^* \hat{B}_s \xi \hat{A}_N^\dagger - \xi^* \hat{A}_N\right),$$

where components of the vector

$$\vec{\alpha} = (\alpha_1, ..., \alpha_p, \beta_1, ..., \beta_p, \xi)$$

are complex numbers, and acting by displacement operator on ground state $\psi_{\vec{0}}(t)(\vec{q}, t)$ the entire family of correlated squeezed states can be obtained. So the wave functions in coordinate representation have the form

$$\psi_{\vec{\alpha}}(\vec{q}, t) = \psi_{\vec{0}}(\vec{q}, t)\exp\left\{-\frac{|\xi|^2}{2} - \frac{\xi^2\epsilon_0^*}{2\epsilon_0}\right.$$

$$+\sqrt{\frac{2}{N}}\frac{\xi}{l_0\epsilon_0}\sum_{m=1}^{N} q_m + \sum_{s=1}^{p}[-\frac{|\alpha_s|^2}{2} - \frac{|\beta_s|^2}{2} - \frac{\epsilon_s^*}{2\epsilon_s}(\alpha_s^2 + \beta_s^2)]$$

$$+\left.\sum_{s=1}^{p}\frac{2}{\epsilon_s l_s \sqrt{N}}\sum_{m=1}^{N} q_m[\alpha_s\cos\left(2\pi sm/N\right) + \beta_s\sin\left(2\pi sm/N\right)]\right\}. \quad (18)$$

The correlated squeezed states satisfy the Schrödinger equation with Hamiltonian (1) and are eigenstates of the integrals of motion (10)-(12), and components of the vector $\vec{\alpha}$ are eigenvalues of operators (10)-(12)

$$\hat{A}_s(t)\psi_{\vec{\alpha}}(\vec{q}, t) = \alpha_s\psi_{\vec{\alpha}}(\vec{q}, t),$$

$$\hat{B}_s(t)\psi_{\vec{\alpha}}(\vec{q}, t) = \beta_s\psi_{\vec{\alpha}}(\vec{q}, t),$$

$$\hat{A}_N(t)\psi_{\vec{\alpha}}(\vec{q}, t) = \xi\psi_{\vec{\alpha}}(\vec{q}, t).$$

One can see that the wave function of the ground state (17) and squeezed correlated states (18) are gaussian states with time-dependent coefficients in quadratic form of exponent function.

Using the property of squeezed correlated states (18) to be a generating function for Fock states

$$\psi_{\vec{\alpha}}(\vec{q}, t) = \exp\left(-\frac{|\xi|^2}{2} - \frac{1}{2}\sum_{s=1}^{p}(|\alpha_s|^2 + |\beta_s|^2)\right)\sum_{\vec{n}=0}^{\infty}\frac{\xi^{n_0}}{\sqrt{n_0!}}\prod_{s=1}^{p}\frac{\alpha_s^{n_s}\beta_s^{m_s}}{\sqrt{n_s!m_s!}}\psi_{\vec{n}}(\vec{q}, t),$$

where vector $\vec{n}$ has the components $\vec{n} = (n_0, n_1, ..., n_p, m_1, ..., m_p)$, the Fock states of quantum parametric chain with damping can be constructed, and are of the form

$$\psi_{\vec{n}}(\vec{q}, t) = \frac{1}{\sqrt{n_0!}} \left(\frac{\epsilon_0^*}{2\epsilon_0} \right)^{n_0/2} H_{n_0} \left(\frac{1}{|\epsilon_0| \sqrt{N}} \sum_{m=1}^{N} q_m \right)$$

$$\prod_{s=1}^{p} \frac{1}{\sqrt{n_s! m_s!}} \left(\frac{\epsilon_s^*}{2\epsilon_s} \right)^{(n_s + m_s)/2} H_{n_s} \left(\frac{2}{|\epsilon_s| \sqrt{N}} \sum_{m=1}^{N} q_m cos(2\pi sm/N) \right)$$

$$H_{m_s} \left(\frac{2}{|\epsilon_s| \sqrt{N}} \sum_{m=1}^{N} q_m sin(2\pi sm/N) \right) \psi_{\vec{0}}(\vec{q}, t), \tag{19}$$

where $H_i(x_j)$ are Hermite polynomials.

The Fock states (19) are the eigenstates of the integrals of motion $\hat{A}_s^\dagger(t)\hat{A}_s(t)$, $\hat{B}_s^\dagger(t)\hat{B}_s(t)$ and $\hat{A}_N^\dagger(t)\hat{A}_N(t)$ and components of the vector $\vec{n}$ are eigenvalues of these operators.

$$\hat{A}_s^\dagger(t)\hat{A}_s(t)\psi_{\vec{n}}(\vec{q}, t) = n_s \psi_{\vec{n}}(\vec{q}, t),$$

$$\hat{B}_s^\dagger(t)\hat{B}_s(t)\psi_{\vec{n}}(\vec{q}, t) = m_s \psi_{\vec{n}}(\vec{q}, t),$$

$$\hat{A}_N^\dagger(t)\hat{A}_N(t)\psi_{\vec{n}}(\vec{q}, t) = n_0 \psi_{\vec{n}}(\vec{q}, t).$$

4   Squeezing and Correlated Coefficients

Let us calculate the dispersions of coordinates and momenta in squeezed correlated states (18). We define the dispersions and correlations by the formulae

$$\sigma_{q_i q_k} = \langle \psi_{\vec{a}}(\vec{q}, t) \mid \hat{q}_i \hat{q}_k \mid \psi_{\vec{a}}(\vec{q}, t) \rangle - \langle \psi_{\vec{a}}(\vec{q}, t) \mid \hat{q}_i \mid \psi_{\vec{a}}(\vec{q}, t) \rangle \langle \psi_{\vec{a}}(\vec{q}, t) \mid \hat{q}_k \mid \psi_{\vec{a}}(\vec{q}, t) \rangle,$$

$$\sigma_{p_i p_k} = \langle \psi_{\vec{a}}(\vec{q}, t) \mid \hat{p}_i \hat{p}_k \mid \psi_{\vec{a}}(\vec{q}, t) \rangle - \langle \psi_{\vec{a}}(\vec{q}, t) \mid \hat{p}_i \mid \psi_{\vec{a}}(\vec{q}, t) \rangle \langle \psi_{\vec{a}}(\vec{q}, t) \mid \hat{p}_k \mid \psi_{\vec{a}}(\vec{q}, t) \rangle,$$

$$\sigma_{q_i p_k} = \frac{1}{2} \langle \psi_{\vec{a}}(\vec{q}, t) \mid \hat{q}_i \hat{p}_k + \hat{p}_k \hat{q}_i \mid \psi_{\vec{a}}(\vec{q}, t) \rangle - \langle \psi_{\vec{a}}(\vec{q}, t) \mid \hat{q}_i \mid \psi_{\vec{a}}(\vec{q}, t) \rangle \langle \psi_{\vec{a}}(\vec{q}, t) \mid \hat{p}_k \mid \psi_{\vec{a}}(\vec{q}, t) \rangle.$$

So one can calculate that the correlation of coordinates and momenta of different oscillators are not equal to zero and have the form

$$\sigma_{q_i q_k} = \frac{l_0^2 |\epsilon_0|^2}{2N} + \sum_{s=1}^{p} \frac{l_s^2 |\epsilon_s|^2}{N} \cos(2\pi sm(i - k)/N),$$

$$\sigma_{p_i p_k} = \frac{\hbar^2 e^{4\Gamma(t)} |\dot{\epsilon}_0|^2}{2N l_0^2 \Omega_0^2(0)} + \sum_{s=1}^{p} \frac{\hbar^2 e^{4\Gamma(t)} |\dot{\epsilon}_s|^2}{N l_s^2 \Omega_s^2(0)} \cos(2\pi sm(i - k)/N).$$

One has for the dispersions of coordinate and momenta of the same oscillator

$$\sigma_{q_i^2} = \frac{l_0^2 |\epsilon_0|^2}{2N} + \sum_{s=1}^{p} \frac{l_s^2 |\epsilon_s|^2}{N}, \tag{20}$$

$$\sigma_{p_i^2} = \frac{\hbar^2 e^{4\Gamma(t)} \mid \dot{\epsilon}_0 \mid^2}{2N l_0^2 \Omega_0^2(0)} + \sum_{s=1}^{p} \frac{\hbar^2 e^{4\Gamma(t)} \mid \dot{\epsilon}_s \mid^2}{N l_s^2 \Omega_s^2(0)}. \tag{21}$$

The correlated squeezed states (18) and ground state (17) minimize the Schrödinger-Robertson uncertainty relation [17, 18]

$$\sigma_{q_k^2} \sigma_{p_k^2} \geq \frac{\hbar^2}{4(1 - r^2)}$$

with the correlation coefficient

$$r = \sigma_{q_k p_k} \left(\sigma_{q_k^2} \sigma_{p_k^2} \right)^{-1/2}$$

equal to

$$r = \left(1 - N^2 e^{-4\Gamma(t)} \left[\mid \epsilon_0 \mid^2 l_0^2 + 2\sum_{s=1}^{p} \mid \epsilon_s \mid^2 l_s^2 \right]^{-1} \left[\frac{\mid \dot{\epsilon}_0 \mid^2}{\Omega_0^2(0) l_0^2} + 2\sum_{s=1}^{p} \frac{\mid \dot{\epsilon}_s \mid^2}{\Omega_s^2(0) l_s^2} \right]^{-1} \right)^{1/2}. \tag{22}$$

One can see from (20)-(21) that changing of the frequencies influence the dispersions and the squeezing coefficient $k = \frac{\sigma_{q_k^2}(t)}{2\sigma_{q_k^2}(0)}$. Namely, by changing the frequencies one can decrease the dispersions of the coordinates due to increasing of the dispersions of momenta, and vice versa, and make squeezing coefficients less then unity. So the squeezing phenomenon arises due to parametric excitation of quantum chain of coupled parametric oscillators with damping. It is necessary to note that due to parametric excitation each oscillator has the additional time-dependent parameter (22), so-called correlation coefficient which is equal to zero in the stationary regime.

5   Acknowledgements

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