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HARMONIC OSCILLATOR STATES IN ABERRATION OPTICS

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Abstract

The states of the three-dimensional quantum harmonic oscillator classify optical aberrations of axis-symmetric systems due to the isomorphism between the two mathematical structures. Cartesian quanta and angular momentum classifications have their corresponding aberration classifications. The operation of concatenation of optical elements introduces a new operation between harmonic oscillator states.

1 Introduction: Optical Phase Space

Geometric optics uses the following 'screen' coordinates for light rays [1]:

$\mathbf{q} = (q_x, q_y)$: Coordinates of position on the 2-dimensional screen. The intersection of the ray with the screen ranges over $\Re^2$.

$\mathbf{p} = (p_x, p_y)$: Coordinates of momentum (with respect to the same screen). The projecton of the ray 3-vector along the ray [of length n, the refractive index of the medium at the point] on the plane of the screen is the momentm 2-vector; the component normal to the screen is the evolution Hamiltonian (below). See Figure 1.

Geometric optics has Hamiltonian evolution equations between the canonically conjugate variables $\mathbf{p}$ and $\mathbf{q}$. The optical Hamiltonian is

$$h = -p_z = -\sqrt{n(\mathbf{q}, z)^2 - |\mathbf{p}|^2}. \tag{1}$$

Plain geometry [2] provides the first Hamilton equation

$$\frac{d\mathbf{q}}{dz} = \frac{\partial h}{\partial \mathbf{p}} = \frac{\mathbf{p}}{p_z} = \{p_z, \circ\}\ \mathbf{q}, \tag{2}$$

while Snell's law yields the dynamics of the second Hamilton equation

$$\frac{d\mathbf{p}}{dz} = -\frac{\partial h}{\partial \mathbf{q}} = -\frac{n}{p_z}\frac{\partial n}{\partial \mathbf{q}} = \{p_z, \circ\}\ \mathbf{p}. \tag{3}$$

We use the Poisson bracket [3]

$$\{f, g\} = \sum_{i=x,y} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right), \tag{4}$$

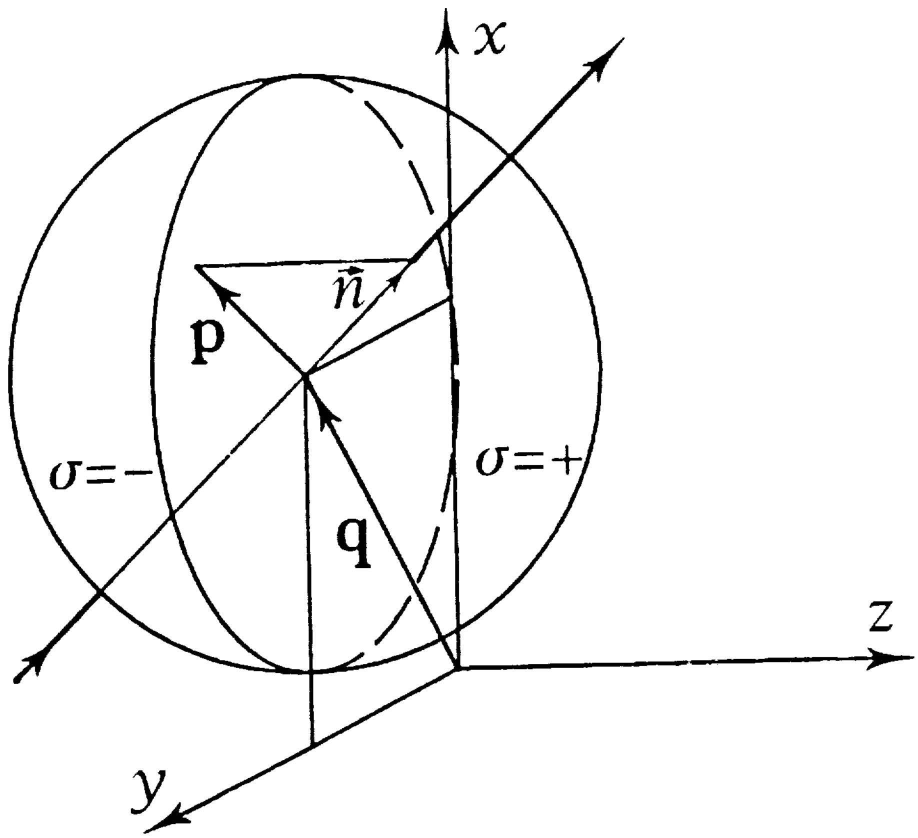

FIG. 1. The coordinates $(\mathbf{p}, \mathbf{q})$ of optical phase space in geometric optics. The z-axis is traditionally the optical axis.

for functions f, g of $\mathbf{p}$ and $\mathbf{q}$, and the corresponding Lie-Poisson operators $\{f, \circ\}$ as generators of Lie algebras [4]. This space is subject to Hamiltonian evolution equations and constitutes *optical phase space*, subject to the same local symplectic structure as the well known mechanical phase space. Optical phase space *differs* from the mechanical one *globally*, however, in that its momentum ranges over a compact domain: (two) *disks* (forward and backward rays) of radius $n(\mathbf{q})$, the refractive index.

2 Examples

Free flight in homogeneous medium (n constant) yields the solutions

$$\mathbf{q}(z) = \exp z\{p_z, \circ\}\, \mathbf{q} = \mathbf{q} + z\frac{\mathbf{p}}{\sqrt{n^2 - |\mathbf{p}|^2}}, \qquad \mathbf{p}(z) = \exp z\{p_z, \circ\}\, \mathbf{p} = \mathbf{p}. \tag{5}$$

This is shown in Figure 2.

One example we may use to distinguish mechanical from optical phase space transformations pertains the mechanical oscillator *versus* —or *vis-à-vis* [5]— light in a fiber whose refractive index is a function of the radius $|\mathbf{q}|$ to the optical axis: $n(\mathbf{q}) = \sqrt{n_o^2 - \nu|\mathbf{q}|^2}$. We call

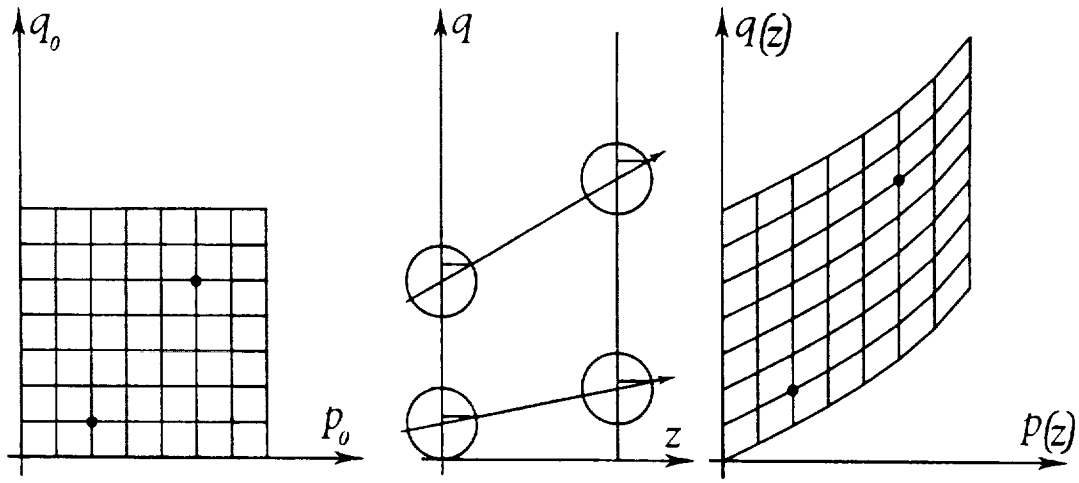

FIG. 2. The deformation of phase space due to free flight between two screens. Since the optical transformation is canonical, the area elements are preserved, and so is the allowed strip of phase space, $|\mathbf{p}| \leq n$. [Figure by G. KRÖTZSCH, IIMAS–UNAM.]

this the **elliptic index-profile fiber**. The evolution Hamiltonian is $h = -\sqrt{n_o^2 - (|\mathbf{p}|^2 + \nu|\mathbf{q}|^2)}$. For every z phase space is the patch $(|\mathbf{p}|^2 + \nu|\mathbf{q}|^2) \leq n_o^2$. Figure 3 shows the resulting evolution of phase space, compared with that of the mechanical harmonic oscillator.

In a homogeneous optical medium, phase space is a strip $|\mathbf{p}| \leq n$. The **Fourier transform** is a fundamental ingredient for much of coherent-state (paraxial) optics; yet, the Fourier transformation ($\mathbf{p} \mapsto -\mathbf{q}$, $\mathbf{q} \mapsto \mathbf{p}$) is not an invariance transformation of this space. Nevertheless, one may define a truly **optical Fourier** transform, that in one dimension has the form

$$p^{\mathrm{F}} = -\frac{(1-p^2)^{3/2}}{\sqrt{1+(1-p^2)^3 q^2}}\,q, \qquad q^{\mathrm{F}} = \frac{[1+(1-p^2)^3 q^2]^{3/2}}{\sqrt{1-p^2}}\,p, \tag{6}$$

that respects the strip of phase space, paraxially rotates by $\frac{1}{2}\pi$ (as the usual Fourier transform), and is nonlinear symplectic. In Figure 4 we show the optical Fourier transform for a quadrant in the phase space strip.

3 Linear Transformations of Phase Space

Although mechanics and optics phase spaces differ in their global properties, their paraxial correspondence motivates that we Taylor-expand all expressions into series where we may truncate the series to some *aberration order* in the powers of the phase space variables. We will thus work with polynomials and so we may ignore the range restriction. In fact, we thus substitute the structure of *Euclidean*-based 4π optics by the *metaxial* Heisenberg–Weyl Lie-algebraic structure suitable for aberration expansions in powers of the phase space coordinates.

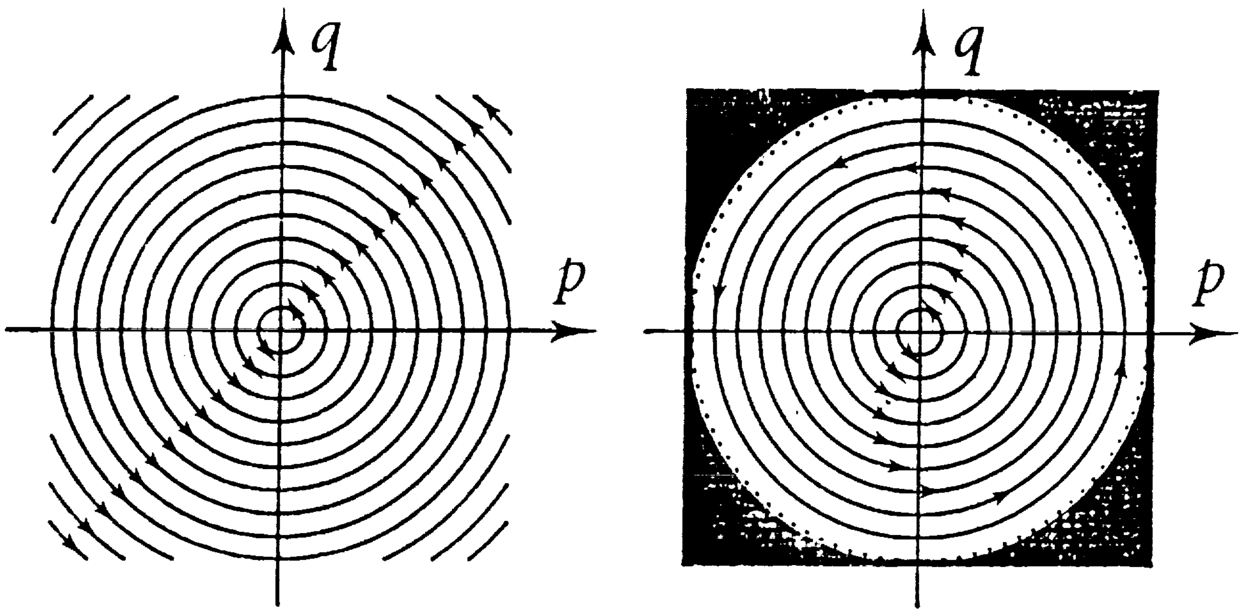

FIG. 3. *Left*: evolution of mechanical phase space under harmonic oscillator time evolution, for one space and one momentum dimension. Phase space rotates rigidly. *Right*: evolution of optical phase space under z-translation. The phase space disk rotates differentially and resembles the harmonic oscillator only for small p and q, *i.e.*, in the *paraxial* optical regime. The circular (in general, elliptic) region is optical phase space: coordinates are meaningless beyond the edge. [Figure by G. KRÖTZSCH, IIMAS–UNAM.]

In effect, we may classify the symplectomorphisms of phase space by the Lie-Poisson generators of the finite transformations.

Translations of phase space are generated by *linear* functions of $(\mathbf{p},\mathbf{q})$ through exponentiated Lie–Poisson operators:

$$\exp(\mathbf{a}\cdot\{\mathbf{p},\circ\})\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}=\begin{pmatrix}\mathbf{p}\\ \mathbf{q}-\mathbf{a}\end{pmatrix},\qquad \exp(\mathbf{b}\cdot\{\mathbf{q},\circ\})\begin{pmatrix}\mathbf{p}\\ \mathbf{q}\end{pmatrix}=\begin{pmatrix}\mathbf{p}+\mathbf{b}\\ \mathbf{q}\end{pmatrix},\tag{7}$$

as shown in Figure 5.

Linear canonical transformations of phase space are generated by the *quadratic* functions p_ip_j, p_iq_j, q_iq_j, $i = x, y$. These functions close into an $\mathrm{sp}(4,\Re)$ algebra under Poisson brackets. Further, optical systems that have a common axis of rotational (and inversion) symmetry are generated by linear combinations of $p^2 = |\mathbf{p}|^2$, $\mathbf{p}\cdot\mathbf{q}$, and $q^2 = |\mathbf{q}|^2$, that close into an $\mathrm{sp}(2,\Re) = \mathrm{so}(2,1) = \mathrm{sl}(2,\Re)$ Lie algebra. The corresponding group transformations of phase space are well known and shown in Figure 6. The quadratic polynomial $p^2 + q^2$ generates rigid rotations, as is well known.

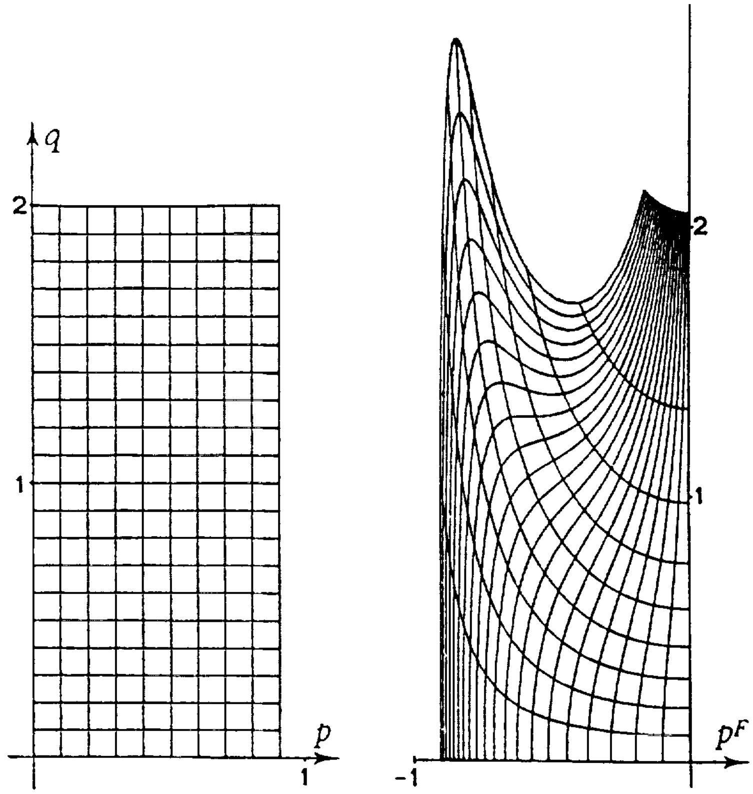

FIG. 4. *Left*: a quadrant in the phase space strip $p < 1$. The usual (mechanical) Fourier transform rotates this patch by $\frac{1}{2}\pi$ in the counterclockwise direction. *Right*: the optical Fourier transform maps points in the strip on points in the same; paraxially it rotates the plane by $\frac{1}{2}\pi$, and is globally symplectic: it conserves area elements. [Figure by G. KRÖTZSCH, IIMAS–UNAM.]

4 General Nonlinear Transformations

Nonlinear symplectic transformations of phase space are generated by polynomials in the components of $(\mathbf{p}, \mathbf{q})$ of degree higher than second. Again, the transformations of axis-symmetric optical systems leads us to concentrate on polynomials of three variables

$$\xi_+ = p^2/\sqrt{2}, \qquad \xi_0 = \mathbf{p}\cdot\mathbf{q}, \qquad \xi_- = q^2/\sqrt{2}. \tag{8a}$$

These variables may be placed into a vector $\vec{\xi}$ of cartesian components

$$\xi_1 = \tfrac{1}{2}(p^2 - q^2), \qquad \xi_2 = \tfrac{i}{2}(p^2 + q^2), \qquad \xi_3 = \mathbf{p}\cdot\mathbf{q}. \tag{8b}$$

They close into an $\mathrm{sp}(2, \Re)$ algebra, as we noted before. The Casimir of this algebra is minus the squared radius of a sphere,

$$(\mathbf{p}\times\mathbf{q})^2 = p^2q^2 - (\mathbf{p}\cdot\mathbf{q})^2 = -\xi_1^2 - \xi_2^2 - \xi_3^2 = -|\vec{\xi}\,|^2, \tag{9}$$

and is the well-known Petzval invariant of geometric optics.

In representing the action of optical elements by means of Lie-Poisson transformations generated by polynomials, we are aided by the following theorem:

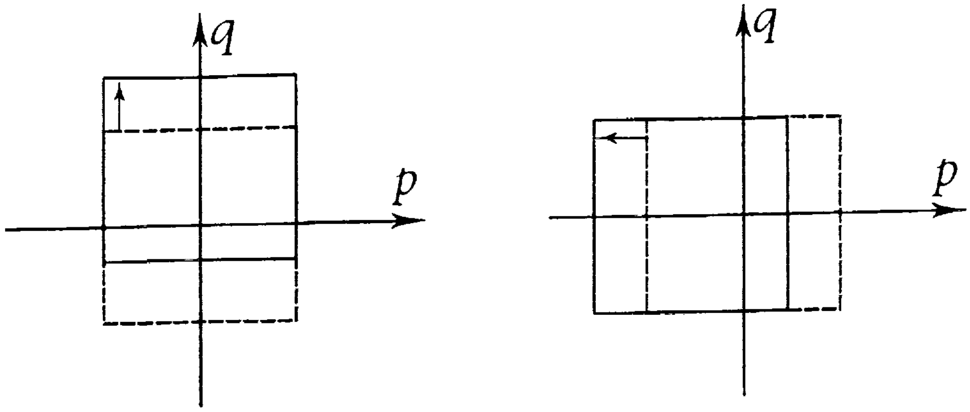

FIG. 5. Translations of phase space are generated by linear functions of the phase space observables. *Left*: $\exp\{p, \circ\}$ translates in q. *Right*: $\exp\{p, \circ\}$ translates in p. The latter does not leave the optical phase space invariant, but serves as a first-order approximation to translation and rotation of the screen. [Figure by G. KRÖTZSCH, IIMAS–UNAM.]

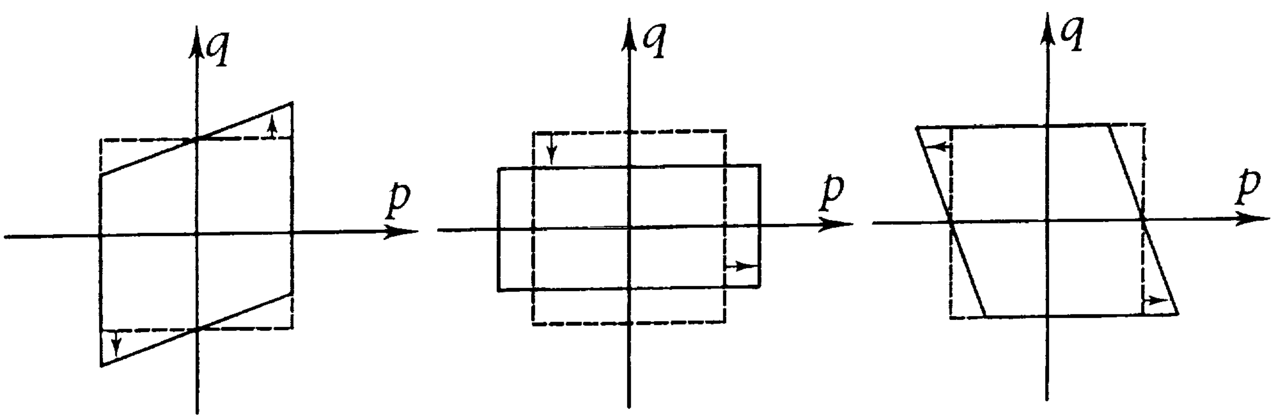

FIG. 6. Linear transformations of phase space are generated by quadratic polynomials in the phase space variables. In one position and one momentum coordinate, the transformations shown above correspond to $\exp\{p^2, \circ\}$, $\exp\{pq, \circ\}$, $\exp\{q^2, \circ\}$, and respectively. [Figure by G. KRÖTZSCH, IIMAS–UNAM.]

Theorem (Dragt & Finn) [6]: Canonical transformations $\mathcal{M}$ leaving the origin invariant, (*i.e.*, *excluding* translations) can be approximated by a truncated product series of Lie transformations

$$\mathcal{M} = \cdots \exp\{f_5, \circ\} \exp\{f_4, \circ\} \exp\{f_3, \circ\} \exp\{f_2, \circ\}, \tag{10}$$

where $f_k(\mathbf{p}, \mathbf{q})$ are homogeneous polynomials of degree k in the components of $\mathbf{p}$ and $\mathbf{q}$.

Correspondingly, for axis-symmetric systems, the approximation is written as

$$\mathcal{A} = \cdots \exp\{a_4, \circ\} \exp\{a_3, \circ\} \exp\{a_2, \circ\} \exp\{a_1, \circ\}, \tag{11}$$

with $a_k(\vec{\xi})$ polynomials of degree k in the components of ξ as defined in Eqs. (8); k is called the *rank* of the phase space generating monomials.

The rightmost factor in (10) and (11) is the subgroup of linear transformations, and factors to the left of it are nonlinear, called generically *aberrations* in optics. Since the degrees of polynomials in Poisson brackets satisfy $\{f_j, f_k\} = f_{j+k-2}$, the *ranks* of the sp$(2,\Re)$–based polynomials satisfy

$$\{a_j, a_k\} = a_{j+k-1}. \tag{12}$$

it follows that for each rank k, the polynomial set a_k is a (reducible) *ideal* under linear (paraxial) transformations. This is the aberration ideal of *order* $A = 2k - 1$. Axis-symmetric optical systems are described by a paraxial approximation modified by aberrations of orders 3, 5, 7, ..., generated by a_2, a_3, a_4, ..., respectively.

There are legitimate questions about the convergence of the product series and global properties of the *group* of nonlinear symplectic transformations. In any case, by taking functions of phase space *modulo* polynomials of degree higher than the aberration order, we may construct a well-defined finite-parameter group of transformations of phase space truncated to that order. Its best parametrization and, especially, the **product law** must be discovered —once and for all. Thus we construct the *A-th order aberration groups*. The theory developed in optical aberrations serves as well in higher approximation theory.

5 The Monomial Classification of Aberrations

It should be quite evident by now that we may classify aberrations of axis-symmetric systems through proposing complete bases of polynomial functions $a_k(\vec{\xi})$ of degree $2k$ in phase space. (This may also be used to classify non-axis-symmetric aberrations as broken symmetry [7]. We have a sp$(2,\Re)$ Lie algebraic graded covering structure with Poisson (Berezin) brackets between polynomial functions of three variables.

The **monomial** basis is

$$M_{k_+,k_0,k_-} = \text{const} \times \xi_+^{k_+}\, \xi_0^{k_0}\, \xi_-^{k_-}, \qquad k_i = 0, 1, \ldots. \tag{13}$$

These monomials have rank $k = k_+ + k_0 + k_-$, and are classified as the harmonic oscillator Cartesian basis states. We may examine the action generated by exponentials of these monomials up to the first Taylor term,

$$\exp\{M_{k_+,k_0,k_-}, \circ\} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{p} + k_0 M_{k_+,k_0-1,k_-}\mathbf{p} + 2k_- M_{k_+,k_0,k_--1}\mathbf{q} + \cdots \\ \mathbf{q} - 2k_+ M_{k_+-1,k_0,k_-}\mathbf{p} - k_0 M_{k_+,k_0-1,k_-}\mathbf{q} + \cdots \end{pmatrix}. \tag{14}$$

In this Dragt [8] recognized the traditional third-order Seidel aberrations, generated by and called

$(p^2)^2$ SPHERICAL ABERRATION,
$p^2\, \mathbf{p}\cdot\mathbf{q}$ CIRCULAR COMA,

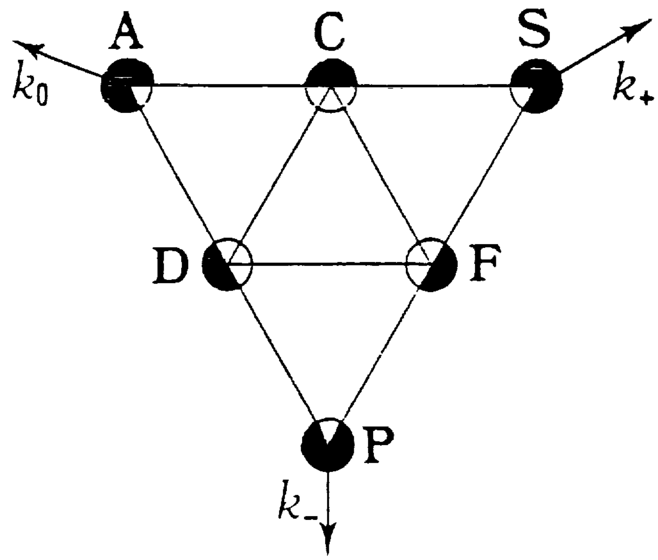

FIG. 7. The third order aberration sextuplet classified in ξ_+, ξ_0, and ξ_- coordinates, and the traditional names. S: spherical aberration, C: coma, A: astigmatism, F: curvature of field, D: distorsion, and P: pocus.

$(\mathbf{p}\cdot\mathbf{q})^2$	ASTIGMATISM,
$p^2 q^2$	CURVATURE OF FIELD,
$\mathbf{p}\cdot\mathbf{q}\, q^2$	DISTORSION,
$(q^2)^2$	POCUS.

Actually, *pocus* was excluded from Dragt's list because it does not produce any change in the screen images; rather it only changes the directions of ray arrival; it has a **p**-unfocusing action that we have playfully called pocus [9]. Yet it is an aberration of phase space on par with the other five, and the Fourier transform of spherical aberration. In Figure 7 we show the familiar harmonic oscillator states with two quanta and its corresponding aberrations of rank two in the monomial basis.

6 The Symplectic Classification of Aberrations

The harmonic oscillator eigenfunction structure naturally suggests another basis, following the number of quanta, total angular momentum, and one projection.

The **symplectic harmonic** basis is [1], [9], [10], [11]

$$ {}^k\chi^j_m(\vec{\xi}) = \text{const} \times [(\vec{\xi})^2]^{(k-j)/2}\, \mathcal{Y}_{j,m}(\vec{\xi}), \tag{15} $$

where $(\vec{\xi})^2$ is the Petzval and $\mathcal{Y}_{j,m}(\vec{\xi})$ the **solid** spherical harmonic of angular momentum j and projection m. The total degree of ${}^k\chi^j_m(\vec{\xi})$ in ξ_i is k —the rank. The index j we may call the *symplectic spin*, and m the Seidel index [12]. The third-order aberration sextuplet thus reduces, under linear transformations, into a spin-2 quintuplet and a spin-0 singlet (that is in effect the Petzval itself). Only the $m = 0$ member of the quintuplet and the singlet 'mix' the

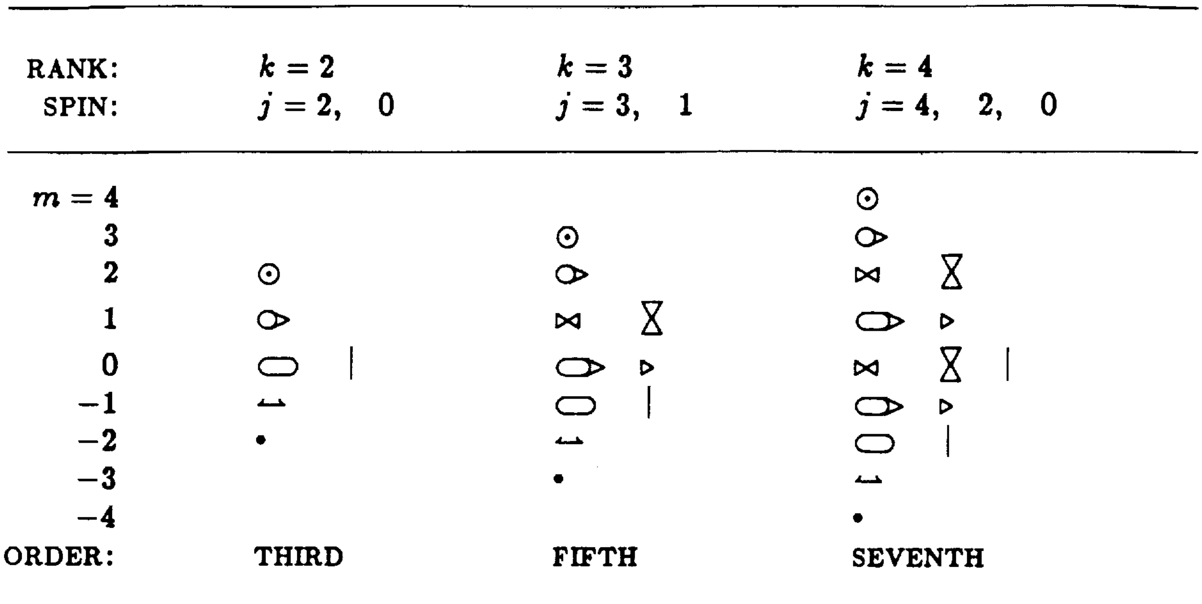

FIG. 8. The aberration multiplets to seventh order classified into symplectic harmonics by rank, spin and Seidel index m. The symbols are ⊙: spherical aberration, ⊳: circular coma, ⊳, ▹: elliptic comas, ⬭, ⋈: curvatisms, |, ⋈: astigmatures, ↼: distorsion, and •: pocus [12].

curvature of field and astigmatism monomials into aberrations that we could call curvatism (in the quintuplet) and astigmature (the singlet, invariant under paraxial transformations). The scheme appears in Figure 8; this is but the harmonic oscillator ℓ–m spectrum from the $k = 2$ level (s–d shell) up, seen sidewise.

The constant in front of the definition of ${}^k\chi^j_m$ in Eq. (15) has been chosen to avoid square roots, because they are time-consuming for symbolic manipulation. Thus we have defined them starting from the normalization of the highest weight ${}^k\chi^k_k = (p^2)^k$, and their lowering through $\{\frac{1}{2}q^2, \circ\}\ {}^k\chi^j_m = (m+j)\ {}^k\chi^j_{m-1}$. They are

$$
\begin{aligned}
{}^k\chi^j_m(\vec{\xi}) &= (\xi^2)^{(k-j)/2} \sqrt{\frac{4\pi(2j+1)\,(j+m)!\,(j-m)!}{(2j-1)!!}}\, \mathcal{Y}^j_m(\xi) \\
&= (\xi^2)^{(k-j)/2} \frac{(j+m)!\,(j-m)!}{2^{m/2}(2j-1)!!} \sum_n \frac{1}{2^n} \frac{\xi_+^{m+n}}{(m+n)!} \frac{\xi_0^{j-m-2n}}{(j-m-2n)!} \frac{\xi_-^n}{n!}.
\end{aligned}
\tag{16}
$$

We give the symplectic harmonics only for $m \geq 0$, because

$$
{}^k\chi^j_{-m}(\xi_+, \xi_0, \xi_-) = {}^k\chi^j_m(\xi_-, \xi_0, \xi_+). \tag{17}
$$

The first two ranks ($k = 0, 1$) correspond to the scalar and the sp$(2, \Re)$ generator functions:

$$
{}^0\chi^0_0 = 1; \qquad {}^1\chi^1_{\pm 1} = \sqrt{2}\xi_\pm, \quad {}^1\chi^1_0 = \xi_0. \tag{18a}
$$

The $k = 2$ basis functions are the generators of **third**-order aberrations:

$$\begin{aligned} {}^2\chi_2^2 &= 2\xi_+^2, \\ {}^2\chi_1^2 &= \sqrt{2}\xi_+\xi_0, \\ {}^2\chi_0^2 &= \tfrac{2}{3}(\xi_+\xi_- + \xi_0^2), \qquad {}^2\chi_0^0 = 2\xi_+\xi_- - \xi_0^2. \end{aligned} \tag{18b}$$

For $k = 3$ we have the generators of **fifth**-order aberrations:

$$\begin{aligned} {}^3\chi_3^3 &= 2\sqrt{2}\xi_+^3, \\ {}^3\chi_2^3 &= 2\xi_+^2\xi_0, \\ {}^3\chi_1^3 &= \tfrac{2}{5}\sqrt{2}(\xi_+^2\xi_- + 2\xi_+\xi_0^2), \qquad {}^3\chi_1^1 = {}^2\chi_0^0\,{}^1\chi_1^1, \\ {}^3\chi_0^3 &= \tfrac{2}{5}(3\xi_+\xi_0\xi_- + \xi_0^3), \qquad {}^3\chi_0^1 = {}^2\chi_0^0\,{}^1\chi_0^1. \end{aligned} \tag{18c}$$

For $k = 4$ we have those of *seventh*-order aberrations:

$$\begin{aligned} {}^4\chi_4^4 &= 4\xi_+^4, \\ {}^4\chi_3^4 &= 2\sqrt{2}\xi_+^3\xi_0, \\ {}^4\chi_2^4 &= \tfrac{4}{7}(\xi_+^3\xi_- + 3\xi_+^2\xi_0^2), \qquad {}^4\chi_2^2 = {}^2\chi_0^0\,{}^2\chi_2^2, \\ {}^4\chi_1^4 &= \tfrac{2}{7}\sqrt{2}(3\xi_+^2\xi_0\xi_- + 2\xi_+\xi_0^3), \qquad {}^4\chi_1^2 = {}^2\chi_0^0\,{}^2\chi_1^2, \\ {}^4\chi_0^4 &= \tfrac{4}{35}(3\xi_+^2\xi_-^2 + 12\xi_+\xi_0^2\xi_- + 2\xi_0^4), \quad {}^4\chi_0^2 = {}^2\chi_0^0\,{}^2\chi_0^2, \quad {}^4\chi_0^0 = ({}^2\chi_0^0)^2. \end{aligned} \tag{18d}$$

7 Spot Diagrams of Harmonic Oscillator States

The **spot diagram** of a transformation $\mathcal{M} : (\mathbf{p}, \mathbf{q}) \mapsto \big(\mathbf{p}'(\mathbf{p}, \mathbf{q}), \mathbf{q}'(\mathbf{p}, \mathbf{q})\big)$ is a projection of $\Re^4$ phase space on the screen plane, that pictures $\mathbf{q}'(\mathbf{p}, \mathbf{q})$, the image of a pencil of rays (range of $\mathbf{p}$) diverging from a *fixed* object point $\mathbf{q}$. If we let $\mathbf{p}$ mark a polar coordinate grid around the optical axis, we obtain the spots of Figures 9.1–9.5 [12]. These are the new "faces" of the harmonic oscillator states that we present in aberration optics.

8 Characterization of Optical Elements

Optical elements may be characterized by

$$\mathcal{G}\{A_4, A_3, A_2, \mathbf{M}\} = \exp\{A_4, \circ\}\ \exp\{A_3, \circ\}\ \exp\{A_2, \circ\}\ \exp\{A_1, \circ\}, \tag{19a}$$

with the coefficients their linear action $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det \mathbf{M} = 1$, and the coefficients $A_{k,j,m}$ of its aberration polynomials

$$A_k = \sum_{j=k,k-2,\ldots}^{1 \text{ or } 0}\ \sum_{m=j,j-1,\ldots}^{-j} A_{k,j,m}\,{}^k\chi_m^j(\vec{\xi}). \tag{19b}$$

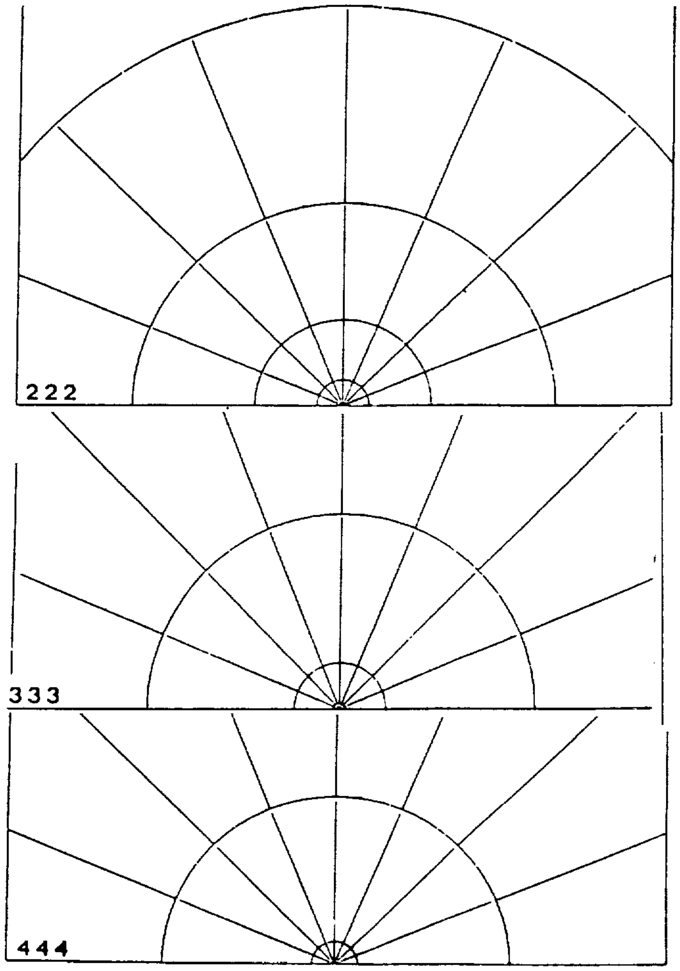

FIG. 9.1. **Spherical aberrations ⊙ generated by ${}^k\chi^k_k$ (at left we indicate the *kjm* classification of the spot diagram).**

Some of the most common optical elements thus represented are the following:

Free propagation: The paraxial part is $\mathbf{F}(z) = \begin{pmatrix} 1 & 0 \\ -z/n & 1 \end{pmatrix}$. The (spherical aberration) coefficients are:

$$f^z_{2,2,2} = -z/(8n^3), \quad f^z_{3,3,3} = -z/(16n^5), \quad f^z_{4,4,4} = -5z/(128n^7).$$

Elliptic-profile fiber free propagation in medium $n = \sqrt{\nu^2 - \mu^2 q^2}$ is treated in [13]. The

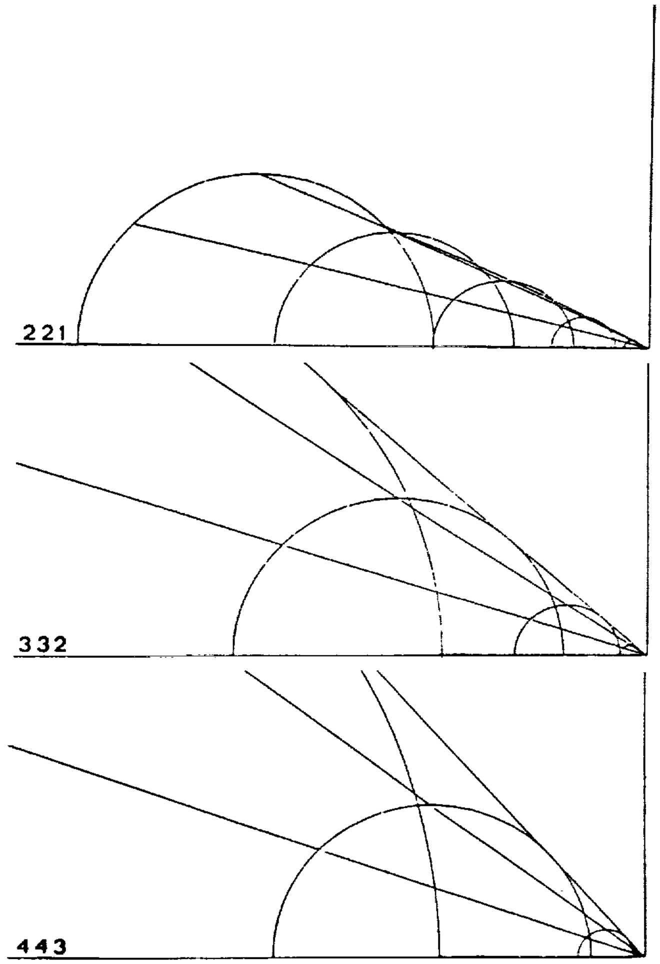

FIG. 9.2. Circular comas ⊙> generated by ${}^{k}\chi^{k}_{k-1}$.

paraxial part is $\mathbf{F}_{n(q)}(z) = \begin{pmatrix} \cos\frac{\mu z}{\nu} & \mu\sin\frac{\mu z}{\nu} \\ -\frac{1}{\mu}\sin\frac{\mu z}{\nu} & \cos\frac{\mu z}{\nu} \end{pmatrix}$. The aberration polynomials are:

$$F_2 = -z(|\mathbf{p}|^2 + \mu^2|\mathbf{q}|^2)^2/8\nu^3,$$
$$F_3 = -z(|\mathbf{p}|^2 + \mu^2|\mathbf{q}|^2)^3/16\nu^5,$$
$$F_4 = -5z(|\mathbf{p}|^2 + \mu^2|\mathbf{q}|^2)^4/128\nu^7.$$

The root transformation [14] indicated by $\mathcal{R}_{n;S}$ (that is the root of refraction —see below) in medium n associated to the surface $S(q^2) = \varsigma_2 q^2 + \varsigma_4 q^4 + \varsigma_6 q^6 + \cdots$. The paraxial part is $\mathbf{R}_{n,S} = \begin{pmatrix} 1 & -2n\varsigma_2 \\ 0 & 1 \end{pmatrix}$. The aberration coefficients $R_{k,j,m}$, arranged as row multi-vectors with components numbered by descending values of j and m, are

${}^2\mathbf{r} = \{\{0, 0, -\varsigma_2/(2n), 0, n\varsigma_4\}, \{-\varsigma_2/(3n)\}\},$

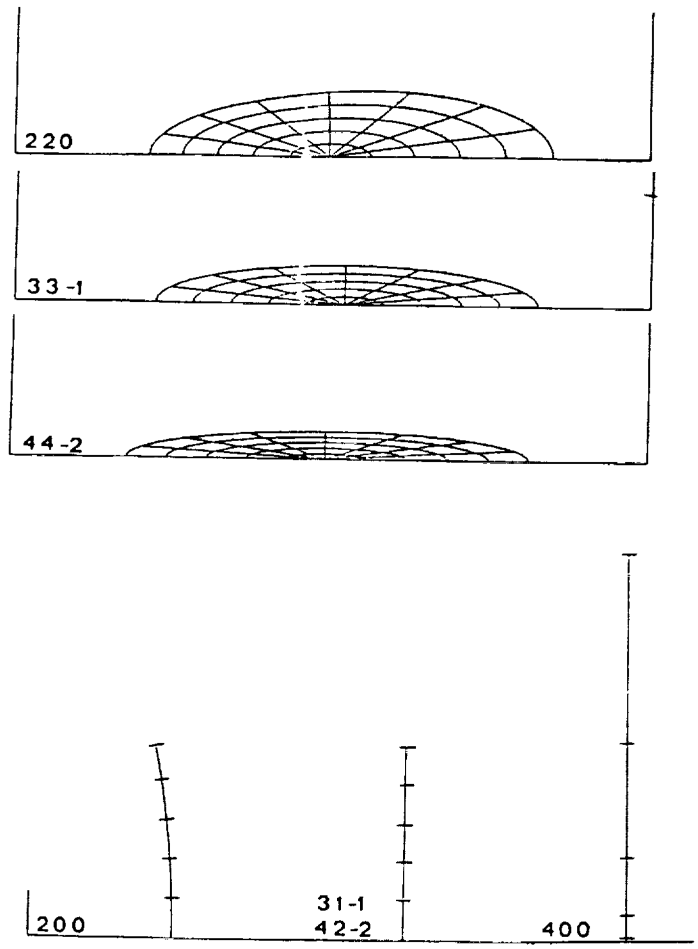

FIG. 9.3. The curvatism-astigmature m-degenerate pairs ⬭—|, generated by ${}^{k}\chi^{k}_{k-2}$ and ${}^{k}\chi^{k-2}_{k-2}$.

$${}^{3}\mathbf{r} = \{\{0, 0, -\varsigma_2/(8n^3), -\varsigma_2^2/(2n^2), -\varsigma_4/(2n), 2\varsigma_2\varsigma_4, n\varsigma_6\}, \{-\varsigma_2/(10n^3), -\varsigma_2^2/(5n^2), -2/5\varsigma_4/n\}\},$$

$${}^{4}\mathbf{r} = \{\{0, 0, -\varsigma_2/(16n^5), -\varsigma_2^2/(4n^4), -\varsigma_4/(8n^3) - 5/6\varsigma_2^3/n^3, -2\varsigma_2\varsigma_4/n^2, -\varsigma_6/(2n) + 16/3\varsigma_2^2\varsigma_4/n, 6\varsigma_2\varsigma_6, 8/3n\varsigma_2\varsigma_4^2 + n\varsigma_8\}, \{-3/56\varsigma_2/n^5, -\varsigma_2^2/(7n^4), -\varsigma_4/(7n^3) - 2/7\varsigma_2^3/n^3, -8/7\varsigma_2\varsigma_4/n^2, -3/7\varsigma_6/n - 16/21\varsigma_2^2\varsigma_4/n\}, \{-\varsigma_4/(15n^3)\}\}.$$

The refracting surface S between medium n and medium m is given by the factorization theorem [14], [15]

$$S_{n,m;S} = \mathcal{R}_{n,S}\, \mathcal{R}^{-1}_{m,S}. \tag{20}$$

The linear part is $\mathbf{S} = \begin{pmatrix} 1 & 2m\varsigma_2 - 2n\varsigma_2 \\ 0 & 1 \end{pmatrix}$ and the aberrations are:

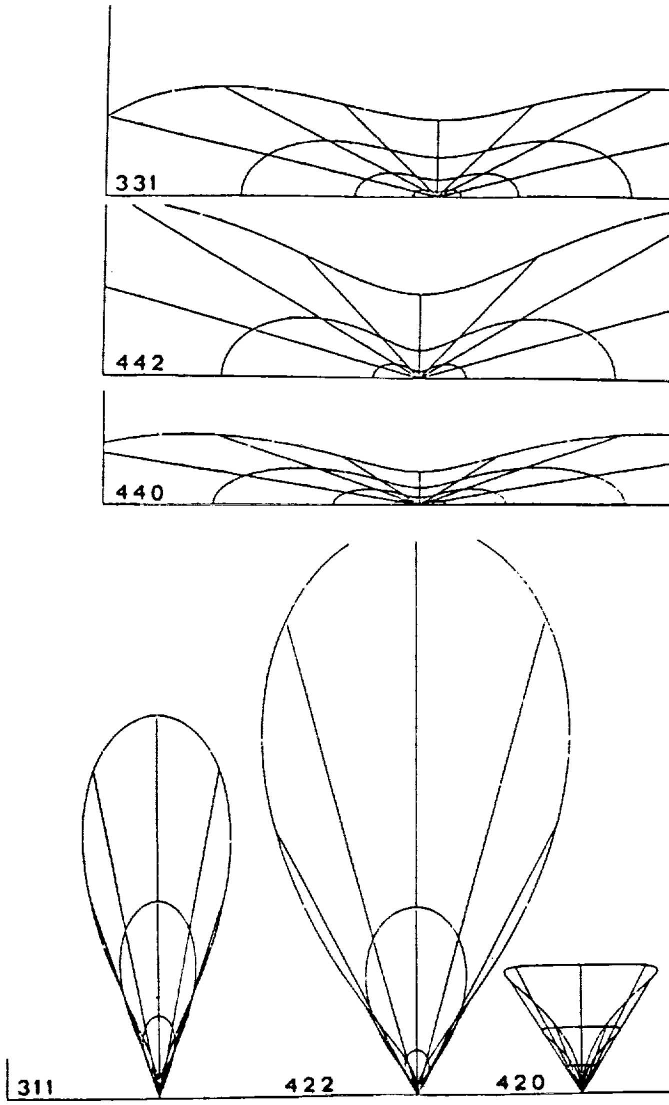

FIG. 9.4. **The m-degenerate pairs ⋈—⧗.**

$^{2}\mathrm{s} = \{\{0, 0, \varsigma_2/(2m) - \varsigma_2/(2n), 2n\varsigma_2^2/m - 2\varsigma_2^2, -m\varsigma_4 + 2m\varsigma_2^3 + n\varsigma_4 - 4n\varsigma_2^3 + 2n^2\varsigma_2^3/m\}, \{\varsigma_2/(3m) - \varsigma_2/(3n)\}\}$,

$^{3}\mathrm{s} = \{\{0,0,\varsigma_2/(8m^3) - \varsigma_2/(8n^3), n\varsigma_2^2/m^3 - \varsigma_2^2/(2m^2) - \varsigma_2^2/(2n^2), 3n^2\varsigma_2^3/m^3 - 3n\varsigma_2^3/m^2 + \varsigma_4/(2m) + 2\varsigma_2^3/m - \varsigma_4/(2n) - 2\varsigma_2^3/n, -2m\varsigma_2\varsigma_4/n + 4m\varsigma_2^4/n - 2\varsigma_2\varsigma_4 + 4n^3\varsigma_2^4/m^3 - 6n^2\varsigma_2^4/m^2 + 4n\varsigma_2\varsigma_4/m + 4n\varsigma_2^4/m - 6\varsigma_2^4, -m\varsigma_6 + 6m\varsigma_2^2\varsigma_4 - 2m\varsigma_2^5 + n\varsigma_6 - 12n\varsigma_2^2\varsigma_4 + 4n\varsigma_2^5 + 2n^4\varsigma_2^5/m^3 - 4n^3\varsigma_2^5/m^2 + 6n^2\varsigma_2^2\varsigma_4/m\}, \{\varsigma_2/(10m^3) - \varsigma_2/(10n^3), 2/5n\varsigma_2^2/m^3 - \varsigma_2^2/(5m^2) - \varsigma_2^2/(5n^2), 2/5n^2\varsigma_2^3/m^3 - 2/5n\varsigma_2^3/m^2 + 2/5\varsigma_4/m - 2/5\varsigma_2^3/m - 2/5\varsigma_4/n + 2/5\varsigma_2^3/n\}\}$,

$^{4}\mathrm{s} = \{\{0,0,\varsigma_2/(16m^5) - \varsigma_2/(16n^5), 3/4n\varsigma_2^2/m^5 - \varsigma_2^2/(4m^4) - \varsigma_2^2/(4m^3n) - \varsigma_2^2/(4n^4), 15/4n^2\varsigma_2^3/m^5 - 5/2n\varsigma_2^3/m^4 + \varsigma_4/(8m^3) - 5/12\varsigma_2^3/m^3 - \varsigma_4/(8n^3) - 5/6\varsigma_2^3/n^3, 10n^3\varsigma_2^4/m^5 - 10n^2\varsigma_2^4/m^4 + 3n\varsigma_2\varsigma_4/m^3 + 8/3n\varsigma_2^4/m^3 - 2\varsigma_2\varsigma_4/m^2 - 4\varsigma_2^4/m^2 + \varsigma_2\varsigma_4/(mn) + 4\varsigma_2^4/(mn) - 2\varsigma_2\varsigma_4/n^2 - 8/3\varsigma_2^4/n^2, -16/3m\varsigma_2^2\varsigma_4/n^2 +$

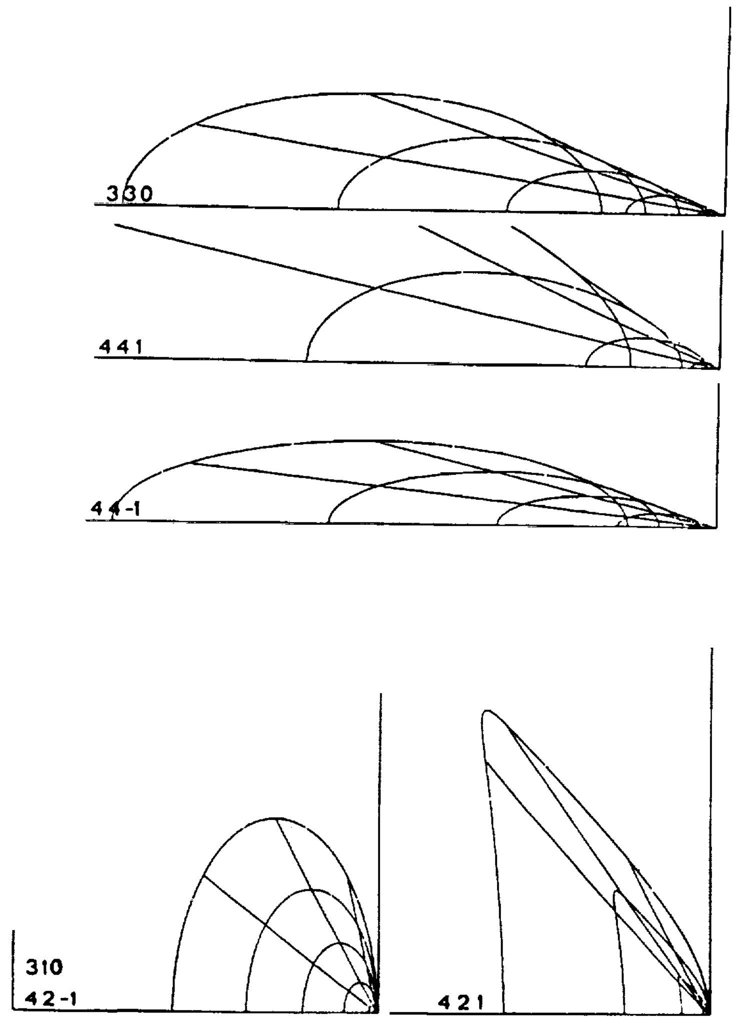

FIG. 9.5. Elliptical comas and their m-degenerate partners ⊂⊃▷—▷.

$32/3m\varsigma_2^5/n^2 + 15n^4\varsigma_2^5/m^5 - 20n^3\varsigma_2^5/m^4 + 15n^2\varsigma_2^2\varsigma_4/m^3 + 6n^2\varsigma_2^5/m^3 - 62/3n\varsigma_2^2\varsigma_4/m^2 -$
$20/3n\varsigma_2^5/m^2 + \varsigma_6/(2m) + 19\varsigma_2^2\varsigma_4/m + 11\varsigma_2^5/m - \varsigma_6/(2n) - 8\varsigma_2^2\varsigma_4/n - 16\varsigma_2^5/n, -6m\varsigma_2\varsigma_6/n +$
$116/3m\varsigma_2^3\varsigma_4/n - 52/3m\varsigma_2^6/n + 4\varsigma_2\varsigma_6 + 12n^5\varsigma_2^6/m^5 - 20n^4\varsigma_2^6/m^4 + 28n^3\varsigma_2^3\varsigma_4/m^3 - 4/3n^3\varsigma_2^6/m^3 -$
$176/3n^2\varsigma_2^3\varsigma_4/m^2 + 64/3n^2\varsigma_2^6/m^2 + 2n\varsigma_2\varsigma_6/m + 72n\varsigma_2^3\varsigma_4/m - 36n\varsigma_2^6/m + 4n\varsigma_4^2/m - 80\varsigma_2^3\varsigma_4 +$
$124/3\varsigma_2^6 - 4\varsigma_4^2, 8m\varsigma_2\varsigma_4^2 - m\varsigma_8 + 2m\varsigma_2^2\varsigma_6 + 86/3m\varsigma_2^4\varsigma_4 - 124/3m\varsigma_2^7 - 24n\varsigma_2\varsigma_4^2 + n\varsigma_8 - 4n\varsigma_2^2\varsigma_6 -$
$40n\varsigma_2^4\varsigma_4 + 200/3n\varsigma_2^7 + 4n^6\varsigma_2^7/m^5 - 8n^5\varsigma_2^7/m^4 + 18n^4\varsigma_2^4\varsigma_4/m^3 - 20/3n^4\varsigma_2^7/m^3 - 152/3n^3\varsigma_2^4\varsigma_4/m^2 +$
$112/3n^3\varsigma_2^7/m^2 + 40/3n^2\varsigma_2\varsigma_4^2/m + 2n^2\varsigma_2^2\varsigma_6/m + 164/3n^2\varsigma_2^4\varsigma_4/m - 188/3n^2\varsigma_2^7/m + 8/3m^2\varsigma_2\varsigma_4^2/n -$
$32/3m^2\varsigma_2^4\varsigma_4/n + 32/3m^2\varsigma_2^7/n\}, \{3/56\varsigma_2/m^5 - 3/56\varsigma_2/n^5, 3/7n\varsigma_2^2/m^5 - \varsigma_2^2/(7m^4) - \varsigma_2^2/(7m^3n) -$
$\varsigma_2^2/(7n^4), 9/7n^2\varsigma_2^3/m^5 - 6/7n\varsigma_2^3/m^4 + \varsigma_4/(7m^3) - \varsigma_2^3/(7m^3) - \varsigma_4/(7n^3) - 2/7\varsigma_2^3/n^3, 12/7n^3\varsigma_2^4/m^5 -$

$12/7n^2\varsigma_2^4/m^4 + 12/7n\varsigma_2\varsigma_4/m^3 - 8/7n\varsigma_2^4/m^3 - 8/7\varsigma_2\varsigma_4/m^2 + 12/7\varsigma_2^4/m^2 + 4/7\varsigma_2\varsigma_4/(mn) -$
$12/7\varsigma_2^4/(mn) - 8/7\varsigma_2\varsigma_4/n^2 + 8/7\varsigma_2^4/n^2, 16/21m\varsigma_2^2\varsigma_4/n^2 - 32/21m\varsigma_2^5/n^2 + 6/7n^4\varsigma_2^5/m^5 -$
$8/7n^3\varsigma_2^5/m^4 + 20/7n^2\varsigma_2^2\varsigma_4/m^3 - 20/7n^2\varsigma_2^5/m^3 - 64/21n\varsigma_2^2\varsigma_4/m^2 + 104/21n\varsigma_2^5/m^2 + 3/7\varsigma_6/m -$
$12/7\varsigma_2^2\varsigma_4/m - 18/7\varsigma_2^5/m - 3/7\varsigma_6/n + 8/7\varsigma_2^2\varsigma_4/n + 16/7\varsigma_2^5/n\}, \{\varsigma_4/(15m^3) - \varsigma_4/(15n^3)\}\}.$

The curved mirror transformation by S can be found from $\mathcal{M}_{n,S} = \mathcal{R}_{n,S}\mathcal{R}_{-n,S}^{-1}$ [16]. The refracting surface between two elliptic-profile fibers has been calculated in Ref. [13].

9 Multiplication Law in the Aberration Group

For aberration orders 1, 3, 5, and 7 (ranks k = 1, 2, 3, and 4), the dimensionality of the basis is, respectively: the 3 generators of $sp(2,\Re)$, the 6 third-order aberrations (separated into a quintuplet and a singlet), the 10 fifth-order aberrations (a septuplet and a triplet), and the 15 seventh-order aberrations (divided into a nonuplet, quintuplet and singlet). The number of parameters of the corresponding aberration group elements in Eqs. (11) and (19) thus accumulates to:

3	SP(2,$\Re$) (ABERRATION ORDER 1)
9	LINEAR + ABERRATION ORDER 3
19	UP TO ABERRATION ORDER 5
34	UP TO ABERRATION ORDER 7

If we indicate the seventh-order *pure aberration* group elements by the coefficients of the polynomials in

$$\mathcal{G}\{\mathbf{A}_4, \mathbf{A}_3, \mathbf{A}_2, \mathbf{1}\} = \exp\{A_4, \circ\}\ \exp\{A_3, \circ\}\ \exp\{A_2, \circ\}, \tag{21}$$

the central problem is to find the **multiplication table** involving the 31 up-to-seventh order aberration coefficients in the product

$$\mathcal{G}\{\mathbf{C}_4, \mathbf{C}_3, \mathbf{C}_2, \mathbf{1}\} = \mathcal{G}\{\mathbf{A}_4, \mathbf{A}_3, \mathbf{A}_2, \mathbf{1}\}\mathcal{G}\{\mathbf{B}_4, \mathbf{B}_3, \mathbf{B}_2, \mathbf{1}\}, \quad i.e. \quad \mathbf{C} = \mathbf{A} \# \mathbf{B}. \tag{22}$$

To find explicitly the *gato* operation # between the individual coefficients, we may use Baker-Campbell-Hausdorff relations. Order 3 is abelian; order 5 is the practical limit for hand calculations, and order 7 is definitely nontrivial and needs symbolic computation [12], [16]. The pure aberration group composition law was calculated once and for all, to find the composition of aberrations. It is:

Aberration order 3:

$$C_{2,j,m} = A_{2,j,m} + B_{2,j,m}, \qquad j = 2, 0, \quad m = j, j-1, \ldots, -j.$$

Aberration order 5:

$$C_{3,3,3} = 2A_{2,2,1}B_{2,2,2} - 2A_{2,2,2}B_{2,2,1} + A_{3,3,3} + B_{3,3,3},$$
$$C_{3,3,2} = 4A_{2,2,0}B_{2,2,2} - 4A_{2,2,2}B_{2,2,0} + A_{3,3,2} + B_{3,3,2},$$

$C_{3,3,1} = 6A_{2,2,-1}B_{2,2,2} + 2A_{2,2,0}B_{2,2,1} - 2A_{2,2,1}B_{2,2,0} - 6A_{2,2,2}B_{2,2,-1} + A_{3,3,1} + B_{3,3,1}$,

$C_{3,3,0} = 8A_{2,2,-2}B_{2,2,2} + 4A_{2,2,-1}B_{2,2,1} - 4A_{2,2,1}B_{2,2,-1} - 8A_{2,2,2}B_{2,2,-2} + A_{3,3,0} + B_{3,3,0}$,

$C_{3,3,-1} = 6A_{2,2,-2}B_{2,2,1} + 2A_{2,2,-1}B_{2,2,0} - 2A_{2,2,0}B_{2,2,-1} - 6A_{2,2,1}B_{2,2,-2} + A_{3,3,-1} + B_{3,3,-1}$,

$C_{3,3,-2} = 4A_{2,2,-2}B_{2,2,0} - 4A_{2,2,0}B_{2,2,-2} + A_{3,3,-2} + B_{3,3,-2}$,

$C_{3,3,-3} = 2A_{2,2,-2}B_{2,2,-1} - 2A_{2,2,-1}B_{2,2,-2} + A_{3,3,-3} + B_{3,3,-3}$;

$C_{3,1,1} = 4/5A_{2,2,-1}B_{2,2,2} - 2/5A_{2,2,0}B_{2,2,1} + 2/5A_{2,2,1}B_{2,2,0} - 4/5A_{2,2,2}B_{2,2,-1} + A_{3,1,1} + B_{3,1,1}$,

$C_{3,1,0} = 16/5A_{2,2,-2}B_{2,2,2} - 2/5A_{2,2,-1}B_{2,2,1} + 2/5A_{2,2,1}B_{2,2,-1} - 16/5A_{2,2,2}B_{2,2,-2} + A_{3,1,0} + B_{3,1,0}$,

$C_{3,1,-1} = 4/5A_{2,2,-2}B_{2,2,1} - 2/5A_{2,2,-1}B_{2,2,0} + 2/5A_{2,2,0}B_{2,2,-1} - 4/5A_{2,2,1}B_{2,2,-2} + A_{3,1,-1} + B_{3,1,-1}$.

Aberration order 7:

$C_{4,4,4} = 8A_{2,2,1}^2B_{2,2,2} + 32/3A_{2,2,2}^2B_{2,2,0} - 16/3A_{2,2,0}B_{2,2,2}^2 - 32/3A_{2,2,0}A_{2,2,2}B_{2,2,2} - 8A_{2,2,1}A_{2,2,2}B_{2,2,1} + 4A_{2,2,1}B_{2,2,1}B_{2,2,2} + 6A_{2,2,1}B_{3,3,3} - 4A_{2,2,2}B_{2,2,1}^2 + 16/3A_{2,2,2}B_{2,2,0}B_{2,2,2} - 4A_{2,2,2}B_{3,3,2} + A_{4,4,4} + B_{4,4,4}$,

$C_{4,4,3} = 32A_{2,2,2}^2B_{2,2,-1} - 16A_{2,2,-1}B_{2,2,2}^2 - 32A_{2,2,-1}A_{2,2,2}B_{2,2,2} + 64/3A_{2,2,0}A_{2,2,1}B_{2,2,2} - 80/3A_{2,2,0}A_{2,2,2}B_{2,2,1} - 8/3A_{2,2,0}B_{2,2,1}B_{2,2,2} + 12A_{2,2,0}B_{3,3,3} + 16/3A_{2,2,1}A_{2,2,2}B_{2,2,0} + 40/3A_{2,2,1}B_{2,2,0}B_{2,2,2} + 2A_{2,2,1}B_{3,3,2} + 16A_{2,2,2}B_{2,2,-1}B_{2,2,2} - 32/3A_{2,2,2}B_{2,2,0}B_{2,2,1} - 8A_{2,2,2}B_{3,3,1} + A_{4,4,3} + B_{4,4,3}$,

$C_{4,4,2} = 64/3A_{2,2,0}^2B_{2,2,2} + 8/3A_{2,2,1}^2B_{2,2,0} + 64A_{2,2,2}^2B_{2,2,-2} - 32A_{2,2,-2}B_{2,2,2}^2 - 64A_{2,2,-2}A_{2,2,2}B_{2,2,2} + 16A_{2,2,-1}A_{2,2,1}B_{2,2,2} - 56A_{2,2,-1}A_{2,2,2}B_{2,2,1} - 20A_{2,2,-1}B_{2,2,1}B_{2,2,2} + 18A_{2,2,-1}B_{3,3,3} - 4/3A_{2,2,0}B_{2,2,1}^2 - 8/3A_{2,2,0}A_{2,2,1}B_{2,2,1} - 64/3A_{2,2,0}A_{2,2,2}B_{2,2,0} + 32/3A_{2,2,0}B_{2,2,0}B_{2,2,2} + 8A_{2,2,0}B_{3,3,2} + 40A_{2,2,1}A_{2,2,2}B_{2,2,-1} + 28A_{2,2,1}B_{2,2,-1}B_{2,2,2} + 4/3A_{2,2,1}B_{2,2,0}B_{2,2,1} - 2A_{2,2,1}B_{3,3,1} - 32/3A_{2,2,2}B_{2,2,0}^2 + 32A_{2,2,2}B_{2,2,-2}B_{2,2,2} - 8A_{2,2,2}B_{2,2,-1}B_{2,2,1} - 12A_{2,2,2}B_{3,3,0} + A_{4,4,2} + B_{4,4,2}$,

$C_{4,4,1} = 16/3A_{2,2,0}^2B_{2,2,1} + 16A_{2,2,1}^2B_{2,2,-1} - 96A_{2,2,-2}A_{2,2,2}B_{2,2,1} - 48A_{2,2,-2}B_{2,2,1}B_{2,2,2} + 24A_{2,2,-2}B_{3,3,3} - 8A_{2,2,-1}B_{2,2,1}^2 + 160/3A_{2,2,-1}A_{2,2,0}B_{2,2,2} - 16A_{2,2,-1}A_{2,2,1}B_{2,2,1} - 176/3A_{2,2,-1}A_{2,2,2}B_{2,2,0} - 8/3A_{2,2,-1}B_{2,2,0}B_{2,2,2} + 14A_{2,2,-1}B_{3,3,2} - 16/3A_{2,2,0}A_{2,2,1}B_{2,2,0} + 16/3A_{2,2,0}A_{2,2,2}B_{2,2,-1} + 88/3A_{2,2,0}B_{2,2,-1}B_{2,2,2} + 8/3A_{2,2,0}B_{2,2,0}B_{2,2,1} + 4A_{2,2,0}B_{3,3,1} - 8/3A_{2,2,1}B_{2,2,0}^2 + 96A_{2,2,1}A_{2,2,2}B_{2,2,-2} + 48A_{2,2,1}B_{2,2,-2}B_{2,2,2} + 8A_{2,2,1}B_{2,2,-1}B_{2,2,1} - 6A_{2,2,1}B_{3,3,0} - 80/3A_{2,2,2}B_{2,2,-1}B_{2,2,0} - 16A_{2,2,2}B_{3,3,-1} + A_{4,4,1} + B_{4,4,1}$,

$C_{4,4,0} = 40A_{2,2,-1}^2B_{2,2,2} + 40A_{2,2,1}^2B_{2,2,-2} - 20A_{2,2,-2}B_{2,2,1}^2 + 160/3A_{2,2,-2}A_{2,2,0}B_{2,2,2} - 40A_{2,2,-2}A_{2,2,1}B_{2,2,1} - 320/3A_{2,2,-2}A_{2,2,2}B_{2,2,0} - 80/3A_{2,2,-2}B_{2,2,0}B_{2,2,2} + 20A_{2,2,-2}B_{3,3,2} + 40/3A_{2,2,-1}A_{2,2,0}B_{2,2,1} - 80/3A_{2,2,-1}A_{2,2,1}B_{2,2,0} - 40A_{2,2,-1}A_{2,2,2}B_{2,2,-1} + 20A_{2,2,-1}B_{2,2,-1}B_{2,2,2} - 20/3A_{2,2,-1}B_{2,2,0}B_{2,2,1} + 10A_{2,2,-1}B_{3,3,1} + 40/3A_{2,2,0}A_{2,2,1}B_{2,2,-1} + 160/3A_{2,2,0}A_{2,2,2}B_{2,2,-2} + 160/3A_{2,2,0}B_{2,2,-2}B_{2,2,2} + 40/3A_{2,2,0}B_{2,2,-1}B_{2,2,1} + 20A_{2,2,1}B_{2,2,-2}B_{2,2,1} - 20/3A_{2,2,1}B_{2,2,-1}B_{2,2,0} - 10A_{2,2,1}B_{3,3,-1} - 20A_{2,2,2}B_{2,2,-1}^2 - 80/3A_{2,2,2}B_{2,2,-2}B_{2,2,0} - 20A_{2,2,2}B_{3,3,-2} + A_{4,4,0} + B_{4,4,0}$,

$C_{4,4,-1} = 16A_{2,2,-1}^2B_{2,2,1} + 16/3A_{2,2,0}^2B_{2,2,-1} + 96A_{2,2,-2}A_{2,2,-1}B_{2,2,2} + 16/3A_{2,2,-2}A_{2,2,0}B_{2,2,1} -$

$176/3A_{2,2,-2}A_{2,2,1}B_{2,2,0}-96A_{2,2,-2}A_{2,2,2}B_{2,2,-1}-80/3A_{2,2,-2}B_{2,2,0}B_{2,2,1}+16A_{2,2,-2}B_{3,3,1}-8/3A_{2,2,-1}B_{2,2,0}^2-16/3A_{2,2,-1}A_{2,2,0}B_{2,2,0}-16A_{2,2,-1}A_{2,2,1}B_{2,2,-1}+48A_{2,2,-1}B_{2,2,-2}B_{2,2,2}+8A_{2,2,-1}B_{2,2,-1}B_{2,2,1}+6A_{2,2,-1}B_{3,3,0}+160/3A_{2,2,0}A_{2,2,1}B_{2,2,-2}+88/3A_{2,2,0}B_{2,2,-2}B_{2,2,1}+8/3A_{2,2,0}B_{2,2,-1}B_{2,2,0}-4A_{2,2,0}B_{3,3,-1}-8A_{2,2,1}B_{2,2,-1}^2-8/3A_{2,2,1}B_{2,2,-2}B_{2,2,0}-14A_{2,2,1}B_{3,3,-2}-48A_{2,2,2}B_{2,2,-2}B_{2,2,-1}-24A_{2,2,2}B_{3,3,-3}+A_{4,4,-1}+B_{4,4,-1},$

$C_{4,4,-2}=64A_{2,2,-2}^2B_{2,2,2}+8/3A_{2,2,-1}^2B_{2,2,0}+64/3A_{2,2,0}^2B_{2,2,-2}-32/3A_{2,2,-2}B_{2,2,0}^2+40A_{2,2,-2}A_{2,2,-1}B_{2,2,1}-64/3A_{2,2,-2}A_{2,2,0}B_{2,2,0}-56A_{2,2,-2}A_{2,2,1}B_{2,2,-1}-64A_{2,2,-2}A_{2,2,2}B_{2,2,-2}+32A_{2,2,-2}B_{2,2,-2}B_{2,2,2}-8A_{2,2,-2}B_{2,2,-1}B_{2,2,1}+12A_{2,2,-2}B_{3,3,0}-8/3A_{2,2,-1}A_{2,2,0}B_{2,2,-1}+16A_{2,2,-1}A_{2,2,1}B_{2,2,-2}+28A_{2,2,-1}B_{2,2,-2}B_{2,2,1}+4/3A_{2,2,-1}B_{2,2,-1}B_{2,2,0}+2A_{2,2,-1}B_{3,3,-1}-4/3A_{2,2,0}B_{2,2,-1}^2+32/3A_{2,2,0}B_{2,2,-2}B_{2,2,0}-8A_{2,2,0}B_{3,3,-2}-20A_{2,2,1}B_{2,2,-2}B_{2,2,-1}-18A_{2,2,1}B_{3,3,-3}-32A_{2,2,2}B_{2,2,-2}^2+A_{4,4,-2}+B_{4,4,-2},$

$C_{4,4,-3}=32A_{2,2,-2}^2B_{2,2,1}+16/3A_{2,2,-2}A_{2,2,-1}B_{2,2,0}-80/3A_{2,2,-2}A_{2,2,0}B_{2,2,-1}-32A_{2,2,-2}A_{2,2,1}B_{2,2,-2}+16A_{2,2,-2}B_{2,2,-2}B_{2,2,1}-32/3A_{2,2,-2}B_{2,2,-1}B_{2,2,0}+8A_{2,2,-2}B_{3,3,-1}+64/3A_{2,2,-1}A_{2,2,0}B_{2,2,-2}+40/3A_{2,2,-1}B_{2,2,-2}B_{2,2,0}-2A_{2,2,-1}B_{3,3,-2}-8/3A_{2,2,0}B_{2,2,-2}B_{2,2,-1}-12A_{2,2,0}B_{3,3,-3}-16A_{2,2,1}B_{2,2,-2}^2+A_{4,4,-3}+B_{4,4,-3},$

$C_{4,4,-4}=32/3A_{2,2,-2}^2B_{2,2,0}+8A_{2,2,-1}^2B_{2,2,-2}-4A_{2,2,-2}B_{2,2,-1}^2-8A_{2,2,-2}A_{2,2,-1}B_{2,2,-1}-32/3A_{2,2,-2}A_{2,2,0}B_{2,2,-2}+16/3A_{2,2,-2}B_{2,2,-2}B_{2,2,0}+4A_{2,2,-2}B_{3,3,-2}+4A_{2,2,-1}B_{2,2,-2}B_{2,2,-1}-6A_{2,2,-1}B_{3,3,-3}-16/3A_{2,2,0}B_{2,2,-2}^2+A_{4,4,-4}+B_{4,4,-4};$

$C_{4,2,2}=-64/21A_{2,2,0}^2B_{2,2,2}-8/21A_{2,2,1}^2B_{2,2,0}+256/21A_{2,2,2}^2B_{2,2,-2}-128/21A_{2,2,-2}B_{2,2,2}^2-256/21A_{2,2,-2}A_{2,2,2}B_{2,2,2}+176/21A_{2,2,-1}A_{2,2,1}B_{2,2,2}-16/3A_{2,2,-1}A_{2,2,2}B_{2,2,1}+32/21A_{2,2,-1}B_{2,2,1}B_{2,2,2}+24/7A_{2,2,-1}B_{3,3,3}+4/21A_{2,2,0}B_{2,2,1}^2+8/21A_{2,2,0}A_{2,2,1}B_{2,2,1}+64/21A_{2,2,0}A_{2,2,2}B_{2,2,0}-32/21A_{2,2,0}B_{2,2,0}B_{2,2,2}-8/7A_{2,2,0}B_{3,3,2}-64/21A_{2,2,1}A_{2,2,2}B_{2,2,-1}+8/3A_{2,2,1}B_{2,2,-1}B_{2,2,2}-4/21A_{2,2,1}B_{2,2,0}B_{2,2,1}+2A_{2,2,1}B_{3,1,1}+24/35A_{2,2,1}B_{3,3,1}+32/21A_{2,2,2}B_{2,2,0}^2+128/21A_{2,2,2}B_{2,2,-2}B_{2,2,2}-88/21A_{2,2,2}B_{2,2,-1}B_{2,2,1}-4A_{2,2,2}B_{3,1,0}-24/35A_{2,2,2}B_{3,3,0}+A_{4,2,2}+B_{4,2,2},$

$C_{4,2,1}=-16/7A_{2,2,0}^2B_{2,2,1}-88/21A_{2,2,1}^2B_{2,2,-1}+64/3A_{2,2,-2}A_{2,2,1}B_{2,2,2}-704/21A_{2,2,-2}A_{2,2,2}B_{2,2,1}-128/21A_{2,2,-2}B_{2,2,1}B_{2,2,2}+96/7A_{2,2,-2}B_{3,3,3}+44/21A_{2,2,-1}B_{2,2,1}^2-32/21A_{2,2,-1}A_{2,2,0}B_{2,2,2}+88/21A_{2,2,-1}A_{2,2,1}B_{2,2,1}-32/21A_{2,2,-1}A_{2,2,2}B_{2,2,0}-32/21A_{2,2,-1}B_{2,2,0}B_{2,2,2}+16/7A_{2,2,0}A_{2,2,1}B_{2,2,0}+64/21A_{2,2,0}A_{2,2,2}B_{2,2,-1}+16/21A_{2,2,0}B_{2,2,-1}B_{2,2,2}-8/7A_{2,2,0}B_{2,2,0}B_{2,2,1}+4A_{2,2,0}B_{3,1,1}-32/35A_{2,2,0}B_{3,3,1}+8/7A_{2,2,1}B_{2,2,0}^2+256/21A_{2,2,1}A_{2,2,2}B_{2,2,-2}+352/21A_{2,2,1}B_{2,2,-2}B_{2,2,2}-44/21A_{2,2,1}B_{2,2,-1}B_{2,2,1}-2A_{2,2,1}B_{3,1,0}+48/35A_{2,2,1}B_{3,3,0}-32/3A_{2,2,2}B_{2,2,-2}B_{2,2,1}+16/21A_{2,2,2}B_{2,2,-1}B_{2,2,0}-8A_{2,2,2}B_{3,1,-1}-96/35A_{2,2,2}B_{3,3,-1}+A_{4,2,1}+B_{4,2,1},$

$C_{4,2,0}=-16/7A_{2,2,-1}^2B_{2,2,2}-16/7A_{2,2,1}^2B_{2,2,-2}+8/7A_{2,2,-2}B_{2,2,1}^2+128/7A_{2,2,-2}A_{2,2,0}B_{2,2,2}+16/7A_{2,2,-2}A_{2,2,1}B_{2,2,1}-256/7A_{2,2,-2}A_{2,2,2}B_{2,2,0}-64/7A_{2,2,-2}B_{2,2,0}B_{2,2,2}+48/7A_{2,2,-2}B_{3,3,2}-24/7A_{2,2,-1}A_{2,2,0}B_{2,2,1}+48/7A_{2,2,-1}A_{2,2,1}B_{2,2,0}+16/7A_{2,2,-1}A_{2,2,2}B_{2,2,-1}-8/7A_{2,2,-1}B_{2,2,-1}B_{2,2,2}+12/7A_{2,2,-1}B_{2,2,0}B_{2,2,1}+6A_{2,2,-1}B_{3,1,1}-48/35A_{2,2,-1}B_{3,3,1}-24/7A_{2,2,0}A_{2,2,1}B_{2,2,-1}+128/7A_{2,2,0}A_{2,2,2}B_{2,2,-2}+128/7A_{2,2,0}B_{2,2,-2}B_{2,2,2}-24/7A_{2,2,0}B_{2,2,-1}B_{2,2,1}-8/7A_{2,2,1}B_{2,2,-2}B_{2,2,1}+12/7A_{2,2,1}B_{2,2,-1}B_{2,2,0}-6A_{2,2,1}B_{3,1,-1}+48/35A_{2,2,1}B_{3,3,-1}+8/7A_{2,2,2}B_{2,2,-1}^2-$

$$64/7A_{2,2,2}B_{2,2,-2}B_{2,2,0} - 48/7A_{2,2,2}B_{3,3,-2} + A_{4,2,0} + B_{4,2,0},$$

$$C_{4,2,-1} = -88/21A^2_{2,2,-1}B_{2,2,1} - 16/7A^2_{2,2,0}B_{2,2,-1} + 256/21A_{2,2,-2}A_{2,2,-1}B_{2,2,2} + 64/21A_{2,2,-2}A_{2,2,0}B_{2,2,1} - 32/21A_{2,2,-2}A_{2,2,1}B_{2,2,0} - 704/21A_{2,2,-2}A_{2,2,2}B_{2,2,-1} - 32/3A_{2,2,-2}B_{2,2,-1}B_{2,2,2} + 16/21A_{2,2,-2}B_{2,2,0}B_{2,2,1} + 8A_{2,2,-2}B_{3,1,1} + 96/35A_{2,2,-2}B_{3,3,1} + 8/7A_{2,2,-1}B^2_{2,2,0} + 16/7A_{2,2,-1}A_{2,2,0}B_{2,2,0} + 88/21A_{2,2,-1}A_{2,2,1}B_{2,2,-1} + 64/3A_{2,2,-1}A_{2,2,2}B_{2,2,-2} + 352/21A_{2,2,-1}B_{2,2,-2}B_{2,2,2} - 44/21A_{2,2,-1}B_{2,2,-1}B_{2,2,1} + 2A_{2,2,-1}B_{3,1,0} - 48/35A_{2,2,-1}B_{3,3,0} - 32/21A_{2,2,0}A_{2,2,1}B_{2,2,-2} + 16/21A_{2,2,0}B_{2,2,-2}B_{2,2,1} - 8/7A_{2,2,0}B_{2,2,-1}B_{2,2,0} - 4A_{2,2,0}B_{3,1,-1} + 32/35A_{2,2,0}B_{3,3,-1} + 44/21A_{2,2,1}B^2_{2,2,-1} - 32/21A_{2,2,1}B_{2,2,-2}B_{2,2,0} - 128/21A_{2,2,2}B_{2,2,-2}B_{2,2,-1} - 96/7A_{2,2,2}B_{3,3,-3} + A_{4,2,-1} + B_{4,2,-1},$$

$$C_{4,2,-2} = 256/21A^2_{2,2,-2}B_{2,2,2} - 8/21A^2_{2,2,-1}B_{2,2,0} - 64/21A^2_{2,2,0}B_{2,2,-2} + 32/21A_{2,2,-2}B^2_{2,2,0} - 64/21A_{2,2,-2}A_{2,2,-1}B_{2,2,1} + 64/21A_{2,2,-2}A_{2,2,0}B_{2,2,0} - 16/3A_{2,2,-2}A_{2,2,1}B_{2,2,-1} - 256/21A_{2,2,-2}A_{2,2,2}B_{2,2,-2} + 128/21A_{2,2,-2}B_{2,2,-2}B_{2,2,2} - 88/21A_{2,2,-2}B_{2,2,-1}B_{2,2,1} + 4A_{2,2,-2}B_{3,1,0} + 24/35A_{2,2,-2}B_{3,3,0} + 8/21A_{2,2,-1}A_{2,2,0}B_{2,2,-1} + 176/21A_{2,2,-1}A_{2,2,1}B_{2,2,-2} + 8/3A_{2,2,-1}B_{2,2,-2}B_{2,2,1} - 4/21A_{2,2,-1}B_{2,2,-1}B_{2,2,0} - 2A_{2,2,-1}B_{3,1,-1} - 24/35A_{2,2,-1}B_{3,3,-1} + 4/21A_{2,2,0}B^2_{2,2,-1} - 32/21A_{2,2,0}B_{2,2,-2}B_{2,2,0} + 8/7A_{2,2,0}B_{3,3,-2} + 32/21A_{2,2,1}B_{2,2,-2}B_{2,2,-1} - 24/7A_{2,2,1}B_{3,3,-3} - 128/21A_{2,2,2}B^2_{2,2,-2} + A_{4,2,-2} + B_{4,2,-2};$$

$$C_{4,0,0} = A_{4,0,0} + B_{4,0,0}.$$

This *gato* operation # is a noncommutative product, here expressed in the basis of the three-dimensional harmonic oscillator states. It has several properties that link the to physical properties of the optical elements with mathematical statements on selection rules [10]. But further, if we count the number of terms in the preceding gato operation in the symplectic bases, and a corresponding count in the monomial bases, we find some economy in the symplectic basis [16]:

BASIS	ORDER 3	ORDER 5	ORDER 7
MONOMIAL	12	54	422
SYMPLECTIC	12	52	318

10 Economy in Aberration Calculations

Two general linear (M) and aberration (A) group elements multiply through

$$\mathcal{G}\{\mathbf{A},\mathbf{M}\}\ \mathcal{G}\{\mathbf{B},\mathbf{N}\} = \mathcal{G}\{\mathbf{A} \# \mathbf{D}(\mathbf{M})\mathbf{B}, \mathbf{MN}\}. \tag{23}$$

Now, the **symplectic basis** ${}^{k}\chi^{j}_{m}$ is block-diagonal under paraxial —linear— transformations, with a matrix composed of the (analytically continued) Wigner D-matrices:

$$
\begin{pmatrix}
\boxed{\mathbf{D}^1_{[3\times 3]}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \boxed{\begin{matrix} \boxed{\mathbf{D}^2_{[5\times 5]}} & \mathbf{0} \\ \mathbf{0} & \boxed{\mathbf{D}^0_{[1\times 1]}} \end{matrix}} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \boxed{\begin{matrix} \boxed{\mathbf{D}^3_{[7\times 7]}} & \mathbf{0} \\ \mathbf{0} & \boxed{\mathbf{D}^1_{[3\times 3]}} \end{matrix}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \boxed{\begin{matrix} \boxed{\mathbf{D}^4_{[9\times 9]}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boxed{\mathbf{D}^2_{[5\times 5]}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boxed{\mathbf{D}^0_{[1\times 1]}} \end{matrix}}
\end{pmatrix}
$$

The number of non-zero matrix elements in the two bases is

BASIS	ORDER 1	ORDER 3	ORDER 5	ORDER 7	$k \to \infty$
MONOMIAL	9	36	100	225	$\sim k^4/4$
SYMPLECTIC	9	26	58	107	$\sim 2k^3/3$

The total number of operations necessary to 7^{th} aberration order, including matrix multiplication by the linear part is, for the two bases,

monomial : 7680, symplectic : 3882.

For this reason we conclude that the most efficient basis to carry through aberration computations in axis-symmetric systems is the angular momentum basis of the harmonic oscillator states.

11 Outlook

This has been a quick revision of the state of the art in seventh order aberration calculations in axis-symmetric geometric-optics systems. We have seen that the state schemes of the harmonic oscillator provide order and symmetry in the classification of aberrations. In these conference proceedings we cannot go much further, so let me state that *wave* optics is the true objective of this quest. We can design and specify systems in geometric optics; once this is done we would like to predict the imaging behavior of such a system when light of a definite color is used.

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IV. SPECIAL RELATIVITY