

# POINT FORM RELATIVISTIC QUANTUM MECHANICS AND RELATIVISTIC $SU(6)$

W. H. Klink

*Department of Physics and Astronomy  
University of Iowa, Iowa City, Iowa 52242*

## Abstract

The point form is used as a framework for formulating a relativistic quantum mechanics, with the mass operator carrying the interactions of underlying constituents. A symplectic Lie algebra of mass operators is introduced from which a relativistic harmonic oscillator mass operator is formed. Mass splittings within the degenerate harmonic oscillator levels arise from relativistically invariant spin-spin, spin-orbit and tensor mass operators. Internal flavor (and color) symmetries are introduced which make it possible to formulate a relativistic  $SU(6)$  model of baryons (and mesons). Careful attention is paid to the permutation symmetry properties of the hadronic wave functions, which are written as polynomials in Bargmann spaces.

## 1 Relativistic Introduction

Despite many successes, one of the main difficulties of the old  $SU(6)$  theory [1] was that the underlying quantum mechanics was nonrelativistic. In this paper we combine what Dirac called the point form of relativistic quantum mechanics [2] with an  $SU(3)$  flavor internal symmetry to formulate a relativistic  $SU(6)$  theory. The goal is to be able to get hadronic bound-state wave functions and then compute form factors, structure functions, decay rates, and even production scattering amplitudes. In this paper we restrict our attention to formulating a relativistic  $SU(6)$  theory, and then introduce a harmonic oscillator mass operator to obtain bound-state wave functions.

We will view hadrons as bound states of underlying spin  $\frac{1}{2}$  constituents which carry internal  $SU(3)$  flavor and  $SU(3)$  color degrees of freedom. Combining a (relativistic)  $SU(2)$  spin with  $SU(3)$  flavor then leads to a (relativistic)  $SU(6)$  spin-flavor group. The relativistic  $SU(2)$  spin structure comes from properties of the Pauli-Lubanski operator; as will be shown in Section 2 properly chosen sets of four vectors dotted into the Pauli-Lubanski operator generate an  $SU(2)$  Lie algebra. Moreover in the point form of relativistic quantum mechanics all Lorentz transformations are kinematic.  $n$ -particle constituent states called velocity states have the property that under Lorentz transformations the internal momenta and spins are uniformly rotated by a Wigner rotation, meaning that the relativistic  $SU(2)$  spin structure can be extended to  $n$ -particle systems.

In the point form of relativistic quantum mechanics, the four-momentum operator supplies the dynamical information. The (interacting) four-momentum operator is written as  $P^\mu = MV^\mu$ , where  $M$  is the mass operator and  $V^\mu$  the four-velocity operator. The Hamiltonian is then  $H = P^0 = MV^0$ . Since Lorentz transformations are kinematic and the mass operator commutes with all Poincaré operators, the theory is Lorentz covariant. As discussed in Section 3 mass operators are self-adjoint operators on the  $n$ -constituent Hilbert space that commute with Lorentz transformations and the velocity operator. Of particular interest for hadron spectroscopy are confining potentials; in Section 3 we will show how to construct relativistic harmonic oscillator potentials algebraically, using generators of an underlying symplectic group.

To obtain a realistic hadronic mass spectrum mass operators that split the degenerate oscillator levels are needed. We show that spin-orbit, spin-spin, and tensor operators are all readily introduced in the context of point form quantum mechanics. Moreover it is straightforward to construct mass operators out of internal symmetry generators; these operators can be used to obtain Gell-Mann-Okubo and Gürsey-Radicati type mass formulae. Such operators have the usual internal symmetry transformation properties, but the mass splittings are not given by Clebsch-Gordan coefficients, but by matrix elements of the appropriate mass operator.

The Hilbert space of  $n$ -constituents are tensor products of representation spaces of the Poincaré and internal symmetry groups. The relativistic kinematics of  $n$ -particle systems is discussed in Section 2. A hadronic wave function is an appropriately symmetrized wave function containing spatial, spin-flavor, and color pieces. As shown in Section 4 it is convenient to carry out the detailed calculations of the wave functions in Bargmann space, rather than the usual Hilbert space. Thus, the color, spin-flavor, and spatial parts of the wave function are all realized as polynomials in Bargmann spaces; the connection between these polynomials and wave functions in the usual Hilbert space is then given in terms of creation operators acting on a vacuum state.

## 2 Relativistic Kinematics

The Hilbert space of  $n$ -constituent particles is the  $n$ -fold tensor product of single-particle spaces which are the representation spaces of the Poincaré group corresponding to particles of mass  $m$  and spin  $j$  ( $j = \frac{1}{2}$ ). In this paper we take the masses of the constituents to be nonzero; in a later paper we will investigate the properties of hadrons as bound states of massless constituents.

For particles of mass  $m > 0$  and spin  $j$  the representations of the Poincaré group are well known [3], with the representation space  $\mathcal{H} = L^2(\mathbf{R}^3) \times V^j$ . The action of unitary operators corresponding to Lorentz transformations and space-time translations is given by

$$U_\Lambda |pj\sigma f\rangle = \sum_{\sigma'} |\Lambda p, j\sigma' f\rangle D_{\sigma',\sigma}^j(p, \Lambda)$$

$$U_a |pj\sigma f\rangle = e^{ip \cdot a} |pj\sigma f\rangle \quad (2.1)$$

where  $\Lambda \in SO(1, 3)$  is a Lorentz transformation,  $a \in \mathbf{R}^4$  is a space-time translation and  $p \cdot a := p^T g a$  is the Lorentz invariant inner product with the metric  $g = \text{diag}(1, -1, -1, -1)$ .  $\sigma$  is a spin projection variable and  $(p, \Lambda) \in SO(3)$  is a Wigner rotation defined by

$$(p, \Lambda) := B^{-1}(\Lambda p) \Lambda B(p) \in SO(3), \quad (2.2)$$

with  $B(p)$  a boost (coset representative of  $SO(1,3)/SO(3)$ ) satisfying  $p = B(p)p^{\text{rest}}$ ,  $p^{\text{rest}} = (m, 0, 0, 0)$ .  $D^j(\ )$  is an  $SO(3)$  matrix element and  $f$  is an internal symmetry label to be discussed below.

The four-vector momentum  $p$  satisfies  $p \cdot p = m^2$ . There is some ambiguity in specifying  $p$  which corresponds roughly to the different forms of relativistic dynamics. In the instant form  $p$  is written as  $(E, \vec{p})$ , with  $E = \sqrt{m^2 + \vec{p} \cdot \vec{p}}$  and wave functions are written as  $\varphi(\vec{p}, \sigma)$ . The kinematic subgroup consists of rotations  $R \in SO(3)$  and space translations  $\vec{a}$ . Interactions are introduced in the Hamiltonian  $\mathcal{H}$  and pure Lorentz generators. The instant form of dynamics has been used to obtain hadronic wave functions and form factors by several groups [4,5].

Another possibility is to write  $p$  as  $p_{\perp} = p_x + ip_y$ ,  $p_+ = E + p_z$ , so that  $p_- = [(|p_{\perp}|^2 + m^2)/p_+]$  and wave functions are written as  $\varphi(p_+, p_{\perp}, \sigma)$ . In the front form of relativistic dynamics the kinematic subgroup is the two-dimensional Euclidean subgroup  $E(2)$  of the Lorentz group, along with the translations  $a_{\perp} = a_x + ia_y$  and  $a_+ = a_0 + a_z$ . In this case the dynamics is introduced in the  $p_+$  generators, as seen in Refs. [6] and [7].

In the point form of relativistic dynamics to be used in this paper,  $p$  is written as  $p = mv$ , with  $v$  the four-velocity satisfying  $v \cdot v = 1$ . In this case wave functions are written as  $\varphi(v, \sigma)$  and the kinematic subgroup is the full Lorentz group  $SO(1,3)$ , while the dynamics is introduced in the four-momentum operator  $P^{\mu}$ .

Interactions will be introduced in Section 3. To see how they came about, it is first necessary to get the free mass and spin operators. The infinitesimal transformations of Eq. (2.1) generate the operators  $J^{\alpha\beta}$  and  $P_0^{\mu}$ , the free Lorentz and four-momentum operators. From these it is possible to form the free mass, velocity, and spin operators:

$$\begin{aligned} M_0^2 &:= P_0 \cdot P_0 \quad , \quad V_{(0)}^{\mu} := P_{(0)}^{\mu} M_0^{-1} \\ W_{\mu}^{(0)} &= \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} J^{\alpha\beta} P_{(0)}^{\nu} \quad (\text{Pauli-Lubanski operator}) \\ \widetilde{W}_{\mu}^{(0)} &:= \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} J^{\alpha\beta} V_{(0)}^{\nu} \quad (\text{modified Pauli-Lubanski operator}) \end{aligned} \quad (2.3)$$

$V_{(0)}^{\mu}$  is the free four-velocity operator and will be used extensively in the following sections. Notice that there is no "0" subscript or superscript on the  $J^{\alpha\beta}$  operators, because in the point form these operators are not modified in the presence of interactions. The labels  $p$ ,  $j$ , and  $\sigma$  appearing in Eq. (2.1) are now seen to be eigenvalues of the operators  $P_{(0)}^{\mu} = M_0 V_{(0)}^{\mu}$ ,  $\widetilde{W} \cdot \widetilde{W}$ , and  $n \cdot \widetilde{W}$ , respectively, with  $n$  a four vector;  $\widetilde{W} \cdot \widetilde{W}$  has eigenvalues  $j(j+1)$  with no mass factor (see Ref. [3] for details).

Besides the space-time and Lorentz transformations of single-particle constituents given in Eq. (2.1), there are also internal symmetry transformations which mix charges and other internal symmetry quantum numbers. Let  $G$  be an internal symmetry group (such as  $SU(2)$  isospin or  $SU(3)$  flavor or  $SU(3)$  color or a direct product) for which there is a unitary representation operator  $U_g$ ,  $g \in G$ , acting on a vector space  $V$  with basis  $|f\rangle$ . Then

$$U_g |p j \sigma f\rangle = \sum_{f'} |p, j \sigma f'\rangle D_{f'f}(g) \quad (2.4)$$

gives the action of  $g \in G$  on the basis state  $|pj\sigma f\rangle$ ;  $D_{f'f}(g)$  is a matrix element of  $G$ . Thus  $|pj\sigma f\rangle$  is a basis state for the one-particle Hilbert space  $L^2(\mathbb{R}^3) \times V^j \times V$ .

As shown in Ref. [3]  $(j, \sigma)$  are eigenvalues of relativistic operators that form a Lie algebra of  $SU(2)$ . Let  $n, n_{\pm}$  be four vectors such that  $n \cdot \widetilde{W}, n_{\pm} \cdot \widetilde{W}$  form the Lie algebra of  $SU(2)$  with  $\sigma$  the eigenvalue of  $n \cdot \widetilde{W}$  and  $j(j+1)$  the eigenvalue of  $\widetilde{W} \cdot \widetilde{W}$ . If  $j$  is chosen to be  $\frac{1}{2}$ , the spin of the constituents, and  $G$  is chosen to be  $SU(3)_{\text{flavor}}$ , the two algebras, relativistic  $SU(2)$  and  $SU(3)_f$ , can be combined to give a larger algebra, namely  $SU(6)$ . The labels  $(\sigma, f)$  in the basis state are jointly transformed under the action of  $SU(6)$ ; that is

$$U_g |p, j = \frac{1}{2}, \sigma, f\rangle = \sum_{\sigma', f'} |p, j = \frac{1}{2}, \sigma', f'\rangle D_{\sigma'f' \sigma f}(g), \quad g \in SU(6), \quad (2.5)$$

so that the group element  $g$  of  $SU(6)$  mixes the relativistic spin variable  $\sigma$  and the flavor variable  $f$ .  $D_{\sigma'f' \sigma f}(g)$  is the six-dimensional  $SU(6)$  matrix element. It is the infinitesimal actions of  $U_g$  that will be used in Section 3 to obtain mass splitting operators.

Hadrons are bound states of confined constituents. An  $n$ -particle constituent particle space is defined to be the  $n$ -fold tensor product of single-particle constituent spaces, with basis and group actions given by

$$\begin{aligned} |p_1 j_1 \sigma_1 f_1, \dots, p_n j_n \sigma_n f_n\rangle &:= |p_1 j_1 \sigma_1 f_1\rangle \cdots |p_n j_n \sigma_n f_n\rangle \\ U_{\Lambda} |p_1 j_1 \sigma_1 f_1, \dots, p_n j_n \sigma_n f_n\rangle &= \prod_{\alpha=1}^n (\Lambda p_{1, j_1 \sigma_1' f_1}, \dots, \Lambda p_{n, j_n \sigma_n' f_n}) D_{\sigma_{\alpha}' \sigma_{\alpha}}^{j_{\alpha}}(p_{\alpha}, \Lambda) \\ U_a |p_1 j_1 \sigma_1 f_1, \dots, p_n j_n \sigma_n f_n\rangle &= e^{i \sum_{\alpha} p_{\alpha} \cdot a} |p_1 j_1 \sigma_1 f_1, \dots, p_n j_n \sigma_n f_n\rangle \\ U_{\pi} |p_1 j_1 \sigma_1 f_1, \dots, p_n j_n \sigma_n f_n\rangle &:= |\pi(p_1 j_1 \sigma_1 f_1, \dots, p_n j_n \sigma_n f_n)\rangle, \end{aligned} \quad (2.6)$$

where in the last equation  $\pi$  is an element of the permutation group on  $n$  letters,  $S_n$ ; permutation symmetry will play an important role in the hadronic wave functions to be discussed in Section 4.

To develop the point form of relativistic dynamics, it is useful to define  $n$ -particle "velocity states" that are eigenstates of the free velocity operator  $V_{(0)}^{\mu}$ . Define

$$|v, \vec{k}_{\alpha} j_{\alpha} \mu_{\alpha} f_{\alpha}\rangle := U_{B(v)} |k_1 j_1 \mu_1 f_1, \dots, k_n j_n \mu_n f_n\rangle \quad (2.7)$$

with  $\sum_{\alpha=1}^n \vec{k}_{\alpha} = \vec{0}$ . We want to show that the spin labels  $\mu_{\alpha}$  transform like nonrelativistic variables. To see this, consider a Lorentz transformation  $\Lambda$  acting on the boost  $B(v)$  defined in Eq. (2.2)ff:

$$\begin{aligned}
U_\Lambda |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha\rangle &= U_\Lambda U_{B(v)} |k_1 j_1 \mu_1 f_1, \dots, k_n j_n \mu_n f_n\rangle \\
&= U_{B(\Lambda v)} U_{R_w} |k_1 j_1 \mu_1 f_1, \dots, k_n j_n \mu_n f_n\rangle \\
&= U_{B(\Lambda v)} \sum_{\mu'_\alpha} |R_w k_1, j_1 \mu'_1 f_1, \dots, R_w k_n, j_n \mu'_n f_n\rangle \prod_{\alpha=1}^n D_{\mu'_\alpha \mu_\alpha}^{j_\alpha}(k_\alpha, R_w) \\
&= \sum_{\mu'_\alpha} |\Lambda v, R_w \vec{k}_\alpha, j_\alpha \mu'_\alpha f_\alpha\rangle \prod_{\alpha=1}^n D_{\mu'_\alpha \mu_\alpha}^{j_\alpha}(R_w) \tag{2.8}
\end{aligned}$$

where use has been made of the fact that the boosts defining the Wigner rotation  $(k_\alpha, R_w)$  are canonical boosts, so that  $(k_\alpha, R_w) = R_w$  (see Ref. [3]).  $R_w$  is itself the Wigner rotation  $B^{-1}(\Lambda v)\Lambda B(v)$  as defined by Eq. (2.2).

Equation (2.8) states that for velocity states, Eq. (2.7), the internal momenta  $\vec{k}_\alpha$  and internal spins  $(j_\alpha \mu_\alpha)$ , transform like nonrelativistic variables. If they were nonrelativistic variables, the Wigner rotation  $R_w$  determined by  $v$  and  $\Lambda$  would be replaced by  $R \in SO(3)$ . But since the Wigner rotations appearing in the velocity state are all the same rotation, the spins  $j_1 \cdots j_n$  can all be coupled together to give an overall spin, the internal momenta  $\vec{k}_\alpha$  can be replaced by  $k_\alpha \ell_\alpha m_{\ell_\alpha}$ , the magnitude of  $\vec{k}$  and the orbital and orbital projection quantum numbers of the  $\alpha^{\text{th}}$  constituent, and these coupled together to give the overall orbital angular momentum of the  $n$ -particle constituents, exactly as is done nonrelativistically. Thus the external variables  $v$ , and  $j, \sigma$  (if the spin and orbital angular momentum are coupled to give the total angular momentum of the  $n$ -particle system) transform as relativistic variables [see Eq. (2.1)] while the internal variables transform as though they were nonrelativistic variables.

Similarly the action of a space-time translation  $a$  on a velocity state gives

$$\begin{aligned}
U_a |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha\rangle &= U_a U_{B(v)} |k_1 j_1 \mu_1 f_1, \dots, k_n j_n \mu_n f_n\rangle \\
&= e^{iB^{-1}(v)a \cdot \sum k_\alpha} |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha\rangle \\
&= e^{ia \cdot B(v)} \sum k_\alpha |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha\rangle \\
&= e^{ia \cdot v m_n} |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha\rangle \tag{2.9}
\end{aligned}$$

which means that

$$\begin{aligned}
P_{(0)}^\mu |v, \vec{k}_\alpha \mu_\alpha f_\alpha\rangle &= v^\mu m_n |v, \vec{k}_\alpha \mu_\alpha f_\alpha\rangle \\
V_{(0)}^\mu |v, \vec{k}_\alpha \mu_\alpha f_\alpha\rangle &= v^\mu |v, \vec{k}_\alpha \mu_\alpha f_\alpha\rangle \\
M_0 |v, \vec{k}_\alpha \mu_\alpha f_\alpha\rangle &= m_n |v, \vec{k}_\alpha \mu_\alpha f_\alpha\rangle \tag{2.10}
\end{aligned}$$

where  $\sum_\alpha k_\alpha = \sum (\omega_\alpha, \vec{k}_\alpha) = (\sum \omega_\alpha, \vec{0}) = (m_n, \vec{0})$ , with  $\omega_\alpha := \sqrt{m^2 + \vec{k}_\alpha \cdot \vec{k}_\alpha}$  and  $m_n := \sum \omega_\alpha$ . It is the free mass operator  $M_0$  acting on the  $n$ -particle space of constituents that will be modified to give the interacting mass operator.

The connection between velocity states, Eq. (2.7) and  $n$ -particle constituent states is

$$\begin{aligned} |v, \vec{k}_\alpha j_\alpha \mu_\alpha f_\alpha\rangle &= U_{B(v)} |k_1 j_1 \mu_1 f_1, \dots, k_n j_n \mu_n f_n\rangle \\ &= \sum_{\sigma_\alpha} |p_1 j_1 \sigma_1 f_1, \dots, p_n j_n \sigma_n f_n\rangle \prod_{\alpha=1}^n D_{\sigma_\alpha \mu_\alpha}^{j_\alpha}(k_\alpha, B(v)) \end{aligned} \quad (2.11)$$

where  $p_\alpha = B(v)k_\alpha$ ,  $\sum_{\alpha=1}^n \vec{k}_\alpha = 0$ .

In the following sections we will set  $j_\alpha = \frac{1}{2}$ , and suppress the label  $j_\alpha$  in the velocity states.  $\mu_\alpha$  is the eigenvalue of  $n_\alpha \cdot \vec{W}_\alpha$  and together with  $n_{\pm\alpha} \cdot \vec{W}_\alpha$  forms an  $SU(2)$  algebra. Hence as seen in Eq. (2.5) for single-particle states, an  $SU(6)$  element mixes the  $(\mu_\alpha, f_\alpha)$  labels:

$$U_g |v, \vec{k}_\alpha \mu_\alpha f_\alpha\rangle = \sum_{\mu'_\alpha f'_\alpha} |v, \vec{k}_\alpha \mu'_\alpha f'_\alpha\rangle \prod_{\alpha=1}^n D_{\mu'_\alpha f'_\alpha \mu_\alpha f_\alpha}(g), \quad g \in SU(6).$$

### 3 Relativistic Dynamics

In the point form of relativistic dynamics the free four-momentum operator is modified to include interactions. The six Lorentz generators do not change when interactions are included and hence the unitary operators  $U_\Lambda$  representing Lorentz transformations retain their form as given in Eqs. (2.6) or (2.8). The easiest way to modify the free four-momentum operator,  $P_{(0)}^\mu = M_0 V_{(0)}^\mu$ , is to change the free mass operator  $M_0$  to the interacting mass operator  $M$  while leaving  $V_{(0)}^\mu$  unchanged:

$$P^\mu := M V_{(0)}^\mu. \quad (3.1)$$

As shown in a succeeding paper dealing with electromagnetic currents and form factors, it is also necessary to modify the free velocity operator, but when dealing with hadronic wave functions it suffices to use only  $V_{(0)}^\mu$ .  $M$  must commute with  $U_\Lambda$  and  $V_{(0)}^\mu$ , for then

$$\begin{aligned} U_\Lambda P^\mu U_\Lambda^{-1} &= U_\Lambda M V_{(0)}^\mu U_\Lambda^{-1} \\ &= M U_\Lambda V_{(0)}^\mu U_\Lambda^{-1} \\ &= M (\Lambda^{-1})^\mu{}_\nu V_{(0)}^\nu \\ &= (\Lambda^{-1})^\mu{}_\nu P^\nu, \end{aligned} \quad (3.2)$$

which along with  $[P^\mu, P^\nu] = 0$  guarantees the commutation relations of the Poincaré group.

The condition that  $M$  commute with  $U_\Lambda$  and  $V_{(0)}^\mu$  is easily satisfied on velocity states. Since  $U_\Lambda$  transforms  $v$  to  $\Lambda v$  and  $\vec{k}_\alpha$  to  $R_w \vec{k}_\alpha$  [see Eq. (2.8)], it follows that if the kernel of  $M$  on the velocity state is independent of  $v$  and rotationally invariant,  $M$  will commute with  $U_\Lambda$  and  $V^\mu$ :

$$\langle v', \vec{k}'_\alpha \mu'_\alpha f'_\alpha | M | v k_\beta \mu_\beta f_\beta \rangle = v^0 \delta^3(\vec{v}' - \vec{v}) K(\vec{k}'_\alpha \mu'_\alpha f'_\alpha, \vec{k}_\beta \mu_\beta f_\beta), \quad (3.3)$$

where  $K(\ )$  is rotationally invariant. Since  $SU(6)$  spin-flavor transformations can be made rotationally invariant, it is clear that mass operators can be formed out of  $SU(6)$  generators, resulting in relativistic  $SU(6)$  mass splitting terms.

To analyze mass operators more carefully, it is convenient to make all the internal momentum variables independent. The internal momenta satisfy  $\sum_{\alpha=1}^n \vec{k}_\alpha = \vec{0}$  and the Hilbert space norm is given from

$$\begin{aligned} \sum_{\mu_\alpha f_\alpha} \int \frac{d^3 p_1}{E_1} \dots \frac{d^3 p_n}{E_n} &= \sum_{\mu_\alpha f_\alpha} \int \frac{d^3 v}{\sqrt{1 + \vec{v} \cdot \vec{v}}} \frac{d^3 k_1}{\omega_1} \dots \frac{d^3 k_n}{\omega_n} m_n^3 \delta^3 \left( \sum \vec{k}_\alpha \right) \\ &= \sum_{\mu_\alpha f_\alpha} \int \frac{d^3 v}{\sqrt{1 + \vec{v} \cdot \vec{v}}} \frac{d^3 k_1}{\omega_1} \dots \frac{d^3 k_{n-1}}{\omega_{n-1}} m_n^3 \end{aligned} \quad (3.4)$$

where  $\vec{k}_n = -\sum_{\alpha=1}^{n-1} \vec{k}_\alpha$ . Wave functions are now written in independent variables as  $\varphi(\vec{k}_\alpha, \mu_\alpha f_\alpha)$ , where it is understood that  $\alpha = 1 \dots n-1$  for the internal momenta, while for spin and flavor,  $\alpha = 1 \dots n$ . With  $n-1$  independent internal momenta, the action of the permutation group  $S_n$  changes its form from Eq. (2.6); for transpositions in which the  $\alpha^{\text{th}}$  and  $n^{\text{th}}$  momenta are interchanged the representation matrix is

$$\begin{aligned} \pi \begin{pmatrix} \vec{k}_1 \\ \vdots \\ \vec{k}_\alpha \\ \vdots \\ \vec{k}_{n-1} \end{pmatrix} &= \begin{pmatrix} \vec{k}_1 \\ \vdots \\ -\vec{k}_1 - \dots - \vec{k}_{n-1} \\ \vdots \\ -\vec{k}_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ -1 & \dots & -1 & \dots & -1 \\ & & & 1 & \dots \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} \vec{k}_1 \\ \vdots \\ \vec{k}_\alpha \\ \vdots \\ \vec{k}_{n-1} \end{pmatrix} = \tilde{D}(\pi) \begin{pmatrix} \vec{k}_1 \\ \vdots \\ \vec{k}_\alpha \\ \vdots \\ \vec{k}_{n-1} \end{pmatrix} \end{aligned} \quad (3.5)$$

where  $\pi = (\alpha, n) \in S_n$ . All of the representation matrices  $\tilde{D}(\pi)$  involving the  $n^{\text{th}}$  label are nonorthogonal; nevertheless, they form an irreducible representation of  $S_n$  with Young diagram  $(n-1, 1)$  ( $n-1$  boxes in the first row, 1 box in the second row).

In analogy with nonrelativistic Hamiltonians, interacting mass operators can be written as perturbations of the free mass operator:

$$M = M_0 + V, \quad (3.6)$$

where the "potential"  $V$  satisfies Eq. (3.3). As in the nonrelativistic case there are one-body, two-body, ...  $n$ -body interactions. From Eq. (3.6) it is possible to define a relativistic Lippman-Schwinger equation, generated by the time translation operator  $H = V^0 M$ :

$$\begin{aligned} e^{-iHt} \psi_{t=0} &= \psi_t \\ i \frac{\partial \psi_t}{\partial t} &= H \psi_t \\ &= V_{(0)}^0 M \psi_t \\ \psi &= \varphi + G_0 V \psi \end{aligned} \quad (3.7)$$

where the free Green function  $G_0(z) := (1/z - M_0)$ .  $M_0$  is of course more complicated than its nonrelativistic counterpart; in internal momentum variables it is  $\sum_{\alpha} \sqrt{m_{\alpha}^2 + \vec{k}_{\alpha} \cdot \vec{k}_{\alpha}}$ .

Because mass operators are any (self-adjoint) operators that commute with  $V_{(0)}^{\mu}$  and are rotationally invariant, spin-spin, spin-orbit, and tensor forces of the kind defined in nonrelativistic quantum mechanics can all be defined in an analogous fashion for relativistic  $n$ -body systems. The relative orbital angular momentum operator is

$$\vec{L} = \sum_{\alpha=1}^{n-1} \vec{k}_{\alpha} \times \frac{1}{i} \frac{\partial}{\partial \vec{k}_{\alpha}} \quad (3.8)$$

and if generators of  $SU(6)$  for the  $\alpha^{\text{th}}$  particle are written  $\lambda_A^{(\alpha)}$ ,  $\vec{\sigma}^{(\alpha)} \lambda_A^{(\alpha)}$ ,  $\vec{\sigma}^{(\alpha)}$ ,  $A = 1 \dots 8$ , where  $\lambda_A^{(\alpha)}$  are the  $SU(3)$  generators and  $\vec{\sigma}^{(\alpha)}$  Pauli matrices, then for example spin-orbit mass operators of the form

$$M_{LS} = \vec{L} \cdot \vec{\sigma}^{(\alpha)} \lambda_A^{(\alpha)} \quad (3.9)$$

are rotationally invariant.

Mass operators may also be obtained from Lie algebra elements which commute with the orbital and spin angular momentum. Since one of the goals of this paper is to formulate a relativistic  $SU(6)$  model and harmonic oscillator wave functions have been used for the unperturbed energy levels (see Ref. [1]), we wish to obtain relativistic harmonic oscillator mass operators. Consider the operators

$$\vec{k}_{\alpha} \cdot \vec{k}_{\beta}, \quad \vec{k}_{\alpha} \cdot \frac{\partial}{\partial \vec{k}_{\beta}}, \quad \frac{\partial}{\partial \vec{k}_{\alpha}} \cdot \frac{\partial}{\partial \vec{k}_{\beta}} \quad (3.10)$$

with  $\alpha, \beta = 1 \dots n-1$ . These operators form a representation of the Lie algebra of  $Sp(2(n-1), \mathbf{R})$ . The middle operators in Eq. (3.10) come from the action of the general linear group  $GL(n-1, \mathbf{R})$ , which is a subgroup of  $Sp(2(n-1), \mathbf{R})$ :

$$(U_g \varphi)(\vec{k}_{\alpha} \mu_{\alpha} f_{\alpha}) = \varphi((g^{-1} \vec{k})_{\alpha} \mu_{\alpha} f_{\alpha}), \quad g \in GL(n-1, \mathbf{R}); \quad (3.11)$$

note that though the permutation group representation  $\tilde{D}(\pi)$ ,  $\pi \in S_n$  [Eq. (3.5)] is a (nonorthogonal) representation of  $S_n$ , the action of  $S_n$  on wave functions  $\varphi$  is unitary as seen in Eq. (3.11) with  $g = \tilde{D}(\pi)$ .

The Lie algebra of  $Sp(2(n-1), \mathbf{R})$  is more evident if the creation and annihilation operators

$$\begin{aligned}\bar{c}_\alpha^\dagger &:= \frac{1}{\sqrt{2}} \left( \vec{k}_\alpha - \frac{\partial}{\partial \vec{k}_\alpha} \right) \\ \bar{c}_\alpha &:= \frac{1}{\sqrt{2}} \left( \vec{k}_\alpha + \frac{\partial}{\partial \vec{k}_\alpha} \right)\end{aligned}\tag{3.12}$$

replace the  $\vec{k}_\alpha$  and  $\partial/\partial \vec{k}_\alpha$  operators. Define

$$\begin{aligned}X_{\alpha\beta}^0 &:= \bar{c}_\alpha^\dagger \cdot \bar{c}_\beta = \frac{1}{2} \left( \vec{k}_\alpha - \frac{\partial}{\partial \vec{k}_\alpha} \right) \cdot \left( \vec{k}_\beta + \frac{\partial}{\partial \vec{k}_\beta} \right) \\ X_{\alpha\beta}^+ &:= \bar{c}_\alpha^\dagger \cdot \bar{c}_\beta^\dagger = \frac{1}{2} \left( \vec{k}_\alpha - \frac{\partial}{\partial \vec{k}_\alpha} \right) \cdot \left( \vec{k}_\beta - \frac{\partial}{\partial \vec{k}_\beta} \right) \\ X_{\alpha\beta}^- &:= \bar{c}_\alpha \cdot \bar{c}_\beta = \frac{1}{2} \left( \vec{k}_\alpha + \frac{\partial}{\partial \vec{k}_\alpha} \right) \cdot \left( \vec{k}_\beta + \frac{\partial}{\partial \vec{k}_\beta} \right).\end{aligned}\tag{3.13}$$

Then  $X_{\alpha\alpha}^0$  is a harmonic oscillator operator that commutes with  $\vec{L}$  and  $\vec{S}$ , and hence is a possible mass operator; it is not of the form  $M_0 + V$ , as is the case nonrelativistically, but nevertheless has an equally spaced discrete spectrum.

With this Lie algebra of mass operators and mass operators of the form Eq. (3.9) breaking the degenerate harmonic oscillator levels, it is possible to formulate a relativistic  $SU(6)$  model in which the mass operators are not given just in terms of their transformation properties under  $SU(6)$ , but as actual mass operators as defined in Eq. (3.3).

## 4 Relativistic $SU(6)$

To formulate a relativistic  $SU(6)$  theory, it is necessary to pay particular attention to the permutation group properties of the spatial, spin-flavor, and color parts of the overall wave function. A hadronic wave function should be (anti)symmetric under interchange of all constituent particle labels. Though the color degrees of freedom have only been implicitly included in the discussion on internal symmetries, we assume that the color part of a hadronic wave function must be a color singlet under  $SU(3)_c$  with a definite permutation symmetry. The possible permutation symmetries for  $n$ -body color singlets, labeled by the Young diagram  $Y_c$  are given in Ref. [8].

Wave functions in the  $n$ -constituent particle Hilbert space can thus be written as  $\varphi(v_0; \vec{k}_\alpha, \mu_\alpha f_\alpha, c_\alpha)$ , where  $v_0$  is the overall four velocity of the  $n$ -constituents,  $\vec{k}_\alpha$ ,  $\alpha = 1 \dots n-1$  are the internal momenta,  $\mu_\alpha f_\alpha$ ,  $\alpha = 1 \dots n$  the spin and flavor labels transforming under  $SU(6)$  transformations, and  $c_\alpha$  the color label transforming under  $SU(3)$  transformations. Under a Lorentz transformation,  $v_0$  goes to  $\Lambda v_0$ ,  $\vec{k}_\alpha \rightarrow R_w \vec{k}_\alpha$  and  $\mu_\alpha \rightarrow \mu'_\alpha$ , as seen in Eq. (2.8); thus  $\mu_\alpha$ , the internal spin label transforms differently under Lorentz and  $SU(6)$  transformations, a property which can be used to generate mass splittings for different spin particles with  $SU(6)$  multiplets.

We now wish to compute relativistic harmonic oscillator wave functions with the appropriate spin-flavor and color symmetry:

$$\begin{aligned}
|v; \underbrace{NY_s \ell m_\ell}_{Sp(2(n-1), \mathbf{R}) \times 0(3)} ; \underbrace{\chi_{SU(6)} Y_f, \chi_{SU(3)} f, s s_3}_{SU(6) \supset SU(3)_f \times SU(2)} ; \underbrace{1 Y_c}_{SU(3)_c} \\
= \delta^3(v - v_0) \varphi_{NY_s \ell m_\ell}(\vec{k}_\alpha) \varphi_{\chi_{SU(6)} Y_f \chi_{SU(3)} f s s_3}(\mu_\alpha f_\alpha) \varphi_{1 Y_c}(c_\alpha)
\end{aligned} \quad (4.1)$$

where  $v$  is the four velocity of the hadron,  $N$  is the harmonic oscillator eigenvalue label,  $Y_s$  is the Young diagram giving the spatial permutation symmetry, and  $\ell, m_\ell$  are the orbital and orbital projection quantum numbers. Similarly  $\chi_{SU(6)}$  are the  $SU(6)$  multiplet labels, with basis labels including the flavor ( $\chi_{SU(3)}, f$ ) and spin ( $s, s_3$ ) labels.  $Y_f$  is the Young diagram giving the spin-flavor permutation symmetry. Finally, "1" designates an  $SU(3)$  color singlet, and  $Y_c$  is the color permutation symmetry. To obtain an overall antisymmetric (for baryons) or symmetric (for mesons) wave function, the permutation types must be coupled together,  $Y_s \otimes Y_f \otimes Y_c \rightarrow A$  (baryons) or  $S$  (mesons). Once these wave functions are known, mass operators arising from spin-orbit, spin-spin, tensor and  $SU(6)$  type forces of the kind discussed in Section 3 can be introduced to split the degenerate harmonic oscillator mass spectrum.

Though the wave functions described in Eq. (4.1) may seem complicated, we want to show that they can be readily computed when realized as polynomials in Bargmann spaces. Reference [8] shows how to realize the spin-flavor and color parts of the wave function as polynomials in Bargmann spaces. Here we show how to realize harmonic oscillator wave functions as polynomials in a Bargmann space.

The holomorphic Hilbert (or Bargmann) space  $\mathcal{B}(C_{n-1 \times 3})$  needed for the spatial part of the wave function consists of holomorphic functions  $F(z)$  in  $n-1 \times 3$  complex variables,  $z \in C_{n-1 \times 3}$ , with the norm given by

$$\|F\|^2 = F \left( \frac{\partial}{\partial z} \right) \overline{F(\bar{z})} |_{z=0}, \quad F \in \mathcal{B}(C_{n-1 \times 3}), \quad (4.2)$$

where  $F(\partial/\partial z)$  means replacing the entries in  $F(z)$  by the differential operators  $\partial/\partial z$ . Creation and annihilation operators are particularly simple, in that

$$\begin{aligned}
c_{\alpha i}^\dagger &= z_{\alpha i}, & \alpha &= 1 \dots n-1 \\
& & i &= 1, 2, 3 \\
c_{\alpha i} &= \frac{\partial}{\partial z_{\alpha i}} \\
[c_{\alpha i}, c_{\beta j}^\dagger] &= \delta_{\alpha\beta} \delta_{ij}.
\end{aligned} \quad (4.3)$$

$\mathcal{B}(C_{n-1 \times 3})$  is isomorphic to the Hilbert space of internal momenta  $\vec{k}_\alpha$  [see Eq. (3.4)] and the creation and annihilation operators, Eq. (3.12), can be used to transform the polynomial harmonic oscillator wave functions to wave functions in the internal momenta; examples will be given at the end of this section.

There are two natural group actions on elements of  $\mathcal{B}$  that will be needed for permutation group and orbital angular momentum operators. Write

$$\begin{aligned}(R_g F)(z) &:= F(zg) \quad , \quad g \in U(3) \supset SO(3) \\ (L_h F)(z) &:= F(h^{-1}z) \quad , \quad h \in U(n-1) \\ [R_g, L_h] &= 0 \quad .\end{aligned}\tag{4.4}$$

That is,  $g \in U(3)$  restricted to elements of  $SO(3)$  gives the orbital angular momentum operators. Infinitesimal operators coming from  $L_h$  give the harmonic oscillator operators defined in Eq. (3.13):

$$X_{\alpha\beta}^0 = \sum_{i=1}^3 c_{\alpha i}^\dagger c_{\beta i} = \sum_{i=1}^3 z_{\alpha i} \frac{\partial}{\partial z_{\beta i}} \quad ,\tag{4.5}$$

which along with the other two sets of operators,

$$\begin{aligned}X_{\alpha\beta}^+ &= \sum_{i=1}^3 c_{\alpha i}^\dagger c_{\beta i}^\dagger = \sum_{i=1}^3 z_{\alpha i} z_{\beta i} \\ X_{\alpha\beta}^- &= \sum_{i=1}^3 c_{\alpha i} c_{\beta i} = \sum_{i=1}^3 \frac{\partial}{\partial z_{\alpha i}} \frac{\partial}{\partial z_{\beta i}} \quad ,\end{aligned}\tag{4.6}$$

give the Lie algebra action of  $Sp(2(n-1), \mathbf{R})$  on  $\mathcal{B}$ , and commute with  $R_g, g \in SO(3)$ , Eq. (4.4).

It is convenient to transform from a Cartesian basis, with  $i = 1, 2, 3$  to a spherical basis with  $\mu = \pm 1, 0$ . The transformation is

$$\begin{aligned}c_{\alpha\pm} &= \frac{1}{\sqrt{2}} (c_{\alpha 1} \mp i c_{\alpha 2}) \quad , \quad c_{\alpha 0} = c_{\alpha 3} \\ c_{\alpha\pm}^\dagger &= \frac{1}{\sqrt{2}} (c_{\alpha 1}^\dagger \pm i c_{\alpha 2}^\dagger) \quad , \quad c_{\alpha 0}^\dagger = c_{\alpha 3}^\dagger \quad .\end{aligned}\tag{4.7}$$

Then

$$\begin{aligned}X_{\alpha\beta}^0 &= c_{\alpha+}^\dagger c_{\beta+} + c_{\alpha-}^\dagger c_{\beta-} + c_{\alpha 0}^\dagger c_{\beta 0} \\ X_{\alpha\beta}^+ &= c_{\alpha+}^\dagger c_{\beta-}^\dagger + c_{\alpha-}^\dagger c_{\beta+}^\dagger + c_{\alpha 0}^\dagger c_{\beta 0}^\dagger \\ X_{\alpha\beta}^- &= c_{\alpha+} c_{\beta-} + c_{\alpha-} c_{\beta+} + c_{\alpha 0} c_{\beta 0} \quad .\end{aligned}\tag{4.8}$$

In the spherical basis the orbital angular momentum operators are

$$\begin{aligned}
 L_0 &= \sum_{\alpha=1}^{n-1} (c_{\alpha+}^\dagger c_{\alpha+} - c_{\alpha-}^\dagger c_{\alpha-}) \\
 L_+ &= \sqrt{2} \sum_{\alpha=1}^{n-1} (c_{\alpha+}^\dagger c_{\alpha 0} - c_{\alpha 0}^\dagger c_{\alpha-}) \\
 L_- &= (L_+)^\dagger .
 \end{aligned} \tag{4.9}$$

We now want to construct a relativistic harmonic oscillator mass operator out of the operators in Eq. (4.8) that commutes with the permutation group  $S_n$  and the orbital angular momentum operators, Eq. (4.9), for then spatial wave functions will be polynomials in  $z$  labeled by  $N, Y_s, \ell$ , and  $m_\ell$ ,  $p_{|NY_s, \ell m_\ell\rangle}(z)$ , as required from Eq. (4.1). By construction the  $X_{\alpha\beta}^0$  commute with the angular momentum operators, Eq. (4.9); we now show that the symmetric group action is a subgroup of  $U(n-1)$ , so that if the harmonic oscillator mass operator  $M_{H0}$  is chosen to be

$$\begin{aligned}
 M_{H0} &= mX^0 \\
 &= m \sum_{\alpha=1}^{n-1} X_{\alpha\alpha}^0 ,
 \end{aligned} \tag{4.10}$$

it will automatically commute with  $S_n$ . The factor  $m$  in Eq. (4.10) is a constant having the dimensions of mass, and sets the mass scale for the hadronic mass spectrum.

As shown in Eq. (3.5), the action of permutation group elements  $\pi \in S_n$  on internal momentum vectors  $\vec{k}_\alpha$ ,  $\alpha = 1 \dots n-1$  results in nonorthogonal  $n-1$  dimensional representation matrices; nevertheless, as can be ascertained by taking traces of these matrices and using character formulae [9], the  $S_n$  representation matrices are irreducible, with Young tableau  $(n-1, 1)$ . The corresponding orthogonal matrices will be denoted by  $D(\pi)$ , so that

$$\begin{aligned}
 \pi \in S_n &\rightarrow D(\pi) \subset O(n-1) \subset U(n-1) \\
 (L_\pi F)(z) &= F(D^{-1}(\pi)z) , \quad F \in \mathcal{B}(C_{n-1 \times 3}) .
 \end{aligned} \tag{4.11}$$

That is, the orthogonal representation matrices of dimension  $n-1$  of the group  $S_n$  act on elements  $F$  in  $\mathcal{B}$  via the  $U(n-1)$  action defined in Eq. (4.4). Since the  $L_\pi$  action is generated by orthogonal matrices  $D(\pi)$ ,  $L_\pi$  will not only commute with  $X^0 = \sum_\alpha X_{\alpha\alpha}^0$ , but with  $X^+ := \sum_\alpha X_{\alpha\alpha}^+$ , and  $X^- := \sum_\alpha X_{\alpha\alpha}^-$ . Thus, we have  $Sp(2, \mathbf{R}) \times S_n$  embedded in  $Sp(2(n-1), \mathbf{R})$  in which each eigenvalue of  $X^0$  carries a definite permutation symmetry and  $X^\pm$  raise and lower the polynomial eigenfunctions of  $X^0$ .

$X^-$  acts as a lowering operator on  $X^0$  eigenfunctions. The simplest polynomial corresponding to  $N=0, Y_s=S$ , and  $\ell=0$  is  $p(z) = 1$ :

$$P_{|N=0, Y_s=S, \ell=0\rangle}(z) = 1 (= |0\rangle) . \tag{4.12}$$

There is then a tower of eigenstates generated by the raising operator  $X^+ = \sum_{\alpha,\mu} z_{\alpha\mu} z_{\alpha\mu}$ :

$$\begin{aligned} P_{|2N,S,\ell=0\rangle}(z) &= (X^+)^N |0\rangle \\ &= \left( \sum_{\alpha,\mu} z_{\alpha\mu} z_{\alpha\mu} \right)^N. \end{aligned} \quad (4.13)$$

Similarly there is a tower of  $\ell = 1$  states, starting with  $N = 1$ , given by the polynomials  $z_{\alpha\mu}$ , with  $z_{\alpha+}$  the polynomials with  $L_3 = +1$ . At higher levels in the angular momentum towers, states cannot be uniquely labeled by  $Y_s$ ; additional operators commuting with  $X^0$  and  $S_n$  must be introduced; the construction of these operators is given in Ref. [10].

Constituent quark models assume that baryons are bound states of three quarks. To conclude this section we exhibit polynomials for baryons consisting of three constituents. The relevant permutation group is  $S_3$ , and the representation matrices  $D(\pi)$  are given on page 224 of Ref. [9]. There are three types of irreducible representations,  $Y_s = S$  (symmetric),  $A$  (antisymmetric) or  $M$  (mixed, two dimensional). We list here some low  $N$  polynomials for  $\ell = 0, 1, 2$  and  $L_3 = \ell$ . (The other angular momenta can be obtained from the lowering operator  $L_-$ , Eq. (4.9), which means differentiating the given polynomials in a prescribed way):

$$\begin{aligned} P_{|N,Y_s,\ell,\ell\rangle}(z) &= c^\dagger \cdots c^\dagger |0\rangle \\ |0, S, 0, 0\rangle &= 1 &= |0\rangle \\ |2, S, 0, 0\rangle &= \frac{1}{\sqrt{12}} \left( \sum_{\mu} z_{1\mu}^2 + \sum_{\mu} z_{2\mu}^2 \right) &= \frac{1}{\sqrt{12}} \sum_{\alpha,\mu} c_{\alpha\mu}^\dagger c_{\alpha\mu}^\dagger |0\rangle \\ |2, M, 0, 0\rangle &= \begin{cases} \frac{1}{\sqrt{12}} \left( \sum_{\mu} z_{1\mu}^2 - \sum_{\mu} z_{2\mu}^2 \right) &= \frac{1}{\sqrt{12}} \left( \sum_{\mu} c_{1\mu}^{2\dagger} - c_{2\mu}^{2\dagger} \right) |0\rangle \\ \frac{1}{\sqrt{3}} \sum_{\mu} z_{1\mu} z_{2\mu} &= \frac{1}{\sqrt{3}} \sum_{\mu} c_{1\mu}^\dagger c_{2\mu}^\dagger |0\rangle \end{cases} \\ |1, M, 1, 1\rangle &= \begin{cases} z_{1+} &= c_{1+}^\dagger |0\rangle \\ z_{2+} &= c_{2+}^\dagger |0\rangle \end{cases} \\ |2, A, 1, 1\rangle &= \frac{1}{\sqrt{2}} (z_{10} z_{2+} - z_{1+} z_{20}) &= \frac{1}{\sqrt{2}} (c_{10}^\dagger c_{2+}^\dagger - c_{1+}^\dagger c_{20}^\dagger) |0\rangle \\ |2, S, 2, 2\rangle &= \frac{1}{2} (z_{1+}^2 + z_{2+}^2) &= \frac{1}{2} (c_{1+}^{\dagger 2} + c_{2+}^{\dagger 2}) |0\rangle \\ |2, M, 2, 2\rangle &= \begin{cases} \frac{1}{2} (z_{1+}^2 - z_{2+}^2) &= \frac{1}{2} (c_{1+}^{\dagger 2} - c_{2+}^{\dagger 2}) |0\rangle \\ (z_{1+} z_{2+}) &= c_{1+}^\dagger c_{2+}^\dagger |0\rangle \end{cases} \end{aligned} \quad (4.14)$$

The coefficients appearing in front of the harmonic oscillator polynomials normalize the polynomials to one; these factors are easily computed using the differentiation inner product, Eq. (4.2). Moreover the polynomial eigenfunctions are easily transformed to harmonic oscillator wave

functions in internal momentum variables. In this case the vacuum state  $|0\rangle$  is realized as  $e^{-\frac{1}{2} \sum_{\alpha,\mu} k_{\alpha\mu}^2}$  and the creation operators in the right-hand column of Eq. (4.14) are given in Eq. (3.12).

When the spatial polynomial wave functions with permutation symmetry  $Y_s$  are combined with the spin-flavor and color (for which  $Y_c = A$ ) wave functions, the resulting symmetry type must be antisymmetric. For a given  $Y_s$ , this fixes  $Y_f$ , namely

$Y_s$	$Y_f$	$\dim SU(6)$
M	M	70
S	S	56
A	A	20

## 5 Conclusion

We have shown how to construct a relativistic quantum mechanics using Dirac's "point form," in which Lorentz transformations are kinematic and interactions appear in the mass operator. The four-momentum operator is then the product of the mass operator and the four-velocity operator. For eigenstates of the four-velocity operator, mass operators are rotationally invariant self-adjoint operators. Mass operators corresponding to spin-orbit, spin-spin, and tensor forces are readily constructed because the internal coordinates of velocity states transform like nonrelativistic coordinates. Nevertheless, the theory is covariant in that four vectors transform in the usual way under the kinematic Lorentz group. A modified Pauli-Lubanski operator, in which the four-velocity operator replaces the four-momentum operator, when dotted into appropriate four vectors, forms a relativistic  $SU(2)$  spin algebra. The eigenvalue of the spin Casimir operator is  $j(j+1)$ . Combining this  $SU(2)$  algebra with an internal symmetry into a larger symmetry produces mixing between spin and internal symmetry quantum numbers in a relativistically invariant way.

When the internal symmetry is  $SU(3)$  flavor, and the spin of the constituents is  $\frac{1}{2}$ , the result is a relativistic  $SU(6)$  theory. In such a theory there are many ways of choosing mass operators (such as QCD inspired mass operators), but the simplest choice is a harmonic oscillator mass operator with equally spaced mass eigenvalues. Such a mass operator is not constructed like its nonrelativistic counterpart, with  $r^2$  potentials between each of the constituents, but rather is constructed algebraically using a symplectic algebra. By using Bargmann spaces it is possible to realize the harmonic oscillator wave functions as polynomials with definite permutation properties. Moreover, the harmonic oscillator mass operator can be modified without changing the polynomial eigenfunctions by adding on the operator  $X^+X^-$ , in which case the eigenvalues  $N = 0, 1, 2, \dots$  become  $(N - \ell)(N + 3\ell + 1)$ , where  $\ell$  is the orbital angular momentum.

Mass operators can also be formed out of  $SU(6)$  generators, which then give Gürsey-Radicati type mass formulae [11]. By adding such mass operators to spin-orbit or tensor mass operators, it should be possible to reproduce the observed baryon mass spectrum. And if constituents and their antiparticles are combined into a larger internal symmetry, it should also be possible to fit the meson spectrum, as well as the spectrum of some of the low-mass nuclei.

Once realistic relativistic wave functions for mesons and baryons are available, it should be possible to compute form factors, structure functions, decays, and the like for hadrons viewed as bound states of spin  $\frac{1}{2}$  constituents. In a succeeding paper [12] we show how to formulate a point form relativistic quantum mechanical impulse approximation, wherein the electromagnetic properties of the hadrons are determined by the electromagnetic properties of their constituents.

It is possible to generalize the relativistic  $SU(6)$  theory to a Fock space theory, where the Fock space is formed by taking the direct sum of the  $n$ -constituent Hilbert spaces discussed in this paper from  $n$  equals zero to infinity. Such a Fock space is the appropriate space on which to compute decays of excited baryons, such as the  $\Delta \rightarrow \pi + N$  decay which was forbidden in the old  $SU(6)$  theory. Finally, we mention that mass operators need not commute with the number operator; for such mass operators hadrons consist of a direct sum of an indefinite number of constituents and correspond to the current quarks in QCD, in contrast to constituent quarks.

## References

- [1] A.J.G. Hey and R. L. Kelly, Phys. Rep. **96**, 71 (1983).
- [2] P. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
- [3] W. H. Klink, Ann. Phys. **213**, 31 (1992).
- [4] Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (D. Reidel Publishing Co., Dordrecht, Holland, 1986).
- [5] S. Capstick and N. Isgur, Phys. Rev. D **34**, 2809 (1986).
- [6] P. L. Chung, F. Coester, B. D. Keister, and W. N. Polyzou, Phys. Rev. C **37**, 2000 (1988).
- [7] F. M. Lev, Fortschr. Phys. **31**, 75 (1983).
- [8] W. H. Klink, Ann. Phys. **213**, 54 (1992).
- [9] M. Hamermesh, *Group Theory* (Addison Wesley, New York, 1962).
- [10] W. H. Klink and T. Ton-That, J. Phys. A **23**, 2751 (1990).
- [11] F. Gürsey and L. A. Radicati, Phys. Rev. Lett. **23**, 173 (1964); see also the discussion in Ref. [1], pp. 98ff.
- [12] W. H. Klink, Point form relativistic quantum mechanics and electromagnetic current operators, University of Iowa preprint (1992).

