PROPER–TIME RELATIVISTIC DYNAMICS

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ABSTRACT

Proper–time relativistic single–particle classical Hamiltonian mechanics is formulated using a transformation from observer time to system proper time which is a canonical contact transformation on extended phase space. It is shown that interaction induces a change in the symmetry structure of the system which can be analyzed in terms of a Lie–isotopic deformation of the algebra of observables.

1. INTRODUCTION

We begin with some historical remarks. In the transition from nonrelativistic to relativistic quantum mechanics, the Hamiltonian $H = \frac{(\underset{\sim}{P} - \frac{e}{c}\underset{\sim}{A})^2}{2m} + V$ is replaced by $H = [c^2(\underset{\sim}{P} - \frac{e}{c}\underset{\sim}{A})^2 + m^2c^4]^{1/2} + V$. It was quite natural to expect that the first choice for a relativistic wave equation would be

$$i\hbar\frac{\partial\psi}{\partial t}(\underset{\sim}{x},t) = ([c^2(\underset{\sim}{P} - \frac{e}{c}\underset{\sim}{A})^2 + m^2c^4]^{1/2} + V)\psi(\underset{\sim}{x},t), \tag{1.1}$$

where $\underset{\sim}{P} = -i\hbar\underset{\sim}{\nabla}$.

In a survey article on relativistic wave equations, Foldy [1] points out that in the absence of interaction, equation 1.1 gives a perfectly good relativistic wave equation for the description of a (spin zero) free particle. When $\vec{A}$ is not zero, the non–commutativity of $\vec{P}$ with $\vec{A}$ appeared to make it impossible to give an unambiguous meaning to the radical operator. Historically, many authors [2] attempted to circumvent this problem by starting with the relationship $(H - V)^2 = m^2c^4 + C^2(\vec{P} - e/c\ A)^2$ which led to the Klein–Gordon equation. The problems with this equation were so great, that all involved became frustrated and it was dropped from serious consideration for a few years. Dirac [3] argued that the proper equation should be first order in both the space and time variables, in order to be a true relativistic wave equation. This lead to the well–known Dirac equation.

In the same paper that Dirac provided the basic ideas which lead to the Feynman integral [4], he noted that "the Hamiltonian method is essentially non–relativistic in form, since it marks out a particular time variable as the canonical conjugate of the Hamiltonian function."

Dirac's position, that the equation should be first order in the space and time variables, emphasizes the relativistic invariance point of view in the merging of special relativity with quantum mechanics. From the quantum mechanical point of view, one could argue that a proper relativistic wave equation would elevate the time coordinate to the same level as the space coordinates, so that all become operators. In the relativistic quantum theory of the present day, the time coordinate does not have equal status with the space coordinates.

The Proper–Time Problem

If one attempts to implement the successful procedures and methods of nonrelativistic quantum mechanics with the special theory of relativity, it is well–known that problems of physical interpretation appear. The problems are well–known, and discussed by many writers [5]. In order to clearly see one apparent problem, let us note that the three fundamental relationships of classical special relativity:

$$\frac{d\tau}{dt} = (1 - \frac{v^2}{c^2})^{1/2},\ E = mc^2(1 - \frac{v^2}{c^2})^{-1/2},$$

$$E = (c^2 \underset{\sim}{P}^2 + m^2c^4)^{1/2},$$

may be uniquely combined to give $\frac{d\tau}{dt} = mc^2(c^2\underset{\sim}{P}^2 + m^2c^4)^{-1/2}$. If we now make the transition to quantum mechanics, $\underset{\sim}{P} \to -i\hbar\underset{\sim}{\nabla}$, we obtain $\frac{d\tau}{dt} = mc^2(-c^2\hbar^2\Delta + m^2c^4)^{-1/2}$. <u>This result is consistent with quantum mechanics but is inconsistent with the many attempts [5] to treat proper time as a parameter</u>.

The Third Postulate Problem

The two postulates of special relativity are:

1. The physical laws of nature and the results of all experiments are independent of the inertial frame of the observer.
2. The speed of light is independent of the motion of the source.

The first postulate abandons the notion of absolute space, while the second postulate abandons the concept of absolute time. It is of interest to note that another postulate is:

3. The correct implementation of postulates 1 and 2 is to require that time be represented as a <u>fourth coordinate</u> (Minkowski space) and to require that the relativistic laws of physics be invariant or covariant under Lorentz transformations.

This third postulate was proposed by H. Minkowski, a well–known mathematician in the early part of the 20th century. Most of the physics community of the time did not accept it, regarding it as a mathematical obstruction without physical content.

The inability to obtain an alternate approach dictated by physical considerations forced acceptance of the current implementation. Although the second postulate eliminated absolute time, the transformation theory associated with postulate 3 revealed a new unique time variable associated with the observed system, its proper time. The purpose of the present paper is to show how the use of this variable in place of the observer time variable leads to a conceptually (and technically) much simpler implementation of the special theory of relativity. To be sure, the use of this variable is not new. However, we treat the transformation from observer time to system proper time as a canonical contact transformation on extended phase space. This approach forces the identification of the canonical Hamiltonian which generates the Lie Algebra bracket. The problem of interaction is discussed for two–particle momentum – independent potentials. These include, of course, the important case of the relativistic harmonic oscillator. We confine our study to the single–particle classical theory. The many–particle classical theory and the quantum case will be explored elsewhere.

In Section 2 we formulate proper–time Hamiltonian dynamics for a single classical massive particle and discuss some properties of the group of proper–time transformations on extended phase space. Section 3 is devoted to the discussion of the case of particle interaction for two–body potentials independent of the particle momenta, and Section 4 contains some concluding remarks.

2. SINGLE–PARTICLE FORMULATION

The dynamics of a classical observable can be conveniently studied by Hamiltonian mechanics using the Poisson bracket $\{A(p,q), B(P,q)\} = \frac{\partial A}{\partial P}\frac{\partial B}{\partial q} - \frac{\partial A}{\partial q}\frac{\partial B}{\partial P}$. The Hamilton equations ensure that the time development of an arbitrary classical function $W(q,p,t)$ is given by

$$\frac{dW}{dt}(q,P,t) = \{H,\ W(q,P,t)\} + \frac{\partial W}{\partial t}(q,P,t).$$

Defining the proper time τ by $dt = \frac{H}{mc^2}\, d\tau$, the proper–time evolution of the function W is given by the chain rule:

$$\frac{dW}{d\tau} = \frac{dW}{dt}\frac{dt}{d\tau} = \frac{H}{mc^2}\{H,W\} + \frac{\partial W}{\partial t}. \tag{2.1}$$

An energy functional K which is conjugate to the proper–time τ will be defined by $\{K,W\} = \frac{H}{mc^2}\{H,W\}$ with $K = mc^2$ when $H = mc^2$. If the mass m remains invariant during the evolution, this functional can be directly determined to be

$$K = \frac{H^2}{2mc^2} + \frac{mc^2}{2}, \tag{2.2}$$

and the evolution of the function W in terms of τ can be expressed as $\frac{dW}{d\tau} = \{K,W\} + \frac{\partial W}{\partial \tau}$.

Consider the behavior of a single noninteracting particle of mass m, with momentum p as measured in some inertial frame. The usual form of the Hamiltonian representing this system is $H = \sqrt{c^2P^2 + (mc^2)^2}$. For this example, the conjugate proper energy is given by $K = \frac{P^2}{2m} + mc^2$. Several interesting points should be noted:

a. The functional form of the energy K is the same as that of the nonrelativistic energy of the system, even though the system is fully relativistic.

b. The momentum parameter in the functional form of the energy K is the momentum as measured in the original inertial frame, not the proper frame of the particle (which of course would measure zero momentum). This emphasizes the form of the transformation as a canonical time transformation, rather than as a Lorentz transformation.

c. If the particle were to interact with external influences, the proper frame would not be an inertial frame, but the proper time is always defined.

Transformation Group

We noted earlier that the proper time is invariant for all inertial observers. However, different observers will use different Hamiltonians to describe the phase flow of the system. In order to relate the phase flows for different inertial observers, we note that the proper–time transformations form a subgroup of the full group of transformations on the extended phase space which, since they do not transform the time, include the group of symplectic diffeomorphisms.

Consider two inertial observers in frames X, X' with extended phase space coordinates (p,q,t), (p',q',t') respectively. Let L denote the set of Lorentz transformations on space–time reference frames, $L(X,X'): X \to X'$, and denote by T the set of canonical proper–time transformations defined on extended phase space. We denote the map $(P,q,t) \longmapsto (P,q,\tau)$ by $T(q,t,\tau)$.

<u>THEOREM</u>. <u>The proper–time coordinates on X are related to those on X′ by the transformation:</u>

$$S_m(q',q,\tau) = T(q',t',\tau)\, L_m(X,X')T^{-1}(q,t,\tau).$$

Proof. The proof follows from the commutativity of the following diagram.

$$\begin{tikzcd}
X(\underset{\sim}{q},\underset{\sim}{P},t) \arrow[r, "L"] \arrow[d, "T_X"'] & X'(\underline{q}',\underline{P}',t') \arrow[d, "T_{X'}"] \\
(\underline{q},\underline{P},\tau) \arrow[r, dashed, "S"'] & (q',P',\tau)
\end{tikzcd}$$

It is easy to prove that, for each fixed system, the set of proper–time transformations between inertial observers is a group which relates the dynamics as viewed by one observer to the dynamics as viewed by any other observer.

We have used the particle mass in the statement of the above theorem to fix the observed system. The group of proper–time transformations depends on 14 parameters $(m,\underline{P},\underline{q},\underline{P}',\underline{q}',\tau)$. It follows that the free–particle laws will be the same for all inertial observers and will be form invariant under a similarity group action on the Lorentz group.

COROLLARY. There exist Poincare transformations that preserve the time coordinate.
Proof. We note that, in the proof of the above theorem, both $(\underline{q},\underline{P},\tau)$ and $(\underline{q}',\underline{P}',\tau)$ are inertial frames in the free–particle case.

Lie–Isotopic Algebras

Prior to studying the case of interactions, we introduce the essential ideas concerning Lie–isotopes and their properties. For a complete review of these objects, we refer to [6]. Let G denote a given Lie algebra with bracket $[A,B] = AB - BA$ and let T be an invertible element in G. A Lie–isotope of G is then defined as G with the bracket $[A,B]^* = A*B - B*A \equiv ATB - BTA$. It is easy to show that $[\ ,\ ]^*$ is a Lie bracket and that

$(G, [\ ,\ ]^*)$ is a Lie algebra. It turns out that two nonisomorphic groups may have isotopic Lie algebras. The standard example concerns the groups SO(3), SO(2,1). These are symmetry groups for the following respective Hamiltonians:

$$H_1(\underline{q},\underline{P}) = \tfrac{1}{2}(P_1^2 + P_2^2 + P_3^2) + \tfrac{1}{2}(q_1^2 + q_2^2 + q_3^2),$$

$$H_2(\underline{q},\underline{P}) = \tfrac{1}{2}(P_1^2 - P_2^2 + P_3^2) + \tfrac{1}{2}(q_1^2 - q_2^2 + q_3^2).$$

These Hamiltonians lead to the same equations of motion and to the same conservation laws (via Noether's theorem) for the components of angular momentum $L_b (b = 1,2,3)$. Using the notation $A\overleftrightarrow{T}B = ATB - BTA$, we have

$$[L_b, L_c] = \frac{\partial L_b}{\partial q_i} \overleftrightarrow{\delta^i_j} \frac{\partial L_c}{\partial p_j}, \ [\delta^i_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$[L_b, L_c]^* = \frac{\partial L_b}{\partial q_i} \overleftrightarrow{\alpha^i_j} \frac{\partial L_c}{\partial p_j}, \ [\alpha^i_j] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for the respective Lie algebras of the groups SO(3) and SO(2,1). In the latter case we have $T = [\alpha^i_j] = T^{-1}$.

In order to understand the requirement that T be invertible, recall that the group SO(3) leaves the standard inner product $<\underline{a},\underline{b}>_3 = a_1b_1 + a_2b_2 + a_3b_3$ invariant while the group SO(2,1) leaves $<\underline{a},\underline{b}>_{2,1} = a_1b_1 - a_2b_2 + a_3b_3$ invariant. We can write $<\underline{a},\underline{b}>_3 = (u\underline{a})^t\ I(u\underline{b}) = (\underline{a})^t\ I(\underline{b}) = (\underline{a})^t\ (\underline{b})$ so that $u^tIu = I$ if $u^t = u^{-1}$, $u \in SO(3)$; while $<\underline{a},\underline{b}>_{2,1} = (\bar{u}\underline{a})^t\ \bar{I}(\bar{u}\underline{b}) = (\underline{a})^t\ \bar{I}(\underline{b})$ if $\bar{u}^t\ \bar{I}\bar{u} = \bar{I}$ for $\bar{u} \in SO(2,1)$ with $\bar{I} = T^{-1}$.

3. Interaction

The question of where to put the potential energy was essentially resolved when it was found to fit perfectly as the scalar component of a four–vector. Since this point of view is being questioned in our approach, we must revisit this issue.

Consider the following Hamiltonians:

Case 1. $H = [c^2 \underset{\sim}{P}^2 + (mc^2 + V)^2]^{1/2}$,

Case 2. $H = [c^2 \underset{\sim}{P}^2 + m^2 c^4]^{1/2} + V$,

corresponding to two different ways of describing particle interactions. Here, V is assumed to be independent of the momenta.

In case 1 we obtain

$$\frac{d\underset{\sim}{q}}{dt} = \frac{c^2}{H} \underset{\sim}{P}, \quad \frac{dt}{d\tau} = \frac{H}{mc^2 + V},$$

$$\frac{d}{dt} \underset{\sim}{P} = \frac{(mc^2 + H)}{H} (-\underset{\sim}{\nabla} V),$$

so that

$$\frac{d\underset{\sim}{q}}{d\tau} = (m + \frac{V}{c^2})^{-1} \underset{\sim}{P} \tag{3.1}$$

and $\frac{d\underset{\sim}{P}}{d\tau} = -\underset{\sim}{\nabla} V$. We note from (3.1) that, when $V << mc^2$,

$$\frac{d\underset{\sim}{q}}{d\tau} = \frac{\underset{\sim}{P}}{m}, \tag{3.2}$$

the corresponding nonrelativistic form relative to the time τ.

We take K as in (2.2) so that, by an analogue of (2.1), we have $\frac{dW}{d\tau} = \frac{H}{mc^2 + V} \{H,W\}$ or

$$\frac{dW}{d\tau} = \frac{H}{mc^2} [\frac{\partial H}{\partial \underset{\sim}{P}} (1 + \frac{V}{mc^2})^{-1} \frac{\partial W}{\partial \underset{\sim}{q}} - \frac{\partial H}{\partial \underset{\sim}{q}} (1 + \frac{V}{mc^2})^{-1} \frac{\partial W}{\partial \underset{\sim}{P}}]. \tag{3.3}$$

Thus, if we set $T = (1 + \frac{V}{mc^2})^{-1}$, which we note is comparable to unity in the nonrelativistic regime (3.2), we obtain

$$\frac{dW}{d\tau} = \frac{\partial K}{\partial \underset{\sim}{P}} T \frac{\partial W}{\partial \underset{\sim}{q}} - \frac{\partial K}{\partial \underset{\sim}{q}} T \frac{\partial W}{\partial \underset{\sim}{P}} = \{K,W\}^*$$

demonstrating that the proper–time dynamics is described by an isotopic Lie algebra. We infer from the above discussion that *the interaction induces a change in the symmetry structure of the system.*

We can formalize this result as follows. Define $\hat{I} = T^{-1}$ and replace the complex number field $\mathbb{C}$ by $\hat{\mathbb{C}} = \{c\hat{I}: c \in \mathbb{C}\}$, so that $\hat{\mathbb{C}}$ is an example of an *isofield* for which $\hat{I}$ is the unit [6]. For example, the multiplication of two isonumbers is defined as

$$\hat{c} * \hat{b} = (c\hat{I})T\,(b\hat{I}) = cb\hat{I} = (cb)^{\hat{}} \text{ for } c,b \in \mathbb{C}.$$

In a similar manner, a Lie algebra G can be "deformed" to obtain a Lie–isotope of G as discussed in Section 2.

For case 2 we obtain

$$\frac{d\underset{\sim}{q}}{dt} = \frac{c^2\underset{\sim}{P}}{H - V}, \frac{d\underset{\sim}{P}}{dt} = -\underset{\sim}{\nabla}V,$$

$$\frac{dt}{d\tau} = \frac{H - V}{mc^2} = \frac{H}{mc^2}(1 - \frac{V}{H}),$$

$$\frac{d\underset{\sim}{q}}{d\tau} = \frac{\underset{\sim}{P}}{m}, \frac{d\underset{\sim}{P}}{d\tau} = (\frac{H - V}{mc^2})\,(-\underset{\sim}{\nabla}V),$$

and the analogue of (3.3):

$$\frac{dW}{d\tau} = \frac{H}{mc^2}\left[\frac{\partial H}{\partial \underset{\sim}{P}}\left(1 - \frac{V}{H}\right)\frac{\partial W}{\partial \underset{\sim}{q}} - \frac{\partial H}{\partial \underset{\sim}{q}}\left(1 - \frac{V}{H}\right)\frac{\partial W}{\partial \underset{\sim}{P}}\right].$$

In the present case we set $T = 1 - \frac{V}{H}$ and $\tilde{I} = T^{-1}$ so that $T = T^{-1} \cong I$ in the region $V << H$. The operator K is again given by (2.4), and we find

$$K = \frac{\underset{\sim}{P}^2}{2m} + mc^2 + V\sqrt{1 + \left(\frac{\underset{\sim}{P}}{mc}\right)^2} + \frac{V^2}{2mc^2}. \tag{3.4}$$

For purposes of comparison, we note that for case 1 we obtain from (2.4):

$$K = \frac{\underset{\sim}{P}^2}{2m} + mc^2 + V + \frac{V^2}{2mc^2}. \tag{3.5}$$

We note that the two Hamiltonians (3.4), (3.5) agree in the nonrelativistic limit but differ from each other in the ultrarelativistic regime.

4. CONCLUDING REMARKS

We have discussed a formulation of single–particle classical relativistic Hamiltonian mechanics in terms of a proper–time implementation of special relativity using a transformation from observer time to system proper–time which is a canonical contact transformation on extended phase space. The problem of interaction was investigated for two–body potentials independent of the particle momenta. It was shown that the interaction induces a change in the symmetry structure of the system which can be analyzed in terms of a Lie–isotopic deformation of the (Lie) algebra of observables.

In both cases considered in Section 3, the total energy of the system is conserved. In the first case we find an easy physical interpretation; viz., the particle is interacting with a comoving force. The second case does not seem to have a simple interpretation. We infer from it the possibility that the particle can tell the difference between a change in mass at each point and an external comoving force which does not depend on its clock. We believe that our approach makes the four–vector concept unnecessary and solves the interpretational problems associated with the second case.

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