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REMARKS ABOUT MASSIVE AND MASSLESS PARTICLES IN SUPERSYMMETRY

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Abstract

The internal space-time symmetry and simple supersymmetry of relativistic particles are briefly discussed in terms of the little group of the Poincaré group. The little group generators in a finite-dimensional matrix representation of the $N = 1$ super-Poincaré algebra are explicitly constructed. The supergeometry of a massive case *continuously* becomes that of a massless case in the infinite-momentum limit. The origin of the gauge transformations associated with the massless supermultiplets becomes transparent in that limit.

1 Introduction

The concept of the little group of the Poincaré group turned out to be very useful in analyzing the internal space-time symmetries of elementary particles and, hence, in assigning quantum numbers for them [1,2]. The internal space-time symmetry groups for massive and massless particles are known to be locally isomorphic to the three-dimensional rotation group $O(3)$ and the two-dimensional Euclidean group $E(2)$, respectively. The little group of the massless particle can also be represented by the cylindrical group, which is isomorphic to the Euclidean group, when the cylindrical axis being parallel to the momentum [3]. The little groups for massive and massless particles are in fact related by the Wigner-Inonü-type group contraction [4]. As was explained recently [5], the little group for massless particles is an infinite-momentum zero-mass limit of the little group for massive particles.

Our purpose is to extend those observations to the case of supersymmetry. Here we will restrict ourselves to the case of simple or $N = 1$ supersymmetry in four space-time dimensions, though the extended supersymmetries with or without central charges [6], as well as higher-dimensional supersymmetries, could also be studied along similar lines. The role of Wigner's little groups in particle theory and supersymmetry is illustrated in Table I.

We denote the generators for translations and Lorentz transformations by P_μ and $M_{\mu\nu}$, respectively, and for global supersymmetry transformations by Q_a. The algebra of global simple supersymmetry is an extension of the ordinary Poincaré algebra, and it is known as the $N = 1$ super-Poincaré algebra [7]. It comprises

$$\frac{1}{i}[M_{\mu\nu}, M_{\lambda\rho}]_- = \eta_{\mu\lambda}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\rho} ,$$

$$\frac{1}{i}[M_{\mu\nu}, P_\lambda]_- = \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu ,$$

$$[M_{\mu\nu}, Q_a]_- = (\Sigma_{\mu\nu})_a{}^b Q_b \ ,$$
$$\{Q_a, Q_b\}_+ = (\gamma^\mu C)_{ab} P_\mu \ ,$$
$$[P_\mu, P_\nu]_- = [P_\mu, Q_a]_- = 0 \ , \qquad (1)$$

where the third line means, in particular, that the Q transforms as a spinor under Lorentz transformations. The most important equation is represented by the fourth line, which allows to interpret the supersymmetry as the square-root of space-time.

We use the conventions in which $x^\mu \equiv (x^i, t) = (x, y, z, t)$ and $\eta = \mathrm{diag}(+++-)$. In eq. (1) the $\Sigma_{\mu\nu}$ denote the Lorentz generators in the spinor representation, $\Sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]_-$, the C is the four-dimensional charge conjugation matrix, and the γ^μ are Dirac matrices in four dimensions.

2 Matrix Representation of Supersymmetry

An explicit 5×5 matrix representation of the $N = 1$ super-Poincaré algebra (1) is known due to Ferrara and van Nieuwenhuizen [8]

$$M_{\mu\nu} = \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & \Sigma_{\mu\nu} & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} , P_\mu = \begin{pmatrix} & & & & 0 \\ & & & & 0 \\ & \gamma_\mu(1-\gamma_5) & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$Q_a = \begin{pmatrix} 0 & 0 & 0 & 0 & [(1+\gamma_5)C]_{1a} \\ 0 & 0 & 0 & 0 & [(1+\gamma_5)C]_{2a} \\ 0 & 0 & 0 & 0 & [(1+\gamma_5)C]_{3a} \\ 0 & 0 & 0 & 0 & [(1+\gamma_5)C]_{4a} \\ (1-\gamma_5)_{a1} & (1-\gamma_5)_{a2} & (1-\gamma_5)_{a3} & (1-\gamma_5)_{a4} & 0 \end{pmatrix} , \qquad (2)$$

In particular, the relation $[P_\mu, P_\nu]_- = 0$ easily follows from the definitions $\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0$, $\gamma_5^2 = 1$. All of the momentum-component operators in eq. (2) are in fact nilpotent and, hence, the representation (2) can serve for the massless case only. Clearly, this finite-dimensional representation of the super-Poincaré group is not unitary. Another convenient representation of the generators of the super-Poincaré group in terms of differential operators is provided by the superspace [7] parametrized by $(x^\mu, \vartheta^\alpha, \bar{\vartheta}_{\dot\alpha})$, where ϑ's represent the Grassmannian anticommuting spinor coordinates in the two-component notation:

$$\vartheta_a = \begin{pmatrix} \vartheta_\alpha \\ \bar{\vartheta}^{\dot\alpha} \end{pmatrix} , \ \gamma_\mu = \begin{pmatrix} 0 & -i\sigma_\mu \\ -i\bar{\sigma}_\mu & 0 \end{pmatrix} , \ \begin{array}{ll} \sigma_\mu = (-\sigma_i, 1); & \alpha = 1,2 \\ \bar{\sigma}_\mu = (+\sigma_i, 1); & \alpha = 1,2 \end{array} \qquad (3)$$

The representation of the super-Poincaré algebra in superspace reads [7]

$$P^\mu = -i\partial^\mu \ , \ M^{\mu\nu} = -i(x^\mu\partial^\nu - x^\nu\partial^\mu) + \tfrac{1}{2}(\vartheta\sigma^{\mu\nu}\frac{\partial}{\partial\vartheta} + \bar{\vartheta}\bar{\sigma}^{\mu\nu}\frac{\partial}{\partial\bar{\vartheta}}) \ ,$$

$$Q_\alpha = -i\frac{\partial}{\partial\vartheta^\alpha} - (\sigma^\mu \partial_\mu \bar{\vartheta})_\alpha \ , \ \bar{\vartheta}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\vartheta}^{\dot{\alpha}}} + (\vartheta\sigma^\mu\partial_\mu)_{\dot{\alpha}} \ , \tag{4}$$

where the $\frac{\partial}{\partial\vartheta}$ and $\frac{\partial}{\partial\bar{\vartheta}}$ are the Grassmannian left derivatives, and

$$\sigma_{\mu\nu} \equiv \sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu \ , \ \bar{\sigma}_{\mu\nu} \equiv \bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu \ . \tag{5}$$

This representation can be used for both massive and massless cases.

3 Little Group and Wigner-Inonü Contraction

According to Wigner [1], the little group is the maximal subgroup of the Poincaré group whose transformations leave the four-momentum of a given particle invariant. For a massive point particle one can choose a Lorentz frame in which the particle is at rest. In this frame, the little group is clearly the three-dimensional rotation group. The whole group of Lorentz transformations is generated by these three rotation generators J_i and, in addition, three Lorentz boost generators K_i [1,2]. Hence, the little group of the moving (say, along the z direction) massive particle can be obtained by boosting with the operator $B(\eta) = \exp(-\eta K_3)$. Then the little group is generated by

$$\begin{aligned} J_1' &= (\cosh\eta)J_1 + (\sinh\eta)K_2 \ , \\ J_2' &= (\cosh\eta)J_2 - (\sinh\eta)K_1 \ , \\ J_3' &= J_3 \ . \end{aligned} \tag{6}$$

The idea is to consider the rapidly moving massive particle for large values of η. Then after renormalizing the generators J_1' and J_2' as $N_1 \equiv -(\cosh\eta)^{-1}J_2'$ and $N_2 \equiv (\cosh\eta)^{-1}J_1'$, in the infinite-$\eta$ limit one obtains

$$\begin{aligned} N_1 &= K_1 - J_2 \ , \\ N_2 &= K_2 + J_1 \ . \end{aligned} \tag{7}$$

These operators and J_3 satisfy the commutation relations of the $E(2)$-like little group for *massless* particles [1,2,9] and, hence, the massless case is not needed to be considered as independent. The supersymmetry representation theory was usually considered separately for the massive and massless cases, while the Wigner-Inonü group contraction provides a connection between them.

In case of the representation (2) of the super-Poincaré algebra, let $\tilde{P} \equiv P_0 + P_3$ be the fixed momentum. Then it is easy to check that the associated little group is generated by the three generators among $M_{\mu\nu}$: $\Sigma_{12} \sim \gamma_1\gamma_2$, $(\Sigma_{01} + \Sigma_{31}) \sim (\gamma_0 + \gamma_3)\gamma_1$ and $(\Sigma_{02} + \Sigma_{32}) \sim (\gamma_0 + \gamma_3)\gamma_2$. Taking the convenient representation of the 4×4 γ-matrices, in which

$$\gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix} \ , \ \gamma_0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \ , \ \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{8}$$

where we have introduced the standard 2×2 Pauli matrices as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{9}$$

we have

$$C = i\gamma_2\gamma_0 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \tag{10}$$

to satisfy the defining equation $C\gamma_\mu = -\gamma_\mu^T C$ for the charge conjugation matrix C. Therefore, we find

$$\gamma_1 C = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \gamma_2 C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\gamma_3 C = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \gamma_0 C = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Now it is easy to calculate the square root $\tilde{Q}$ of the given momentum $\tilde{P}$ in the supersymmetry algebra:

$$\tilde{Q} = \sum_{a=1}^{4} d_a Q_a \;, \; \{\tilde{Q}, \tilde{Q}\}_+ = \sum_{a,b=1}^{4} d_a d_b (\gamma^\mu C)_{ab} P_\mu = 2\tilde{P} \;, \tag{12}$$

i.e. find the appropriate numerical coefficients d_a. The result is given by

$$\tilde{Q} = 2e^{-i\pi/4}(Q_2 + Q_3) = 2e^{-i\pi/4} \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{13}$$

so that $\tilde{Q}^2 = \tilde{P}$ indeed. One should emphasize that no such notion as the little group of $\tilde{Q}$, can be introduced, since the Q-operators are defined in the *spinor* representation space and that group would be trivial. Now it becomes clear why the generators of the little group of $\tilde{P}$ do *not* commute with its square root $\tilde{Q}$.

With each massless particle one can associate a circular cylinder whose axis is parallel to the momentum. Then one can rotate a point on the surface of this cylinder around the axis or translate along the direction of the axis. As is well known, the rotational degree of freedom is associated with the helicity, while the translation corresponds to a gauge transformation [3,9]. This translational degree of freedom is shared by all massless particles. In case of supersymmetry, we can extend the contents of gauge transformations to all massless supermultiplets by considering again the massive supermultiplets in the infinite-momentum limit.

Taking the mass value to be equal to 1 for convenience, the massive particle at rest is characterized by the four-momentum $P^\mu_{\rm m,r} = (0,0,0,1)$. The same particle moving with the momentum p along the z direction, has the four-momentum $P^\mu_{\rm m} = (0,0,p,\sqrt{p^2+1})$. Renormalizing this operator as $P^\mu_{\rm m} \rightarrow p^{-1}P^\mu_{\rm m} \equiv P^\mu_{\rm r}$, we obtain in the infinite-p limit that $P^\mu_{\rm r} = (0,0,1,\sqrt{1+p^{-2}}) \rightarrow P^\mu_0 \equiv (0,0,1,1)$, which is just the conventional choice of the four-momentum in the massless case.

These very simple observations are still very useful in the case of supersymmetry. The supersymmetry algebra can conveniently be represented for our purposes here in the two-dimensional notation of eq. (3) as

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\}_+ = \sigma^\mu_{\alpha\dot\beta} P_\mu \, ,$$

$$\{Q_\alpha, Q_\beta\}_+ = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \, . \tag{14}$$

Now, on the one hand, we immediately see that in the massive case at rest we obtain a Clifford algebra of the form

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\}_+ = \mathbf{1}_{\alpha\dot\beta} \, , \tag{15}$$

where all of the Q-operators are active. They can be interpreted as the operators of creation and destruction, and then used to develop the massive supermultiplets structure [6,7].

On the other hand, in the massless case we obtain instead

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\}_+ = (\mathbf{1} + \sigma_3)_{\alpha\dot\beta} \, ,$$

$$\mathbf{1} + \sigma_3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \, , \tag{16}$$

which means the degeneracy of the supersymmetry algebra. Eq. (16) can be obtained from eq. (15) in the infinite-momentum limit after the renormalization $Q_{\text{mass}} \to Q_{\text{massless}}$ induced by the transition $P^\mu_{\text{m,r}} \to P^\mu_0$ discussed above. This gives rise to the reduced supermultiplets structure since only *a half* of the Q-operators are now active. The rest represents the supersymmetric gauge transformations which always accompany the massless supermultiplets containing photino or gravitino in this picture (their role is to kill the redundant degrees of freedom), just like the invariance under the translational gauge symmetry is associated with photons and gravitons [2,10].

The main point of our brief discussion is that the massive and massless cases in supersymmetry should be considered on equal footing, the connection between them being provided by the Wigner-Inonü contraction, which has a clear physical meaning. Of course, this fact is already known and can be read off, in particular, from the contents of Refs. [11,12]. Nevertheless, we would like to stress its conceptual simplicity in this paper, and give it in the most obvious way, which was not presented in the past.

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TABLE I. *Symmetries of massive and massless particles*

The first two rows display the unification of the energy-momentum relations and the internal symmetries of massive and massless particles, as given in Ref. [5]. The third row means that supersymmetry can also be included into this picture.

	Massive *or* Slow	← between →	Massless *or* Fast
Energy *and* Momentum	$E = \frac{p^2}{2m}$	$\leftarrow E = \sqrt{m^2 + p^2} \rightarrow$	$E = p$
Spin, Gauge *and* Helicity	S_3 $S_1 \quad S_2$	← Little Groups →	S_3 Gauge Transformations
Supersymmetry	$Q_1, \bar{Q}_{\dot{1}}$ $Q_2, \bar{Q}_{\dot{2}}$	Square Root ← of Space-Time → Translations	$Q_1, \bar{Q}_{\dot{1}}$ Non-Active Charges