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THE ALGEBRA OF SUPERTRACES FOR 2+1 SUPER DE SITTER GRAVITY

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Abstract

We calculate the algebra of the observables for 2+1 super de Sitter gravity, for one genus of the spatial surface. The algebra turns out to be an infinite Lie algebra subject to non-linear constraints. We solve the constraints explicitly in terms of five independent complex supertraces. These variables are the true degrees of freedom of the system and their quantized algebra generates a new structure which we refer to as a "central extension" of the quantum algebra $SU(2)_q$.

1 Introduction

The discovery by Witten that many gravity theories in $2+1$ dimensions are equivalent to Chern Simons theories, and are in principle exactly quantizable, has sparked a great deal of interest in their study [1]. Perhaps the key obstacle in carrying out this quantization explicitly has been our poor understanding of the observable phase space. Pure Chern-Simons theories in vacuum are locally trivial and interesting situations arise either in the presence of sources or when the topology of the space-time manifold is non-trivial. In either case, the observable degrees of freedom for the field theory are the traces of the holonomies (alternatively called the integrated connections) associated to non-contractible loops of the space-time manifold M, which are classified by the fundamental group $\pi_1(M)$. These traces span the reduced phase space of the theory in a highly redundant way. Indeed, the group $\pi_1(M)$ is infinite, while the dimension of the reduced phase space is known to be $(2g-2) \times dim(\mathcal{A})$, where $\mathcal{A}$ is the Lie algebra considered in the Chern-Simons action. The traces are subject to non-linear constraints (NLC) which depend on the characteristic equation for the matrices in the defining representation. Our approach here is to first reduce the classical system to a finite-dimensional observable phase space and then quantize. Unfortunately, this is an extremely difficult task, which has only recently been solved for arbitrary genus in de Sitter gravity [2].

The reduced phase space is well-understood for any genus in Poincaré gravity [3], but in terms of inhomogeneous variables which have not yet been generalized to curved spacetimes. The purpose of this contribution is to provide the reduced phase space for one genus in 2+1 super de Sitter gravity.

2 The Algebra of Supertraces

Following previous discussions, we will consider the case when the space-time manifold has the topology $M = \Sigma \times R$, where R is the time and Σ is an arbitrary closed, orientable two-dimensional surface of genus g. Also we will restrict the discussion to only one genus of such a surface. The Poincaré [4,5], de Sitter [6], and conformal [7] cases have been previously discussed along these lines and the Poisson bracket algebra of the traces calculated. The quantized version of the algebra of observables for the de Sitter case provides a realization of a pair of commuting $SU(2)_q$ quantum algebras [8].

Witten's formulation of 2+1 dimensional gravity theories as Chern-Simons theories has been extended to the supersymmetric case in Ref. [9], where the super de Sitter case is studied by considering the orthosymplectic group $OSp(1|2;\mathbb{C})$ as the gauge group. The system is described by the Chern Simons action [10],

$$I = \frac{1}{2} Tr \int_M \left(dA - \frac{2}{3} A \wedge A \right) \wedge A, \tag{1}$$

where $A = A_\mu dx^\mu$ ($\mu = 0,1,2$) is the superconnection

$$A = A^A T_A = e^a P_a + \omega^a J_a + \chi^\alpha U_\alpha + \Theta^\alpha V_\alpha, \tag{2}$$

which takes values on the Lie algebra of $OSp(1|2;\mathbb{C})$. Here $T_A = (P_a, J_a, U_\alpha, V_\alpha)$, where $P_a, J_a (a = 0,1,2)$ are the bosonic generators and $U_\alpha, V_\alpha (\alpha = 1,2)$ are the fermionic ones. The fields $\chi^\alpha, \Theta^\alpha$ are spinors whose components are odd Grassmann numbers.The trace in Eq.(1) is defined in terms of the group-invariant non-degenerate bilinear tensor

$$Tr(T_A T_B) = D_{AB} = \begin{bmatrix} 0 & \eta_{ab} & 0 & 0 \\ \eta_{ab} & 0 & 0 & 0 \\ 0 & 0 & -2\epsilon_{\alpha\beta} & 0 \\ 0 & 0 & 0 & 2\epsilon_{\alpha\beta} \end{bmatrix}, \tag{3}$$

where $\eta_{ab} = diag(-1,1,1), \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, with $\epsilon_{12} = +1$ and $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha^\beta$. The generators satisfy the superalgebra of $OSp(1 \mid 2);\mathbb{C})$ which is given in Ref. [9].

The constraints equations that follow from (1) imply that A is a pure gauge, that is $A = d\psi\psi^{-1}$ where $\psi \in OSp(1|2;\mathbb{C})$. The Poisson brackets of A are easily calculated from the action (1) [11],

$$\{A_i(x), A_j(y)\}_{P.B.} = -2\epsilon_{ij}\, M\, \delta^2(x-y), \tag{4a}$$

where x, y are generic points on the Σ, $i,j = 1,2$ are spatial vector indices on Σ, $\epsilon_{ij} = -\epsilon_{ji}$ with $\epsilon_{12} = +1$ and

$$M \equiv D^{AB} T_A \otimes T_B = P_a \otimes J^a + J_a \otimes P^a + \frac{1}{2}\left(U_\alpha \otimes U^\alpha - V_\alpha \otimes V^\alpha\right), \qquad (4b)$$

with $D_{AC} D^{CB} = \delta_A^B$.

Let us consider two generic points P, Q on Σ and a path ρ joining them, parametrized by $x(t)$, $t \in [0,1]$ with $x(0) = P$ and $x(1) = Q$. The solution to the differential equation

$$\frac{d\psi}{dt} = A_t \psi \qquad (5)$$

subject to the boundary condition $\psi(0) = 1$, where $A_t \equiv A_\alpha T^\alpha$ is a tangent vector along ρ, will depend only on the homotopy class of ρ and it is denoted by $\psi(\rho)$ (see Ref. [6] for details). For a second path ρ' with end points Q, R we have the solution $\psi(\rho')$ of (5). The solution for the path $\rho'\rho$, with end points P, R is then

$$\psi(\rho'\rho) = \psi(\rho')\psi(\rho). \qquad (6)$$

By restricting to closed paths, this equation defines a group homomorphism $\psi : \pi_1(\Sigma) \to OSp(1|2;\mathbb{C})$. The fundamental group of the surface Σ based on the point B, $\pi_1(\Sigma, B)$, is presented via 2g generators u_i, v_i $i = 1, ..., g$ which satisfy the relation $u_1 v_1 u_1^{-1} v_1^{-1} ... u_g v_g u_g^{-1} v_g^{-1} = 1$.

Let ψ, ϕ be generic elements of $OSp(1|2;\mathbb{C})$. The Poisson brackets of the integrated connections $\psi(\rho)$, $\psi(\sigma)$ of two elements of $\pi_1(\Sigma)$, with base points P, Q respectively, which have a single intersection may be calculated from (4) by a procedure already established in Refs. [4,6]. The result is

$$\begin{aligned}\left\{\psi_\alpha{}^\beta(\rho),\ \psi_\mu{}^\nu(\sigma)\right\}_{P.B.} = 2s\,(-1)^{\,[(g(\beta)-g(\eta))(g(\alpha)-g(\eta))+(g(\mu)-g(\theta))(g(\nu)-g(\theta))]}& \\ M_{\xi\theta}{}^{\eta\varphi}\,\psi_\eta{}^\beta(\rho_i)\psi_\alpha{}^\xi(\rho_f)\psi_\varphi{}^\nu(\sigma_i)\psi_\mu{}^\theta(\sigma_f)& \qquad (7)\end{aligned}$$

where $M_{\xi\theta}{}^{\eta\varphi} = D^{AB}(T_A)_\xi{}^\eta\,(T_B)_\theta{}^\varphi$. The subindex i (f) labels that part of the path before (after) the intersection and $s = s(\rho,\sigma) = -s(\sigma,\rho) = \pm 1$ is called the intersection number.

The integrated connection $\psi(\rho)$ is not gauge invariant, but the supertrace $C(\rho) \equiv Str\psi(\rho) = (-1)^{g(\alpha)}\,\psi_\alpha{}^\alpha$ is, namely:

$$C(\rho) = C(\nu\rho\nu^{-1}) \qquad (8)$$

with $\rho \in \pi_1(\sigma)$ and ν being any open path. Equation (8) expresses the invariance of $C(\rho)$ under a change of the base point of $\pi_1(\sigma)$. Thus, one can calculate the Poisson bracket of two closed paths ρ and σ based on two different points P and Q respectively and make $P = Q$ after the calculation, so that ρ, σ become elements of $\pi_1(\sigma; Q)$. By supertracing (7) one obtains [13]

$$\{C(\rho), C(\sigma)\}_{P.B.} = is\sqrt{\Lambda}\,\left(C(\rho\sigma) - C(\rho\sigma^{-1})\right), \qquad (9)$$

for paths with a single intersection or with no intersection ($s = 0$). This result is the same that has been obtained for the de Sitter, Poincaré and conformal groups [4-7].

By repeated use of (7) and with the help of (6) we obtain the following general formula for the Poisson brackets of elements ρ, σ of $\pi_1(\Sigma;\, Q)$ with n intersections

$$\{\, C(\rho),\, C(\sigma)\,\}_{P.B.} = i\sqrt{\Lambda}\sum_{k=1}^{n} s_k\,\left(C(\rho_k\sigma_k) - C(\sigma_k\rho_k^{-1})\right), \tag{10}$$

where s_k is the intersection number of the k-th intersection and the subindex k on each path means that the product of them is constructed by taking the k-th intersection point as the base point, instead of the point Q.

Any matrix $\psi(\rho)$ which is an element of $OSp(1|2;\mathbb{C})$ satisfies the generalized Cayley-Hamilton identity

$$\psi(\rho^3) - (C(\rho)+2)\left(\psi(\rho^2) - \psi(\rho)\right) - 1 = 0. \tag{11}$$

Multiplying (11) by $\psi(\sigma\rho^{-1})$ and supertracing one obtains the non linear constraint

$$R(\rho,\, \sigma) \equiv C(\rho)C(\rho\sigma) - C(\rho)C(\sigma) - C(\rho^2\sigma) + C(\rho\sigma^{-1}) + 2C(\rho\sigma) - 2C(\sigma) = 0. \tag{12}$$

In order to obtain the algebra of observables we must take into account the relation (12). This relation appears to be an ideal of the traces algebra. Although we were not able to obtain an algebraic proof, computer calculations in various examples indicate that R has zero Poisson bracket with the traces, as in the ordinary de Sitter case [6]. This implies that the relations (12) hold "strongly", i.e. that they can be used within the Poisson brackets (10).

Fortunately, it is possible to solve the relations $R(u,v) = 0$ explicitly, by expressing all traces on one genus in terms of five fundamental ones, which can be chosen as $C(u) = A$, $C(v) = B, C(uv) = C$, $C(uv^2) = D$ and $C(uvu^2v^2) = E$. This property can be shown to be a direct consequence of the identity (12).

Finally, we can calculate the algebra satisfied by these variables. To this end it is more convenient to define the following combinations of the basic traces previously introduced

$$\begin{aligned} X &= \frac{1+A}{2}, \quad Y = \frac{1+B}{2}, \quad Z = \frac{1+C}{2}, \\ V &= \frac{1}{2}\left(\frac{1+D}{2} + X - 2YZ\right), \\ U &= \frac{1+E}{2} - Z(1 + 8XYZ - 4X^2 - 4Y^2 \\ &\qquad + 8(X+Y)V + 4V). \end{aligned} \tag{13}$$

This choice is dictated by the property that in the de Sitter limit (fermionic variables equal to zero) X, Y and Z go into the variables used in Ref. [6], while U and V go to zero. The Poisson brackets of these variables can be computed with the help of (10), assuming that the relations (12) are indeed an ideal of the algebra. We find

$$\begin{aligned} &\{X,Y\}_{P.B.} = is\sqrt{\Lambda}(Z - XY - V), \\ &\{X,V\}_{P.B.} = \{X,U\}_{P.B.} = \{U,V\}_{P.B.} = 0, \end{aligned} \tag{14}$$

plus cyclical permutations of X, Y, Z.

We quantize the above system using the correspondence principle $XY - YX = i\hbar \times \{X, Y\}_{P.B.}$ and symmetrising the XY product. The result can be written as

$$e^{i\theta/2}XY - e^{-i\theta/2}YX = 2isin\theta/2\ (Z - V), \tag{15}$$

and cyclical, where $tan\theta/2 \equiv \frac{i\hbar s\sqrt{\Lambda}}{2}$ and U, V are central elements. The de Sitter limit ($U = V = 0$) on Ref. [6] is clearly recovered from Eqs. (15) now in terms of arbitrary complex variables X, Y and Z. The algebra (15) provides a central extension of $SU(2)_q$ [14], with V being the central charge.

3 The NLC Constraints

These are relations among the supertraces (see for example Eq. (12)), which constitute the basic tool for reducing the original infinite dimensional supertraces algebra to a finite one. A general way of obtaining such relations is starting from a Cayley-Hamilton type identity satisfied by the matrix. In the case of a supermatrix M, the characteristic polynomial is not given by $p(x) = Sdet(M - xI)$, and the problem of constructing such polynomial in the general case seems to be still an open one. The basic definition is $p(x) = \Pi_i(\lambda_i - x)$, where λ_i are the eigenvalues of M and the idea is to translate this into "simpler" operations which would bypass the explicit calculation of the eigenvalues. In the case of an arbitrary 2×2 supermatrix with entries $M_{12} = a$, $M_{12} = \alpha$, $M_{21} = \beta$, $M_{22} = b$, where a, b (α, β) are even (odd) Grassmann numbers, the characteristic polynomial is

$$p(x) = (a - b)x^2 - (a^2 - b^2 + 2\alpha\beta)x + (ab(a - b) + (a + b)\alpha\beta), \tag{16}$$

and one can verify that $p(M) \equiv 0$ as a matrix identity. Another explicit example of such polynomials is Eq. (11) which corresponds to a particular case of a 3×3 supermatrix.

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