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NONLINEAR MODES OF THE TENSOR DIRAC EQUATION AND CPT VIOLATION

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Abstract

Recently, it has been shown that Dirac's bispinor equation can be expressed, in an equivalent tensor form, as a constrained Yang-Mills equation in the limit of an infinitely large coupling constant. It was also shown that the free tensor Dirac equation is a completely integrable Hamiltonian system with Lie algebra type Poisson brackets, from which Fermi quantization can be derived directly without using bispinors. In this paper we investigate the Yang-Mills equation for a finite coupling constant. We show that the nonlinear Yang-Mills equation has exact plane wave solutions in one-to-one correspondence with the plane wave solutions of Dirac's bispinor equation. We apply the theory of nonlinear dispersive waves to establish the existence of wave packets. We investigate the CPT violation of these nonlinear wave packets, which could lead to new observable effects consistent with current experimental bounds.

1 Introduction

In a recent paper [1] it was shown that square-integrable positive energy bispinor fields in a Minkowski space-time cannot be physically distinguished from constrained tensor fields. It was also shown [1], [2] that the free tensor Dirac equation is a completely integrable Hamiltonian system with (non-canonical) Lie algebra type Poisson brackets, from which Fermi quantization can be derived directly without using bispinors.

Also, it was shown [1] that the tensor Dirac Lagrangian may be derived from the following Yang-Mills Lagrangian for SL(2,C) $\times$ U(1) gauge potentials A^K_α and complex scalar field ρ:

$$L = -\tfrac{1}{4}\,\mathrm{Re}\,[A^{\alpha\beta}_K A^K_{\alpha\beta}] + \overline{(D_\alpha\sigma)}\,(D^\alpha\sigma) - V(|\rho|^2) \tag{1}$$

where $\sigma = \rho + c$ where c is a constant, and $V = V(|\rho|^2)$ is a smooth (at least twice differentiable) function of $|\rho|^2$. The gauge potentials A^K_α satisfy the orthogonal constraint:

$$A^K_\alpha A_{K\beta} = -|\rho|^2 g_{\alpha\beta} \tag{2}$$

where $g_{\alpha\beta}$ is the metric tensor. More detailed discussion of formulas (1) and (2) is given in Section 2. With the further condition:

$$\lim_{g\to\infty} g^{-2}\,V = \tfrac{1}{2}|\rho|^4 \tag{3}$$

where g is the Yang-Mills coupling constant, and setting $c = 2m_e/g$ where m_e is the fermion mass, then the Dirac Lagrangian equals the Yang-Mills Lagrangian (1) in the limit that the coupling constant g tends to infinity.

The fact that the free Dirac equation is a constrained Yang-Mills equation in the limit of a large coupling constant is significant for both classical and quantized theories. The classicial theory, which we regard as the first quantized theory, is characterized by the classical observables given by the electric current J^α, the energy-momentum tensor $T^{\alpha\beta}$, and the spin-polarization tensor $S^{\alpha\beta\gamma}$. These classical observables are sufficient to describe many experiments with electron beams [3]. The present paper addresses these classical observables.

In Section 2, we consider the Lagrangian (1) for the case of a finite coupling constant g. We show that for any smooth self-interaction $V(|\rho|^2)$, the constrained Euler-Lagrange equations have *exact* plane wave solutions in one-to-one correspondence with the plane wave solutions of Dirac's equation. We apply the theory of non-linear dispersive waves [4] to establish the existence of wave packets.

For the special case $V = \frac{1}{2} g^2 |\rho|^4$ we show that:

a) The mass, m, of each plane wave is independent of amplitude and equals $\frac{1}{2} g|c|$.

b) The wave packets are identical with bispinor wave packets.

c) The wave packets are covariant under the CPT operation (defined in Section 3).

When $V \neq \frac{1}{2} g^2|\rho|^4$, the properties (a), (b), and (c) are all violated. However, we show in Section 3 that these violations could lead to new experimental observations consistent with present bounds for CPT violations.

2 Plane Waves and Wave Packets

In [1] we showed that Dirac's bispinor Lagrangian equals the Yang-Mills Lagrangian (1) in the limit of an infinitely large coupling constant g. In the remainder of this paper we will investigate the possibly observable effects of a finite coupling g.

First, we show in this section that the Euler-Lagrange equations for (1) and (2) have exact plane wave solutions in one-to-one correspondence with the plane wave solutions of Dirac's equation. For finite coupling g, and general V, the mass of each plane wave depends on its amplitude given in formula (2) as $|\rho|$. However, for the special case in which $V = \frac{1}{2} g^2 |\rho|^4$, we will see that the mass equals the constant $\frac{1}{2} g |c|$, and hence is independent of amplitude.

Second, we will establish the existence of wave packets using results from the theory of nonlinear dispersive waves [4]. The most significant departure from linearity is the splitting of the group velocity for finite g and general V. However, again for the special case in which $V = \frac{1}{2} g^2 |\rho|^4$, the velocity splitting does not occur, and the wave packets are identical to the bispinor wave packets which are derived from Dirac's equation.

The Euler-Lagrange equations for (1) and (2) are given by:

$$\begin{aligned} D^\alpha \vec{A}_{\alpha\beta} &= -2\lambda^\alpha_\beta \vec{A}_\alpha \\ D^\alpha A^0_{\alpha\beta} - 2g \,\mathrm{Re}\left[i\,\bar{\sigma} D_\beta \sigma\right] &= -2\,\mathrm{Re}\left[\lambda^\alpha_\beta\right] A^0_\alpha \\ D^\alpha D_\alpha \sigma + V'\rho &= \mathrm{Re}\left[\lambda^\alpha_\alpha\right] \rho \end{aligned} \tag{4}$$

where $\lambda^{\alpha\beta} = \lambda^{\beta\alpha}$ are the Lagrange multipliers for the constraint (2), where V' denotes the derivative of V with respect to $|\rho|^2$, and where the Yang-Mills covariant derivatives D_γ acting on $A^K_{\alpha\beta}$ and σ are given by:

$$
\begin{aligned}
D_\gamma A^0_{\alpha\beta} &= \nabla_\gamma A^0_{\alpha\beta} \\
D_\gamma \overrightarrow{A}_{\alpha\beta} &= \nabla_\gamma \overrightarrow{A}_{\alpha\beta} - g\, \overrightarrow{A}_\gamma \times \overrightarrow{A}_{\alpha\beta} \\
D_\gamma \sigma &= \nabla_\gamma \sigma + ig\, A^0_\gamma\, \sigma
\end{aligned}
\tag{5}
$$

Again, $\sigma = \rho + c$ and solutions A^K_α and ρ of equations (4) are required to satisfy the constraint (2).

Plane wave solutions of equations (4) are defined by:

$$
\begin{aligned}
A^0_\alpha(x^\beta) &= A^0_\alpha(0) \\
\overrightarrow{A}_\alpha(x^\beta) &= e^{2i\theta(x^\beta)T}\, \overrightarrow{A}_\alpha(0) \\
\rho(x^\beta) &= \rho(0)
\end{aligned}
\tag{6}
$$

where $x^\beta \varepsilon R^4$ denotes the space-time coordinates, T generates a one-parameter subgroup of SL(2, C) gauge transformations, and $\theta(x^\beta) = p_\beta x^\beta$ where $p_\beta \varepsilon R^4$ denotes the momentum variables. Note that if $A^K_\alpha(0)$ and $\rho(0)$ satisfy the orthogonal constraint (2), then the same is true for $A^K_\alpha(x^\beta)$ and $\rho(x^\beta)$ for all $x^\beta \varepsilon R^4$, since in formula (6), the SL(2, C) gauge transformations generated by T preserve the orthogonal constraint. Note also that

$$
T(\overrightarrow{A}_\alpha) = i\, \overrightarrow{\omega} \times \overrightarrow{A}_\alpha \tag{7}
$$

for some $\overrightarrow{\omega} \varepsilon C^3$ satisfying $\overrightarrow{\omega} \cdot \overrightarrow{\omega} = 1$. (The reader is reminded that SL(2, C) is the complexification of SU(2) for which we can take $\overrightarrow{\omega} \varepsilon R^3$.)

On differentiating formula (6) we get using (7):

$$
\begin{aligned}
\nabla_\beta A^0_\alpha &= 0 \\
\nabla_\beta\, \overrightarrow{A}_\alpha &= -2p_\beta\, \overrightarrow{\omega} \times \overrightarrow{A}_\alpha \\
\nabla_\beta \rho &= 0
\end{aligned}
\tag{8}
$$

Note in formula (8) that the $\overrightarrow{A}_\alpha$ have twice the rotation rate of bispinors, and $p^\alpha p_\alpha = m^2$ where m is the mass in Dirac's equation. Suppose that the plane waves (6) satisfy the same conditions which are satisfied by bispinor plane waves, given as follows:

$$
\begin{aligned}
p^\alpha A^0_\alpha &= 0 \\
p^\alpha \overrightarrow{A}_\alpha &= \pm m\, |\rho|\, \overrightarrow{\omega}
\end{aligned}
\tag{9}
$$

where the positive sign is used for particles and the negative sign for antiparticles. Since ρ is constant by (8), formula (9) can be regarded as the initial conditions for the fields A^K_α. Note that formula (9) is consistent with $p^\alpha p_\alpha = m^2$, $\overrightarrow{\omega} \cdot \overrightarrow{\omega} = 1$, and the constraint (2), and moreover, p_α for particles becomes $-p_\alpha$ for antiparticles. Conversely, with p_α so defined, formula (9) defines $\overrightarrow{\omega}$ and hence the gauge generator T in formulas (6) and (7).

Substituting (8) into the first two equations (4), using (2), (5), and (9), we get:

$$
\lambda_{\alpha\beta} = -2\, p_\alpha\, p_\beta + g^2 \frac{A^0_\alpha A^0_\beta}{|\rho|^2} |\sigma|^2 + \frac{\overrightarrow{A}_\alpha \cdot \overrightarrow{A}_\beta}{|\rho|^2} \left(2m^2 \pm 2\, mg|\rho| + g^2|\rho|^2\right) \tag{10}
$$

Note that $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}$ and by the constraint (2), $\lambda_{\alpha\beta}$ is real. (Recall that A^0_α is real and $\overrightarrow{A}_\alpha$ is complex.)

Now substituting (10) into the last equation (4), we get:

$$4m^2 \pm 6\,mg\,|\rho| = V' + g^2 c\bar{\rho} + g^2\left(|\rho+c|^2 - 2|\rho|^2\right) \tag{11}$$

Clearly, since all other terms are real, $c\bar{\rho}$ is also real. Without loss of generality, we assume that $c \geq 0$, hence ρ is real. Choosing $\rho \geq 0$ for particle plane waves and $\rho \leq 0$ for antiparticle plane waves, formula (11) becomes:

$$4\left(m^2 - \frac{g^2c^2}{4}\right) + 6g\left(m - \frac{gc}{2}\right)\rho = V' - g^2\rho^2 \tag{12}$$

with the obvious solution:

$$\begin{aligned} V &= \frac{g^2}{2}|\rho|^4 \\ m &= \frac{gc}{2} \end{aligned} \tag{13}$$

We see in this case that the mass $m = gc/2$ is independent of amplitude.

Wave packets are defined to be plane waves with slowly changing parameters (e.g., amplitude, spin, and momentum). To describe such wave packets we introduce "slow" coordinates $y^\beta = \varepsilon x^\beta$, where ε is a small parameter, into formula (6) as follows [4]:

$$\begin{aligned} A^0_\alpha(x^\beta) &= \underset{\sim}{A}^0_\alpha(y^\beta) \\ \vec{A}_\alpha(x^\beta) &= e^{2i\varepsilon^{-1}\underset{\sim}{\theta}(y^\beta)T}\,\vec{\underset{\sim}{A}}_\alpha(y^\beta) \\ \rho(x^\beta) &= \underset{\sim}{\rho}(y^\beta) \end{aligned} \tag{14}$$

where $\theta(x^\beta) = \underset{\sim}{\theta}(\varepsilon x^\beta)$, etc. The resulting equations governing the wave packets [4] are given by:

$$\begin{aligned} p^\alpha p_\alpha &= m^2 \\ \nabla_\alpha p_\beta &= \nabla_\beta p_\alpha \\ \nabla_\alpha J^\alpha &= 0 \end{aligned} \tag{15}$$

where now $p_\alpha = \nabla_\alpha \theta$, where $m = m(\rho)$ is given in formula (12), and

$$J_\alpha = F\,v_\alpha, \quad v_\alpha = p_\alpha/m, \quad F = \frac{8m}{g}\rho^2 + 4\rho^3 \tag{16}$$

To analyze equations (15) we now consider a space-time with one space dimension. Then $v_\alpha = (v_0, v_1)$ and the group velocity is $v = v_1/v_0$.

That is,

$$\begin{aligned} v_0 &= \gamma = \frac{1}{\sqrt{1-v^2}} \\ v_1 &= v\,\gamma \end{aligned} \tag{17}$$

Similarly, $x^{\alpha} = (t, x)$. Formula (15) becomes:

$$\frac{\partial}{\partial t}(F\gamma) + \frac{\partial}{\partial x}(Fv\gamma) = 0$$

$$\frac{\partial}{\partial t}(mv\gamma) + \frac{\partial}{\partial x}(m\gamma) = 0 \tag{18}$$

The equations for the characteristic curves for (18) are easily derived [4], and are given by:

$$\frac{dx}{dt} = \frac{v \pm \delta}{1 \pm v\delta} \tag{19}$$

where

$$\delta = \sqrt{\frac{Fm'}{F'm}} \tag{20}$$

where F' and m' denote the derivatives of F and m with respect to ρ. On the curves (19) we have:

$$\gamma^2\, dv = \pm \frac{F'\delta}{F}\, d\rho \tag{21}$$

When $V = \frac{1}{2} g^2 \,|\rho|^4$, $m' = 0$ by formula (13) so that by (21), $dv = 0$, that is, v is constant on the characteristic curves (19). Since then $\delta = 0$, the curves (19) are straight lines. It is then straightforward to show that the wave packets are identical to bispinor wave packets.

In general, for wave packets to exist, δ, in formula (20) must be real. If $V \neq \frac{1}{2} g^2 \,|\rho|^4$, a general wave packet will split into two wave packets that propagate along the characteristic curves (19).

3 CPT and Velocity Splitting

By the Cartan map, the CPT operation which, for bispinors, is given by [5]:

$$\psi(x^{\beta}) \rightarrow i\gamma_5\, \psi(-x^{\beta}) \tag{22}$$

becomes for the tensor fields A^K_{α} and ρ:

$$A^K_{\alpha}(x^{\beta}) \rightarrow A^K_{\alpha}(-x^{\beta})$$

$$\rho(x^{\beta}) \rightarrow -\rho(-x^{\beta}) \tag{23}$$

Note that because of the constant c, the Yang-Mills Lagrangian L in formula (1) is not covariant under the CPT operation (23). Nevertheless in the limiting case that the coupling constant g tends to infinity, the Euler-Lagrange equations commute with CPT. In this section we examine the question of CPT violation for finite coupling g.

One of the main tests for CPT covariance is the equality of particle and antiparticle masses [6]. According to formula (13), when $V = \frac{1}{2} g^2 \,|\rho|^4$ the masses are equal. Therefore, suppose instead that

$$V = \frac{g^2}{2} |\rho|^4 + \varepsilon|\rho|^4 \tag{24}$$

where ε is a small parameter. Then to first order in ε, formula (12) gives:

$$m = \frac{gc}{2} + \frac{\varepsilon}{3g}\rho \tag{25}$$

Since $\rho \geq 0$ for particle plane waves and $\rho \leq 0$ for antiparticle plane waves, the mass difference Δm is:

$$\Delta m = \frac{2\varepsilon}{3g} |\rho| \tag{26}$$

On substituting formula (25) into (20), the velocity splitting 2δ becomes, to lowest order in ε and g^{-1} (ignoring factors close to one; i.e., $\sqrt{2/3}$):

$$2\delta = \sqrt{\frac{\Delta m}{m}} \tag{27}$$

Assuming a fractional mass difference for electrons and positrons of one part in a million, the velocity splitting would be $2\delta = 10^{-3}$ or 3×10^5 meters per second, which should be observable in experiments that measure the spreading of low energy electron wavepackets. CPT violations of 10^{-6} are consistent with current observations of particle-antiparticle mass difference and suggest new experiments to observe velocity splittings of 3×10^5 meters per second, or less [6].

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