N93-27347

SYMMETRY ALGEBRA OF A GENERALIZED ANISOTROPIC HARMONIC OSCILLATOR

O. Castaños and R. López-Peña
Instituto de Ciencias Nucleares, UNAM
Apdo. Postal 70-543, México, D. F., 04510 México

Abstract

It is shown that the symmetry Lie algebra of a quantum system with accidental degeneracy can be obtained by means of the Noether's theorem. The procedure is illustrated by considering a generalized anisotropic two dimensional harmonic oscillator, which can have an infinite set of states with the same energy characterized by an $u(1,1)$ Lie algebra.

1 Introduction

We are going to study the accidental degeneracy [1,2] of the Hamiltonian

$$H = \frac{1}{2}\sum_i (p_i^2 + x_i^2) + \lambda M \tag{1.1}$$

which is a two dimensional harmonic oscillator plus the projection of the angular momentum in the z direction, M. We use atomic units in which $\hbar = m = e = 1$ and λ is a constant parameter. This quantum system, for $\lambda = 1$, describes the motion of an electron in a constant magnetic field [3, 4] and its corresponding symmetry Lie algebra has been discussed by Moshinsky *et al* [4]. A procedure that use the Noether's theorem [5] is established to get the symmetry algebra of the hamiltonian systems (1.1), for rational values of the parameter lambda. We show that (1.1) represents a generalization of the degeneracies present in the anisotropic two dimensional harmonic oscillator [6,7].

For the purpose of the paper it is convenient to introduce appropriate combinations of the creation η_i and annihilation ξ_i operators, with $i = 1, 2$, *i.e.*

$$\eta_\pm = \frac{1}{\sqrt{2}}(\eta_1 \pm i\eta_2) , \quad \xi_\pm = \frac{1}{\sqrt{2}}(\xi_1 \mp i\xi_2) , \tag{1.2}$$

with the properties

$$[\xi_a, \xi_b] = [\eta_a, \eta_b] = 0; \qquad [\xi_a, \eta_b] = \delta_{ab}, \qquad (\eta_a)^\dagger = \xi_a, \quad a = +, - . \tag{1.3}$$

It is straightforward to find the expression of the hamiltonian (1.1) in terms of these operators

$$H = (1+\lambda)N_+ + (1-\lambda)N_-, \tag{1.4}$$

where a constant term was neglected and N_a, denotes the number of quanta in direction a. The eigenstates of (1.4) are well known [4] and its eigenvalues are given by $E_{\nu m} = \nu + \lambda m$; with $|m| = \nu, \nu - 2 \ldots 1$ or 0 and ν denoting the total number of quanta. From this expression, it is immediate that there is degeneracy for rational values of λ, which can be defined as follows

$$\lambda = -\frac{\Delta\nu}{\Delta m} = -\frac{\nu_f - \nu_i}{m_f - m_i} . \tag{1.5}$$

Thus the accidental degeneracy associated to the hamiltonian (1.4) can be classified according to the strength of the parameter λ in three groups

$$\{\lambda = \pm 1\} , \quad \{\lambda > 1, \ \ \lambda < -1\} , \quad \{-1 < \lambda < 1\} . \tag{1.6a, b, c}$$

For the cases (1.6a, b), there are an infinite number of levels with the same energy, while for the case (1.6c), there is a finite number of levels with the same energy .

In the second section, we find the classical symmetry Lie algebra of the generalized two–dimensional anisotropic harmonic oscillator. In the section three, we discuss for all the cases of λ the corresponding symmetry Lie algebras which are responsible of the accidental degeneracy of the hamiltonian (1.4). Finally some conclusions and remarks are made.

2 Classical Symmetry Lie Algebra for the Hamiltonian

In this section we apply Noether's theorem in its active version [8] to the system described by (1.4), its corresponding lagrangian is given by

$$L = \frac{1}{2\lambda_a}(\dot{x}_a^2 - \lambda_a^2 x_a^2) , \tag{2.1}$$

where we associate indices 1 and 2 to the labels + and −, and we define $\lambda_1 = 1 + \lambda$ and $\lambda_2 = 1 - \lambda$. From now onwards we adopt the convention: repeated indices are summed except when one of them appears with λ_a. Let us propose a symmetry transformation in terms of an arbitrary function of coordinates and velocities, $\delta x_a = F_a(x_b, \dot{x}_b)$. The corresponding variation of the lagrangian (2.1) is given by

$$\delta L = (\ddot{x}_b \frac{\partial F_a}{\partial \dot{x}_b} + \dot{x}_b \frac{\partial F_a}{\partial x_b})\frac{1}{\lambda_a}\dot{x}_a - F_a \lambda_a x_a . \tag{2.2}$$

Because δx_a is a symmetry transformation, (2.2) must be a total time derivative of a function Ω. This implies that the following system of equations must be satisfied

$$\frac{\partial \Omega}{\partial \dot{x}_a} = \frac{1}{\lambda_b}\dot{x}_b \frac{\partial F_b}{\partial \dot{x}_a} , \qquad \dot{x}_a \frac{\partial \Omega}{\partial x_a} = \frac{1}{\lambda_b}\dot{x}_b \dot{x}_a \frac{\partial F_b}{\partial x_a} - F_a \lambda_a x_a . \tag{2.3a, b}$$

In order to establish the integrability conditions for this system, we derive (2.3a) with respect to x_c and (2.3b) with respect to $\dot{x}_c$, and compare the results. Thus we get

$$\frac{\partial \Omega}{\partial x_b} = \frac{1}{\lambda_b}\frac{\partial F_b}{\partial x_c}\dot{x}_c + \frac{1}{\lambda_c}\frac{\partial F_c}{\partial x_b}\dot{x}_c - \lambda_c x_c \frac{\partial F_c}{\partial \dot{x}_b} . \tag{2.4}$$

Now we set up the equality between the five crossed partial derivatives of Ω, and give rise to the following system of second order partial differential equations

$$\frac{\partial F_a}{\partial p_b} = \frac{\partial F_b}{\partial p_a}, \tag{2.5a}$$

$$\frac{1}{2}\mathcal{O}\Big(\frac{\partial F_a}{\partial p_b} + \frac{\partial F_b}{\partial p_a}\Big) - \Big(\lambda_a \frac{\partial F_b}{\partial x_a} + \lambda_b \frac{\partial F_a}{\partial x_b}\Big) = 0, \tag{2.5b}$$

$$\mathcal{O}\Big(\lambda_a \frac{\partial F_b}{\partial x_a} - \lambda_b \frac{\partial F_a}{\partial x_b}\Big) - \Big(\lambda_a^2 \frac{\partial F_a}{\partial p_b} - \lambda_b^2 \frac{\partial F_b}{\partial p_a}\Big) = 0, \tag{2.5c}$$

where the change from velocities to momenta $\dot{x}_a = \lambda_a p_a$ was made, and have defined the differential operator

$$\mathcal{O} = \lambda_c\Big(x_c \frac{\partial}{\partial p_c} - p_c \frac{\partial}{\partial x_c}\Big).$$

From Eq. (2.5a) it is immediate that $F_k = \frac{\partial G}{\partial p_k}$, which means that G is the generator of the symmetry transformation. Through the change of variables $z_k = \frac{1}{\sqrt{2}}(x_k + ip_k)$, and its complex conjugate, z_k^*, it is straightforward to show that the operator $\mathcal{O} = i(\mathcal{N} - \mathcal{N}^*)$, with $\mathcal{N} = \lambda_k z_k \frac{\partial}{\partial z_k}$. Using these results, we arrive to a set of partial differential equations which has a solution of the form

$$G(z_k, z_k^*) = z_1^{n_1} z_2^{n_2} z_1^{*n_3} z_2^{*n_4} , \tag{2.6}$$

if the n_i are integer numbers and satisfy that $n_1 = n_3$ and $n_2 = n_4$ or the condition

$$\frac{(n_1 - n_3)}{(n_2 - n_4)} = -\frac{\lambda_2}{\lambda_1} = \frac{\Delta m + \Delta\nu}{-\Delta m + \Delta\nu} \equiv \frac{k_1}{\epsilon k_2} , \tag{2.7}$$

where the Eq. (1.5) was used. The integers k_1 and k_2 are relatively prime integers, and the parameter ϵ takes the value 1 or -1. It takes the value 1 when $\Delta m + \Delta\nu$ and $\Delta\nu - \Delta m$ have the same sign, and -1 otherwise. Thus we get, besides the trivial solution, six fundamental solutions, although only three of them are independent. Then the corresponding conserved quantities are given by

$$\begin{aligned} N_1 &= z_1 z_1^*, \qquad K_3 = z_1^{*k_1} z_2^{-\epsilon k_2}, \qquad K_5 = z_1^{*k_1} z_2^{*\epsilon k_2}, \\ N_2 &= z_2 z_2^*, \qquad K_4 = z_1^{k_1} z_2^{*-\epsilon k_2}, \qquad K_6 = z_1^{k_1} z_2^{\epsilon k_2}. \end{aligned} \tag{2.8}$$

From this set we must find a symmetry algebra for the classical system. It is important to realize that to build the algebra once we select a conserved quantity its complex conjugate must be included. To do this, we find separately for the cases indicated in Eqs. (1.6) the corresponding expressions for the constants of the motion and from them select the independent ones which allows its extension to the quantum case.

For $\lambda = 1$ and $\lambda = -1$ the sets are given by $\{1, N_1, z_2, z_2^*\}$ and $\{1, N_2, z_1, z_1^*\}$, respectively. In order to identify the symmetry algebra, we calculate its Poisson brackets, and clearly they correspond to the direct sum of one-dimensional Weyl and unitary algebras, $w(1) \oplus u(1)$.

For $\lambda > 1$ and $\lambda < -1$, the constants of the motion are identical and we choose the set

$$h_1 = \frac{1}{k_1}N_1 - \frac{1}{k_2}N_2 , \quad m_1 = \frac{1}{k_1 - k_2}(N_1 - N_2) , \tag{2.9a, b}$$

$$K_5 = F_5(N_1, N_2) z_1^{*k_1} \; z_2^{*k_2} , \quad K_6 = F_6(N_1, N_2) z_1^{k_1} \; z_2^{k_2} . \tag{2.9c, d}$$

The F_5 and F_6 functions are defined in such a way to obtain that the Poisson bracket

$$\left\{K_5, K_6\right\} = iCm_1, \tag{2.10}$$

where C is a constant that can be ± 1. This condition implies that

$$F_5 F_6 = \frac{C}{2(k_1 - k_2)^2}(N_1 - N_2)^2 N_1^{-k_1} N_2^{-k_2}. \tag{2.11}$$

Then it is easy to prove that the set of constants of the motion $\{h_1, m_1, K_5, K_6\}$ constitute the classical symmetry Lie algebra which, depending on the value of C, can be identified with an $u(2)$ or $u(1,1)$ algebra.

For $-1 < \lambda < 1$ we select the following independent constants of the motion:

$$h_2 = \frac{1}{k_1}N_1 + \frac{1}{k_2}N_2 , \quad m_2 = \frac{1}{k_1 + k_2}(N_1 - N_2) , \tag{2.12a, b}$$

$$K_3 = F_3(N_1, N_2) \; z_1^{*k_1} z_2^{k_2} , \quad K_4 = F_4(N_1, N_2) \; z_1^{k_1} z_2^{*k_2}, \tag{2.12c, d}$$

where as in the previous case the F_3 and F_4 functions are defined to give the Poisson bracket

$$\left\{K_3, K_4\right\} = iCm_2, \tag{2.13}$$

with C equal to ± 1. This condition implies that

$$F_3 F_4 = \frac{C}{2(k_1 + k_2)^2}(N_1 - N_2)^2 N_1^{-k_1} N_2^{-k_2}. \tag{2.14}$$

Therefore the set of constants of the motion $\{h_2, m_2, K_3, K_4\}$ generates the classical symmetry Lie algebras $u(2)$ or $u(1,1)$, depending if the value of C is $+1$ or -1, respectively.

3 Quantum Symmetry Lie Algebra for the Hamiltonian

To quantize the system we replace the classical variables x and p by the corresponding quantum operators in definitions (2.8), and Poisson brackets by commutators, *i.e.*, $\{\} \rightarrow \frac{1}{i}[]$. Then the classical variables z_k and z_k^* are replaced by the operators

$$\hat{z}_k = \frac{1}{\sqrt{2}}(\hat{x}_k + i\hat{p}_k) , \quad \hat{z}_k^\dagger = \frac{1}{\sqrt{2}}(\hat{x}_k - i\hat{p}_k) , \tag{3.1}$$

which satisfy the standard commutation relations of creation and annihilation operators.

We choose as a base for the physical space the simultaneous eigenstates of $\{N_1, N_2\}$, which we label as $|n_1, n_2>$, because they form a complete set of commuting operators. This let us see that not all operators in (2.8) make sense all the time. According to the previous section we consider three cases:

(i) For $\lambda = \pm 1$, we have two sets of operators, $\{I, \hat{N}_1, \hat{z}_2, \hat{z}_2^\dagger\}$ and $\{I, \hat{N}_2, \hat{z}_1, \hat{z}_1^\dagger\}$, whose commutation relations correspond to the direct sum $w(1) \oplus u(1)$.

(ii) When $\lambda > 1$ and $\lambda < -1$, the set of constants of the motion (2.9), must be replaced by its quantum version, however this is ambiguous for the constants (2.9c,d) and so we eliminate from them the F_5 and F_6 functions. It is easy to evaluate their commutators and get an algebra but to identify a Lie algebra a redefinition of the constants of the motion must be done. This is achieved by constructing the new operators [7]

$$\tilde{z}_i^\dagger = \left(\lfloor\frac{\hat{N}_i}{k_i}\rfloor \frac{(\hat{N}_i - k_i)!}{(N_i)!}\right)^{\frac{1}{2}} (\hat{z}_i^\dagger)^{k_i}, \tag{3.2a}$$

$$\tilde{z}_i = (\hat{z}_i)^{k_i} \left(\lfloor\frac{\hat{N}_i}{k_i}\rfloor \frac{(\hat{N}_i - k_i)!}{(N_i)!}\right)^{\frac{1}{2}}. \tag{3.2b}$$

where $\lfloor x \rfloor$ denotes the largest integer $\leq x$. From (3.2) it is easy to check that

$$\tilde{N}_i = \tilde{z}_i^\dagger \tilde{z}_i = \lfloor\frac{\hat{N}_i}{k_i}\rfloor. \tag{3.3}$$

Then the Lie algebra is identified by considering the following operators

$$\hat{h}_1 = \tilde{N}_1 - \tilde{N}_2 \ , \quad \tilde{K}_5 = \tilde{z}_1^\dagger \tilde{z}_2^\dagger \ , \quad \tilde{K}_6 = \tilde{z}_1 \tilde{z}_2 \ , \quad \hat{C}_1 = \frac{1}{2}(\tilde{N}_1 + \tilde{N}_2 + 1) \ . \tag{3.4}$$

that satisfy the commutation relations

$$[\hat{C}_1, \tilde{K}_5] = \tilde{K}_5 \ , \quad [\hat{C}_1, \tilde{K}_6] = -\tilde{K}_6 \ , \quad [\tilde{K}_5, \tilde{K}_6] = -2\hat{C}_1 \ . \tag{3.5}$$

These were evaluated by using that $[\tilde{z}_i, \tilde{z}_j^\dagger] = \delta_{ij}$, which is valid for any state $|n_1, n_2>$ of the Hilbert space of the system, and they are the generators of a $u(1,1)$ Lie algebra, with $\hat{h}_1$ generating the invariant subalgebra.

(iii) Finally for $-1 < \lambda < 1$, the symmetry algebra can be found by considering the operators

$$\tilde{K}_3 = \tilde{z}_1^\dagger \tilde{z}_2, \qquad \tilde{K}_4 = \tilde{z}_2^\dagger \tilde{z}_1 \ , \qquad \hat{h}_2 = \tilde{N}_1 + \tilde{N}_2 \ , \quad \hat{C}_2 = \frac{1}{2}(\tilde{N}_1 - \tilde{N}_2) \ . \tag{3.6}$$

Evaluating the commutation relations between these operators we have

$$[\hat{C}_2, \tilde{K}_3] = \tilde{K}_3, \qquad [\hat{C}_2, \tilde{K}_4] = -\tilde{K}_4, \qquad [\tilde{K}_3, \tilde{K}_4] = 2\hat{C}_2, \tag{3.7}$$

and the operator $\hat{h}_2$ is the ideal of the algebra. Thus we get for this case a $u(2)$ symmetry Lie algebra.

4 Conclusions

We have established a procedure that uses Noether's theorem to find the symmetry Lie algebra of a quantum system with accidental degeneracy. First, we solve the differential equations that determine the constants of the motion. Second, once we have chosen the minimal set of constants of the motion that close under Poisson brackets, to identify the classical Lie algebra we need in general to form combinations of the selected Noether charges. And third, to find the corresponding quantum counterparts. Afterwards, the identification of the quantum symmetry Lie algebra can be done immediately by making the standard replacement of Poisson brackets by commutators. However, this is true if there are not ambiguities in establishing the associated quantum operators for the constants of motion which form a Lie algebra under the Poisson bracket operation. If this is not the case, it is more convenient to choose the minimal set of constants of the motion that allows a quantum extension, and make the necessary redefinitions to build the associated Lie algebra of the system. Following this procedure we get for the generalized anisotropic two dimensional harmonic oscillator (1.4) the symmetry algebra which determine the degeneracy of the system. The symmetry Lie algebras are, depending on the value for $\lambda, w(1) \oplus u(1), u(2)$, and $u(1,1)$. However with the generators of the first one a Holstein–Primakoff realization [4] of a $u(1,1)$ Lie algebra can be obtained.

5 Acknowledgment

This work was supported in part by project UNAM-DGAPA, IN10-3091. We would like to thank to A. Frank, S. Hojman and G. Rosensteel for important comments on this work.

References

[1] V. Fock, Z. Phys. **98**, 145 (1935).
V. Bargmann, Z. Phys. **99**, 576 (1936).

[2] J. M. Jauch, and E. L. Hill, Phys. Rev. **57**, 641 (1940).

[3] H. V. McIntosh, *Symmetry and Degeneracy* in Group Theory and Applications, Vol. II, p. 75. (E. M. Loebl, Ed., Academic Press (New York, 1971).

[4] M. Moshinsky, C. Quesne, and G. Loyola, Ann. Phys. **198**, 103 (1990).

[5] P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer–Verlag, Berlin, 1986).

[6] J. D. Louck, M. Moshinsky, and K. B. Wolf, J. Math. Phys., **14**, 692 (1973).

[7] G. Rosensteel, and J. P. Draayer, J. Phys. A: Math. Gen., **22**, 1323 (1989).

[8] R. Jackiw, Ann. Phys. **129**, 183 (1980).