N93-27348

FERMION REALIZATION OF EXCEPTIONAL LIE ALGEBRAS FROM MAXIMAL UNITARY SUBALGEBRAS

A. SCIARRINO
Universitá di Napoli Federico II
Dipartimento di Scienze Fisiche
Mostra d'Oltremare Pad.19
80125 NAPOLI ITALY

Abstract

From the decomposition of the exceptional Lie algebras (ELAs) under a maximal unitary subalgebra a realization of the ELAs is obtained in terms of fermionic oscillators.

1 Introduction

Realizations of classical Lie algebras (LAs) in terms of bosonic and/or fermionic oscillators are known long since and are very useful in several physical contexts. Via the embedding of $SO(8) \oplus SO(8) \subset E_8$ a realization of ELAs in terms of fermionic oscillators has been obtained by the author [1]. However it is more convenient to dispose of several different realizations of ELAs which allow to describe in a more appropriate way different subalgebras embedding chains. Moreover, e.g., the embeddings $G_2 \subset SO(7)$ and $F_4 \subset E_6$ are not "deformable", while the embeddings $SU(3) \subset G_2$ and $SO(9) \subset F_4$ are "deformable". The proposal of this contribution is to present a realization of ELAs in terms of multilinears in fermionic oscillators via the embedding of a maximal unitary subalgebra. It should be quoted that constructions of ELAs as bilinears in fermionic fields in the basis $SU(9)$ and $SU(3)^4$ has been obtained by Koca [2]. While Koca's approach makes a more evident connection with physical applications in a GUT framework, the multilinear approach keeps a closer connection with the algebraic structure of LAs (roots, weights, etc.). Moreover this formalism allows to obtain multilinear realizations for all the fundamental representations and for generators and vector spaces of all maximal embeddings of ELAs [3].

2 Composition law for fermionic multilinears

Let us introduce a set of N fermionic oscillators a_i^+, a_i satisfying: $(i, j = 1,2,...,N)$

$$\{a_i^+, a_j^+\} = \{a_i, a_j\} = 0 \qquad \{a_i, a_j^+\} = \delta_{ij}. \tag{1}$$

A fermionic multilinear (f.m.) X is defined by the following formula:
$(f_i = a_i^+, f_{-i} = a_i, i > 0)$

$$X = \prod_i f_i \qquad i \in I \subset Z^* \tag{2}$$

The number of f_i will be called the order of X.

We define the **contraction** of two bilinears X and Y of, resp., order N and N' as a operation giving a f.m. ($\widehat{XY}$) obtained from the m. XY by deleting the couples (if any) (f_i, f_{-i}) with f_i "in" X and f_{-i} "in" Y, multiplied by a factor $(-1)^n$, n being the number of transpositions necessary to obtain all the f_i near to f_{-i} in XY, and by a rational coefficent $C(N, N', Z)$, Z being the number of **contractions**.

We define a **composition law** $(X \circ Y)$ of two f.m. by the following equation $(i_k \in I, j_l \in J)$

$$X \circ Y = \frac{1}{2} \times (\widehat{XY} - \widehat{YX}) + \delta_{I,-J} \times \frac{1}{N} \sum_k \sum_l (f_{i_k} f_{j_l} - f_{j_l} f_{i_k}) \times (-1)^{k-1} \delta_{i_k j_l} \tag{3}$$

We remark:

- $X \circ Y$ = - $(Y \circ X)$
- $X \circ Y$ = [X, Y] (N, N'$\in$ 1,2)

We put (N,N' = 1, 2, 3, 6; N_T = order of $\widehat{XY}$):

- C(N, N', 0) = 1
- C(N, N', 1) = $\delta_{N_T,N}$ or $\delta_{N_T,N'}$
- C(N, N, N-1) = $\frac{2}{N}$
- C(N, 2N, N) = $\frac{1}{2}$ $(N > 1)$
- C(N, N, $\frac{N}{2}$) = -1 (N even)

3 Realization of E_8

We consider the embedding SU(9) $\subset E_8$. The adjoint representation of E_8 decomposes as :

$$248 \Longrightarrow 80 + 84 + \overline{84} \tag{4}$$

Introducing a set of 9 fermionic creation and annihilation operators and we can write (i,j = 1,2,..,9):

$$80 \equiv \{ \{a_i^+ a_j\} \quad (i \neq j), \quad a_k^+ a_k - a_{k+1}^+ a_{k+1} = h_k - h_{k+1} \quad (k \neq 9)\} \tag{5}$$

$$84 \equiv \{a_i^+ a_j^+ a_k^+ + \frac{1}{6!} \varepsilon_{ijklmnpqr} a_l a_m a_n a_p a_q a_r\} \tag{6}$$

$$\overline{84} \equiv \{a_i a_j a_k + \frac{1}{6!} \varepsilon_{ijklmnpqr} a_l^+ a_m^+ a_n^+ a_p^+ a_q^+ a_r^+\} \tag{7}$$

In the following we call:

- a_i^+ "hermitian coniugate" (h.c.) of a_i;
- $\varepsilon_{ijklmnpqr} a_l^+ a_m^+ a_n^+ a_p^+ a_q^+ a_r^+$ "dual coniugate" (d.c.) of $a_i a_j a_k$.

Proposition 1 *The above set of bilinears and trilinears in the fermionic oscillators closes and satisfies the Jacobi identity under the composition law ($\circ$) defined in Sec. 2.*

The generators corresponding to the simple roots are:

$$\alpha_1 \to a_1^+ a_2, \quad \alpha_2 \to a_1 a_2 a_3 + d.c., \quad \alpha_k \to a_{k-1}^+ a_k \quad (3 \le k \le 8) \tag{8}$$

The generator corresponding to the highest root is $a_8^+ a_9$.

4 Realization of E_7

In the embedding SU(8) $\subset E_7$ the adjoint representation decomposes as:

$$133 \Longrightarrow 63 + 70 \tag{9}$$

The SU(8) $\subset E_7$ is not contained in the SU(9) $\subset E_8$, Exploiting the property that the two unitary algebras have a common maximal subalgebra SU(6), the following realization of E_7 is obtained (i,j,k = 1,2,..6 ; r = 1,2,..,5):

$$\begin{aligned} 63 \;\equiv\; & \{a_i^+ a_j, \quad a_7^+ a_8^+ a_9^+ + d.c., \quad a_i^+ a_7, \quad a_i^+ a_8^+ a_9^+ + d.c., \quad h.c. \\ & h_r - h_{r+1}, \quad \frac{2}{3}(h_7 + h_8 + h_9) - \frac{1}{3}\sum_i h_i, \quad 2h_7 - h_8 - h_9\} \end{aligned} \tag{10}$$

$$70 \equiv \{a_i a_j a_7 + d.c., \quad a_i a_j a_k + d.c., \quad h.c.\} \tag{11}$$

5 Realization of E_6

In the embedding $SU(6) \oplus SU(2) \subset E_6$ the adjoint representation decomposes as:

$$78 \Longrightarrow (35,1) + (1,3) + (20,2) \tag{12}$$

We have (i,j,k = 1,2,..6; r = 1,2,..5):

$$(35,1) \equiv \{a_i^+ a_j, \quad h_r - h_{r+1}\} \tag{13}$$

$$(1,3) \equiv \{a_7^+ a_8^+ a_9^+ + d.c., \quad h.c., \quad \frac{2}{3}(h_7 + h_8 + h_9) - \frac{1}{3}\sum h_i\} \tag{14}$$

$$(20,2) \equiv \{a_i a_j a_k + d.c., \quad h.c.\} \tag{15}$$

6 Realization of F_4

In the embedding $SU(4) \oplus SU(2)' \subset F_4$ the adjoint representation decomposes as:

$$52 \Longrightarrow (15,1) + (1,3) + (4,2) + (\bar{4},2) + (6,3) \tag{16}$$

The most convenient way to identify the elements of F_4 is the following:

i) draw the Dynkin diagram of E_6;

ii) from i) draw, by folding, the Dynkin diagram of F_4, identify the corresponding simple roots and the highest root;

iii) draw the extended Dynkin diagram of F_4 and then, by deleting a dot, identify $SU(4) \oplus SU(2)'$.

We get for the 52 (i,j,k = 1,2,..6):

$$\begin{aligned}
&a_7^+ a_8^+ a_9^+ + d.c., \quad a_i^+ a_j \quad (i+j=7), \quad h.c. \\
&a_j^+ a_j + (-1)^{k+l-1} a_k^+ a_l \quad (i \neq j \neq k \neq l; i+j \leq k+l; i+j+k+l=14), \quad h.c. \\
&a_i a_j a_k + d.c. \quad (i<j<k; i+j+k=M; M=6,7,9,10,..14), \quad h.c. \\
&a_i a_j a_t + a_k a_l a_t + d.c. \quad (t=1,3.4; i \neq j \neq k \neq l; i+j+k+l=7), \quad h.c. \\
&-\frac{2}{3}(h_1+h_2+h_3) + \frac{1}{3}\sum_i h_i, \qquad h_3+h_4 \\
&h_1+h_5-h_2-h_6, \qquad h_2+h_4-h_3-h_5
\end{aligned} \tag{17}$$

7 Realization of G_2

In the emdedding SU(3) $\subset G_2$ the adjoint representation decomposes as

$$14 \Longrightarrow 8 + 3 + \bar{3} \tag{18}$$

where (i = 1,2,..9; j = 1,2...6):

$$\begin{aligned}
8 \equiv \; &\{a_1 a_2 a_3 + d.c., \quad a_7 a_8 a_9 + d.c., \quad a_4^+ a_3, \quad h.c. \\
&-\frac{2}{3}(h_1+h_2+h_3) + \frac{1}{3}\sum_{i>3} h_i \quad -\frac{2}{3}(h_7+h_8+h_9) + \frac{1}{3}\sum_j h_j\}
\end{aligned} \tag{19}$$

$$3 + \bar{3} \equiv \{a_1^+ a_6, \quad a_2 a_3 a_6 + d.c., \quad a_1^+ a_4^+ a_5^+ + d.c., \quad h.c.\} \tag{20}$$

8 Conclusions

One of the advantages of the oscillators construction of LAs is the knowledge of the Fock space which becomes the carrier space of irrep. of the the LAs.

In the case of construction of LAs SU(N) by using fermionic oscillators it is well known that the carrier space of antisymmetric irreps. can be realized on the Fock space. As the fundamental irreps. of G_2, E_6, E_7, of dimension, resp. $7, 27, 56$, decompose under the maximal unitary subalgebras as a sum of antisymmetric irreps. as:

$$\begin{aligned} 7 &\Longrightarrow 3 + \overline{3} + 1 \\ 27 &\Longrightarrow (15,1) + (\overline{6},2) \\ 56 &\Longrightarrow 28 + \overline{28} \end{aligned} \tag{21}$$

one can think that on the Fock space of the fermionic oscillators it is possible build up the fundamental representations, at least, of these ELAs.
Indeed for G_2 this has already been obtained [4].

References

[1] A. Sciarrino, J.Math.Phys. **30**, 1674 (1989)

[2] M. Koca, Phys. Rev. D **24**, 2645 (1981)

[3] A. Sciarrino, "Fermionic multilinear realization of exceptional Lie algebras", in preparation

[4] L. Frappat and A. Sciarrino, J.Phys A 25, L383 (1992)