ORBIT-ORBIT BRANCHING RULES

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Abstract

We show how the derivation of group-subgroup (IR to IR) branching rules is facilitated by the use of the much simpler (Weyl) orbit-orbit branching rules.

1 Introduction

We must begin by relating our subject to harmonic oscillators. This is easy if we use as basis states for an IR polynomials in the states of the fundamental IR's of the group under consideration. The variables representing the fundamental states can be replaced by creation operators for similarly labeled harmonic oscillators—the Schwinger boson calculus.

The audience does not need to be convinced of the utility of "building blocks" of larger objects of interest in physics. The role of (Weyl) orbits as constituents of IR's (we use IR as an abbreviation for basis of an irreducible representation) has not been exploited much. Their use simplifies considerably the derivation of group-subgroup (IR) branching rules.

The problem is broken into three steps: I decomposition of the group IR into group orbits, II decomposition of each group orbit into subgroup orbits and III the assembly of subgroup orbits into subgroup IR's.

In § 2 we discuss steps I and III; in § 3 we discuss step II. Most of the material presented here appears in articles by Patera and Sharp[1] and by Gingras, Patera and Sharp[2].

2 Orbits expanded in IR's and vice versa.

Let λ_n be a set of weights of a semisimple algebra and c_n the multiplicity of the weight λ_n; we suppose the weights have Weyl symmetry. Then the weights can be written as a superposition of weights of IR's:

$$\sum_n \Lambda^{\lambda_n} c_n = \sum_a \chi_a g_a. \tag{1}$$

χ_a is the character of the IR (a) and g_a is an integer which we call the multiplicity of (a) even though it may now take negative as well as positive values. The dummies Λ_i carry weight components λ_i as exponents: $(\Lambda^\lambda \equiv \prod_i \Lambda_i^{\lambda_i})$.

To find g_a use Weyl's character formula

$$\chi_a = \xi_a/\xi_0 \tag{2}$$

where ξ_a is the Weyl characteristic function

$$\xi_a = \sum_W (-1)^W \Lambda^{W(a+R)}. \tag{3}$$

The sum is over Weyl reflections, $(-1)^W$ is the determinant of the matrix of W, i.e., ± 1 according to whether W is a product of an even or odd number of relections and R is half the sum of the positive roots, or the sum of the fundamental weights; ξ_0 is the characteristic of the scalar IR ($a = 0$). Then

$$\sum_n \Lambda^{\lambda_n} c_n \xi_0 = \sum_a \xi_a g_a \tag{4}$$

Now ξ_a has just one term Λ^{a+R} in the dominant Weyl sector, so g_a is the coefficient of Λ^a in $\sum_n \Lambda^{\lambda_n - R} c_n \xi_0$. We take the λ_n in (1) to be the weights of the Weyl orbit $[\lambda]$. Then g_a is the multiplicity of the IR (a) in the expansion of the orbit $[\lambda]$.

The sum in (4) can be visualized graphically in the spirit of a Speiser[3] diagram. But since Speiser's methodology is impracticable for rank higher than 2 we prefer a numerical approach, effecting Weyl reflections with the help of the Cartan matrix. For illustration we use the G_2 orbit [2,1]. The instructions given in the caption of Table I apply unchanged for any simple group. An orbit weight is recognized as lying on a reflection hyperplane if any of its weight components is zero at any stage of the reflection process (e.g. in Column 2 or 3). This completes our description of step III, in which subgroup orbits are converted to subgroup IR's.

TABLE I. The G_2 orbit [2,1] expanded in modules.

1	2	3	4
{2,1}	{3,2}	{3,2}	(2,1)
$\{\bar{2},7\}$	$\{\bar{1},8\}$	$-\{1,5\}$	$-(0,4)$
$\{\bar{3},8\}$	$\{\bar{2},9\}$	$-\{2,3\}$	$-(1,2)$
$\{\bar{5},8\}$	$\{\bar{4},9\}$	{1,3}	(0,2)
$\{5,\bar{8}\}$	$\{6,\bar{7}\}$	{1,4}	(0,3)
$\{3,\bar{8}\}$	$\{4,\bar{7}\}$	$-\{1,2\}$	$-(0,1)$

Column 1 contains those orbit weights which do not lie on a reflection line. Column 2 contains the weights of Column 1 augmented by R. Column 3 contains the weights of Column 2 reflected to the dominant sector with sign ± 1 according to whether an even or odd member of relections is involved. Column 4 contains the IR's in the expansion of the orbit [2,1], obtained from Column 3 by subtracting R.

We now turn to step I, in which group IR's are decomposed in group orbits. We first carry out step III for all group orbits no further from the origin than the highest weight of the IR under discussion. Only orbits of the same congruence class as the IR in question need to be considered. These orbit $\rightarrow$ IR expansions define the triangular orbit-IR matrix which is easily inverted to give the IR-orbit matrix; it gives the orbit content of IR's. This procedure is simpler to implement than other methods such as Freudenthal's recursion formula for weight multiplicities, or the character formulas of Weyl or Demazure.

3 Orbit-orbit branching rules.

Orbit-orbit branching rules are always much simpler to derive and to describe than the usual IR-IR branching rules. One approach which has general applicability makes use of the orbit-weight generating function $F(M,\Lambda)$. Its power-expansion

$$F(M,\Lambda) = \sum_{\mu,\lambda} M^{\mu}\Lambda^{\lambda}c_{\mu\lambda} \tag{5}$$

gives the multiplicity $c_{\mu\lambda}$ of the weight $\{\lambda\}$ in the orbit $[\mu]$. As a simple example the SU(3) orbit-weight generating function is

$$\begin{aligned} F(M_1,M_2;\Lambda_1,\Lambda_2) = {} & \frac{1}{(1-M_1\Lambda_1)(1-M_2\Lambda_2)} \\ & + \frac{M_1\Lambda_1^{-1}\Lambda_2}{(1-M_2\Lambda_2)(1-M_1\Lambda_1^{-1}\Lambda_2} + \frac{M_2\Lambda_1^{-1}}{(1-M_1\Lambda_1^{-1}\Lambda_2)(1-M_2\Lambda_1^{-1})} \\ & + \frac{M_1\Lambda_2^{-1}}{(1-M_2\Lambda_1^{-1})(1-M_1\Lambda_2^{-1})} + \frac{M_2\Lambda_1\Lambda_2^{-1}}{(1-M_1\Lambda_2^{-1})(1-M_2\Lambda_1\Lambda_2^{-1})} \\ & + \frac{M_1M_2\Lambda_1^2\Lambda_2^{-1}}{(1-M_2\Lambda_1\Lambda_2^{-1})(1-M_1\Lambda_1)} . \end{aligned} \tag{6}$$

Λ_1,Λ_2 carry weight components in a fundamental weights basis. To convert the orbit-weight generating function to an orbit-orbit branching rules generating function it is necessary only to replace the dummies Λ which carry weight components with new dummies carrying subgroup weight components and then retain the part that contains only non-negative powers of the new dummies. For example for SU(3)$\supset$SU(2)$\times$U(1) the replacements are $\Lambda_1 \rightarrow NY^{\frac{1}{3}}, \Lambda_2 \rightarrow Y^{\frac{2}{3}}$. We remark that for a dummy like Y carrying a U(1) label one should retain both negative and positive powers. For SU(3)$\supset$SO(3) the replacements are $\Lambda_1 \rightarrow N^2, \Lambda_2 \rightarrow 1$.

The method described in the preceding paragraph becomes laborious for higher rank groups because of the large order of the Weyl group. It is usually simpler to determine an integrity basis for subgroup orbits by examining low

group orbits. The integrity basis consists of "elementary" orbits, from which all can be obtained as stretched products (orbit labels additive).

Two types of subgroup may be distinguished according to whether the Weyl sectors of group and subgroup do or do not "line up." When we compare regions of group and subgroup weight space, a region of subgroup weight space, say a Weyl chamber, means the region of group weight space which projects into the subgroup region in question. The simpler situation is that in which group and subgroup chambers line up, i.e., each subgroup chamber contains only complete group chambers, N/N' of them, where N and N' are the orders of the group and subgroup Weyl groups.

Consider the lining up case. Let W be one of the N/N' group Weyl elements which carry the dominant group chamber within the dominant subgroup chamber. Then a group-subgroup orbit pair (a,b) corresponding to the terms $A^a B^b$ in the orbit-orbit generating function can be written (a,PWa) corresponding to $\prod_i \left(A_i \prod_j B_j^{(PWM_i)_j} \right)^{a_i}$ where P is the projection onto subgroup weight space. Thus the elementary orbits correspond to $A_i \prod_j B_j^{(PWM_i)_j}$, i.e., they are the subgroup orbits contained in the fundamental group orbits. The compatibility rules for elementary orbits can be stated as follows: two elementary orbits are compatible if and only if the two weights WM_i and WM_k can be obtained by the same Weyl element W; in particular two subalgebra orbits belonging to the same fundamental group orbit are incompatible. SU(3)⊃SO(3) and SU(4)⊃SU(2)×SU(2)×U(1) are examples of group and subgroup chambers' lining up. A sufficient but not necessary condition for the lining up is that group and subgroup have equal ranks; for all known maximal subjoint algebras that is always the case. The Weyl chambers line up for a regular subgroup.

Examples of cases where the chambers do not line up are SO(5)⊃SU(2) and SU(4)⊃SU(2)×SU(2) (Wigner supermultiplet). When a dominant subgroup weight lies inside a chamber of group weight space that is only partly in the dominant subalgebra sector, it cannot be compounded from elementary orbits belonging to fundamental algebra orbits; hence composite elementary orbits (more than one algebra label nonzero) arise.

We conclude with an example where group and subgroup have equal rank, $F_4 \supset SO(9)$.

The decomposition of the fundamental group orbits is as follows:

$$
\begin{aligned}
[1000] &\supset \overset{a}{[0100]}, \qquad [0100] \supset \overset{b}{[1010]}, \\
[0010] &\supset \overset{c}{[1001]} + \overset{d}{[0010]}, \\
[0001] &\supset \overset{e}{[1000]} + \overset{f}{[0001]}.
\end{aligned}
$$

"Names" for the elementary orbits have been written above them. Compatibility rules are found by looking at orbits with two labels non zero; in this case we

need examine only the composite F_4 orbit [0011] (since a and b are the only subgroup orbits in their respective group orbits they are compatible with all other elementary orbits).

$$[0011] \supset \overset{ce}{[2001]} + \overset{cf}{[1002]} + \overset{df}{[0011]}.$$

The interpretation of subalgebra orbits as products of elementary ones gives us the compatibility rules. There are three ($= N/N'$) sets of mutually compatible orbits, *abce*, *abcf* and *abdf*. Hence the general F_4 orbit $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ decomposes into SO(9) orbits $[\lambda_2 + \lambda_3 + \lambda_4, \lambda_1, \lambda_2, \lambda_3]$, $[\lambda_2 + \lambda_3, \lambda_1, \lambda_2, \lambda_3 + \lambda_4]$ and $[\lambda_2, \lambda_1, \lambda_2 + \lambda_3, \lambda_4]$.

We remark that the methods and results here apply equally to Kac–Moody algebras.

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VII. ATOMIC, NUCLEAR, AND PARTICLE PHYSICS