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MASS SPECTRA OF THE PARTICLE-ANTIPARTICLE SYSTEM WITH A DIRAC OSCILLATOR INTERACTION

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Abstract

The present view about the structure of measons is that they are a quark-antiquark system. The mass spectrum corresponding to this system should, in principle, be given by chromodynamics, but this turns out to be a complex affair. Thus it is of some interest to consider relativistic systems of particle-antiparticle, with a simple type of interaction, which could give some insight on the spectra we can expect for mesons.In the present paper we carry this analysis when the interaction is of the Dirac oscillator type. We show that the Dirac equation of the antiparticle can be obtained from that of the particle by just changing the frequency ω into $-\omega$. Following a procedure suggested by Barut we derive the equation for the particle-antiparticle system and solve it by a perturbation procedure. We thus obtain explicit expressions for the square of the mass spectra and discuss its implications in the meson case.

1 Introduction and summary

It is well known[1] that mesons are considered as formed by a quark-antiquark systems where, in many cases,[1] the particle and antiparticle are of the same mass *i.e.* $u\bar{u}, u\bar{d}, d\bar{u}, d\bar{d}; s\bar{s}$ etc.

The calculation of the mass spectra of mesons within the framework of quantum chromodynamics would be a complex affair[2]. Thus it is of some interest to consider relativistic systems of particle-antiparticle, with a simple type of interaction, that could give us some insight in the type of spectra that we can expect for mesons.

In the present paper we intend to carry this analysis when the interaction is of a Dirac oscillator type[3,4]. We begin in section 2 by considering the positive and negative energy solutions of the one particle Dirac oscillator problem[3,4], and show that the equation for the anti-particle can be derived from that of the particle if we change the frequency ω of the oscillator to $-\omega$.

In section 3 we consider the Barut[5] procedure for deriving a single equation for n-free relativistic particles of spin 1/2, and generalize it to n particles with Dirac oscillator interactions with different frequencies $\omega_s, s = 1, 2, ...n$. We then apply it to the particle-antiparticle case where $n = 2$ and $\omega_1 = -\omega_2 \equiv \omega$.

In section 4 we reduce our equation, which has four components, to just a single one, and proceed to show how to solve the latter by perturbation theory.

In section 5 we derive explicitly the square of the mass spectra of our particle-antiparticle system to first order perturbation theory, and proceed to draw the square of the mass level scheme as function of the total angular momentum j, of the parity $(-1)^j$ or $-(-1)^j$ as well as of the number of quanta N of the oscillator, for different values of ω.

Finally, in the concluding section, we also give the square of the mass spectra of the mesons and show that, while quite different from our present theoretical analysis, it could, as in the three quark case of baryons[6,7], give a better agreement if other interactions are also considered.

2 The Dirac oscillator equation for a particle and for an antiparticle

The single particle Dirac oscillator equation was suggested by the replacement[3]

$$\mathbf{p} \rightarrow \mathbf{p} - i\omega\mathbf{x}\beta, \tag{2.1}$$

in the Dirac free particle expression[8] giving rise to

$$i(\partial\psi/\partial x^0) = \left[\boldsymbol{\alpha}\cdot(\mathbf{p} - i\omega\mathbf{x}\beta) + \beta\right]\psi, \tag{2.2}$$

where x^0 is the time and ω the frequency of the oscillator, all in the units

$$\hbar = m = c = 1 \tag{2.3}$$

where m is the mass of the particle and c the velocity of light. Note furthermore that

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.4a, b}$$

where $\boldsymbol{\sigma}$ is the Pauli spin vector.

We require now the solutions of (2.2) both for positive and negative energy where E will denote its absolute value. For positive energy we can write

$$\psi^+ = \begin{pmatrix} \psi_1^+ \\ \psi_2^+ \end{pmatrix} \exp(-iEx^0), \tag{2.5}$$

where ψ_1^+, ψ_2^+ are the large and small components depending only on the coordinates and, from (2.4a,b), we obtain

$$(E-1)\psi_1^+ = \left[\boldsymbol{\sigma}\cdot(\mathbf{p} + i\omega\mathbf{x})\right]\psi_2^+, \tag{2.6a}$$

$$(E+1)\psi_2^+ = \left[\boldsymbol{\sigma}\cdot(\mathbf{p} - i\omega\mathbf{x})\right]\psi_1^+, \tag{2.6b}$$

so that from the second equation

$$\psi_2^+ = (E+1)^{-1}[\boldsymbol{\sigma}\cdot(\mathbf{p} - i\omega\mathbf{x})]\psi_1^+, \tag{2.7}$$

and substituting in the first we get

$$(E^2 - 1)\psi_1^+ = [\mathbf{p}^2 + \omega^2\mathbf{x}^2 - 3\omega - 4\omega\mathbf{L}\cdot\mathbf{S}]\psi_1^+, \tag{2.8}$$

where

$$\mathbf{L} = \mathbf{x}\times\mathbf{p} \quad , \quad \mathbf{S} = \boldsymbol{\sigma}/2. \tag{2.9a, b}$$

Clearly then ψ_1^+ is given by

$$\psi_1^+ \equiv |N\Big(\ell, \frac{1}{2}\Big)jm\rangle \equiv \phi = R_{N\ell}(r)\sum_{\sigma=-1/2}^{1/2}\Big[\langle \ell m - \sigma, \frac{1}{2}\sigma|jm\rangle Y_{\ell m-\sigma}(\theta,\varphi)\chi_{1/2\sigma}\Big], \tag{2.10}$$

while to obtain ψ_2^+ we have to carry out the operation (2.7). In (2.10) $R_{N\ell}(r)$ is the radial function given in terms of Laguerre polynomials, $<\,|\,>$ a Clebsch-Gordan coefficient, $Y_{\ell m-\sigma}(\theta,\varphi)$ a spherical harmonic, and $\chi_{1/2\sigma}$ a spin 1/2 state with projection σ. The N indicates the total number of quanta while $\ell, j = \ell \pm \frac{1}{2}$ are the orbital and total angular momentum.

Now we turn our attention to the negative energy states where we can take

$$\psi^- = \begin{pmatrix}\psi_1^- \\ \psi_2^-\end{pmatrix}\exp(iEx^0), \tag{2.11}$$

from which

$$-(E+1)\psi_1^- = \Big[\boldsymbol{\sigma}\cdot(\mathbf{p}+i\omega\mathbf{x})\Big]\psi_2^-, \tag{2.12a}$$

$$(-E+1)\psi_2^- = \Big[\boldsymbol{\sigma}\cdot(\mathbf{p}-i\omega\mathbf{x})\Big]\psi_1^-, \tag{2.12b}$$

so that

$$\psi_1^- = -(E+1)^{-1}\Big[\boldsymbol{\sigma}\cdot(\mathbf{p}+i\omega\mathbf{x})\Big]\psi_2^-, \tag{2.13}$$

while ψ_2^- satisfies

$$(E^2-1)\psi_2^- = \Big[\mathbf{p}^2 + \omega^2\mathbf{x}^2 + 3\omega + 4\omega\mathbf{L}\cdot\mathbf{S}\Big]\psi_2^-, \tag{2.14}$$

so it is again given by the ket $|N(\ell,\frac{1}{2})jm\rangle \equiv \phi$ of (2.10).

The particle state ψ_p is the positive energy one ψ^+ which from (2.5) can be written as[10]

$$\psi_p \equiv \psi^+ = \begin{bmatrix}\phi \\ (E+1)^{-1}[\boldsymbol{\sigma}\cdot(\mathbf{p}-i\omega\mathbf{x})]\phi\end{bmatrix}\exp(-iEx^0), \tag{2.15}$$

with ϕ given by (2.10).

For the antiparticle state ψ_a we follow Bjorken and Drell[9] by taking the conjugate of ψ^- of (2.11) and apply to it

$$i\gamma^2 = i\beta\alpha_2 = i\begin{pmatrix}0 & \sigma_2 \\ -\sigma_2 & 0\end{pmatrix}, \tag{2.16}$$

thus getting[10]

$$\psi_a \equiv i\gamma_2\psi^{-*} = \begin{bmatrix} i\sigma_2\phi^* \\ (E+1)^{-1}\boldsymbol{\sigma}\cdot(\mathbf{p}+i\omega\mathbf{x})(i\sigma_2\phi^*) \end{bmatrix} \exp(-iEx^0) \tag{2.17}$$

as

$$(-i\sigma_2)\boldsymbol{\sigma}^*(-i\sigma_2) = \boldsymbol{\sigma} \quad , \quad (-i\sigma_2)(i\sigma_2) = I. \tag{2.18}$$

Furthermore as

$$i\sigma_2\chi_{\frac{1}{2}\frac{1}{2}} = -\chi_{\frac{1}{2}-\frac{1}{2}} \quad , \quad i\sigma_2\chi_{\frac{1}{2}-\frac{1}{2}} = \chi_{\frac{1}{2}\frac{1}{2}} \tag{2.19a, b}$$

we see that

$$i\sigma_2\phi^* = R_{N\ell}(r)\sum_{\sigma=-1/2}^{1/2}\left[\langle \ell m-\sigma,\frac{1}{2}\sigma|jm\rangle(-1)^{m+\frac{1}{2}}Y_{\ell,-m+\sigma}(\theta,\varphi)\chi_{\frac{1}{2}-\sigma}\right] \tag{2.20}$$

so changing σ, m into $-\sigma, -m$ and using properties of the Clebsch-Gordan coefficients[11] we obtain

$$i\sigma_2\phi^* = (-1)^{m+j-\ell}\phi. \tag{2.21}$$

Thus, (except for the phase factor $(-1)^{m+j-\ell}$ which is irrelevant) the state ψ_a of the antiparticle is the solution of the Dirac oscillator equation (2.2) when we change ω by $-\omega$.

In the next section we consider a Poincaré invariant equation for the two body system of particle and antiparticle.

3 Equation for the particle-antiparticle system with Dirac oscillator interaction

As in previous publications[4,10] we start from the Dirac equation for n-free particles

$$\sum_{s=1}^{n}(\boldsymbol{\alpha}_s\cdot\mathbf{p}_s+\beta_s)\psi = E\psi, \tag{3.1}$$

where E is the total energy for the system, $\boldsymbol{\alpha}_s, \beta_s$ are direct products such as

$$\beta_s = I\otimes I ... I\otimes\beta\otimes I ... \otimes I, \tag{3.2}$$

with β in s^{th} position, while $\mathbf{p}_s$ is the momentum of the s^{th} particle.

Following the analysis of Barut[5], we showed that the Poincaré invariant form of the equation (3.1) is[4,10]

$$\sum_{s=1}^{n}\left[\Gamma_s(\gamma_s^\mu p_{\mu s}+1)\right]\psi = 0. \tag{3.3}$$

We first explain all the symbols appearing in (3.3). The index μ takes now the values $\mu = 0, 1, 2, 3$, and

$$\gamma^0 = \beta, \gamma^i = \beta\alpha_i, i = 1, 2, 3, \tag{3.4a, b}$$

with β, α_i given by (2.4a,b). For n particles we have $\gamma_s^\mu, s = 1,2,...n$, given by direct products such as (3.2).

We also introduce the concept[5] of unit time like four vector $(u_\mu) = (u_0, u_1, u_2, u_3)$, which means that in some reference frame it can tkae the value (1,0,0,0). With the help of u_μ we define the Lorentz scalars

$$\Gamma = \prod_{r=1}^{n}(\gamma_r^\mu u_\mu), \tag{3.5a}$$

$$\Gamma_s = (\gamma_s^\mu u_\mu)^{-1}\Gamma, \tag{3.5b}$$

where repeated indeces are summed over $\mu = 0,1,2,3$. Note that $(\gamma_s^\mu u_\mu)^{-1}$ in (3.5b) just eliminates the corresponding term in Γ of (3.5a) so Γ_s is still in product form.

The terms in our equation (3.3) are then fully defined and we proceed now to look at it in the frame of reference where $(u_\mu) = (1,0,0,0)$, where it takes the form

$$\left\{\Gamma^0 \sum_{s=1}^{n} p_{0s} + \sum_{s=1}^{n}\left[\Gamma_s^0(\boldsymbol{\gamma}_s \cdot \mathbf{p}_s + 1)\right]\right\}\psi = 0, \tag{3.6}$$

where

$$\Gamma^0 = \prod_{r=1}^{n}\gamma_r^0 = \beta \otimes \beta \otimes .... \otimes \beta \equiv B, \tag{3.7a}$$

$$\Gamma_s^0 = (\gamma_s^0)^{-1}\Gamma^0 = \beta \otimes \beta....\beta \otimes I \otimes \beta.... \otimes \beta, \tag{3.7b}$$

Multiplying (3.6) by Γ^0 and using $\beta^2 = I, \gamma_i = \beta\alpha_i$ and $(\Gamma^0)^2 = I$ we obtain

$$\left[-P^0 + \sum_{s=1}^{n}(\boldsymbol{\alpha}_s \cdot \mathbf{p}_s + \beta_s)\right]\psi = 0, \tag{3.8}$$

where we put the time like component P_0 of the four momentum

$$P_\mu = p_{\mu 1} + p_{\mu 2} + ... + p_{\mu n}, \tag{3.9}$$

in its contravariant form $-P^0$ as our metric tensor is

$$g_{\mu\nu} = 0 \text{ if } \mu \neq \nu,\ g_{11} = g_{22} = g_{33} = -g_{00} = 1. \tag{3.10}$$

Clearly we then recover equation (3.1) if we interpret P^0 as E, as is usually done.

The Barut equation (3.3) will then provide the starting point for the one involving n-particles with Dirac oscillator interactions of frequency $\omega_s, s = 1,2,...n$. To proceed in this direction we could replace $p_{\mu s}; \mu = 0,1,2,3; s = 1,2,...n$ in (3.3) by a linear function of $p_{\mu s}$ and $x_{\mu s}$ as was done in (2.1) for the one particle problem. We note though that while $p_{\mu s}$ commutes with the total four momentum P_ν of (3.9), which is a genereator of the Poincaré group, $x_{\mu s}$ does not. Thus it is more suggestive[4] to use the translationally invariant coordinates $x'_{\mu s}$ defined by

$$x'_{\mu s} = x_{\mu s} - X_\mu, \tag{3.11a}$$

where

$$X_\mu = n^{-1}(x_{\mu 1} + x_{\mu 2} + ... + x_{\mu n}). \tag{3.11b}$$

Furthermore we would like that the resulting equation, in the frame of reference where the center of mass is at rest *i.e.* $P_i = 0$, $i = 1,2,3$, should depend only on a single time as is the case in Eq. (3.8). Thus it is convenient to use the transverse coordinates $\hat{x}'_{\mu s}$ defined by

$$\hat{x}'_{\mu s} = x'_{\mu s} - (P^\nu x'_{\nu s})P_\mu(P_\tau P^\tau)^{-1/2}, \tag{3.12}$$

which has the property that in the center of mass frame, where $P_i = 0, i = 1,2,3, \hat{x}'_{0s} = 0$.

With the above restrictions we could obtain from (3.3) a Poincaré invariant equation with a Dirac oscillator interaction if we make in it the replacement

$$p_{\mu s} \rightarrow p_{\mu s} - i\omega_s \hat{x}'_{\mu s}\Gamma, \tag{3.13}$$

where Γ is defined as in (3.5a), and we assignate a different frequency ω_s for each particle. We then arrive at the equation

$$\sum_{s=1}^{n}\left\{\Gamma_s\left[\gamma_s^\mu(p_{\mu s} - i\omega_s\hat{x}'_{\mu s}\Gamma) + 1\right]\right\}\psi = 0, \tag{3.14}$$

where we make the choice[4] for the unit time like vector u_μ in Γ and Γ_s of (3.5) as

$$u_\mu = P_\mu(-P_\nu P^\nu)^{-1/2} \tag{3.15}$$

In the center of mass frame of reference, where $P_i = 0, i = 1,2,3$, we have $\{u_\mu\} = (1000)$ and so by a reasoning similar to the one that leads from (3.3) to (3.8) we obtain the equation

$$\left\{-P^0 + \sum_{s=1}^{n}\left[\boldsymbol{\alpha}_s \cdot (\mathbf{p}'_s - i\omega_s \mathbf{x}'_s B) + \beta_s\right]\right\}\psi = 0, \tag{3.16}$$

where B is given by (3.7a) while $\boldsymbol{\alpha}_s, \beta_s$ are direct products of the form (3.2) and

$$\mathbf{p}'_s \equiv \mathbf{p}_s - n^{-1}\mathbf{P}, \tag{3.17}$$

becomes identical to $\mathbf{p}_s$ in the center of mass frame.

As P^0 is the total energy of the system, the rest of the expression (3.16) is then the mass operator[4], which we will designate by $\mathcal{M}$, for the n particles interacting with Dirac oscillators of frequencies $\omega_s, s = 1,2,...n$. If we are dealing with the particle-antiparticle system $n = 2$ and, from the discussion of the previous section $\omega_1 = -\omega_2 \equiv \omega$, so we get[4]

$$\mathcal{M} = (1/\sqrt{2})\left\{(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2)\cdot\mathbf{p} - i\omega\left[(\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2)\cdot\mathbf{x}\right]B\right\} + (\beta_1 + \beta_2) \tag{3.18}$$

where[4]

$$\mathbf{p} = (1/\sqrt{2})(\mathbf{p}_1 - \mathbf{p}_2), \mathbf{x} = (1/\sqrt{2})(\mathbf{x}_1 - \mathbf{x}_2), \tag{3.19a, b}$$

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1, & 0 & 0 \\ \boldsymbol{\sigma}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}_1 \\ 0 & 0 & \boldsymbol{\sigma}_1 & 0 \end{pmatrix}, \tag{3.20a}$$

$$\boldsymbol{\alpha}_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} 0 & \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \boldsymbol{\sigma}_2 & 0 \\ 0 & 0 & 0 & \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_2 & 0 & 0 & 0 \\ 0 & \boldsymbol{\sigma}_2 & 0 & 0 \end{pmatrix}, \tag{3.20b}$$

$$\beta_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}, \tag{3.21a}$$

$$\beta_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}, \tag{3.21b}$$

$$B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \tag{3.22}$$

and the wave function can be written as[4)]

$$\psi = \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{12} \\ \psi_{22} \end{pmatrix} \tag{3.23}$$

4 Solution of the eq. (3.18) by a perturbative procedure

Denoting by μ the eigenvalue of the mass operator of (3.18) and making use of (3.19-3.23) we obtain the equation

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \cdot (\mathbf{p} + i\omega\mathbf{x}) & -\boldsymbol{\sigma}_2 \cdot (\mathbf{p} - i\omega\mathbf{x}) & 0 \\ \boldsymbol{\sigma}_1 \cdot (\mathbf{p} - i\omega\mathbf{x}) & 0 & 0 & -\boldsymbol{\sigma}_2 \cdot (\mathbf{p} + i\omega\mathbf{x}) \\ -\boldsymbol{\sigma}_2 \cdot (\mathbf{p} + i\omega\mathbf{x}) & 0 & 0 & \boldsymbol{\sigma}_1 \cdot (\mathbf{p} - i\omega\mathbf{x}) \\ 0 & -\boldsymbol{\sigma}_2 \cdot (\mathbf{p} - i\omega\mathbf{x}) & \boldsymbol{\sigma}_1 \cdot (\mathbf{p} + i\omega\mathbf{x}) & 0 \end{pmatrix} \begin{pmatrix} \psi_{11} \\ \psi_{21} \\ \psi_{12} \\ \psi_{22} \end{pmatrix}$$
$$= \begin{pmatrix} (\mu - 2) & \psi_{11} \\ \mu & \psi_{21} \\ \mu & \psi_{12} \\ (\mu + 2) & \psi_{22} \end{pmatrix}, \tag{4.1}$$

where $\mathbf{p} = -i\nabla$.

Introducing now the creation and annihilation operators by the definitions

$$\boldsymbol{\eta} = (1/\sqrt{2})(\omega^{1/2}\mathbf{x} - i\omega^{-1/2}\mathbf{p}), \boldsymbol{\xi} = (1/\sqrt{2})(\omega^{1/2}\mathbf{x} + i\omega^{-1/2}\mathbf{p}), \quad (4.2a, b)$$

we get the equations

$$i\omega^{1/2}\begin{pmatrix} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\eta} & \boldsymbol{\sigma}_2 \cdot \boldsymbol{\xi} \\ \boldsymbol{\sigma}_2 \cdot \boldsymbol{\xi} & \boldsymbol{\sigma}_1 \cdot \boldsymbol{\eta} \end{pmatrix}\begin{pmatrix} \psi_{21} \\ \psi_{12} \end{pmatrix} = \begin{pmatrix} (\mu - 2) & \psi_{11} \\ (\mu + 2) & \psi_{22} \end{pmatrix}, \quad (4.3a)$$

$$-i\omega^{1/2}\begin{pmatrix} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\xi} & \boldsymbol{\sigma}_2 \cdot \boldsymbol{\eta} \\ \boldsymbol{\sigma}_2 \cdot \boldsymbol{\eta} & \boldsymbol{\sigma}_1 \cdot \boldsymbol{\xi} \end{pmatrix}\begin{pmatrix} \psi_{11} \\ \psi_{22} \end{pmatrix} = \mu\begin{pmatrix} \psi_{21} \\ \psi_{12} \end{pmatrix}, \quad (4.3b)$$

Multiplying (4.3a) by μ and substituting in it (4.3b), we obtain, after some straightforward algebra, that

$$\omega\begin{bmatrix} A & D \\ D & A \end{bmatrix}\begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix} = \begin{bmatrix} \mu^2 - 2\mu & 0 \\ 0 & \mu^2 + 2\mu \end{bmatrix}\begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix}, \quad (4.4)$$

where

$$A \equiv 2\boldsymbol{\eta} \cdot \boldsymbol{\xi} + 3 - \mathbf{L} \cdot (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2), \quad (4.5a)$$

$$D \equiv 2(\mathbf{S} \cdot \boldsymbol{\eta})^2 + 2(\mathbf{S} \cdot \boldsymbol{\xi})^2 - (\boldsymbol{\eta} \cdot \boldsymbol{\eta}) - (\boldsymbol{\xi} \cdot \boldsymbol{\xi}), \quad (4.5b)$$

while

$$\mathbf{L} = \mathbf{x} \times \mathbf{p} = -i(\boldsymbol{\eta} \times \boldsymbol{\xi}), \mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2), \quad (4.6a, b)$$

and extensive use was made of the relation between Pauli spin matrices *i.e.*

$$\sigma_i \sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k. \quad (4.7)$$

It is convenient to substitute ψ_{11}, ψ_{22} by ϕ_+, ϕ_-, through the relation

$$\begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}, \quad (4.8)$$

so that equation (4.4) becomes

$$\omega\begin{bmatrix} A - D & 0 \\ 0 & A + D \end{bmatrix}\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix} = \begin{bmatrix} \mu^2 & -2\mu \\ -2\mu & \mu^2 \end{bmatrix}\begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}. \quad (4.9)$$

Writing the two equations in ϕ_+, ϕ_- explicitly and eliminating ϕ_- between them we obtain for ϕ_+, which from now on we denote simply by ϕ, the equation

$$\left[\mu^4 - (4 + 2A\omega)\mu^2 + \omega^2(A^2 - D^2 - AD + DA)\right]\phi = 0 \quad (4.10)$$

Unfortunately, because of the term D of (4.5b), this problem is not exactly soluble as was the case of two particles, *i.e.* $\omega_1 = \omega_2 = \omega$, discussed in reference 4. We note though that the operator in (4.10) contains the frequency ω as a parameter. As this frequency is given in units of the rest mass 1 of the particle, we expect ω to be small as compared with 1, as is the case in nuclear physics. We can then begin by disregarding the term in ω^2 and so our equation becomes

$$\mu^2\Big[\mu^2 - (4 + 2A\omega)\Big]\phi = 0, \tag{4.11}$$

so our first objective will be to find the eigenstates and eigenvalues of the operator A given by (4.5a).

To achieve our purpose we start by introducing the ket

$$|N(\ell, s)jm\rangle = R_{N\ell}(r)\sum_\sigma \langle \ell m - \sigma, s\sigma|jm\rangle Y_{\ell m-\sigma}(\theta,\varphi)\chi_{s\sigma}, \tag{4.12}$$

where all the functions and symbols are defined as in the paragraph following (2.10), except that now the spin function $\chi_{s\sigma}$ corresponds to $s = 0$ or 1 and not the 1/2 appearing in (2.10).

We note from (4.5) that the operators in (4.10a,b) are invariant under reflections, *i.e.* change of $\boldsymbol{\eta}, \boldsymbol{\xi}$ into $-\boldsymbol{\eta}, -\boldsymbol{\xi}$ and thus the parity of the states (4.12), which is $(-1)^\ell$, is a good quantum number. Considering then separately the states (4.12) in which $\ell = j$, and those in which $\ell = j \pm 1$, we find by straightforward Racah algebra[11] that the eigenstates of A of (4.5a), which we shall denote by ϕ_0, are

$$\phi_0 \equiv (1/\sqrt{2})\Big[|N(j,0)jm\rangle \pm |N(j,1)jm\rangle\Big] \text{ for parity } (-1)^j, \tag{4.13a}$$

$$\phi_0 \equiv |N(j \pm 1, 1)jm\rangle \quad \text{for parity } -(-1)^j, \tag{4.13b}$$

The corresponding eigenvalues of μ^2, which we denote by μ_0^2, are given by

$$\mu_0^2 = 4 + 2\omega\Big\{(2N+3) \pm [j(j+1)]^{1/2}\Big\}, \tag{4.14a}$$

for parity $(-1)^j$, while for parity $-(-1)^j$ we get

$$\mu_0^2 = 4 + 2\omega(2N+3), \tag{4.14b}$$

and thus we have a complete solution of the problem (4.11).

Our interest though is in the equation (4.13) which we can solve by a perturbation procedure. We first define

$$W \equiv \mu^2, \tag{4.15a}$$

$$H_0 = 4 + 2A\omega, \tag{4.15b}$$

$$H' = \omega^2(A^2 - D^2 - AD + DA), \tag{4.15c}$$

so equation (4.10) becomes

$$(W^2 - WH_0 + H')\phi = 0. \tag{4.16}$$

We then, as for example in Schiff book[12], replace H' by $\lambda H'$ where λ is a parameter and write

$$W = W_0 + \lambda W_1 + \lambda^2 W_2 + ..., \tag{4.17a}$$

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + ..., \tag{4.17b}$$

where $W_0 = \mu_0^2$ of (4.14) and ϕ_0 is given by (4.13).

From (4.17a) we obtain

$$W^2 = W_0^2 + \lambda(2W_0 W_1) + \lambda^2(2W_0 W_2 + W_1^2) + \ldots, \tag{4.17c}$$

so that using (4.17) we see that, up to first order in λ, (4.16) takes the form

$$\Big[W_0(W_0 - H_0)\phi_0\Big] + \lambda\Big[(2W_0W_1 - W_1H_0 + H')\phi_0 + W_0(W_0 - H_0)\phi_1\Big] + \ldots = 0, \tag{4.18}$$

where each of the square brackets must vanish[12]. For the first one this is automatic as from (4.11) we have

$$H_0\phi_0 = W_0\phi_0. \tag{4.19}$$

From the second square bracket, when we take its scalar product[12] with ϕ_0, we obtain

$$W_1 = -W_0^{-1}(\phi_0, H'\phi_0), \tag{4.20}$$

where we made use of the hermitian character of H_0 and of Eq. (4.19).

Thus to first order in perturbation theory, when we take, as usual[12], $\lambda = 1$, we have that

$$\mu^2 = \mu_0^2 - \mu_0^{-2}(\phi_0, H'\phi_0) + \ldots \tag{4.21}$$

where μ_0 is given by (4.14), ϕ_0 by (4.13) and H' is (4.15c). In the next section we calculate this μ^2 explicitly.

5 Square of the mass spectra of the particle-antiparticle system

To determine the square of the mass μ^2, given to first order perturbation theory by (4.21), we need to calculate the scalar product $(\phi_0, H'\phi_0)$. As ϕ_0 has a definite number of quanta N, which is indicated in (4.13), we need only to consider that part of H' in (4.15c) that does not change the number of quanta. The terms AD, DA in (4.15) change the number of quanta by ± 2, as indicated in (4.5), so we can disregard them. The A^2 is diagonal in the basis ϕ_0 of (4.13) and its contribution to the scalar product in (4.20) is

$$(\phi_0, A^2\phi_0) = \Big\{(2N+3) \pm [j(j+1)]^{1/2}\Big\}^2 \text{ for parity}(-1)^j \tag{5.1a}$$

$$(\phi_0, A^2\phi_0) = (2N+3)^2 \quad \text{for parity} - (-1)^j \tag{5.1b}$$

For the D^2 operator, where D is given by (4.5b), the only terms that contribute to its expectation value with respect to ϕ_0 *i.e.* that do not change the number of quanta N, are

$$\begin{aligned}&4(\mathbf{S}\cdot\boldsymbol{\eta})^2(\mathbf{S}\cdot\boldsymbol{\xi})^2 + 4(\mathbf{S}\cdot\boldsymbol{\xi})^2(\mathbf{S}\cdot\boldsymbol{\eta})^2 - 2(\mathbf{S}\cdot\boldsymbol{\eta})^2(\boldsymbol{\xi}\cdot\boldsymbol{\xi}) - 2(\mathbf{S}\cdot\boldsymbol{\xi})^2(\boldsymbol{\eta}\cdot\boldsymbol{\eta})\\ &- 2(\boldsymbol{\eta}\cdot\boldsymbol{\eta})(\mathbf{S}\cdot\boldsymbol{\xi})^2 - 2(\boldsymbol{\xi}\cdot\boldsymbol{\xi})(\mathbf{S}\cdot\boldsymbol{\eta})^2 + (\boldsymbol{\eta}\cdot\boldsymbol{\eta})(\boldsymbol{\xi}\cdot\boldsymbol{\xi}) + (\boldsymbol{\xi}\cdot\boldsymbol{\xi})(\boldsymbol{\eta}\cdot\boldsymbol{\eta}).\end{aligned} \tag{5.2}$$

To evaluate the matrix element $(\phi_0, H'\phi_0)$ we need then to determine the matrix elemtns of $(\mathbf{S}\cdot\boldsymbol{\eta})^2, (\mathbf{S}\cdot\boldsymbol{\xi})^2, (\boldsymbol{\eta}\cdot\boldsymbol{\eta}), (\boldsymbol{\xi}\cdot\boldsymbol{\xi})$ with respect to states of the form $|N(\ell,s)jm\rangle$ of (4.12). This is done in the Appendix using results for the matrix elements of $(\mathbf{S}\cdot\boldsymbol{\eta}), (\mathbf{S}\cdot\boldsymbol{\xi})$ given in reference 13 as well as the operator form of the harmonic oscillator states of reference 14.

We finally arrive at the following results for the μ^2 of (4.21): For the states of parity $(-1)^j, \mu^2$ can be expressed as a function of N, j, ω, using both signs $\pm$ in the ϕ_0 of (4.13a) *i.e.*

$$\mu_\pm^2(N,j,\omega) = 4 + 2\omega\Big\{(2N+3) \pm [j(j+1)]^{1/2}\Big\}$$
$$-(\omega^2/4)\Big\{[2N+3) \pm j^{1/2}(j+1)^{1/2}]^2 - 2[N(N+3) - j(j+1) + 3]\Big\} + \dots \tag{5.3}$$

For the states of parity $-(-1)^j$, the two cases of orbital angular momentum $\ell = j+1$ or $\ell = j-1$, have to be written separately. We shall distinguish them from (5.3) by putting a bar above the μ^2 and an index + or - when ℓ is respectively $j+1$ or $j-1$. Thus we obtain

$$\bar{\mu}_+^2 = 4 + 2\omega(2N+3) - (\omega^2/4)\Big\{(2N+3)^2 - 2[j(j-1) + N(N+3) + 1]\Big\} + \dots \tag{5.4a}$$

$$\bar{\mu}_-^2 = 4 + 2\omega(2N+3) - (\omega^2/4)\Big\{(2N+3)^2 - 2[j(j+3) + N(N+3) + 3]\Big\} + \dots \tag{5.4b}$$

In all of these cases we keep only terms up to ω^2 so that μ_0^{-2} appearing as a coefficient of the scalar product in (4.21) is replaced by (1/4). Note furthermore that as $N = 2n + \ell$ where ℓ is the *orbital* angular momentum, we see that for $\mu_\pm^2(N,j,\omega)$ of (5.3) the N takes the values $N = j, j+2, j+4, \dots$. On the other hand for $\bar{\mu}_+^2(N,j,\omega)$ of (5.4a) N takes the values $N = j+1, j+3, j+5, \dots$, while for $\bar{\mu}_-^2(N,j,\omega)$ of (5.4b) we have $N = j-1, j+1, j+3, j+5, \dots$.

In Fig. 1 we graph $\mu_\pm^2$ of (5.3) for $\omega = 0.1$ as function of j, indicating the sign $\pm$ to which the level corresponds on its left hand side while on the right hand side we give the value of N. Note that when $j = 0$ the state $|N(0,1)00\rangle$ does not exist so that the normalized ϕ_0 of (4.13a) reduces to $|N(0,0)00\rangle$. In this case the spin is 0 as indicated int the left hand side of the level $j = 0$, and not the mixture $\pm$ in (4.13a).

In Fig. 2 we graph $\bar{\mu}_\pm^2$ of (5.4) for $\omega = 0.1$ as function of j. The orbital angular momentum $\ell = j \pm 1$ is indicated on the left of the levels and the total number of quanta N is given on the right. Note that for $j = 0$, $|N(-1,1)00\rangle$ does not exist so that we have only $|N(1,1)00\rangle$ corresponding to $\ell = 1$ indicated on the left hand side of the levels with $j = 0$. Also the levels with $N = \ell = j-1$ are unique as indicated in the corresponding left hand side of the levels. For the other states $|N(j\pm 1, 1)jm\rangle$ both values $\ell = j+1$ and $j-1$ are possible, and because of the first order corrections, *i.e.* the term in ω^2 of (5.4) they are separated by $\omega^2(2j+1)$. In Fig. 2 we give on the left hand side the values $\ell = j+1$ and $\ell = j-1$ of the paired levels.

The parity denoted by a script $\mathcal{P}$ is given in all three figures *i.e.* $\mathcal{P} = (-1)^j$ or $\mathcal{P} = -(-1)^j$.

In the next section we discuss the comparison of our reuslts with squares of the mass spectra for non-strange mesons[1)].

6 Comparison with the meson spectra

In page 37 of reference 1 there is Meson Summary Table in which, in the first two columns, are given the non-strange mesons with the code name (a single letter with an index indicating the total angular momentum j), mass μ in MeV, j, isospin I, and parity defined there[1)] as $P = -(-1)^{\ell}$, as well as a charge conjugation number $C = (-1)^{\ell+s}$, where ℓ is the orbital angular momentunm and s the total spin of the quark-antiquark system.

From the above information we can get the square of the mass μ^2 in units $(GeV)^2$, the j, s, I as well as $(-1)^{\ell}$, so that, $\ell = j$ or $\ell = j \pm 1$. As mentioned before our notation for parity will be the script $\mathcal{P} = (-1)^{\ell}$ *i.e.* $\mathcal{P} = (-1)^j$ or $-(-1)^j$.

The information given in the previous paragraph is summarized in Figs. 3 to 6, where in the abscissa we have $j = 0, 1, 2, 3, 4, 5$ and μ^2 in the ordinate. We note first that for comparison with the particle-antiparticle system, with a Dirac oscillator interaction, we have divided the information according to parity $\mathcal{P} = (-1)^j$ or $-(-1)^j$ in Figs. 3,4 or Figs. 5,6 as $\mathcal{P}$ is an integral of motion of our problem. Furthermore the isospin $I = 0, 1$ is completely independent from the Poincaré group and thus of the analysis previous sections. We could then consider that in our theoretical $\mu_{\pm}^2$ of (5.3) or $\bar{\mu}_{\pm}$ of (5.4), ω is a function of I and another one could be added to these $\mu_{\pm}, \bar{\mu}_{\pm}$ variables. Thus it is convenient to graph separately the levels for $I = 1, \mathcal{P} = (-1)^j$ (Fig. 3); $I = 0, \mathcal{P} = (-1)^j$ (Fig. 4); $I = 1, \mathcal{P} = -(-1)^j$ (Fig. 5); and $I = 0, \mathcal{P} = -(-1)^j$ (Fig. 6).

In figures 3 to 6 we put on the left hand side the name of the meson and where there are several of the same name we distinguish them by primes *i.e.* π_j, π'_j, π''_j etc. In Figs. 3,4 we put on the right hand side the total spin $s = 0$ or 1 and furthermore we differentiate the two values by using either a full or dashed line. In Figs. 5,6 only the names of the mesons appear on the levels as the spin s is always 1.

We now wish to make a qualitative comparison between the theoretical figures 1 to 2 and the experimental ones for mesons in figures 3 to 6.

We begin with parity $\mathcal{P} = (-1)^j$ where we should compare Fig. 1 with Figs. 3,4. For angular momentum $j = 0$ the comparison in Figs. 1 and 3 is quite good and the spin is $s = 0$. For all other levels we cannot compare because in Fig. 3 they have definite spin, while in Fig. 1 we have 50% each of admixtures of $s = 0$ and $s = 1$, in or out of phase, as shown in (4.13a). This clearly shows the need to add other interactions in our equation (3.14) that are Poincaré invariant, which we shall discuss below. Note also that even for $j = 0$ the comparison between Fig. 1 and Fig. 4 is not good, even if we change the scale in Fig. 1, but this can be attributed to the fact that the η_0 mesons may have and admixture of $s\bar{s}$ where s has different mass from u, d.

Turning now our attention to parity $\mathcal{P} = -(-1)^j$ we could compare Fig. 2 with Fig. 5 or 6. In Fig. 5 *i.e.* $I = 1$ the information is sparse though for $j = 1$ we see what may be a pair ρ', ρ'' which, with a change of scale, could resemble the pair for $j = 1, N = 2$ in Fig. 2. This is more noticeable when we compare, for $j = 2$, Fig. 2 with Fig. 6, where we see a number of pairs *i.e.* $N = 3$ corresponding f'_2, f''_2; $N = 5$ to f'''_2, f^{IV}_2; and if we had graphed it, $N = 7$ to f^{V}_2, f^{VI}_2; $N = 9$ to f^{VII}_2, f^{VIII}_2. Note also that for parity $\mathcal{P} = -(-1)^j$ the lowest level in Figs. 5, 6 appears for $j = 1$ and not $j = 0$, and this is also true in the theoretical calculations of Fig. 2.

Clearly though in all cases we would have to modify our starting hypothesis in a similar way as we did in the baryon example[7)]. We want to do this in a Poincaré invariant way and thus it is convenient to introduce the four vector[7)]

$$\mathcal{W}_\mu = \frac{1}{2}\mathcal{E}_{\mu\nu\sigma\tau}P^\nu K^{\sigma\tau}, \tag{6.1}$$

where $K^{\sigma\tau}$ could be either $J^{\sigma\tau}, L^{\sigma\tau}, S^{\sigma\tau}$ defined respectively by

$$J^{\sigma\tau} = L^{\sigma\tau} + S^{\sigma\tau}, \tag{6.2a}$$

$$L^{\sigma\tau} = \sum_{s=1}^{2}(x_s^\sigma p_s^\tau - x_s^\tau p_s^\sigma), \tag{6.2b}$$

$$S^{\sigma\tau} = (i/4)\sum_{s=1}^{2}(\gamma_s^\sigma\gamma_s^\tau - \gamma_s^\tau\gamma_s^\sigma), \tag{6.2c}$$

where $x_s^\tau, p_s^\tau, \gamma_s^\tau; \tau = 0,1,2,3; s = 1,2$ are the ones in section 3 of this paper with $n = 2$.

We consider now the Poincaré invariants

$$\mathcal{W}^2 = \mathcal{W}^\mu\mathcal{W}_\mu \quad , \quad P^2 = -P_\mu P^\mu, \tag{6.3a, b}$$

as well as the Γ of (3.5a), which we combine in the form

$$\Gamma(\mathcal{W}^2/P^2), \tag{6.4}$$

and in the center of mass frame *i.e.* when $P_i = 0, i = 1,2,3$, it reduces to[7)]

$$J^2, L^2, S^2, \tag{6.5a, b, c}$$

depending on whether $K^{\sigma\tau}$ is equal to $J^{\sigma\tau}, L^{\sigma\tau}, S^{\sigma\tau}$.

Thus, as indicated in Eq. (2.9a) of reference 7, our equation (3.16) could be modified to

$$\left\{ -P^0 + \mathcal{M} + aJ^2 + bL^2 + cS^2 \right\}\psi = 0 \tag{6.6}$$

where $\mathcal{M}$ is given by (3.18) and a, b, c are, so far, arbitrary real constants.

If $b = c = 0$, as J^2 is an integral of motion of the operator $\mathcal{M}$, we have that the new mass, which we call μ', is related with the old one by

$$\Big[\mu' - aj(j+1)\Big]^2 = \mu^2, \tag{6.7}$$

and thus the new mass spectra is

$$\mu' = \mu + aj(j+1), \tag{6.8}$$

where we have at our disposal the parameter a with which we can adjust the spectra corresponding to different j's.

When we have b, c also different from zero, as L^2, S^2 are not integrals of motion, our only way to proceed is by considering the matrix of the operator

$$\mathcal{M} + aJ^2 + bL^2 + cS^2, \tag{6.9}$$

with respect to the states $|N(\ell, s)jm\rangle$, where j is fixed, and diagonalizing this matrix up to certain maximum number of quanta N_{max}.

Calculations of this type, done by Luis Benet, when $a = b = 0$, *i.e.* only with an S^2 term, allow us to break the wave functions of (4.13a) into states of definite spin *i.e.* $s = 0$ or 1, but keeping some of the ordering as in Fig. 1, so that its more comparable with the meson spectra in which the spin is given.

We do not wish though to consider the more general operator (6.9) in this paper. To begin with, when dealing only with the operator $\mathcal{M}$ of (3.18), we have a single parameter, the frequency ω, and also as a scale in our calcualtions the mass m of the quarks. If we go to the operator (6.9) we have the parameters ω, a, b, c, plus the m. Thus we can of course adjust the meson spectra better, but it may be meaningless.

Thus we conclude by stating that the particle-antiparticle system with a Dirac oscillator interaction, may give some insight on the meson spectra, and that is all we aspire to achieve in the present paper.

7 Appendix

To obtain the matrix elements of $(\phi_0, D^2\phi_0)$ we need first those of the factors in each product appearing in (5.2). Those of $(\mathbf{S}\cdot\boldsymbol{\eta})^2$ can be obtained from the ones of $(\mathbf{S}\cdot\boldsymbol{\eta})$ given in (3.20) and (3.24) of reference 13, and thus we have

$$\langle N+2(j,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j,1)jm\rangle = -\Big[(N+j+3)(N+2-j)\Big]^{1/2} \tag{A.1a}$$

$$\langle N+2(j+1,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j+1,1)jm\rangle = -\Big[(N+j+4)(N+1-j)\Big]^{1/2}\Big[j/(2j+1)\Big] \tag{A.1b}$$

$$\langle N+2(j-1,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j+1,1)jm\rangle = \Big[(N-j+3)(N+1-j)\Big]^{1/2}\Big[j(j+1)\Big]^{1/2}(2j+1)^{-1} \tag{A.1c}$$

$$\langle N+2(j+1,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j-1,1)jm\rangle = \Big[(N+j+2)(N+j+4)\Big]^{1/2}\Big[j(j+1)\Big]^{1/2}(2j+1)^{-1} \tag{A.1d}$$

$$\langle N+2(j-1,1)jm|(\mathbf{S}\cdot\boldsymbol{\eta})^2|N(j-1,1)jm\rangle = -\Big[(N+j+2)(N-j+3)\Big]^{1/2}\Big[(j+1)/(2j+1)\Big]. \tag{A.1e}$$

From the hermitian conjugates of (5.1) we obtain those of $(\mathbf{S}\cdot\boldsymbol{\xi})^2$ *i.e.*

$$\langle N-2(j,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j,1)jm\rangle = -\Big[(N+j+1)(N-j)\Big]^{1/2} \tag{A.2a}$$

$$\langle N-2(j+1,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j+1,1)jm\rangle = -\Big[(N+j+2)(N-j-1)\Big]^{1/2}\Big[j/(2j+1)\Big] \tag{A.2b}$$

$$\langle N-2(j+1,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j-1,1)jm\rangle = \Big[(N-j+1)(N-j-1)\Big]^{1/2}\Big[j(j+1)\Big]^{1/2}(2j+1)^{-1} \tag{A.2c}$$

$$\langle N-2(j-1,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j+1,1)jm\rangle = \Big[(N+j)(N+j+2)\Big]^{1/2}\Big[j(j+1)\Big]^{1/2}(2j+1)^{-1} \tag{A.2d}$$

$$\langle N-2(j-1,1)jm|(\mathbf{S}\cdot\boldsymbol{\xi})^2|N(j-1,1)jm\rangle = -\Big[(N+j)(N-j+1)\Big]^{1/2}\Big[(j+1)/(2j+1)\Big]. \tag{A.2e}$$

Finally, from the operator form of the harmonic oscillator states, given in reference 14, we have that

$$\langle N+2(\ell,s)jm|\boldsymbol{\eta}\cdot\boldsymbol{\eta}|N(\ell,s)jm\rangle = -\Big[(N+\ell+3)(N-\ell+2)\Big]^{1/2}, \tag{A.3a}$$

and from its hermitian conjugate we get

$$\langle N-2(\ell,s)jm|\boldsymbol{\xi}\cdot\boldsymbol{\xi}|N(\ell,s)jm\rangle = -\Big[(N+\ell+1)(N-\ell)\Big]^{1/2}. \qquad (A.3b)$$

With the help of these expressions we obtain straighforwardly the $\mu_\pm^2$ of (5.3) and $\bar{\mu}_\pm^2$ of (5.4).

8 Acknowledgements

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9 Figure Captions

- Fig. 1. We graph $\mu_{\pm}^2$ of (5.3) for $\omega = 0.1$ as function of j, indicating the sign $\pm$ to which the level corresponds on its left hand side while on the right hand side we give the value of N. Note that when $j = 0$ the state $|N(0,1)00>$ does not exist so that the normalized ϕ_o of (4.13a) reduces to $|N(0,0)00>$. In this case the spin is 0 as indicated in the left hand side of the level $j = 0$, and not the mixture $\pm$ in (4.13a).

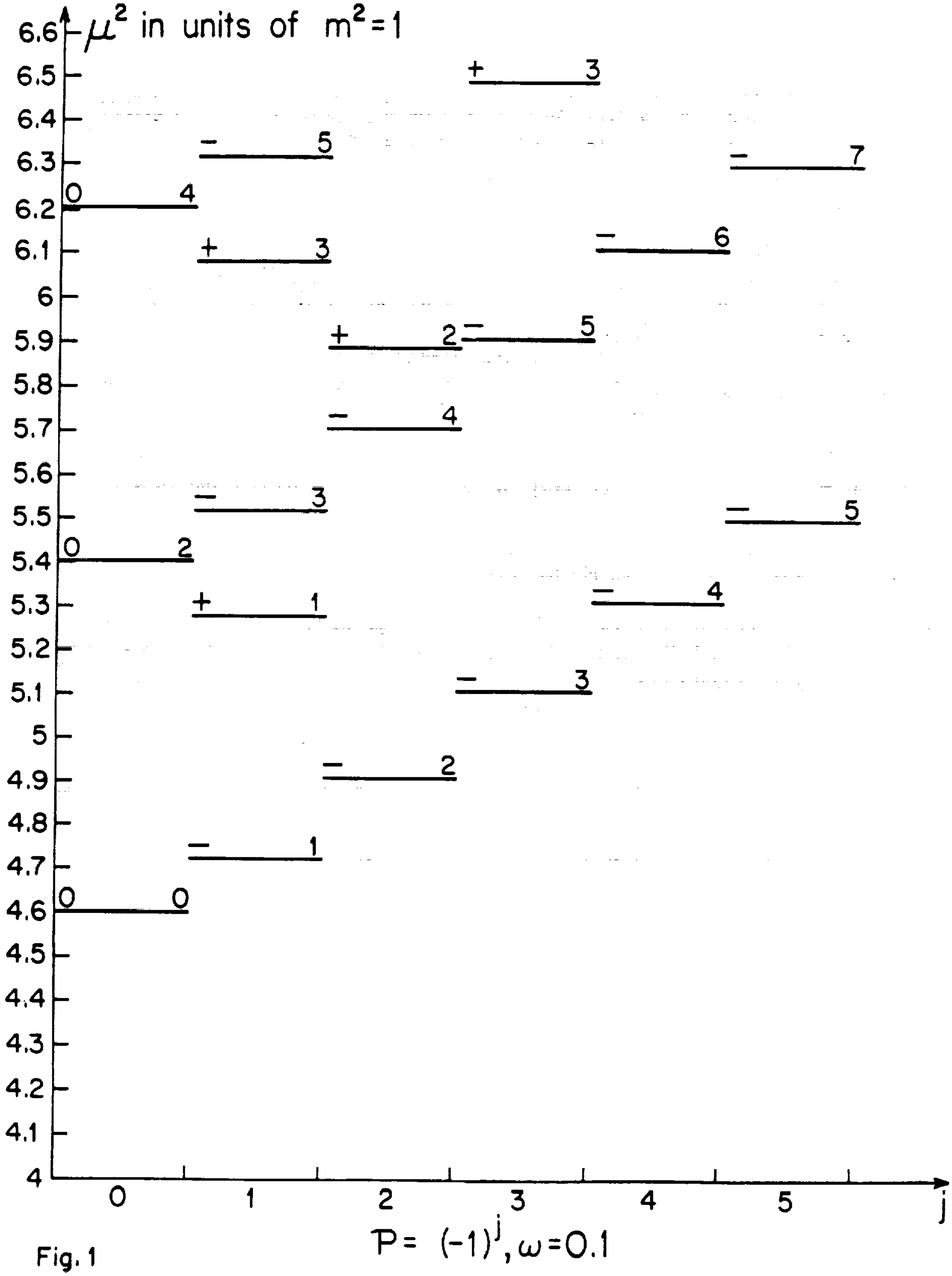
- Fig. 2. We graph $\bar{\mu}_{\pm}^2$ of (5.4) for $\omega = 0.1$ as function of j. The orbital angular momentum $\ell = j \pm 1$ is indicated on the left of the levels and the total number of quanta N is given on the right. Note that for $j = 0, |N(-1,1)00>$ does not exist so that we have only $|N(1,1)00>$ that corresponds to $\ell = 1$ indicated on the left hand side of the levels with $j = 0$. Also the levels with $N = \ell = j - 1$ are unique as indicated in the corresponding left hand side of the levels. For the other states $|N(j \pm 1, 1)jm>$ both values $\ell = j + 1$ and $j - 1$ are possible, and their separation is given by $\omega^2(2j + 1)$, thus both values $j - 1, j + 1$ appear on the left hand side of those levels.

- Fig. 3. The square of the masses of the mesons are given as functions of j for isospin $I = 1$ and parity $\mathcal{P} = (-1)^j$. The name of the meson is given on the left hand side with an index j and upper primes if there are several of them. Full lines correspond to spin 0 and dashed to spin 1 as indicated on the right hand side.

- Fig. 4. The square of the masses of the mesons are given as function of j for isospin $I = 0$ and parity $\mathcal{P} = (-1)^j$. The name of the meson is given on the left hand side with an index j and upper primes if there are several of them. Full lines correspond to spin 0 and dashed to spin 1 as indicated on the right hand side.

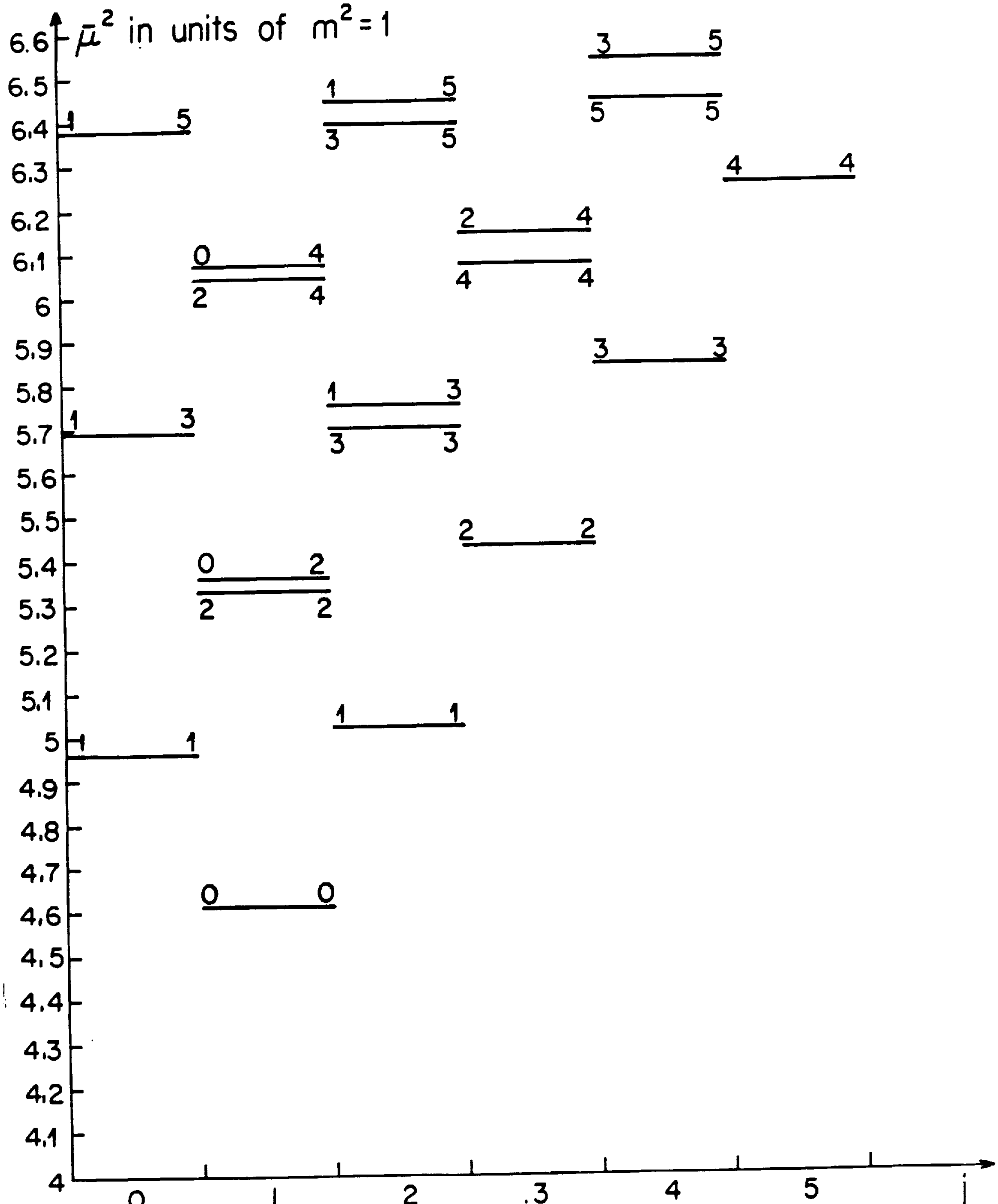
- Fig. 5. The square of the masses of the mesons are given as function of j for isospin $I = 1$ and parity $\mathcal{P} = -(-1)^j$. The name of the meson is given on the left hand side with an index j and upper primes if there are several of them. The spin is always 1.

- Fig. 6. The square of the masses of the mesons are given as function of j for isospin $I = 0$ and parity $\mathcal{P} = -(-1)^j$. The name of the meson is given on the left hand side with an index j and upper primes if there are several of them. The spin is always 1.



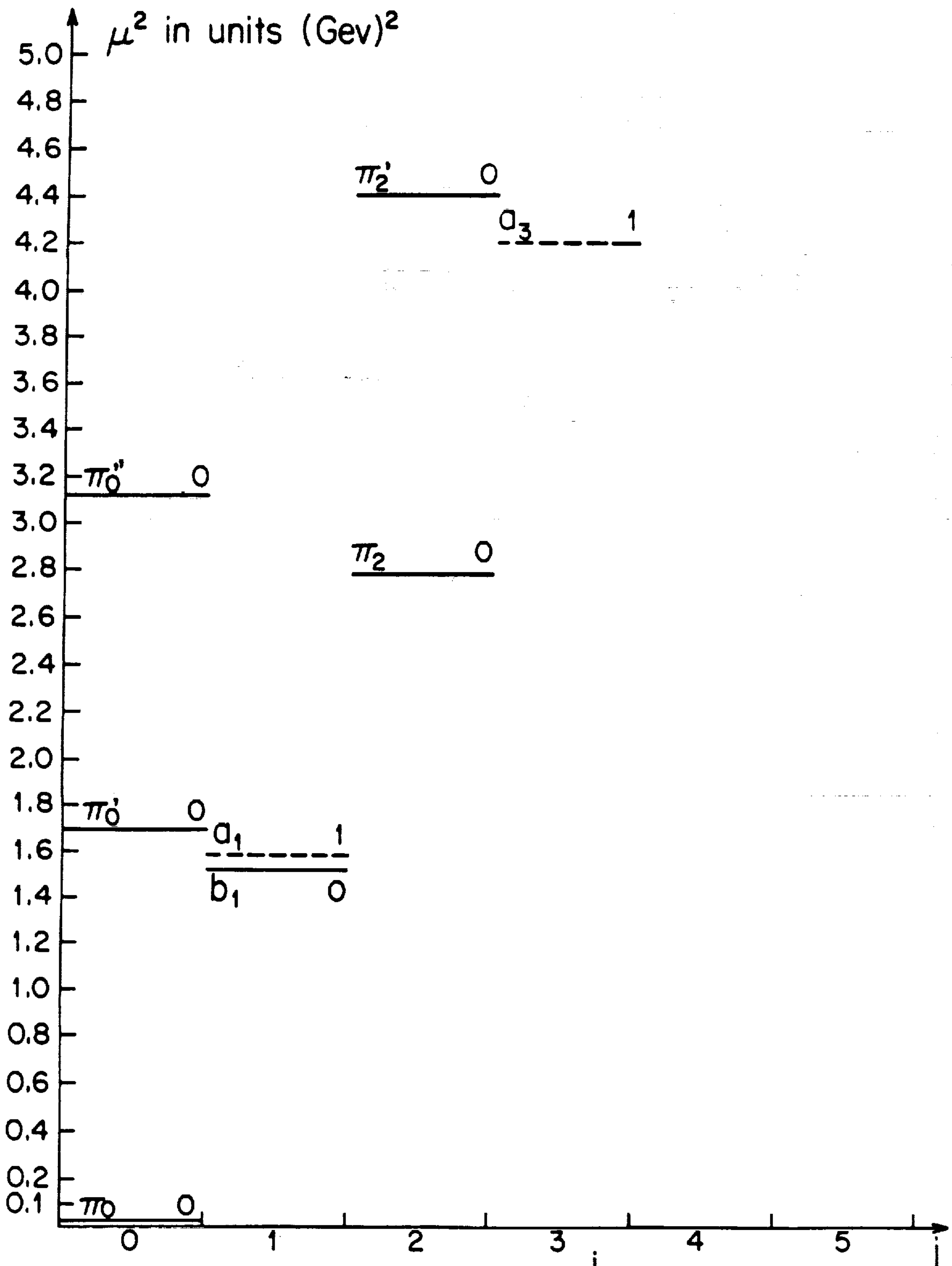
μ^2 in units of $m^2=1$
6.6
6.5
6.4
6.3
6.2
6.1
6
5.9
5.8
5.7
5.6
5.5
5.4
5.3
5.2
5.1
5
4.9
4.8
4.7
4.6
4.5
4.4
4.3
4.2
4.1
4
0
1
2
3
4
5
j
$P = (-1)^j, \omega = 0.1$

Fig. 1



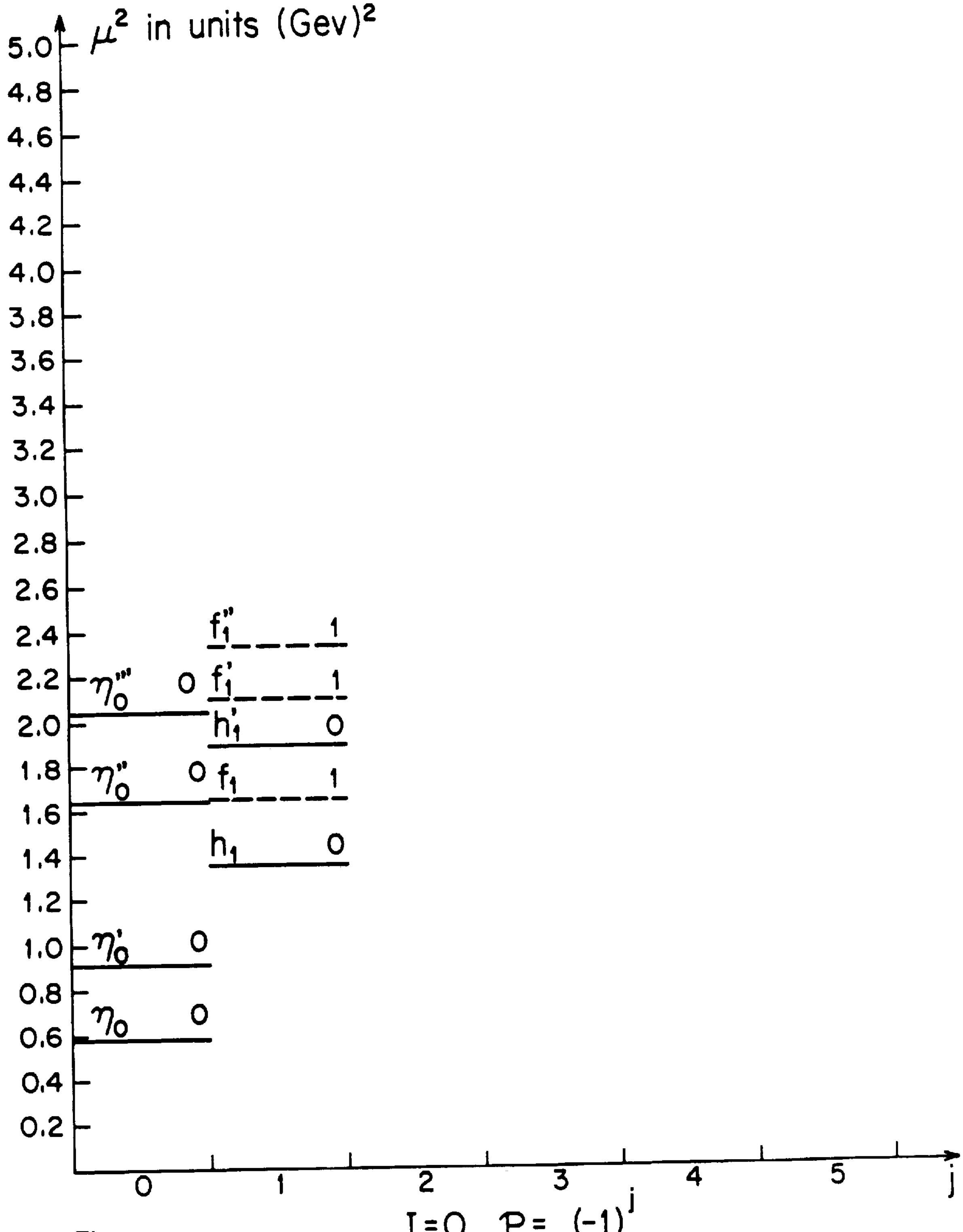
$\bar{\mu}^2$ in units of $m^2 = 1$
6.6
6.5
6.4
6.3
6.2
6.1
6
5.9
5.8
5.7
5.6
5.5
5.4
5.3
5.2
5.1
5
4.9
4.8
4.7
4.6
4.5
4.4
4.3
4.2
4.1
4
0
1
2
3
4
5
j
$P = -(-1)^j, \omega = 0.1$

Fig. 2



μ^2 in units $(\mathrm{Gev})^2$
5.0
4.8
4.6
4.4
4.2
4.0
3.8
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.1
π_2' 0
a_3 1
π_0'' 0
π_2 0
π_0' 0
a_1 1
b_1 0
π_0 0
0
1
2
3
4
5
j

Fig. 3 $I=1,\ P=(-1)^j,\ S$



μ^2 in units $(\text{Gev})^2$
5.0
4.8
4.6
4.4
4.2
4.0
3.8
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
f_1'' 1
η_0''' 0
f_1' 1
h_1' 0
η_0'' 0
f_1 1
h_1 0
η_0' 0
η_0 0
0
1
2
3
4
5
j
$I=0,\ P=(-1)^j$

Fig. 4

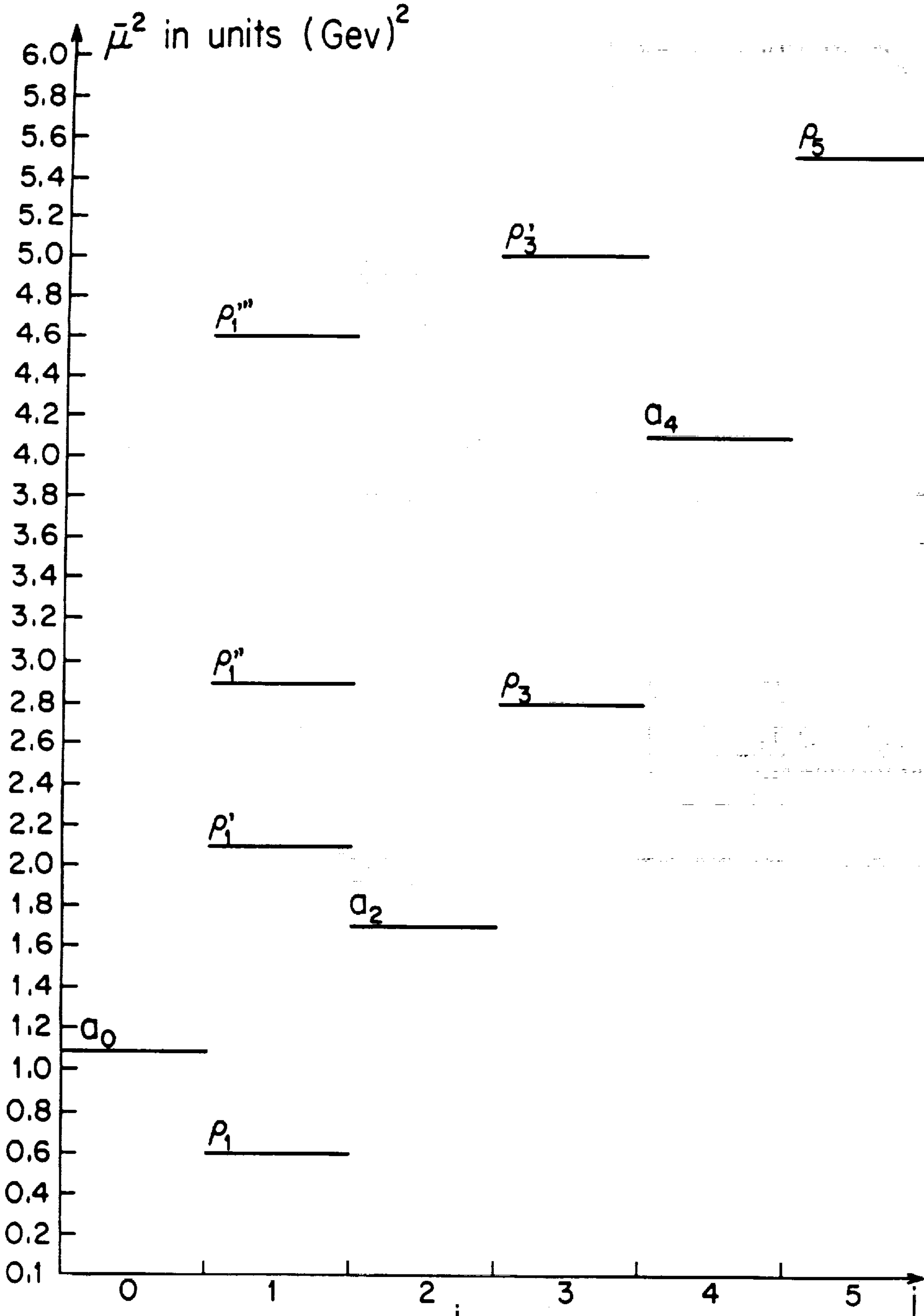


Fig. 5 $\quad I=1,\ \mathcal{P}=-(-1)^j \quad S=1,\ \ell=j\pm1$

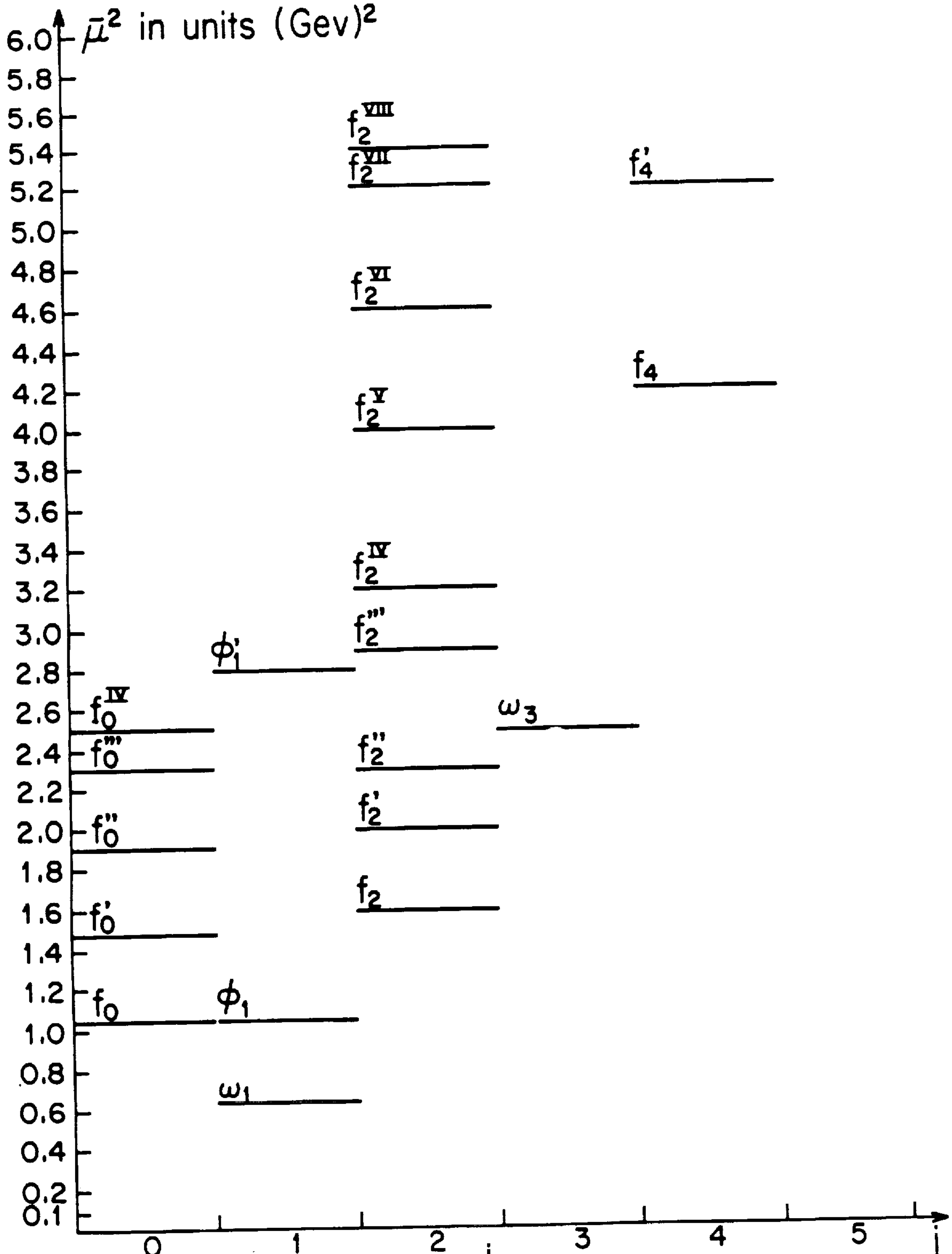


Fig.6 $I=0$, $P=-(-1)^j$ $S=1$, $\ell=j\pm1$