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THE HARMONIC OSCILLATOR AND NUCLEAR PHYSICS

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Abstract

The three-dimensional harmonic oscillator plays a central role in nuclear physics. It provides the underlying structure of the independent-particle shell model and gives rise to the dynamical group structures on which models of nuclear collective motion are based. It will be shown that the three-dimensional harmonic oscillator features a rich variety of coherent states, including vibrations of the monopole, dipole and quadrupole types, and rotations of the rigid flow, vortex flow and irrotational flow types. Nuclear collective states exhibit all of these flows. It will also be shown that the coherent state representations, which have their origins in applications to the dynamical groups of the simple harmonic oscillator, can be extended to vector coherent state representations with a much wider range of applicability. As a result, coherent state theory and vector coherent state theory become powerful tools in the application of algebraic methods in physics.

1 Introduction

Harmonic oscillators are important in many-body physics for many reasons. The following are some of the reasons:

(i) Small amplitude normal modes of a system about a configuration of stable equilibrium are harmonic both in classical and quantum mechanics.

(ii) Harmonic oscillators have non-dispersive coherent states. Thus, they exhibit a perfect correspondence between classical and quantum mechanics.

(iii) The boson quanta of the harmonic oscillator provide important building blocks for the boson second quantization of the observables of a large number of systems; e.g., the Hamiltonian is often usefully expressed in the second quantized form

$$h = \sum \hbar\omega_\nu a^\dagger_\nu a_\nu + \sum \left(V^{\mu\nu\mu'\nu'} a^\dagger_\mu a^\dagger_\nu a^\dagger_{\mu'} a^\dagger_{\nu'} + V^{\mu\nu\mu'}_{\nu'} a^\dagger_\mu a^\dagger_\nu a^\dagger_{\mu'} a_{\nu'} \ldots \right) . \tag{1}$$

(iv) The classical Lie algebras all have boson (Weil) representations; e.g., an abstract element X of GL(n,C) can be realised as a matrix (X_{ij}) or as the boson operator

$$X = \sum_{ij} X_{ij} a^\dagger_i a_j . \tag{2}$$

It is shown in the following that the three-dimensional harmonic oscillator has a rich variety of symmetries and coherent states all of which feature in the theory of nuclear collective motions.

2 Symmetries and coherent states of the simple harmonic oscillator

The symmetries and dynamical group structures of the harmonic oscillator are most easily recognized when the latter is expressed in terms of the Heisenberg-Weyl algebra. The Heisenberg-Weyl algebra is a Lie algebra spanned by harmonic oscillator raising and lowering operators (also called boson operators) and the identity operator; i.e.,

$$\mathrm{hw}(1) = \langle a^\dagger, a, I\rangle . \tag{3}$$

The elements of hw(1) satisfy the boson commutation relations

$$[a, a^\dagger] = I, \quad [a, I] = [a^\dagger, I] = 0 . \tag{4}$$

The simple harmonic oscillator Hamiltonian is given by

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2}) . \tag{5}$$

One finds that the symmetry group of the Hamiltonian H is the one-dimensional unitary group U(1) and that H has three dynamical groups: viz., the Heisenberg-Weyl group HW(1), the symplectic group Sp(1,R) (strictly the metaplectic group) and the inhomogeneous symplectic group ISp(1,R). The Lie algebra, u(1), of U(1) is spanned by a single element

$$\mathrm{u}(1) = \langle a^\dagger a\rangle . \tag{6}$$

The Lie algebras of the symplectic and inhomogeneous symplectic groups are given, respectively, by

$$\begin{aligned} \mathrm{sp}(1,\mathrm{R}) &= \langle a^\dagger a^\dagger, aa, a^\dagger a + aa^\dagger\rangle \\ \mathrm{isp}(1,\mathrm{R}) &= \langle a^\dagger a^\dagger, aa, a^\dagger a + aa^\dagger, a^\dagger, a, I\rangle . \end{aligned} \tag{7}$$

The coherent states of the dynamical groups are all of considerable interest. First recall that a coherent state of a group is, by definition [1, 2] a state obtained by applying a group transformation to a particular state of a (usually irreducible) representation space on which the group acts. The standard (Glauber) coherent states, for example, are obtained by applying elements of the Heisenberg-Weyl group to the harmonic oscillator ground state [3].

An arbitrary element g of the Heisenberg-Weyl group is represented as the operator

$$T(g) = e^{\alpha a^\dagger - \alpha^* a + i\varphi I} , \tag{8}$$

where α and φ are complex and real parameters, respectively, and I is the identity operator. Thus, we have

$$T(g)|0\rangle = e^{\alpha a^\dagger - \alpha^* a + i\varphi I}|0\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle e^{i\varphi} . \tag{9}$$

The state

$$|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle \tag{10}$$

is well known to have a wave function that is of the same form as the ground state wave function but with its centroid displaced from the harmonic oscillator equilibrium position and given some non-zero mean value of momentum. A HW(1) coherent state for a real value of α is illustrated, for example, in Fig. 1. The phase factor $e^{i\varphi}$ can be regarded as a U(1) gauge factor.

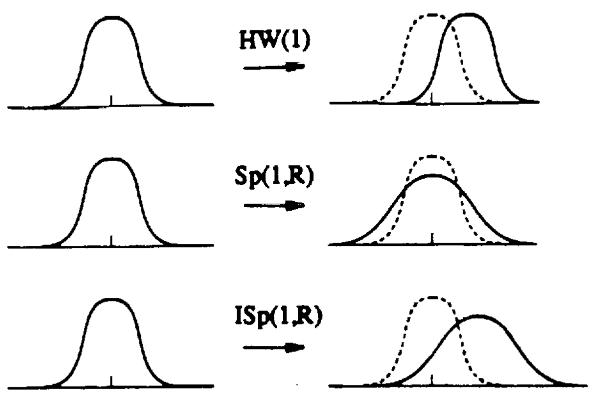

Fig. 1. Harmonic oscillator coherent states induced by HW(1), Sp(1,R) and ISp(1,R) transformations, respectively.

Coherent states of the symplectic group are constructed in a similar way; i.e., a Sp(1,R) group element is represented in factored form

$$T(g) = e^{za^\dagger a^\dagger - z^* aa} e^{i\varphi(a^\dagger a + aa^\dagger)}, \quad g \in \mathrm{Sp}(1, \mathrm{R}) \tag{11}$$

and symplectic coherent states are given by

$$T(g)|0\rangle = e^{za^\dagger a^\dagger - z^* aa}|0\rangle e^{i\varphi}. \tag{12}$$

Again there is a U(1) gauge factor. Coherent states of the symplectic group [4, 5, 6] are often described as *squeezed states.*

There has been much interest in squeezed states in optics in recent years. One of their predicted properties [7] which, as far as I know, has never been investigated is that they should exhibit enhanced non-linear phenomena. This is expected because non-linear properties require the simultaneous presence of two photons of light and one may anticipate, therefore, that the squeezed light emitted, for example, from a two-photon laser should be particularly effective at providing photons in pairs. Squeezed coherent states are also of paramount importance in nuclear physics as we discuss in the following.

3 The three-dimensional harmonic oscillator

The Heisenberg-Weyl algebra

$$\mathrm{hw}(3) = \langle a_i^\dagger, a_i, I; i = 1, 2, 3\rangle \tag{13}$$

satisfies the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij} I, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = [a_i, I] = [a_i^\dagger, I] = 0 \tag{14}$$

and the Hamiltonian of the three-dimensional harmonic oscillator is given by

$$h = \sum_i \hbar\omega_i a_i^\dagger a_i. \tag{15}$$

This Hamiltonian has an energy spectrum as shown in Fig. 2, where the standard spectroscopic notation is used to label the radial and angular momentum quantum numbers of the energy levels.

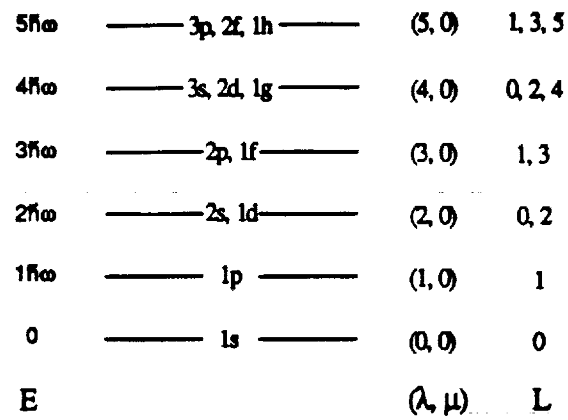

Fig. 2. The energy level spectrum for a single particle in a three-dimensional harmonic oscillator potential. Each energy levels consists of a degenerate multiplet of states which span an irrep of the symmetry group U(3).

The three-dimensional harmonic oscillator underlies the shell-model theory of nuclear physics. One finds that if one were to assume that the neutrons and protons inside a nucleus obeyed an independent-particle Schrödinger equation in which the potential energy is that of a harmonic oscillator then, because of the Pauli exclusion principle, one can assign a given harmonic oscillator wave function to at most two neutrons, one with spin up and one with spin down. The same is true for the protons. Thus, one constructs the ground state of a nucleus by *filling* the harmonic oscillator energy levels starting from the bottom and progressing upwards. Because of the multiplicity of harmonic oscillator states of a given energy, the harmonic oscillator energy of a many-nucleon state increases discontinously as each level is filled and the next level starts to be populated. The nucleon numbers at which this happens are called *magic numbers* and the nuclei at which it happens are called *closed-shell* nuclei. Such nuclei are expected to be particularly stable, like the inert gases of atomic physics. Now, experimentally observed magic numbers differ from those of the harmonic oscillator Hamiltonian. However, if one adjusts the harmonic oscillator Hamiltonian by the addition of an angular momentum term proportional to the square of the orbital angular momentum, to simulate the effects of a more realistic shell model potential, and includes a spin-orbit interaction, then the single-particle levels become of the type shown in Fig. 3 and the experimentally observed magic numbers are reproduced [8].

A typical shell model Hamiltonian is therefore of the form

$$H = \sum_n h_n + V\,, \tag{16}$$

where h_n is a single-particle Hamiltonian for the n'th nucleon of the form

$$h = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 r^2 + C\mathbf{l}\cdot\mathbf{s} + D\mathbf{l}^2 \tag{17}$$

and V is the residual interaction between the nucleons.

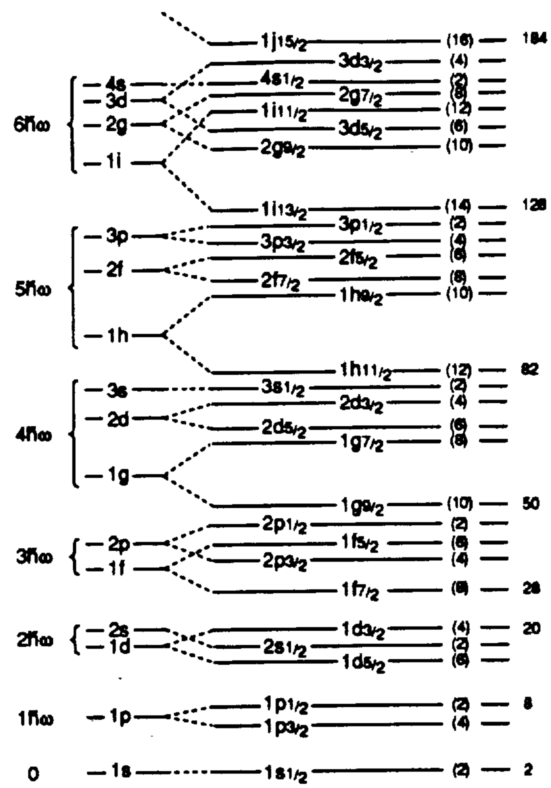

Fig. 3. The single-particle energy level spectrum for a shell model potential with an l^2 term and a spin-orbit interaction. The numbers in parenthesis indicate the multiplicity of states of a given energy level. The numbers to the right of an energy level indicates the cumulative number of states up to that energy. These numbers correspond to the experimentally observed magic numbers.

The degeneracies, and hence the symmetries, of the harmonic oscillator are broken in nuclear physics. Nevertheless, the states of the harmonic oscillator provide a basis in which realistic Hamiltonians can be diagonalized and in which the symmetry breaking effects can be described.

It will be shown in the following that the symmetries and dynamical group structures of the harmonic oscillator Hamiltonian are also of vital importance for identifying and separating the important collective degrees of freedom of a strongly interacting system of nucleons.

The symmetry group of the three-dimensional harmonic oscillator is the unitary group U(3) whose Lie algebra is spanned by the bilinear combinations of boson operators $\{a_i^\dagger a_j; i,j = 1,2,3\}$. The dynamical groups of the three-dimensional harmonic oscillator are the Heisenberg-Weyl group, the symplectic group Sp(3,R) and the inhomogeneous symplectic group ISp(3,R). The Lie algebras of these groups are given by the natural extensions of their one-dimensional counterparts; e.g.,

$$\mathrm{sp}(3,\mathrm{R}) = \langle a_i^\dagger a_j^\dagger, a_i a_j, a_i^\dagger a_j + a_j a_i^\dagger \rangle\,. \tag{18}$$

The coherent states of a single particle in a three-dimensional harmonic oscillator potential will not be discussed here. Instead, we proceed directly to the coherent states of a many-particle nucleus. It will be shown that different kinds of coherent states are generated depending on the symmetries of the initial (undisplaced) state. In particular, the coherent states of open-shell nuclei have a richer structure than those of closed-shell nuclei.

4 Heisenberg-Weyl coherent states of a nucleus

The degenerate (equal energy) states of a three-dimensional harmonic oscillator energy level span an irreducible representation of the symmetry group U(3). The same is true of the SU(3) $\subset$ U(3) subgroup; i.e., the states of the N'th harmonic oscillator level span an SU(3) irrep $(N,0)$ as shown in Fig. 3. It follows that the equal energy states of a nucleus with an independent-particle harmonic oscillator Hamiltonian span the reducible representation of SU(3) given by the Kronecker product of all the $(N,0)$ irreps to which the nucleons separately belong.

4.1 Closed-shell nuclei

The ground state of a closed shell nucleus is characterized by a single closed-shell state which must, therefore, span the one-dimensional identity SU(3) representation $(0,0)$. Nuclei for which such harmonic oscillator closed-shell states are believed to provide a good approximation to their ground states are the light nuclei ^{16}O and ^{40}Ca which, respectively, close the $N=1$ and $N=2$ harmonic oscillator shells.

We now consider coherent states of the Heisenberg-Weyl group obtained by applying a group transformation to a closed-shell state. Note, however, that a straightforward Heisenberg-Weyl transformation simply displaces the centre-of-mass of the whole nucleus or gives the whole nucleus centre-of-mass momentum without exciting it or changing its intrinsic structure in any way. Thus it is not very interesting. However, there is another representation of the Heisenberg-Weyl Lie algebra that is interesting. It is the representation in which the neutrons are displaced in one direction while the protons are displaced in the opposite direction in such a way that the centre-of-mass position and momentum remain fixed. In this representation the boson operators of the Heisenberg-Weyl Lie algebra are the linear combinations of the elementary (harmonic oscillator) boson operators for neutrons and protons

$$a_i^\dagger = \frac{1}{\sqrt{2Z}}\sum_{p=1}^{Z} a_{pi}^\dagger - \frac{1}{\sqrt{2N}}\sum_{n=1}^{N} a_{ni}^\dagger , \tag{19}$$

where n indexes the neutrons and p indexes the protons. These combinations satisfy the commutation relations of Eq. (14) and, therefore, belong to a Heisenberg-Weyl Lie algebra.

The coherent states of this representation of the Heisenberg-Weyl Lie group are of the form

$$T(g)|0\rangle = \exp\left[\sum_i \alpha_i \left(\frac{1}{\sqrt{2Z}}\sum_p a_{pi}^\dagger - \frac{1}{\sqrt{2N}}\sum_n a_{ni}^\dagger\right) - \text{h.c.}\right]|0\rangle e^{i\varphi} . \tag{20}$$

Thus, the group transformation $T(g)$ is seen to displace the ground state distributions of neutrons and protons in opposite directions as illustrated in Fig. 4.

Such a coherent state is of major interest in nuclear physics. It corresponds to a coherent collective motion of the nucleus in which neutrons and protons oscillate in antiphase and thereby generate an oscillating electric dipole moment. It is the so-called *Goldhaber-Teller mode* [9] of the *giant dipole resonance*. The mass associated with this mode is given by the reduced mass of the separate neutron and proton centres of mass. The restoring force for a dipole displacement can be estimated from the nuclear symmetry energy (i.e., the energy associated with a neutron-proton

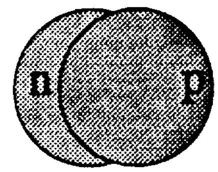

Fig. 4. The displaced neutron and proton density distributions for a coherent state of the Heisenberg-Weyl group in which neutrons are displaced one way while protons are displaced the opposite way.

density difference). Thus, one deduces the frequency and the rate of electric dipole radiation for such a coherent oscillation. The requantization of this mode leads to a relatively high energy (one quantum) excited state which decays rapidly to the ground state by electric dipole radiation.

4.2 Open-shell nuclei

An open-shell nucleus does not have a unique lowest harmonic oscillator energy state; there is a multiplicity of lowest energy states. As already observed, such a multiplet of states spans a generally reducible representation of the harmonic oscillator symmetry group U(3).

Consider, for example, the nucleus ^{20}Ne. The first 16 nucleons form an ^{16}O closed-shell core and combine to give a $(0,0)$ SU(3) (identity) irrep. Thus, the lowest energy states of ^{20}Ne have four nucleons, two neutrons and two protons, in the $N = 2$ shell (the so-called sd shell). They span a reducible SU(3) representation given by the Kronecker product of four copies of the (2,0) representation; i.e.,

$$(2,0) \times (2,0) \times (2,0) \times (2,0) = (8,0) + (4,2) + \ldots \tag{21}$$

The residual interaction of the shell-model Hamiltonian (16) will cause the states constructed in this way to be mixed and the SU(3) symmetry to be broken. However, although the energies of states become non-degenerate, the states retain their SU(3) quantum numbers to a first approximation. One says that SU(3) is an approximate dynamical symmetry for light nuclei.

In the Elliott model [10], it is assumed that states of different SU(3) irreps do not mix and that the energies of states are given by a Hamiltonian which is a sum of SU(3) and SO(3) Casimir invariants

$$H = AC_2 + BL^2\,; \tag{22}$$

L^2 is the square of the angular momentum of the SO(3) subgroup of SU(3). This Hamiltonian has an energy spectrum characteristic of a rotor

$$E = A(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu) + BL(L+1)\,, \tag{23}$$

where (λ, μ) are the labels of an SU(3) irrep. The coefficient A of C_2 is assumed to be negative so that the states of the SU(3) irrep with largest value of the Casimir invariant lie lowest in energy. The parameter B is then adjusted to give the lowest band of energy levels for ^{20}Ne as shown in comparison to the experimentally observed energy spectrum in Fig. 5. The agreement is far from

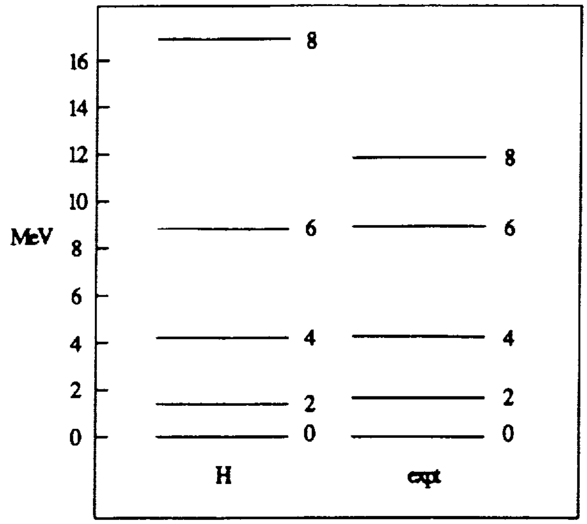

Fig. 5. The low energy spectrum for ^{20}Ne as calculated in the Elliott model and as observed experimentally.

perfect but it does describe the rotational properties of ^{20}Ne to a first approximation. Moreover, from the SU(3) quantum numbers, one can infer the shape of the deformed ^{20}Ne nucleus which gives rise to the observed rotational band.

We now consider the coherent states corresponding to the giant dipole vibrations of such a deformed nucleus. Since there are many degenerate harmonic oscillator ground states for an open-shell nucleus, it is appropriate to consider coherent states of the semi-direct product group comprising both the Heisenberg-Weyl group and the SU(3) group. For this group, there is a uniquely defined lowest weight state for each irrep. We can then construct simultaneous coherent states of the combined groups in which we have both rotations and dipole vibrations of the type illustrated in Fig. 6. As one can see, the energy level spectrum of such coupled rotations and vibrations can be relatively complex. Nevertheless, it is simply described in terms of the SU(3) and Heisenberg-Weyl dynamical groups.

5 Symplectic coherent states in nuclear physics

A many-particle representation of an element of the symplectic group Sp(3,R) can be expressed in the form

$$T(g) = \exp\Big[\sum_{nij} z_{ij} a^\dagger_{ni} a^\dagger_{nj} - \text{h.c.}\Big] \exp\Big[\mathrm{i}\sum_{nij} \alpha_{ij}\left(a^\dagger_{ni} a_{nj} + a_{nj} a^\dagger_{ni}\right)\Big]. \qquad (24)$$

5.1 Closed-shell nuclei

When acting on a closed-shell state $|0\rangle$, such an operator generates the coherent state

$$T(g)|0\rangle = \exp\Big[\sum_{nij} z_{ij} a^\dagger_{ni} a^\dagger_{nj} - \text{h.c.}\Big]|0\rangle e^{\mathrm{i}\varphi}, \qquad (25)$$

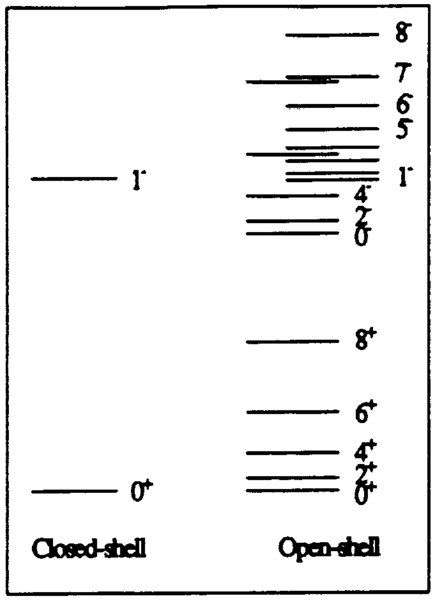

Fig. 6. The low energy spectrum rotational spectrum and the combined rotation-giant resonance vibration of a nucleus like ^{20}Ne.

where φ is given by

$$\sum_{nij} \alpha_{ij} \left(a^\dagger_{ni} a_{nj} + a_{nj} a^\dagger_{ni} \right) |0\rangle = \varphi |0\rangle \, . \tag{26}$$

Thus, as for the one-dimensional case, there is a gauge angle φ associated with such coherent states. However, the phase factor $\exp(i\varphi)$ now comes from a one-dimensional representation of the symmetry group U(3) of the three-dimensional harmonic oscillator.

The coherent states of the symplectic group Sp(3,R) constructed in this way are natural extensions of the squeezed states of the one-dimensional harmonic oscillator. The difference is that there are now three different directions and the squeezing or dilation in the three directions need not all be the same. Fig. 7(a) shows a coherent state deformation of the ground state density distribution of a nucleus in which the squeezing/dilation is the same in all directions. Such a coherent state deformation preserves the spherical symmetry of the density distribution of the (closed-shell) nucleus and is described as a monopole or breathing mode vibration. Fig. 7(b) shows a coherent state deformation in which the nucleus is squeezed in one direction and dilated in another. The result is an ellipsoidal (i.e., quadrupole) deformation.

5.2 Open-shell nuclei

For an open-shell nucleus, we may construct symplectic coherent states by applying symplectic group transformations to the lowest weight state of an Sp(3,R) irrep. It can be shown that an Sp(3,R) lowest weight state is also an SU(3) lowest weight state for the SU(3) $\subset$ Sp(3,R) subgroup discussed in the previous section and, like it, has a non-spherical density distribution. Application of a symplectic transformation to such a state can effect the changes shown in Fig. 8. It can cause

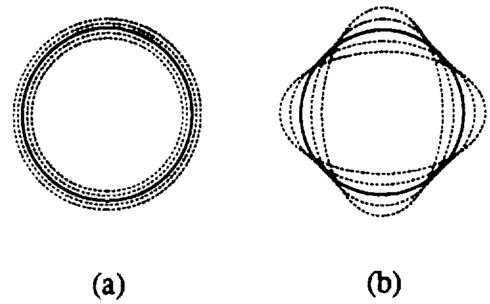

Fig. 7. Coherent state deformations of a spherical closed-shell density distribution: (a) shows a monopole (breathing mode) deformation and (b) shows a quadrupole deformation.

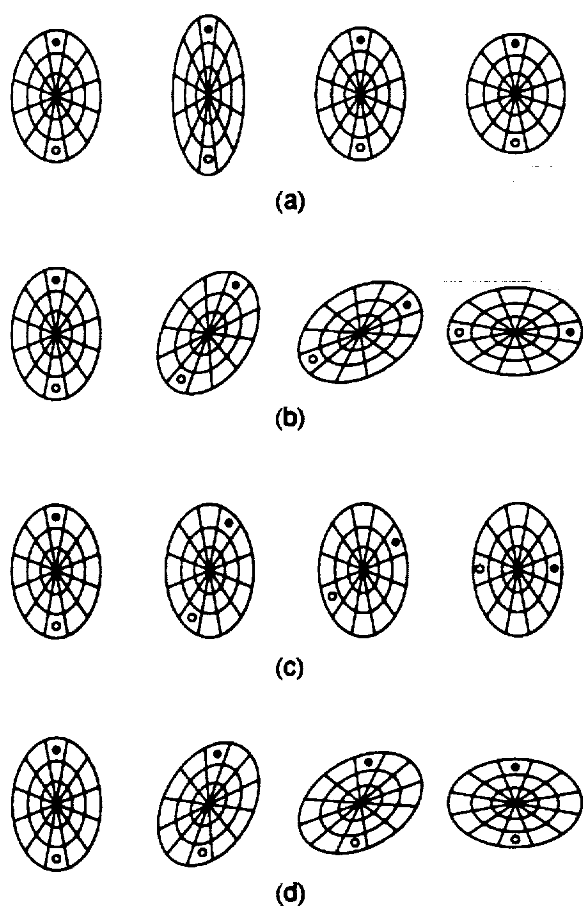

Fig. 8. Coherent states of a deformed (open-shell) nucleus: (a) shows a vibration; (b) a rigid rotation; (c) a vortex rotation; and (d) an irrotational-flow rotation.

squeezing and dilation of the nucleus thereby changing the deformation of the density distribution as illustrated in Fig. 8(a). Since the symplectic group Sp(3,R) contains the group SO(3) as a subgroup, symplectic group transformations include pure rotations, i.e., rigid body rotations of the type illustrated in Fig. 8(b). Figs. 8(c) and (d) show other kinds of rotations that are possible for a fluid but would not be possible for a rigid body. Fig. 8(c) shows a flow, called *vortex rotation,* in which the fluid circulates but leaves the quadrupole shape of the deformed nucleus fixed. Fig. 8(d) is a combination of rigid and vortex flow, called *irrotational flow,* in which the shape rotates but elements of the fluid move as little as possible consistent with the rotating shape.

It is of interest to note that a pure vortex rotation is not observable if one looks only at the density distribution of the nucleus. Thus, the vortex degree of freedom is naturally regarded as a non-abelian gauge degree of freedom.

To observe vortex flows, one needs a probe of the nucleus that is sensitive to currents and not just densities. Electron scattering is a natural choice. One is acustomed to think of an electron microscope as giving snapshots of the density distribution of a microscopic object. However, electrons are also sensitive to charge currents. Thus, electron scattering experiments have the potential for probing the contributions of vortex flows in nuclear (and perhaps other) collective motions. Some experimental beginnings have already been made in this direction. One thing is clear. The moments of inertia needed to describe the rotational energy spectra of nuclei are smaller, by approximately a factor of two, than those for rigid-body rotations and larger, by approximately a factor of five, than for irrotational flow. This suggests that nuclear rotational flows have a vortex component similar to that of a slightly viscous fluid.

We next consider the construction of a model of nuclear vibrations and rotations that admits rotational flows with arbitrary amounts of vorticity, ranging from none, for a viscous-free fluid, up to that of a rigid body. The fact that the coherent states of the symplectic group span the full range of possibilities, suggests that the symplectic group is a suitable dynamical group for such a model.

6 The nuclear symplectic model

The nuclear symplectic model [11, 12, 13] is based on the observation that the symplectic group Sp(3,R) is a suitable dynamical group for a microscopic theory of nuclear collective rotations, vibrations and vortex rotations. The important feature of a dynamical group is that a Hamiltonian which can be expressed as a polynomial in the Lie algebra of the dynamical group cannot mix states belonging to different irreps of the group.

A suitable Hamiltonian for the symplectic model is of the form given by Eq. (16) with the parameters C and D of Eq. (17) set equal to zero and

$$V = c_2 \mathrm{Tr} Q^2 + c_3 \mathrm{Tr} Q^3 + \dots ; \tag{27}$$

Q is the Cartesian quadrupole tensor with components given in terms of the nucleon coordinates $\{x_{ni}\}$ by

$$Q_{ij} = \sum_n x_{ni} x_{nj} . \tag{28}$$

Note that by taking traces of powers of Q, we construct rotationally invariant polynomials. Note also that the trace of Q itself is the sum of the squared nuclear radii

$$\mathrm{Tr}Q = \sum_{ni} x_{ni}^2 = \sum_n r_n^2 . \tag{29}$$

Thus, the harmonic oscillator Hamiltonian

$$\sum_n h_n = \sum_n \left[\frac{1}{2m} p_n^2 + \frac{1}{2} m\omega^2 r_n^2\right] \tag{30}$$

already contains a term in $\mathrm{Tr}Q$.

By a suitable choice of the parameters, $c_2, c_3, \ldots$, one can construct physically relevant potential energy functions for a deformed (rotational) nucleus. An important point is that polynomials in the traces of powers of Q are functions of three observable quantities which characterize the shape of the nucleus: the three quantities can be denoted as α, which measures the mean square radius of the nucleus, β which, measure the magnitude of the quadrupole moment of a nucleus, and γ, which measures the magnitude of the axial asymmetry of the deformation. Thus, for a spherical nucleus, β and γ would be zero. For a spheroidal (axially symmetric) nucleus, γ would vanish but β would be non-zero. But, for a generic ellipsoidal nuclear shape, both β and γ would be non-zero. It can be shown that

$$\begin{aligned} &\mathrm{Tr}Q \propto \alpha \\ &\mathrm{Tr}Q^2 \propto \beta^2 \\ &\mathrm{Tr}Q^3 \propto \beta^3 \cos 3\gamma . \end{aligned} \tag{31}$$

A simple two-parameter potential of this kind, with only c_2 and c_3 non-zero, is shown in Fig. 9. The potential shown has a minimum at a non-zero, but axially symmetric, deformation.

A calculation within the shell model space for an Sp(3,R) irrep was carried out for the low energy states of each of four heavy nuclei by Park *et al.* [14]. Their results for the energy levels and electric quadrupole (E2) radiative transition rates is shown in Fig. 10 in comparison with experimentally measured results. The agreement with experiment is not perfect but it is remarkably good considering that there is very little flexibility in the choice of the two parameters of the potential. The minimum of the potential is fixed at the known experimental deformation of the nucleus and the strength of the potential is fixed such that the potential is just strong enough to ensure that the wave function has the same deformation as the potential minimum. The most remarkable feature of the results is that one gets the correct moment of inertia (i.e., the energy level spacing comes out correctly) even though there is no adjustable momentum of inertia in the Hamiltonian; the kinetic energy of the Hamiltonian is the known microscopic kinetic energy for a system of nucleons. This is a major success of the model because it suggests that the amount of vorticity predicted by the symplectic model is just about right.

7 Coherent state and vector coherent state representations

In the application of algebraic models in physics, like the symplectic model, it is necessary to construct a basis for an appropriate irrep of the dynamical algebra and calculate the matrix

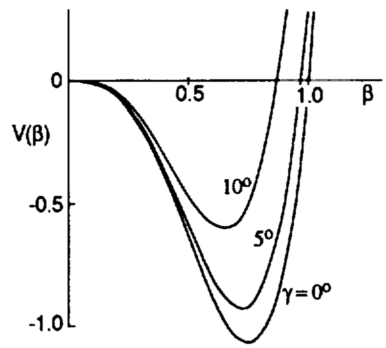

Fig. 9. A two-parameter potential for the symplectic model.

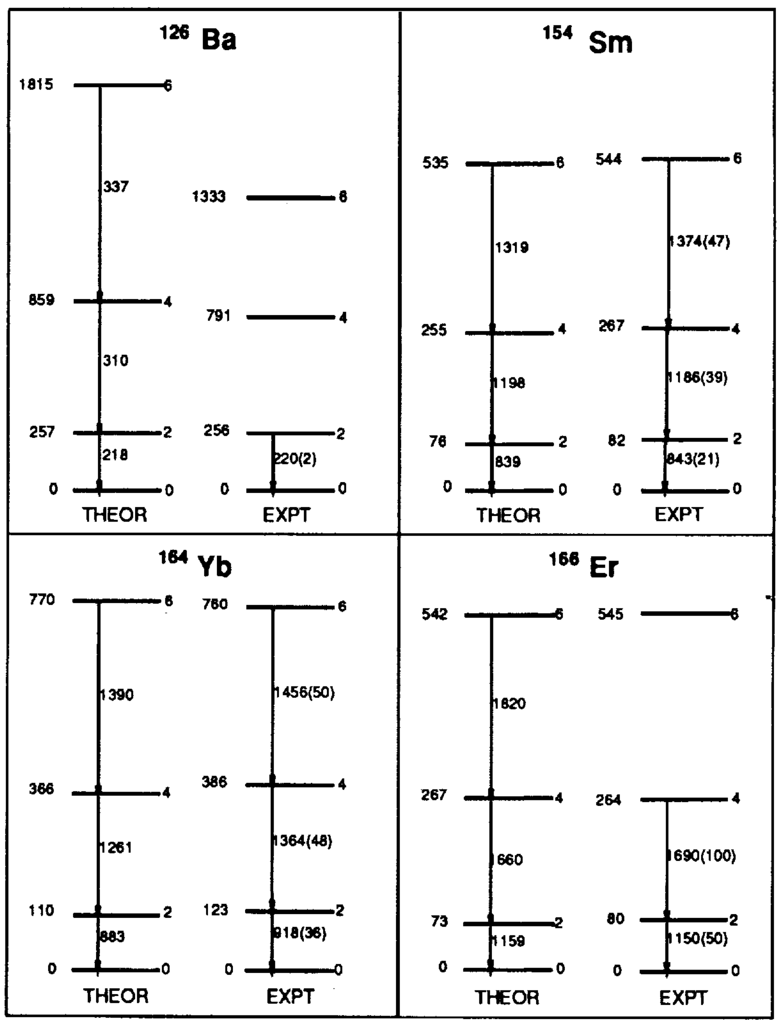

Fig. 10. The low energy spectrum and E2 transition rates between states as calculated by Park *et al.* [14] in the symplectic model and as determined experimentally.

elements of observable quantities. I do not wish to go into the technology of this subject here. However, I do wish to point out that the solution to the problem is given by a straightforward application of coherent-state and more genereally, vector-coherent-state representation theory.

7.1 Coherent state representations

A coherent state representation of a group G can be defined, following Perelomov [1] and Onofri [2] as follows.

Let T be the desired representation of the group G and suppose it is carried by a module (carrier space) V. Let $|0\rangle$ be a particular reference state in the space V; usually we choose V to be the lowest (or highest) weight state if such exists but it can also be chosen in other ways. Then, an arbitrary state $|\Psi\rangle \in V$ can be represented as a function ψ over the group, with ψ defined by

$$\psi(g) = \langle 0|T(g)|\Psi\rangle, \quad g \in G. \tag{32}$$

The remarkable fact is that without actually knowing the representation T, i.e., knowing only some properties of the special state $|0\rangle$, one can determine the kinds of coherent state functions that can occur. Furthermore, one is able to construct an explicit *coherent state* realization of the desired representation.

The standard example is the Bargmann representation of the Heisenberg-Weyl group. If the state $|0\rangle$ is the lowest weight state of the Heisenberg-Weyl Lie algebra, i.e., it satisfies

$$a|0\rangle = 0, \quad I|0\rangle = |0\rangle\,, \tag{33}$$

then

$$\psi(g) = \langle 0|e^{za-z^*a^\dagger+i\varphi I}|\Psi\rangle = \langle 0|e^{za}|\Psi\rangle e^{-\frac{1}{2}|z|^2}e^{i\varphi}\,. \tag{34}$$

Dropping the factor $e^{-\frac{1}{2}|z|^2}e^{i\varphi}$, one obtains the familiar Bargmann representation [15] of a harmonic oscillator state $|\Psi\rangle$ by the holomorhpic function ϕ of a complex variable z with

$$\phi(z) = \langle 0|e^{za}|\Psi\rangle. \tag{35}$$

For example, a harmonic oscillator state

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{36}$$

is represented by the function

$$\phi_n(z) = \frac{1}{\sqrt{n!}}z^n\,. \tag{37}$$

In the Bargmann representation, the harmonic oscillator raising and lowering operators are simply the differential operators

$$a^\dagger = z, \quad a = \frac{\partial}{\partial z}, \quad I = 1\,. \tag{38}$$

Coherent state representations of the symplectic groups Sp(1,R) and Sp(3,R) can be constructed in a similar way. Consider, for example, the representation of Sp(3,R) with lowest weight state given by a closed-shell state; i.e., a state which satisfies

$$\sum_{nij} z_{ij} a_{ni} a_{nj} |0\rangle = 0$$

$$\sum_{nij} \alpha_{ij} \left(a^\dagger_{ni} a_{nj} + a_{nj} a^\dagger_{ni} \right) |0\rangle = \varphi |0\rangle . \tag{39}$$

A coherent state representation of a state $|\Psi\rangle$ is then given by a function ψ over Sp(3,R) with

$$\psi(g) == \langle 0 | \exp \sum_{nij} \left[z_{ij} a_{ni} a_{nj} - z^*_{ij} a^\dagger_{ni} a^\dagger_{nj} \right] |\Psi\rangle e^{i\varphi} , \tag{40}$$

which is proportional to the holomorphic function of six complex variables

$$\phi(z) = \langle 0 | \exp \sum_{nij} z_{ij} a_{ni} a_{nj} |\Psi\rangle . \tag{41}$$

The expression of symplectic operators in this representation is simple and enables one to calculate their matrix elements in analytic form.

Such a construction of coherent state representations is an explicit realization of the Borel-Weil theory (see, for example, ref. [16]) of the representations of semi-simple Lie groups.

7.2 Vector coherent state representations

The direct application of the above construction of a general representation, i.e., one whose lowest weight state does not span a trivial one-dimensional representation of the SU(3) ⊂ Sp(3,R) subgroup, is much more complicated and, therefore, not so useful. However, it is possible to construct a so-called *vector coherent state* representation which is simple.

First observe, from eq. (39), that the gauge factor $e^{i\varphi}$ is a representation of a U(3) transformation; i.e.,

$$\exp \sum_{nij} \alpha_{ij} \left(a^\dagger_{ni} a_{nj} + a_{nj} a^\dagger_{ni} \right) \to e^{i\varphi} . \tag{42}$$

Now, the lowest weight state of a generic Sp(3,R) irrep does not by itself span an irrep of the U(3) subgroup of Sp(3,R). However, it is one state of a multidimensional irrep. This suggests that more general Sp(3,R) irreps can be constructed in which the one-dimensional U(3) irrep of Eq. (42) is replaced by a general multidimensional U(3) irrep. This is correct and one finds that a state Ψ of any discrete series representation of Sp(3,R) can be realized as a holomorphic vector-valued wave function ψ with

$$\psi(z) = \sum_{\nu} |\nu\rangle \langle \nu | \exp \sum_{nij} z_{ij} a_{ni} a_{nj} |\Psi\rangle , \tag{43}$$

where $\{|\nu\rangle\}$ is a basis for a lowest weight irrep of the subgroup U(3) ⊂ Sp(3,R).

The calculation of matrix elements of the sp(3,R) Lie algebra in such a representation is a simple task. When there are no missing quantum numbers, one obtains analytic expressions for the matrix elements. When there are missing quantum numbers, which is the generic situation, one has to do relatively small numerical calculations to construct orthonormal basis states.

The vector coherent state techniques apply to all the semi-simple Lie groups. They are, in fact, an explicit realization of the Harish-Chandra theory [17] of induced holomorphic representations.

8 Concluding remarks

I hope to have shown that the harmonic oscillator in three dimensions has a rich structure and that its many-particle representations and coherent states provide the framework for both independent-particle and collective models of nuclear states. Moreover, the coherent state and vector coherent state representations, which originated in applications of the dynamical groups of the harmonic oscillator, have much wider applicability and are now essential tools in the hands of those who use algebraic methods in physics.

References

[1] A. Perelomov, Commun. Math. Phys. **26**, 222 (1972).

[2] E. Onofri, J. Math. Phys. **16**, 1087 (1975).

[3] R. Y. Glauber, Phys. Rev. **131**, 2766 (1963).

[4] D. Stoler, Phys. Rev. **D1**, 3217 (1970).

[5] A. O. Barut and L. Girardello, Commun. Math. Phys. **21**, 41 (1971).

[6] H. P. Yuen, Phys. Rev. **A13**, 2226 (1976).

[7] D. J. Rowe, Can. J. Phys. **56**, 442 (1978).

[8] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure,* (Wiley, N.Y., 1955).

[9] M. Goldhaber and E. Teller, Phys. Rev. **74**, 1046 (1948).

[10] J. P. Elliott, Proc. Roy. Soc. **A245**, 128, 562 (1958).

[11] G. Rosensteel and D. J. Rowe, Phys. Rev. Lett. **38**, 10 (1977).

[12] G. Rosensteel and D. J. Rowe, Ann. Phys., N.Y. **126**, 343 (1980).

[13] D. J. Rowe, Rep. Prog. Phys. **48**, 1419 (1985).

[14] P. Park, J. Carvalho, M. Vassanji, D. J. Rowe and G. Rosensteel, Nucl. Phys. **A414**, 93 (1984).

[15] V. Bargmann, Commun. Math. Phys. **14**, 187 (1961).

[16] A. W. Knapp, *Representation Theory of Semi-simple Groups,* (Princeton Univ. Press, 1986).

[17] Harish-Chandra, Am. J. Math. **78**, 1, 564 (1956).