ON THE SPRING AND MASS OF THE DIRAC OSCILLATOR

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Abstract

The Dirac oscillator is a relativistic generalization of the quantum harmonic oscillator. In particular, the square of the hamiltonian for the Dirac oscillator yields the Klein-Gordon equation with a potential of the form: $(ar^2 + b\mathbf{L} \cdot \mathbf{S})$, where a and b are constants. To obtain the Dirac oscillator, a "minimal substitution" is made in the Dirac equation, where the ordinary derivative is replaced with a covariant derivative. However, an unusual feature of the covariant derivative in this case is that the potential is a non-trivial element of the Clifford algebra. A theory which naturally gives rise to gauge potentials which are non-trivial elements of the Clifford algebra is that based on local automorphism invariance. I present an exact solution of the automorphism gauge field equations which reproduces both the potential term and the mass term of the Dirac oscillator.

1 Introduction

The Dirac oscillator exhibits many interesting features. It is the relativistic generalization of the classic non-relativistic harmonic oscillator Schrödinger equation to the Dirac equation in the sense that the square of the Dirac hamiltonian yields the relativistic Klein-Gordon equation for the spinor fields with a potential of the form: $(ar^2 + b\mathbf{L} \cdot \mathbf{S})$ where a and b are constants [1,2,3]. The equation is exactly solvable as in the non-relativistic case [4], and exhibits a hidden supersymmetry [4,5]. In addition, this particular form of potential has been used to model the inter-quark interactions in the hope of obtaining a realistic model of the hadrons [6,7]. Finally, an interesting version involving a "scalar" coupling has been investigated [8].

A highly unusual feature of the Dirac oscillator is that the potential which is introduced as a "minimal substitution" is a non-trivial element of the Clifford algebra. This is to be contrasted with all "usual" gauge theories where the potentials are Clifford scalars (that is, the potentials multiply the unit element of the algebra). A theory which naturally incorporates gauge potentials which are general elements of the Clifford algebra is that based upon local automorphism invariance [9,10]. The basic idea behind automorphism gauge theory is the observation that the particular matrix representation chosen for the Clifford algebra generators should not effect the physical predictions of the theory. If we then demand that this freedom of choice be allowed locally we obtain automorphism gauge theory.

In this paper I present a set of exact "chiral" solutions of the automorphism gauge field equations which reproduces both the potential term and the mass term of the Dirac oscillator as a special case. Additional details and further discussion of these topics may be found in reference [11], upon which this paper is based.

2 The Dirac Oscillator

The connection between the Dirac oscillator and the automorphism gauge theory is most easily seen by considering the "minimal substitution" that is made to obtain the Dirac oscillator [3,4]:

$$\mathbf{p} \rightarrow \mathbf{p} - im\omega\beta\mathbf{r} \tag{1}$$

This "minimal substitution" has the interesting property of being dependent upon the Clifford algebra generators (β is the Dirac matrix), and suggests that the theory can be derived from the automorphism gauge theory, since in this case the gauge potentials naturally occur as general elements of the Clifford algebra

To obtain the covariant form of the Dirac oscillator equation we introduce a unit timelike fourvector u_μ and an antisymmetric tensor $r_{\mu\nu}$ formed from the timelike unit vector and the spacetime coordinate vector:

$$u^\mu u_\mu = 1 \quad , \quad r_{\mu\nu} \equiv (u_\mu x_\nu - u_\nu x_\mu) \tag{2}$$

In the "rest frame" these take the form:

$$u_\mu = (1,0,0,0) \quad , \quad r_{0i} = x_i \quad , \quad r_{ij} = 0 \tag{3}$$

Now the covariant Dirac equation may be written as:

$$\left(\gamma^\mu p_\mu - m + \tfrac{1}{2}m\omega r_{\mu\nu}\gamma^{\mu\nu}\right)\Psi = 0 \tag{4}$$

where the matrices $\gamma_{\mu\nu}$ are the bivector elements of the Clifford algebra basis [12]. This equation has an electromagnetic interpretation as a particle with **zero charge** interacting via a **magnetic dipole moment** with a **radial electric field**. In this case the vector u_μ may be considered the four-velocity of the center of the electric field. Note that the electromagnetic interpretation is valid as long as we take equation (4) as our starting point. However, if we wish to view this equation as arising from a minimal substitution of a covariant derivative for an ordinary derivative, then the electromagnetic interpretation is untenable.

3 Local Automorphism Invariance

I now approach the problem from the point of view of local automorphism invariance [9,10]. Although the theory may be developed in spaces of arbitrary dimension and signature, we will restrict our attention to the case of four-dimensional spacetime. If we assume that the particular matrix representation of the Clifford algebra generators may be chosen arbitrarily **at each point in space**, then we obtain a gauge theory based on the automorphism group U(2,2). To incorporate this local invariance into the theory, the ordinary derivative must be replaced with the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu + igA_\mu \tag{5}$$

where the gauge potential is given by [9,12]:

$$A_\mu = a_\mu 1 + a^\rho{}_\mu \gamma_\rho + \tfrac{1}{2}a^{\rho\sigma}{}_\mu \gamma_{\rho\sigma} - b^\rho{}_\mu \tilde{\gamma}_\rho - b_\mu \tilde{\gamma} \tag{6}$$

and for the field strength tensor we find:

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \\
&= f_{\mu\nu}1 + f^\rho{}_{\mu\nu}\gamma_\rho + \tfrac{1}{2}f^{\rho\sigma}{}_{\mu\nu}\gamma_{\rho\sigma} - h^\rho{}_{\mu\nu}\tilde{\gamma}_\rho - h_{\mu\nu}\tilde{\gamma}
\end{aligned} \tag{7}$$

Making the minimal substitution into the Dirac lagrangian we find:

$$\begin{aligned}
L_\Psi &= \tfrac{i}{2}\overline{\Psi}\left(\gamma^\mu \overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu}\gamma^\mu\right)\Psi \\
&\quad - g\overline{\Psi}\left(\Phi 1 + a^\mu\gamma_\mu + 3\tilde{a}^\mu\tilde{\gamma}_\mu - \tilde{b}^{\mu\nu}\gamma_{\mu\nu}\right)\Psi
\end{aligned} \tag{8}$$

where we have made the definitions:

$$\Phi \equiv a^\rho{}_\rho \quad , \quad \tilde{a}_\mu \equiv \tfrac{1}{3!}\varepsilon_{\mu\nu\rho\sigma}a^{\nu\rho\sigma} \quad , \quad \tilde{b}_{\mu\nu} \equiv \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}b^{\rho\sigma} \tag{9}$$

and we see that the automorphism gauge fields couple to the fermion field through scalar, vector, pseudovector, and bivector (spin) interactions. Notice that we have not included the mass term in the basic lagrangian since the scalar coupling (Yukawa interaction) will give rise to mass. The explicit form for the field strength tensor in terms of the gauge potentials and a more detailed derivation of the Dirac part of the lagrangian can be found in my notes on local automorphism invariance [9].

The equations for the gauge fields in the absence of sources is found in the usual manner by demanding stationary action with respect to arbitrary variations of the fields. We find:

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + ig[A_\mu, F^{\mu\nu}] = 0 \tag{10}$$

Notice that we have not included the fermion source term in this equation. This is in concordance with the original approach to the Dirac oscillator in that the potential is introduced as an **external field**. To be entirely consistent, however, we must demand that the gauge potentials satisfy equation (10).

We now make the observation that this interacting lagrangian density will yield the equation for the Dirac oscillator (equation (4)) if the following conditions are met:

$$g\Phi = m \quad , \quad g\tilde{b}_{\mu\nu} = \frac{1}{2}m\omega r_{\mu\nu} \tag{11a}$$

$$a_{\mu} = 0 \quad , \quad \tilde{a}_{\mu} = 0 \tag{11b}$$

It is remarkable that these particular expressions for the potentials form a subset of exact "chiral" solutions to the pure gauge field equations.

4 Chiral Solutions

We now consider a special subset of solutions to equation (13). Consider the "chiral ansatz" defined as:

$$a_{\mu} = 0 \quad , \quad a^{\rho\sigma}{}_{\mu} = 0 \quad , \quad b_{\mu} = 0$$
$$b^{\rho}{}_{\mu} = \pm a^{\rho}{}_{\mu} \tag{12}$$

and the field equations become:

$$f^{\kappa}{}_{\mu\nu} = \partial_{\mu}a^{\kappa}{}_{\nu} - \partial_{\nu}a^{\kappa}{}_{\mu} = h^{\kappa}{}_{\mu\nu} \quad , \quad \partial_{\mu}f_{\kappa}{}^{\mu\nu} = 0 \tag{13a,b}$$

There are, of course, many solutions to equations (13), but consider the special case in which the field strength tensor is constant and uniform. In this case equation (13b) is clearly satisfied. To satisfy equation (13a) an obvious choice is to assume that the potential is linear in the spacetime coordinate. Therefore we write:

$$a^{\kappa}{}_{\mu} = c_1 m(1 + d_1 m(u \cdot x))g^{\kappa}{}_{\mu} + c_2 m(1 + d_2 m(u \cdot x))u^{\kappa}u_{\mu}$$
$$+ c_3 m^2 r^{\kappa}{}_{\mu} + c_4 m^2 \tilde{r}^{\kappa}{}_{\mu} + c_5 m^2 s^{\kappa}{}_{\mu} \tag{14}$$

where the coefficients c_i and d_i are arbitrary dimensionless constants, and we have defined:

$$\tilde{r}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\kappa\tau}r^{\kappa\tau} \quad , \quad s_{\mu\nu} \equiv (u_{\mu}x_{\nu} + u_{\nu}x_{\mu}) \tag{15}$$

The parameter m is a quantity with the dimension of mass and it may be conveniently chosen to be the mass appearing in the Dirac oscillator. Finally note that equation (14) is the most general form which is both linear in the spacetime coordinate and which involves only one arbitrary constant vector.

The field strength tensor may be calculated directly from equation (13a) and we find:

$$f^{\kappa}{}_{\mu\nu} = m^2(c_1 d_1 + c_3 - c_5)(u_{\mu}g^{\kappa}{}_{\nu} - u_{\nu}g^{\kappa}{}_{\mu}) + 2m^2 c_4 \varepsilon^{\kappa}{}_{\mu\nu\tau}u^{\tau} \tag{16}$$

As expected, the field strength tensor is constant and uniform, and therefore trivially satisfies equation (13b). For the part of the gauge potential which interacts directly with the fermion (see equations (8) and (9)) we find:

$$\Phi = (4c_1 + c_2)m + (4c_1 d_1 + c_2 d_2 + 2c_5)m^2(u \cdot x) \tag{17a}$$

$$\tilde{b}_{\mu\nu} = -c_3 m^2 \tilde{r}_{\mu\nu} + c_4 m^2 r_{\mu\nu} \tag{17b}$$

$$a_\mu = 0 \quad , \quad \tilde{a}_\mu = 0 \tag{17c}$$

Notice that only the symmetric part of the gauge potential contributes to the scalar interaction (equation (17a)), and only the antisymmetric part of the gauge potential contributes to the spin interaction (equation (17b)). This statement is generally true as may be seen by inspection of equation (9).

We may now recover the Dirac oscillator interactions as a special case if we make the following choices (compare equations (11) with equations (17)):

$$g(4c_1 + c_2) = 1 \quad , \quad 2gmc_4 = \omega \tag{18a}$$

$$c_3 = 0 \quad , \quad (4c_1 d_1 + c_2 d_2 + 2c_5) = 0 \tag{18b}$$

Now we state without proof (see [11]) that equations (18) can always be satisfied by an appropriate choice of gauge. In other words, the potentials appearing in the Dirac oscillator (where we are including the mass as a constant potential) are essentially unique chiral solutions of the automorphism gauge field equations, since the potentials may always be brought into this form (equations (11)) by a chiral gauge transformation.

5 Summary and Conclusions

I have shown that both the mass and the potential introduced into the Dirac equation to produce the Dirac oscillator may be viewed as a special case of a chiral solution to the automorphism gauge field equations. In addition, this chiral solution is essentially unique in that a gauge transformation can always be found which puts the potential in the form displayed in the Dirac oscillator.

To gain insight into the physical interpretation of this system consider the more familiar situation of an electron interacting with a constant magnetic field. In this case, since the field strength is constant and uniform, the electromagnetic potential will be a linear function of the spacetime coordinate. As is well known [13], this system exhibits harmonic oscillator behavior in the two spatial directions perpendicular to the magnetic field. Therefore, the point of view considered in this paper is actually more like this situation in that there is a constant and uniform field strength and a corresponding linear potential. This should be contrasted with the direct electromagnetic interpretation of a particle with zero charge and non-zero magnetic moment interacting with a linear electric field. Notice, however, that the case of a constant automorphism

field strength does not lend itself easily to the construction of hadrons as advocated by Moshinsky et al [5,6] since we do not view each particle as giving rise to the automorphism field (though they certainly must contribute to the automorphism field as does the electron to the magnetic field, but this is taken here to be a "higher order effect"). In other words, to build the mesons (for example) we may consider a linearly rising potential between the quark-antiquark pair, but the situation with a constant automorphism field is more akin to putting several electrons in a constant magnetic field and neglecting the interactions between them. Each electron undergoes cyclotron motion but the centers of the individual cyclotron orbits are uncorrelated. These two pictures are clearly at odds, and we therefore do not necessarily expect local automorphism invariant gauge field theory to lend itself easily to a model of the hadrons. In fact, the motivation for considering local automorphism invariance is more one of aesthetics in that it may be considered to arise from a generalization of the Principle of Equivalence, and an important anticipated goal is a truly unified approach to the electroweak and gravitational interactions, but these issues will be discussed elsewhere [14].

The approach to the Dirac oscillator discussed in this paper naturally generalizes to spacetimes of arbitrary dimension and signature. In particular, the specific cases of two, three, five, and six dimensions are likely to generate interesting results. In addition, as this application shows, the general theory of local automorphism invariance should be a worthwhile and interesting avenue of exploration.

References

[1]   D. Itō, K. Mori, and E. Carrieri, Nuovo Cimento **51A**, 1119 (1967).

[2]   P. A. Cook, Lett. Nuovo Cimento **1**, 419 (1971).

[3]   M. Moshinsky and A. Szczepaniak, J. Phys. A **22**, L817 (1989).

[4]   J. Benitez, R. P. Martínez y Romero, H. N. Núñez-Yépez, and A. L. Salas-Brito, Phys. Rev. Lett. **64**, 1643 (1990); and Phys. Rev. Lett. **65**, 2085E (1990).

[5]   J. Beckers and N. Debergh, Phys. Rev. D **42**, 1255 (1990).

[6]   M. Moshinsky, G. Loyola, A. Szczepaniak, C. Villegas, and N. Aquino, "The Dirac Oscillator and its Contribution to the Baryon Mass Formula" in "Relativistic Aspects of Nuclear Physics," Editors T. Kodama et al. (World Scientific , Singapore (1990)) pp. 271 - 307.

[7]   M. Moshinsky, G. Loyola, and C. Villegas, "Relativistic Mass Formula for Baryons" in "Notas de Fisica 13", Proceedings of the Oaxtepec Conference of Nuclear Physics 1990, pp. 187 - 196.

[8]   V. V. Dixit, T. S. Santhanam, and W. D. Thacker, J. Math. Phys. **33**, (1992) to appear.

[9]   J. P. Crawford, "Local Automorphism Invariance" in preparation.

[10]   J. P. Crawford, J. Math. Phys. **31**, 1991 (1990).

[11]   J.P. Crawford, "Notes on the Dirac Oscillator and Local Automorphism Invariance" submitted to Foundations of Physics, April 9, 1992.

[12]   J. P. Crawford, J. Math. Phys. **32**, 576 (1991).

[13]   L. D. Huff, Phys. Rev. **38**, 501 (1931). M. H. Johnson and B. A. Lippman, Phys. Rev. **77**, 702 (1950).

[14]   J. P. Crawford, "A Generalization of the Principle of Equivalence and Unified Field Theories" in preparation.