

HOW TO SELECT COMBINATION OPERATORS FOR
FUZZY EXPERT SYSTEMS USING CRI

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Abstract: A method to select combination operators for fuzzy expert systems using Compositional Rule of Inference(CRI) is proposed from the consideration of basic requirement for fuzzy reasoning. First, fuzzy inference processes based on CRI are classified into three categories in terms of their inference results, i.e., the Expansion Type Inference, the Reduction Type Inference, and Other Type Inferences. Further, implication operators under Sup-T composition are classified as the Expansion Type Operator, the Reduction Type Operator, and the Other Type Operators. Finally combination of rules or their consequences is investigated for inference processes based on CRI. It is suggested that for inference processes using Sup-T composition in the context of CRI, the combination operator be "min" if the implication operator $a \rightarrow b = F(a, b)$ is an Expansion Type and is an inversely proportional function of a , i.e., if $a_1 \geq a_2$, then $F(a_1, b) \leq F(a_2, b)$, and the combination operator be "max" if the implication operator $F(a, b)$ is a Reduction Type and is a proportional function of a , i.e., if $a_1 \geq a_2$, then $F(a_1, b) \geq F(a_2, b)$.

Keywords: Compositional Rule of Inference, Inference Processes, Expansion, Reduction, Implication, Composition, Combination.

1. INTRODUCTION

Suppose there are Ω fuzzy rules in the rule base of a fuzzy expert system as follows:

$$\begin{aligned}
 & \text{IF } X \text{ is } A_1 \text{ THEN } Y \text{ is } B_1 \\
 & \text{IF } X \text{ is } A_2 \text{ THEN } Y \text{ is } B_2 \\
 & \dots \\
 & \text{IF } X \text{ is } A_\omega \text{ THEN } Y \text{ is } B_\omega \\
 & \dots \\
 & \text{IF } X \text{ is } A_\Omega \text{ THEN } Y \text{ is } B_\Omega
 \end{aligned} \tag{1.1}$$

where A_ω and B_ω , $\omega = 1, 2, \dots, \Omega$, are fuzzy sets defined in the universe of discourses V and W , respectively.

For a given system observation, in order to obtain a meaningful inference result based on Zadeh's Compositional Rule of Inference(CRI)[25], there are two basic approaches. The first one is called **FIRST INFER - THEN AGGREGATE** approach, "FITA" for short. In this first approach, for a given system observation A' , we first perform inference using CRI on each of the rules in the rule base, and then combine all these intermediate results as follows:

$$B' = \bigcup_{\omega=1}^{\Omega} B_{\omega}' \quad (1.2)$$

where B_{ω}' is the inference result based on rule ω , i.e., $B_{\omega}' = A' \circ R_{\omega}$, where $R_{\omega} = A_{\omega} \rightarrow B_{\omega}$ is the fuzzy implication relation for rule ω and \circ represents composition within the context of CRI, for example, Sup-min composition, and \bigcup is a combination operator, i.e., $\bigcup \in \{S, T\}$, in particular, $\bigcup \in \{\vee, \wedge\}$.

The second one is called **FIRST AGGREGATE - THEN INFER** approach, "FATI" for short. In this second approach, we first aggregate all the rules by forming an overall fuzzy relation R which is the combination of all the fuzzy implication relations as follows:

$$R = \bigcup_{\omega=1}^{\Omega} R_{\omega} \quad (1.3)$$

where $R_{\omega} = A_{\omega} \rightarrow B_{\omega}$, is the fuzzy implication relation for rule ω , \bigcup is a combination operator as specified above.

Then an inference is performed for a given observation A' as follows:

$$B'' = A' \circ R \quad (1.4)$$

where \circ represents composition within the context of CRI.

Therefore, it is clear that an inference process based on CRI includes several stages. More specifically, it includes **implication, composition, and combination** for FITA, and **implication, combination, and composition** for FATI. In the context of CRI, the comparison and selection of implication and composition operators have been widely studied for one rule case. For example, in [2], [10], and [22], applicability of implication operators is studied under Sup-min composition based on experiments for certain given problems. In [5], it is shown that implication is determined by composition operator, and that Gödel implication is a good implication under Sup-min composition in CRI[6]. In [9] and [23], implication operators are classified into three categories, i.e., S-implication, R-implication, and neither, and their properties are investigated based on some criteria which a Modus Ponens generation function[14] should satisfy. In [15, 16, 17], Interval-Valued Fuzzy Sets are used to represent fuzzy implications and reasoning results. Based on the bounds analysis of fuzzy reasoning, a linkage between CRI and AAR[21] is

established[17].

Inference with multiple rules are investigated by some researchers[1, 2, 3, 10]. In [1], combination operators are suggested for different implications from the consideration of interpretation of ELSE in "IF THEN ELSE" rule. In [3], combination is studied in the domain of fuzzy relational equations. In [2] and [10], both "max" and "min" operators are used in the combination for all implication operators in their experiments.

In this paper, issues of combination of rules or their consequences in fuzzy expert systems using CRI are investigated. A method is proposed for the selection of combination operators from the consideration of the basic requirement for fuzzy reasoning, i.e., if we have a system observation which is the same as the left hand side of a rule in the rule base, then the reasoning result should be the same as the right hand side of the rule. As a result of our analysis, we suggest that for an inference process using Sup-T composition in the context of CRI, "min" be used for combination if the implication $a \rightarrow b = F(a, b)$ is an Expansion Type and is an inversely proportional function of a, i.e., if $a_1 \geq a_2$, then $F(a_1, b) \leq F(a_2, b)$, and "max" be used for combination if the implication $F(a, b)$ is a Reduction Type and is a proportional function of a, i.e., if $a_1 \geq a_2$, then $F(a_1, b) \geq F(a_2, b)$.

This paper is organized as follows. In Section 2, Compositional Rule of Inference is reviewed, and inference processes are classified into three categories, i.e., Expansion Type Inference, Reduction Type Inference, and Other Types. Further, implication operators under Sup-T composition are classified as Expansion Type, Reduction Type, and Other Types. Finally, in Section 3, two general classes of implication operators are identified to be appropriate for "max" and "min" combinations. Conclusions are stated in the last section. We use either $\mu_{A \rightarrow B}(a, b)$, or $a \rightarrow b$, or $F(a, b)$, or $R(\rightarrow)$, or just r to represent the implication operator in CRI for the convenience of discussion where it is applicable.

2. CLASSIFICATION OF INFERENCE PROCESSES

In this section fuzzy inference based on CRI is reviewed. Inference processes based on CRI change the membership function grades of the right hand sides of the corresponding rules either by reducing or by increasing the membership grades. Here we consider reasoning with one rule using CRI.

CRI is also called Generalized Modus Ponens (GMP). With a single rule and a system observation, an inference result can be deduced as follows:

Rule:	IF X is A THEN Y is (should be) B
Observation:	X is A'
Consequence:	Y is (should be) $A' \circ (A \rightarrow B)$

where $A, A' \subset V$ and $B \subset W$ are fuzzy sets defined in the universe of discourses V and W ,

respectively, $(A \rightarrow B)$ denotes the implication relation, $R(\rightarrow)$, which is a fuzzy set of Cartesian product universe $V \times W$, and \circ denotes the composition between A' and $(A \rightarrow B)$.

The most notable is Zadeh's Sup-min composition in CRI[25], which has the form(in the membership domain) as follows:

$$\mu_{B'}(y_j) = \bigvee_i \mu_{A'}(x_i) \wedge \mu_{A \rightarrow B}(x_i, y_j), \quad i = 1, 2 \dots I, j = 1, 2 \dots J, \quad (2.1)$$

where B' is the inference result which is a fuzzy set defined in the universe of discourse W , $\mu_{B'}(y_j)$ is the membership value of j th element of B' , $\mu_{A'}(x_i)$ is the membership value of the i th element of A' , and $\mu_{A \rightarrow B}(x_i, y_j)$ is the membership value of the ij th element of the implication relation $R(\rightarrow)$.

2.1 Expansion vs. Reduction Inferences

In this subsection, we present our classification of the inference processes based on their inferred results. More specifically, we classify the inference processes into three categories, i.e., Expansion Type Inference, Reduction Type Inference, and Other Types. Following this point of view, we propose the selection of a proper combination operator such as "max" and "min" as will be discussed in detail later.

Definition 1. For a given rule: $A \rightarrow B$, and a system observation: A' , where $A, A' \subset V$ and $B \subset W$ are fuzzy sets defined in the universe of discourses V and W , respectively, suppose the deduced consequence through an inference process is denoted as B' , if for any A' , we always have:

$$B \subseteq B', \quad (2.2)$$

then the inference process is called the "Expansion Type Inference". Suppose, on the other hand, the deduced consequence is denoted as B^* , if for any A' , we always have:

$$B^* \subseteq B, \quad (2.3)$$

then the inference process is called the "Reduction Type Inference". Further, if the deduced consequence is at some times $B \subseteq B'$, and at some times $B^* \subseteq B$, then the inference process is called the "Other Type Inferences".

After Zadeh's Sup-min composition in CRI was proposed, Sup-T composition has been studied by many researchers[e.g., 6, 12, 15]. In [2, 10], the behaviours of many implication operators are studied using Sup-min composition in the context of CRI for certain specific problems. In this paper, it is assumed that Sup-T is used for composition in CRI in order to cover the general cases, and that all fuzzy sets are normalized.

Without proof here, we have the following theorem for the classification of inference processes.

Theorem 1. For Sup-T composition in the context of CRI, if the implication $a \rightarrow b = F(a, b) \geq b$ for all $a \in [0, 1]$, then the inference process is "Expansion Type Inference". If the implication $a \rightarrow b = F(a, b) \leq b$ for all $a \in [0, 1]$, then the inference is "Reduction Type Inference". If the implication $a \rightarrow b = F(a, b) \geq b$ for some $a \in [0, 1]$, but $F(a, b) \leq b$ for some other $a \in [0, 1]$, then the inference is "Other Type Inference".

According to Theorem 1, for a given implication operator, we can determine whether an inference process is an Expansion or a Reduction Type Inference under Sup-T composition. Thus, if we use Sup-T composition, those implication operators can be classified into three categories: the Expansion Type implication, Reduction Type implication, and Other Types. If Sup-T composition is used, then it is easy to show some implication operators proposed in the literature are Expansion Type implications, e.g., $\min(1, 1-a+b)$; some are Reduction Type ones, e.g., $\min(a,b)$; and some are Other Type implications, e.g., $\max(1-a, \min(a,b))$.

3. PROPER COMBINATION OPERATOR

Unless we have an exact match between a system observation and the antecedent of a rule, we need more than one rule to deduce a meaningful result by combining the intermediate results based on each of the rules. In this section, we first discuss the basic requirement for an inference process. We then propose a method to select combination operators for both Expansion and Reduction inference processes from the consideration of the basic requirement for fuzzy reasoning.

3.1 Basic Requirement for Fuzzy Reasoning

The basic requirement for fuzzy reasoning with one rule is that: given a rule $A \rightarrow B$, if the system observation is $A' = A$, then the deduced result should be B . Some researchers have studied this property[e.g., 4, 5, 6, 13, 14]. For example, in [5], for a given composition m , an implication operator I is derived such that $A \circ_m (A \rightarrow B) = B$. It is shown [5] that for Sup-T composition, denoted as $\circ_{s,t}$, and R-implication where the same t-norm operator as in the Sup-T is used, denoted as \rightarrow_R , we have $A \circ_{s,t} (A \rightarrow_R B) = B$. For example, in CRI, if Sup-min composition is used, Gödel implications have this property[6]. In [14], for a given implication function I , a modus ponens function m is derived, such that $A \circ_m (A \rightarrow B) = B$.

As mentioned previously, we need more than one rule to perform inference unless we have an exact match between the system observation and the left hand side of a rule. Suppose there are Ω rules in the rule base. For each of the rules, we have a reasoning result which we need to combine to obtain an overall inference result. We propose that a fuzzy inference process, with multiple rules, should satisfy the basic requirement for fuzzy reasoning stated as follows.

Criterion 1. The basic requirement for fuzzy reasoning, with multiple rules, is that given Ω

rules: IF X is A_ω THEN Y is B_ω , $\omega = 1, 2, \dots, \Omega$, if observation is $A' = A_\omega$, then reasoning result $B' = B_\omega$.

This criterion is important to the reliability of an expert system. More specifically, this criterion requires that when given a system observation which is one of the left hand sides of the rules, a fuzzy expert system will return the same conclusion as in the rule.

With the presentation of multiple rules, we have to deal with the combination problem as mentioned previously. In [1], combination operators are suggested for different implications from the consideration of interpretation of ELSE in "IF THEN ELSE" rule. In [3], the problem is studied in the domain of fuzzy relational equations. In [2] and [10], both "max" and "min" operators are used in the combination for all implication operators in their experiments. In what follows, from the consideration of the requirement for fuzzy reasoning processes stated above, we propose a method for the selection of combination operators for both the Expansion Type Inference and the Reduction Type Inference processes.

3.2 Combination: min vs. max

For a given system observation, we can perform inference by CRI with two approaches as indicated in Section 1, i.e., "FITA" and "FATI" approaches. The question is "what must be the proper combination operator for (1.2) and (1.3)?", i.e., "must \cup be max or min"? As discussed in Section 2, if Sup-T composition is used, then the category of an inference process can be determined by the implication operator, i.e., if $a \rightarrow b = F(a,b) \geq b$, then the process is an Expansion Type Inference, and if $a \rightarrow b = F(a,b) \leq b$, then the inference process is a Reduction Type Inference. Therefore, in this sense, (1.2) and (1.3) are consistent in terms of reasoning results.

3.2.1 Expansion Inference Process

In an Expansion Inference process, with Definition 1 in Section 2.1, for any system observation A' , we always have:

$$B \subseteq B'.$$

For an expansion inference process based on CRI, we have Necessary condition 1 as follows.

Necessary condition 1. Suppose there are Ω rules in the rule base of a fuzzy expert system. For a system observation and an inference process using Sup-T composition in the context of CRI, if implication $a \rightarrow b = F(a, b) \geq b$ for all $a \in [0,1]$, and is an inversely proportional function of a , i.e., if $a_1 \geq a_2$, then $F(a_1, b) \leq F(a_2, b)$, then "min" is needed for the combination.

The proof of Necessary condition 1 is based on the following idea: for a very low level of similarity(matching)[e.g., 26] between the observation and the left hand side of a rule and in the limit including the case of no match at all, i.e., no overlap, the membership function grade of the inferred result based on that rule has a value equal to(approaching) 1.0 in the limit at each

support point, i.e., this rule creates "unknown". Hence the use of this rule is useless and in this case it does not infer any useful information. Thus, considering the "Criterion 1" and getting a meaningful result for any system observation, we must use "min" for the combination, which will eliminate this useless information.

3.2.2 Reduction Inference Process

In a Reduction Type Inference process, with Definition 1 in Section 2.1, for any observation A' , we always have:

$$B^* \subseteq B.$$

For a reduction inference process based on CRI, we have Necessary condition 2 as follows.

Necessary condition 2. Suppose there are Ω rules in the rule base of a fuzzy expert system. For a system observation and an inference process using Sup-T composition in the context of CRI, if implication $a \rightarrow b = F(a, b) \leq b$ for $a \in [0,1]$, and is a proportional function of a , i.e., if $a_1 \geq a_2$, then $F(a_1, b) \geq F(a_2, b)$, then "max" is needed for the combination.

The proof of Necessary condition 2 is based on the following idea: for a very low level of similarity(matching) between the observation and the left hand side of a rule and in the limit including the case of no match at all, i.e., no overlap, the membership function grade of the inferred result has a value equal to 0 in the limit at each support point. That is, the use of this rule generates "meaningless". Considering the "Criterion 1" of the fuzzy inference and getting a meaningful result for any system observation, we must use "max" for the combination.

Necessary conditions 1 and 2 establish the choice of a combination operator for both Expansion and Reduction inference processes. In other words, after we select the implication and composition operator in CRI, then we could determine the combination operator in accordance with Necessary conditions 1 and 2.

4. CONCLUSIONS

In this paper, we analyzed fuzzy inference method of CRI in terms of inference results. Inference processes are classified into three categories, i.e., the Expansion Type Inference, Reduction Type Inference, and other types, which can be determined based on the implication operator under Sup-T composition in CRI. Based on the basic requirement of fuzzy reasoning stated as Criterion 1, we suggest that for an inference process using Sup-T composition in the context of CRI, "min" be used for the combination if the implication $F(a, b)$ is an Expansion Type and is an inversely proportional function of a , and "max" be used for combination if the implication is a Reduction Type and is a proportional function of a . Therefore, we have general conclusions for both Expansion and Reduction inference processes based on the reasoning results no matter which inference process is used. This proposed principle is also consistent with the existing results in the literature[e.g., 1, 2, 3, 10]. Our method can be used as a guidance to select operators in the

design of fuzzy expert systems and fuzzy controllers. More specifically, for an inference process using Sup-T composition, we first identify the class of an implication operator as discussed in Section 2; then select the combination operator according to Necessary condition 1 or 2. For example, in [2] and [10], the pair of operators: μ_{R12} and μ_{R12^*} , is not necessary since they are Reduction Type but are not directly proportional(not non-decreasing) functions of a; the pairs of operators: μ_{R14} and μ_{R14^*} , μ_{R23} and μ_{R23^*} , μ_{R30} and μ_{R30^*} , and μ_{R32} and μ_{R32^*} , are not necessary since they are Expansion Type but are not inversely proportional(not non-increasing) functions of a; because μ_{R3} , μ_{R4} , μ_{R5} , μ_{R6} , μ_{R22} , μ_{R27} , and μ_{R29} are the Expansion Type, therefore "min" must be used for the combination for each of these processes. In other words, μ_{R3^*} , μ_{R4^*} , μ_{R5^*} , μ_{R6^*} , μ_{R22^*} , μ_{R27^*} , and μ_{R29^*} are "appropriate" candidates. And because μ_{R8^*} , μ_{R25^*} , and μ_{R31^*} are the Reduction Type, "max" must be used for the combination for each of these processes. In other words, μ_{R8} , μ_{R25} , and μ_{R31} are "appropriate" candidates. Since Necessary conditions 1 and 2 establish the selection of combination operators for the Expansion and the Reduction Type inferences, we suggest that appropriate combination operators be selected in the design of fuzzy expert systems.

It should be noted that in order to satisfy Criterion 1, the membership functions of the linguistic terms of a rule in the rule base of an expert system must satisfy some constraints or conditions[19].

In this paper we always make reference to CRI in one or another to remind the readers that there are other approximate reasoning methods such as, for example, Approximate Analogical Reasoning method[21]. Issues of combination for these other methods should also be investigated in a similar manner in the future.

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