Information Compression in the Context Model

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Abstract

The Context Model provides a formal framework for the representation, interpretation, and analysis of vague and uncertain data. The clear semantics of the underlying concepts makes it feasible to compare well-known approaches to the modeling of imperfect knowledge like given in Bayes Theory, Shafer's Evidence Theory, the Transferable Belief Model, and Possibility Theory.

In this paper we present the basic ideas of the Context Model and show its applicability as an alternative foundation of Possibility Theory and the epistemic view of fuzzy sets.

1 Introduction

One origin of imperfect data is due to situations, where the incompleteness of the available information does not support state-dependent specifications of objects by their characterizing tuples of elementary or set-valued attributes.

The most important kinds of imperfect knowledge to be investigated are vagueness and uncertainty. Within the Context Model [Gebhardt, Kruse 1992a, Gebhardt 1992, Kruse et. al. 1992] vagueness is referred to the specification of so-called vague characteristics, which formalize imprecise, possibly contradicting and partial incorrect observations of attribute values with respect to a finite number of conflicting consideration contexts.

The integration of conflicting contexts is related to the phenomenon of competition, whereas imprecision shows that a specialization of the context-dependent non-elementary characteristics attached to a vague characteristic is unjustified without having further information about the corresponding vaguely specified object. Hence, vagueness is the combination of two types of partial ignorance, which are the existence of conflicting contexts (to be called competition) and imprecision.

Uncertainty, on the other hand, is connected with the valuation of vague characteristics: When we have defined a vague characteristic to specify a vague observation of an inaccessible characteristic of an object's attribute in a given state, a decision maker should be enabled to quantify his or her degree of belief in this vague observation — either by objective measurement or by subjective valuation. Since we restrict ourselves to numerical, non-logical approaches to partial ignorance, the theory of measurement seems to be the adequate formal environment for the representation of uncertainty aspects.

The mentioned approach to vagueness and uncertainty modelling leads canonically to the concept of a valuated vague characteristic which is introduced in seticon 2 and serves as one of the foundations of the Context Model.

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Since we intend to focus our attention to *information compression aspects,* we show in which way valuated vague characteristics, and the important notions of correctness-, contradiction-, and sufficiency-preservation turn out to be helpful for establishing richer underlying semantics of Possibility Theory and the epistemic view of fuzzy sets. For this reason section 3 deals with an appropriate definition of possibility functions, while section 4 clarifies how to operate on possibility functions with the requirement of coming to most specific correct results, when correctness assumptions on the composed possibility functions are fulfilled. As an example we refer to some foundations of Fuzzy Control. Finally section 5 shows an interpretation of fuzzy sets and a justification of Zadeh's extension principle by the Context Model.

2 The Context Model: Basic Concepts

In this section we outline basic concepts of the Context Model as far as they are important for the other sections. The following definitions have already been motivated by the general idea of a valuated vague characteristic mentioned in the introduction.

Definition 2.1 *Let D be a nonempty* universe of discourse *(frame of discernment, domain of a data type) and C a nonempty finite* set of contexts.

$\Gamma_C(D) \overset{Df}{=} \{\gamma \mid \gamma : C \rightarrow 2^D\}$ *is defined to be the* set of all vague characteristics of D w.r.t. C.

Ignoring the contexts, $\Gamma(D) \overset{Df}{=} 2^D = \{A \mid A \subseteq D\}$ *designates the* set of all (imprecise) characteristics of D.

Let $\gamma, \nu \in \Gamma_C(D)$ *and* $A \in \Gamma(D)$.

(a) γ empty, *iff* $\gamma(C) = \{\gamma(c) \mid c \in C\} = \{\emptyset\}$;

(b) γ elementary, *iff* $(\forall\ c \in C)\,(|\gamma(c)| = 1)$;

(c) γ precise, *iff* $(\forall\ c \in C)\,(|\gamma(c)| \leq 1)$;

(d) γ contradictory, *iff* $(\exists\ c \in C)\,(\gamma(c) = \emptyset)$;

(h) γ specialization *of ν (ν generalization of γ, γ more specific than ν, ν correct w.r.t. γ), iff* $(\forall\ c \in C)\,(\gamma(c) \subseteq \nu(c))$;

Definition 2.2 *Let $(C, 2^C, P_C)$ be a finite measure space that is referred to a given context set C. Each vague characteristic $\gamma \in \Gamma_C(D)$ is called* valuated w.r.t. $(C, 2^C, P_C)$.

Remark Obviously there are formal analogies, but even semantical differences to the concept of a random set recommended by Matheron [Matheron 1975] and Nguyen [Nguyen 1978]. Considering the original idea of a random set, if $\gamma \in \Gamma_C(D)$, then for all $c \in C$, $\gamma(c)$ should be interpreted as an indivisible *set-valued datum* attached to an outcome c of an underlying random experiment which is formalized by a probability space $(C, 2^C, P_C)$.

Following a reasonable interpretation of Nguyen's approach, $\gamma(c)$ specifies the *set of single-valued data* which are possible in a context c, where $P_C(\{c\})$ quantifies the (objective or subjective) probability that c is the "true" context.

On the other hand, using γ as a valuated vague characteristic, $P_C(\{c\})$ reflects the degree of *reliability* that the context c delivers a *correct* specification of an original characteristic $Orig_\gamma \subseteq D$ (i.e.: $Orig_\gamma \subseteq \gamma(c)$), where $Orig_\gamma$ is an (inaccessible) state-dependent characterization of an object of interest.

Whenever $P_C(\{c\})$ stands for a reliability degree, then P_C in general will neither be defined as a probability measure nor be normalized to a probability measure. Furthermore the interpretation of a valuated vague characteristic does not require that one of the available contexts is the "true" one which has to be selected.

3 Possibility Functions

The main application of (valuated) vague characteristics $\gamma \in \Gamma_C(D)$ refers to the specification of a vague observation of an (inaccessible) characteristic $Orig_\gamma \subseteq D$, the so-called *original* of γ which — generally speaking — characterizes an object in its acutal state. As an example consider a control system with a single input variable and a single output variable taking thier values on the domains X and Y, respectively. The state of this control system may be defined by the actual input value $x_0 \in X$ and the control function $g : X \rightarrow Y$ that relates the possible input values $x \in X$ to their corresponding output values $y \in Y$.

The behaviour of the system can be specified by the inference mechanism that transfers x_0 to the actual output value $y_0 = g(x_0)$, which is

$$\text{infer} : \Gamma(X) \times \Gamma(X \times Y) \rightarrow \Gamma(Y),$$
$$\text{infer}(X_0, R) \stackrel{Df}{=} \{y | (x, y) \in R \cap X_0 \times Y\}.$$

In the special case $X_0 = \{x_0\}$ and $R = g \subseteq X \times Y$ we in fact obtain

$$\text{infer}(X_0, R) = \text{infer}(\{x_0\}, g) = \{g(x_0)\}.$$

In the situation (well-known from fuzzy control) when g and sometimes even x_0 are not available, but only vaguely observed, the context model suggests the specification of vague characteristics $\gamma_1 \in \Gamma_{C_1}(X)$ and $\gamma_2 \in \Gamma_{C_2}(X \times Y)$ based on appropriate context measure spaces $\mathcal{M}_1 = (C_1, 2^{C_1}, P_{C_1})$ and $\mathcal{M}_2 = (C_2, 2^{C_2}, P_{C_2})$.

The adequate choice of context measure spaces is an application-dependent problem, but for our example *it seems to be convincing that the contexts have to be defined by their maximum measurement tolerance*, namely the maximum distance between the measured input value and the original input value that should have been taken.

In practical applications incomplete information and the complexity of required operations will often advise us to avoid the detailed consideration of the underlying context measure spaces, but to use an *information compressed* specification of valuated vague characteristics, as done — from the context model's point of view — in Possibility Theory [Dubois, Prade 1988] and Fuzzy Set Theory [Klir, Folger 1988].

Viewing a valuated vague characteristic $\gamma \in \Gamma_C(D)$ in a pure formal sense as a *genralized random set*, one promising way of coming to an information compressed representation of γ is the choice of the *contour function* of γ, which we prefer — for semantical reasons — to be denoted as the *possibility function* of γ.

Definition 3.1 *Let $\gamma \in \Gamma_C(D)$ be valuated w.r.t. $\mathcal{M} = (C, 2^C, P_C)$.. Then,*

$$\pi_\mathcal{M}[\gamma] : D \rightarrow \mathbb{R}_0^+, \ \pi_\mathcal{M}[\gamma](d) \stackrel{Df}{=} P_C(\{c \in C \mid d \in \gamma(c)\})$$

is called the possibility function of γ, *where* $(\mathbb{R}_0^+ \stackrel{Df}{=} \{r \in \mathbb{R} \mid r \geq 0\})$.

$POSS(D) \stackrel{Df}{=} \{\pi \mid \pi : D \rightarrow \mathbb{R}_0^+ \wedge |\pi(D)| \in \mathbb{N}\}$ *is defined to be the* set of all possibility functions w.r.t. D.

For $\pi \in POSS(D)$, $Repr(\pi) \stackrel{Df}{=} \{(\alpha, \pi_\alpha) \mid \alpha \in \mathbb{R}_0^+\}$ *with the* α-cuts $\pi_\alpha \stackrel{Df}{=} \{d \in D \mid \pi(d) \geq \alpha\}$ *denotes the identifying* set representation of π.

Let $\gamma \in \Gamma_C(D)$ be the vague characterization of an elementary original $Orig_\gamma \in \Gamma(D)$. Obviously, for all $d \in D$, $\pi_\mathcal{M}[\gamma](d)$ quantifies the measure of all contexts $c \in C$, for which a specialization of $\gamma(c)$ into the elementary characteristic $\{d\}$ is feasible. In other words: $\pi_\mathcal{M}[\gamma](d)$ is the measure of all context that do not contradict $\{d\}$ to be the original of γ and therefore expresses the *possibility* that $Orig_\gamma = \{d\}$ is valid. That is one reason why we call $\pi_\mathcal{M}[\gamma]$ a *possibility function*.

But there is even more behind $\pi_\mathcal{M}[\gamma]$ than only measuring possibility degrees. Whenever each context valuation $P_C(\{c\})$ is expected to be the presupposed *reliability degree* with which c delivers a *correct* imprecise characterization $\gamma(c)$ w.r.t. $Orig_\gamma$ (which means that $Orig_\gamma \subseteq \gamma(c)$), then, for all $\alpha \geq 0$, the α-cut $\pi_\mathcal{M}[\gamma]_\alpha$

is the most specific characteristic that is for sure correct w.r.t. $Orig_\gamma$, if the *α-correctness* of γ w.r.t. $Orig_\gamma$ is given (which means that the measure of all contexts $c \in C$ that are correct w.r.t. $Orig_\gamma$ equals α or is greater than α).

Definition 3.2 *Let $\gamma \in \Gamma_C(D)$ be valuated w.r.t. $(C, 2^C, P_C)$ and $A, B \subseteq D$ two characteristics. Furthermore let $\alpha \geq 0$.*

(a) B is correct *w.r.t. A, iff $A \subseteq B$.*

(b) γ is α-correct *w.r.t. A, iff $P_C(\{c \in C \mid A \subseteq \gamma(c)\}) \geq \alpha$.*

The choice of an appropriate correctness level α^* depends on the semantical environment in which $\gamma \in \Gamma_C(D)$ is used. If C is a set of outcomes of an underlying random experiment, then $P_C(\{c\})$ quantifies the probability of the outcome c.

In this case exactly one of the contexts contained in C is selected to be the "true" context, and P_C should be seen as a probability measure (i.e. $P_C(C) = 1$).

In a more general sense C is a set of contexts that represent distinguishable consideration points of view (e.g. experts, sensors). Then it is of course not always reasonable to talk about the existence of a single true context, but rather to interpret $P_C(\{c\})$ as the degree of success with which the context $c \in C$ has delivered correct imprecise charaterizations $\gamma_i(c)$ w.r.t. a number of checkable representative vague observations $\gamma_i \in \Gamma_C(D)$ of original characteristics $Orig_{\gamma_i} \subseteq D, i = 1, \ldots, n$.

If we define

$$\begin{aligned}
\alpha^{(i)} &\stackrel{Df}{=} \max\{\alpha \mid Orig_{\gamma_i} \subseteq \pi_{\mathcal{M}}[\gamma_i]_\alpha\}, \; i = 1, \ldots, n, \text{ and} \\
\alpha_{\min} &\stackrel{Df}{=} \min\{\alpha^{(i)} \mid i \in \{1, \ldots, n\}\}, \\
\alpha_{\max} &\stackrel{Df}{=} \max\{\alpha^{(i)} \mid i \in \{1, \ldots, n\}\},
\end{aligned}$$

then $\alpha^* \in [\alpha_{\min}, \alpha_{\max}]$ seems to be an acceptable choice for the postulation of the correctness degree of future vague characterizations $\gamma \in \Gamma_C(D)$ w.r.t. their (inaccessible) original $Orig_\gamma \subseteq D$.

4 Operating on Possibility Functions

In the previous section we introduced the concepts of a possibility function and the correctness of (vague) characteristics $\gamma \in \Gamma_C(D)$ with respect to their underlying original characteristics $Orig_\gamma \subseteq D$.

Now we change over to the important question how to operate on possibility functions. For this reason let us again come back to our control system example. We assumed to have the vague characterization $\gamma_1 \in \Gamma_{C_1}(X)$ of the actual input value $x_0 \in X$ and the vague characterization $\gamma_2 \in \Gamma_{C_2}(X \times Y)$ of the control function $g \subseteq X \times Y$, referred to the context measure spaces $\mathcal{M}_1 = (C_1, 2^{C_1}, P_{C_1})$ and $\mathcal{M}_2 = (C_2, 2^{C_2}, P_{C_2})$, respectively.

Following the notion of the context model, the starting point in fuzzy control is to neglect γ_1 and γ_2, and to restrict the attention to the induced possibility functions $\pi_{\mathcal{M}_1}[\gamma_1]$ and $\pi_{\mathcal{M}_2}[\gamma_2]$. Postulating α_1-correctness of γ_1 w.r.t. $\{x_0\}$ and α_2-correctness of γ_2 w.r.t. g, we intend to calculate the most specific set $Y_0 \subseteq Y$ of output values which is correct w.r.t. $\{g(x_0)\}$.

In the final decision making process one of the elements contained in Y_0 has to be selected as the adequate output value of the system. Note that — as we handle imprecision as well as conflicting contexts — in the normal case we have no chance to obtain a single output value from the inference mechanism. The choice of an element of Y_0 as the actual output value corresponds to the defuzzification step in applied Fuzzy Control.

For the calculation of Y_0 we consider the more general environment, where $\gamma_i \in \Gamma_{C_i}(D_i)$ are valuated w.r.t. $\mathcal{M}_i = (C_i, 2^{C_i}, P_{C_i})$, $i = 1, \ldots, n$. Each γ_i is interpreted as a valuated α_i-correct specification of a vague observation of an inaccessible non-empty characteristic $A_i \subseteq D_i$. Furthermore let $f : \bigtimes_{i=1}^{n} \Gamma(D_i \rightarrow \Gamma(D)$ be a function of imprecise characteristics. Suppose to have the task to determine the most specific characteristic in $\Gamma(D)$ which is correct w.r.t. $f(A_1,\ldots,A_n)$. This charateristic is called *sufficient* for f w.r.t. $(\gamma_1,\ldots,\gamma_n)$ and $(\alpha_1,\ldots,\alpha_n)$. We now formalize the notion of sufficiency and show how to evaluate sufficient characteristics.

Definition 4.1 *Let $\gamma_i \in \Gamma_{C_i}(D_i)$, $i = 1,2,\ldots,n$ be valuated w.r.t. $(C_i, 2^{C_i}, P_{C_i})$. Consider correctness-levels $\alpha_i > 0$, $i = 1,2,\ldots,n$, a function $f : \bigtimes_{i=1}^{n}\Gamma(D_i) \rightarrow \Gamma(D)$, and a characteristic $F \in \Gamma(D)$.*

(a) F is correct *for f* w.r.t. $(\gamma_1,\ldots,\gamma_n)$ and $(\alpha_1\ldots,\alpha_n)$, *iff*

$$\left(\forall\ (A_1,\ldots,A_n) \in \bigtimes_{i=1}^{n}\Gamma(D_i)\right)$$
$$((\forall\ i \in \{1,\ldots,n\})(\gamma_i \text{ is } \alpha_i - \text{correct w.r.t. } A_i) \Longrightarrow F \text{ correct w.r.t } f(A_1,\ldots,A_n))\,;$$

(b) F is sufficient for f w.r.t. $(\gamma_1,\ldots,\gamma_n)$ and $(\alpha_1,\ldots,\alpha_n)$, iff F fulfils (a) and

$$\left(\forall\ F^* \subsetneqq F\right)\ (F^* \text{ is not correct for } f \text{ w.r.t. } (\gamma_1,\ldots,\gamma_n) \text{ and } (\alpha_1,\ldots,\alpha_n))\,.$$

It turns out that under weak conditions there is an efficient computation of sufficient characteristics by application of the induced possibility functions $\pi_{\mathcal{M}_i}[\gamma_i]$, without explicitly referring to the underlying valuated vague characteristics and the context measure spaces $\mathcal{M}_i$.

Before coming to that result we state the following four (technical) definitions.

Definition 4.2 *Let $D_1, D_2, \ldots D_n, D$ be universes of discourse and $f : \bigtimes_{i=1}^{n} \Gamma(D_i) \rightarrow \Gamma(D)$ a function.*

(a) f is called correctness-preserving, *iff*
$f(A_1,\ldots,A_n) \subseteq f(B_1,\ldots,B_n)$ for all A_i, B_i with $A_i \subseteq B_i \subseteq D_i$, $i = 1,2,\ldots,n$.

(b) f is called contradiction-preserving, *iff*

$$(\forall A_1,\ldots,A_n)((\exists i \in \{1,\ldots,n\})(A_i = \emptyset) \Longrightarrow f(A_1,\ldots,A_n) = \emptyset)$$

Definition 4.3 *Let $D_1,\ldots,D_n, D$ be universes of discourse and $f : \bigtimes_{i=1}^{n}\Gamma(D_i) \rightarrow \Gamma(D)$ a contradiction-preserving mapping. f is* sufficiency–preserving, *iff*

$$f(A_1 \cup B_1,\ldots,A_n \cup B_n) =$$
$$\bigcup\{F \mid (\exists\ C_1,\ldots,C_n)\,(F = f(C_1,\ldots,C_n)\ \wedge (\forall\ j \in \{1,\ldots,n\})\,(C_j = A_j \vee C_j = B_j))\}$$

for all $A_i, B_i \in \Gamma(D_i), i = 1,2,\ldots,n$.

Definition 4.4 *Let $\pi \in POSS(D)$. π is* correct (sufficient) *for f w.r.t. $(\gamma_1,\ldots,\gamma_n)$, iff*
$(\forall\ \alpha > 0)$ (π_α correct(sufficient) *for f w.r.t. $(\gamma_1,\ldots,\gamma_n)$ and $(\alpha_1,\ldots,\alpha_n)$).*

Definition 4.5 *Let $\pi_i \in POSS(D_i)$, $i = 1,\ldots,n$, and $f : \bigtimes_{i=1}^{n}\Gamma(D_i) \rightarrow \Gamma(D)$. The possibility function $f[\pi_1,\ldots,\pi_n] : D \rightarrow \mathbb{R}_0^+$ which is determined by its identifying set representation*
$Repr\,(f[\pi_1,\ldots,\pi_n]) \overset{\text{Df}}{=} \{(\alpha, f[\pi_1,\ldots,\pi_n]_\alpha) \mid \alpha \in \mathbb{R}_0^+\}$ with
$f[\pi_1,\ldots,\pi_n]_0 \overset{\text{Df}}{=} D$ and $(\forall\ \alpha > 0)$ $(f[\pi_1,\ldots,\pi_n]_\alpha = f\,((\pi_1)_\alpha,\ldots,(\pi_n)_\alpha))$
is called the image of $(\pi_1,\ldots,\pi_n)$ under f .

Theorem 4.6 *Let $\mathcal{M}_i = (C_i, 2^{C_i}, P_{C_i})$, $|C_i| \geq 2$, be context measure spaces.*
Additionally let $f : \bigtimes_{i=1}^{n} \Gamma(D_i) \rightarrow \Gamma(D)$ be a correctness- and contradiction-preserving mapping.
f is sufficiency-preserving, iff

$$\left(\forall\ (\gamma_1, \ldots, \gamma_n) \in \bigtimes_{i=1}^{n} \Gamma_{C_i}(D_i)\right) \left(f\left[\pi_{\mathcal{M}_1}[\gamma_1], \ldots, \pi_{\mathcal{M}_n}[\gamma_n]\right] \text{ sufficient for } f \text{ w.r.t. } (\gamma_1, \ldots, \gamma_n)\right)$$

The result is especially related to possibility functions, where $\alpha = \alpha_1 = \alpha_2 = \ldots = \alpha_n$, but an analogous theorem holds in the case when the levels α_i are chosen arbitrarily.

Since the function *infer* is sufficiency-preserving, applying the theorem to our example, the characteristic $Y_0 \overset{Df}{=} \text{infer}(\pi_{\mathcal{M}_1}[\gamma_1]_\alpha, \pi_{\mathcal{M}_2}[\gamma_2]_\alpha)$ is sufficient w.r.t. $\{g(x_0)\}$, if α-correctness of γ_1 w.r.t. $\{x_0\}$ and α-correctness of γ_2 w.r.t. g is given. Hence the output value of the control system has to be selected from Y_0.

5 Fuzzy Sets

Within the Context Model the interpretation of fuzzy sets [Dubois, Prade 1989, Dubois, Prade 1991] and the most important operations on fuzzy sets are based on the concept of valuated vague characteristics in the following way:
Let $F(D) \overset{Df}{=} \{\mu \mid \mu : D \rightarrow [0,1] \land |\mu(D)| \in I\!N\}, D \neq \emptyset$, be the set of all fuzzy sets with finite codomain. Then $\mu \in F(D)$ is considered to be the information compression $\pi_{\mathcal{M}}[\gamma]$ of an underlying vague characteristic γ valuated w.r.t. an appropriate context measure space $\mathcal{M} = (C, 2^C, P_C)$, where P_C is a probability measure. Since the aim of fuzzy sets is the modelling of vague concepts like "young" and "tall", we now abstract from the existence of a vaguely observed original characteristic $\text{Orig}_\gamma \in \Gamma(D)$ by interpreting γ as the specification of a vague property [Kruse, Meyer 1987, Kruse et. al. 1991a]. Nevertheless $F(D)$ equals – at least at the formal level – a set of possibility functions, and therefore all results obtained in section 5 are applicable to fuzzy sets without affecting their special interpretation.
As examples we will investigate the union and intersection of fuzzy sets and Zadeh's extension principle [Zadeh 1975] by application of the following theorem.

Theorem 5.1 *Let $\gamma_i \in \Gamma_{C_i}(D_i)$, $i = 1, \ldots, n$ be non-empty and valuated w.r.t. $\mathcal{M}_i = (C_i, 2^{C_i}, P_{C_i})$.*
Furthermore let $f : \bigtimes_{i=1}^{n} \Gamma(D_i) \rightarrow \Gamma(D)$ be a mapping.

f sufficiency-preserving $\Longrightarrow$

$$(\forall\ d \in D) \Big(f\left[\pi_{\mathcal{M}_1}[\gamma_1], \ldots, \pi_{\mathcal{M}_n}[\gamma_n]\right](d) =$$
$$\sup\left\{\min\{\pi_{\mathcal{M}_1}[\gamma_1](d_1), \ldots, \pi_{\mathcal{M}_n}[\gamma_n](d_n)\} \mid (d_1, \ldots, d_n) \in \bigtimes_{i=1}^{n} D_i \land d \in f(\{d_1\}, \ldots, \{d_n\})\right\}\Big)$$

Union and Intersection of Fuzzy Sets

Let $\mu_1, \mu_2 \in F(D)$ be fuzzy sets and $\gamma_1 \in \Gamma_{C_1}(D)$, $\gamma_2 \in \Gamma_{C_2}(D)$ their underlying vague characteristics; γ_i is assumed to be valuated w.r.t. $\mathcal{M}_i = (C_i, 2^{C_i}, P_{C_i})$, where $P_{C_i}(C_i) = 1, i = 1, 2$. Furthermore suppose that $\mu_1 = \pi_{\mathcal{M}_1}[\gamma_1]$ and $\mu_2 = \pi_{\mathcal{M}_2}[\gamma_2]$. Consider the contradiction-preserving union of characteristics, defined by

$$f_\cup : \Gamma(D) \times \Gamma(D) \rightarrow \Gamma(D),$$
$$f_\cup(A, B) \overset{Df}{=} \begin{cases} A \cup B, & \text{iff } A \neq \emptyset \land B \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

Since $f_\cup$ is sufficiency-preserving, we know by application of Theorem 4.6 that $f_\cup[\mu_1, \mu_2]$ is sufficient for $f_\cup$ w.r.t. (γ_1, γ_2). Applying Theorem 5.1 it is easy to calculate $f_\cup[\mu_1, \mu_2](d) = \max\{\mu_1(d), \mu_2(d)\}$, $d \in D$. In an analogous way we obtain $f_\cap[\mu_1, \mu_2](d) = \min\{\mu_1(d), \mu_2(d)\}$, $d \in D$, with respect to the intersection

f_n of characteristics; (min, max) appears as the well-known pair of t-norm and t-conorm often applied to define intersection and union of fuzzy sets [Klir, Folger 1988]. Using alternative assumptions regarding the underlying context measure spaces, additional t-norms and t-conorms are motivated by the Context Model.

Extension Principle

Zadeh's extension principle [Zadeh 1975] arises as a special case of Theorem 5.1. This principle is defined as follows:
Let $n \in \mathbb{N}$ and $(\mu_1, \ldots, \mu_n) \in [F(\mathbb{R})]^n$. Furthermore let $f : \mathbb{R}^n \to \mathbb{R}$.

The fuzzy set $f^*[\mu_1, \ldots, \mu_n] \in F(\mathbb{R})$, defined by
$f^*[\mu_1, \ldots, \mu_n](t) \stackrel{\text{Df}}{=} \sup \{\min\{\mu_1(t), \ldots, \mu_n(t)\} \mid (t_1, \ldots, t_n) \in \mathbb{R}^n \wedge f(t_1, \ldots, t_n) = t\}$, $t \in \mathbb{R}$ is called the *image of* $(\mu_1, \ldots, \mu_n)$ *under* f, where $\sup \emptyset \stackrel{\text{Df}}{=} 0$.

If we interprete $\mu_1, \ldots, \mu_n$ as possibility functions of valuated vague characteristics, then there exist $\gamma_i \in \Gamma_{C_i}(\mathbb{R})$ and context measure spaces $\mathcal{M}_i = (C_i, 2^{C_i}, P_{C_i})$ fulfilling $P_{C_i}(C_i) = 1$ and $\mu_i = \pi_{\mathcal{M}_i}[\gamma_i]$, $i = 1, 2, \ldots, n$. We define the sufficiency-preserving mapping $g : \Gamma(\mathbb{R})^n \to \Gamma(\mathbb{R})$, $g(A_1, \ldots, A_n) \stackrel{\text{Df}}{=} f(A_1 \times \cdots \times A_n)$ and obtain by application of Theorems 4.6 and 5.1 that $f^*[\mu_1, \ldots, \mu_n] \equiv f[\mu_1, \ldots, \mu_n]$, i.e. $f^*[\mu_1, \ldots, \mu_n]$ is sufficient for g w.r.t. $(\gamma_1, \ldots, \gamma_n)$. We infer that within the Context Model the extension principle is nothing else than the description of how to get sufficiency-preserving mappings of a restricted class of sufficient possibility functions.

6 Concluding Remarks

In this paper we have outlined the application of the Context Model for a new interpretation of Possibility Theory and fuzzy sets. Based on context measure spaces, valuated vague characteristics, induced possibility functions, and the very important concepts of *correctness* and *sufficiency* we demonstrated how to operate on possibilistic data and how to get a new justification of the extension principle.

A short example of fuzzy control was taken to show the practical use of the mentioned ideas. The in-depth look at the whole theory will be distributed on different papers. A comprehensive presentation of the basic semantical aspects of the Context Model, and its relationships to random sets [Nguyen 1978], Dempster-Shafer-Theory [Shafer 1976, Shafer, Pearl 1990], the Transferable Belief Model [Smets, Kennes 1991], and Bayes-Theory [Pearl 1988] is already given in [Gebhardt, Kruse 1992a], whereas [Gebhardt 1992] and [Gebhardt, Kruse 1992b] contain the more detailed approach to a modified view of Possibility Theory. Concerning the semantical foundation of the heuristic methods of Fuzzy Control it turns out that under weak restrictions the well-known if-then-rules should be interpreted by their induced Gödel relations and composed by intersection. Except from the composition mechanism for the rules (which from the Context Model's point of view is rather conjunctive than disjunctive, and therefore coincides with similar composition techniques known from the field of knowledge based systems), the resulting fuzzy controller partly behaves like Mamdani's controller, but — as a consequence of the strict formal and semantical environment — it does not suffer from the inconsistencies of max-min-inference and the problem of justifying the combination of different mathematical formalisms as they are used for fuzzification, fuzzy-inference, and defuzzification (e.g. center of gravity method).

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