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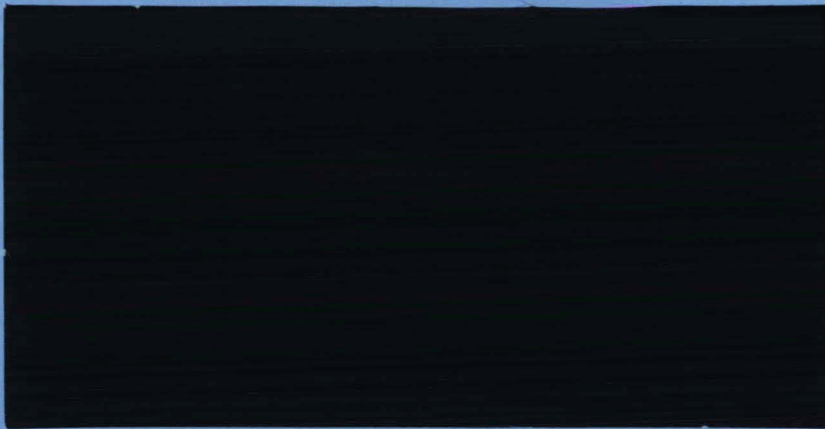
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Efficient Reorientation Maneuvers for Spacecraft
with Multiple Articulated Payloads

Principal Investigator: N. H. McClamroch

Summary

A final report is provided which describes the research program under the indicated NASA support during the period March 3, 1992, to June 3, 1993. A summary of the technical research questions that have been studied and of the main results that have been obtained is given. The specific outcomes of the research program, including both educational impacts as well as research publications, are listed.

The Research Accomplishments

Our research has been concerned with efficient reorientation maneuvers for spacecraft with multiple articulated payloads. This work is closely related to control problems for nonholonomic mechanical systems and involves, in a fundamental way, a number of new ideas in nonlinear control theory. Hence, our research has emphasized both the development of an underlying theory as well as specific spacecraft reorientation maneuvers. During the indicated time period, the following issues have been studied.

Modeling of space multibody systems as nonholonomic dynamic systems: Many examples of nonholonomic dynamic systems occur for mechanical systems which exhibit nonintegrable motion integrals; these include interconnections of rigid and flexible bodies in space where the control inputs are consistent with conservation of angular momentum; under certain assumptions even the attitude dynamics of a single rigid spacecraft is an example of a nonholonomic dynamic system. The key common ingredient in all of these examples is the existence of motion integrals which are nonholonomic or nonintegrable in a precise mathematical sense [1,3]. This whole line of research is motivated by an interest in carrying out reorientation maneuvers which avoid the use of gas jet thrusters or reaction devices; consequently, these reorientation maneuvers are highly efficient in terms of energy or fuel requirements.

Control of spacecraft with multiple articulated payloads systems: It is only within the last several years that research on control of nonholonomic dynamic systems has been initiated. There have been a number of important recent advances, both in the theory for control of nonholonomic dynamic systems and in the applications of that theory to reorientation problems of spacecraft with multiple articulated payloads. The results that we and others have obtained indicate the inherent difficulty of such problems: we have shown that there is no smooth (i.e. differentiable) feedback controller which can stabilize such dynamic systems [1,3]. Consequently there is no linear controller which can stabilize such systems. Until recently, there has been absolutely no stabilization theory which could be applied to these difficult problems. We have introduced the correct mathematical formulation of such control problems; we have demonstrated the fact that smooth stabilizing (time invariant) controllers cannot exist, and we have begun to develop control design approaches for obtaining (nonsmooth) stabilizing controllers [1,3].

Specific reorientation maneuvers for spacecraft with multiple articulated payloads: A large part of our research effort has been directed at the study of specific examples of spacecraft reorientation maneuvers. These examples are of interest in their own, and they provide much insight for us in our general studies. We have studied the following classes of problems:

Planar multibody systems [1,4,7]. Planar spacecraft maneuvers illustrate the fundamental maneuver characteristics and the associated mathematical framework is in the simplest form to study these manevuers. There is a striking similarity with the maneuvers carried out by "a falling cat which reorients itself."

Spacecraft attitude control [5,6,8,9,10]. Reorientation maneuvers of rigid spacecraft fit within the general framework studied if control torques can be applied about only two of the principal axes of the spacecraft. Our results provide an essentially complete analysis of all possible situations under which attitude stabilization can be achieved using two control torques.

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<u>Free-free flexible beam in space</u> [11,12]. This is a simple beam model of a spacecraft system with distributed flexibility which can be reoriented with respect to a fixed inertial frame by appropriate excitation of the deformable shape of the beam.

<u>Space station attitude disturbances arising from internal motions</u> [13]. The effects of internal periodic motions in a space station are demonstrated to lead to attitude disturbances of the space station.

<u>Redesign of the existing manned maneuvering unit</u>. This research, not yet published, uses only internal motions to accomplish reorientation of an astronaut manned maneuvering unit; these results suggest that such a design are significantly more fuel efficient than the current design, allowing substantially increased mission duration.

Outcomes of the Research Program

<u>Educational Impacts</u>: There have been several important educational outcomes as a consequence of the research support.

Three Ph.D. students have received partial financial support.

1. Mr. M. Reyhanoglu was partially supported to complete his Ph.D. research. This support resulted in completion of his Ph.D. degree in the Department of Aerospace Engineering in June, 1992. Mr. Reyhanoglu was a coauthor on several research papers as indicated below.

2. Mr. H. Krishnan was partially supported to complete his Ph.D. research. This support resulted in completion of his Ph.D. degree in the Department of Aerospace Engineering in September, 1992. Mr. Krishnan was a coauthor on several research papers as indicated below.

3. Mr. I. Kolmanovsky was partially supported to complete his M.S. degree in the Department of Aerospace Engineering; he is currently beginning his Ph.D. research. This support resulted in the research papers indicated below.

An additional Ph.D. student, Mr. P. McNally, has been involved in research that is closely associated with the NASA project, but they has not directly received project support. Research papers in which he has been involved are indicated below.

During the past year, the principal investigator and his graduate student colleagues gave several presentations on subject matter related to this project; this includes presentations both at university colloquiums as well as at special workshops:

M. Reyhanoglu, N. H. McClamroch and H. Krishnan, "Nonlinear Control of Mechanical Systems with Nonholonomic Motion Invariants: Theory and Physical Examples," Washington U. - NSF Workshop on Nonlinear Control, St. Louis, May, 1992.

N. H. McClamroch, "Nonlinear Attitude Control of Planar Structures in Space using only Internal Controls," Problems in Sensing, Identification and Control of Flexible Structures Workshop, The Fields Institute for Research in Mathematical Sciences, Waterloo, Ont., June, 1992.

M. Reyhanoglu and N. H. McClamroch, "Geometric Phase Computations Arising in Control Problems for Nonholonomic Caplygin Systems," SIAM Conference on Control, September, 1992.

M. Reyhanoglu, N. H. McClamroch and H. Krishnan, "Attitude Stabilization of a Rigid Spacecraft using Gas Jet Actuators Operating in a Failure Mode," SIAM Conference on Control, September, 1992.

N. H. McClamroch, "Attitude Stabilization of a Rigid Spacecraft Operating in an Actuator Failure Mode," The University of Michigan, October, 1992.

N. H. McClamroch, "Nonlinear Attitude Control of Space Structures using only Internal Controls," Stanford University, November, 1992.

N. H. McClamroch, "Control of Nonholonomic Dynamic Systems," University of California, Berkeley, November, 1992.

H. Krishnan, M. Reyhanoglu and N. H. McClamroch, "Attitude Stabilization of a Rigid Spacecraft using Gas Jet Actuators in a Failure Mode," IEEE Conference on Decision and Control, December, 1992.

I. Kolmanovsky and N. H. McClamroch, "Planar Reorientation of a Free-Free Beam in Space using Embedded Electromechanical Actuators," SPIE Conference on Smart Structures and Materials '93, February, 1993.

I. Kolmanovsky, "Planar Reorientation of a Free-Free Beam in Space using Embedded Electromechanical Actuators," The University of Michigan, February, 1993.

I. Kolmanovsky and N. H. McClamroch, "Planar Reorientation of a Free-Free Beam in Space using Embedded Electromechanical Actuators," 1993 American Control Conference, June, 1993.

P. J. McNally and N. H. McClamroch, "Space Station Attitude Disturbances Arising from Internal Motions," 1993 American Control Conference, June, 1993.

Consequently, the results of the supported research have been disseminated widely via personal presentations as well as through written publications.

Research Publications: The results of our research have been documented in written form and published in archival journals and in conference proceedings. A summary of these outcomes is indicated:

Ph.D. Dissertations

[1] M. Reyhanoglu, "Control and Stabilization of Nonholonomic Dynamic Systems," Department of Aerospace Engineering, June, 1992.

[2] H. Krishnan, "Control of Nonlinear Systems with Applications to Constrained Robots and Spacecraft Attitude Stabilization," Department of Aerospace Engineering, September, 1992.

Publications in Archival Journals

[3] A. M. Bloch, M. Reyhanoglu and N. H. McClamroch, "Control and Stabilization of Nonholonomic Dynamic Systems," IEEE Transactions on Automatic Control, Vol. 37, No. 11, 1992, 1746-1757.

[4] M. Reyhanoglu and N. H. McClamroch, "Reorientation Maneuvers of Planar Multibody Systems in Space using Internal Controls," AIAA Journal of Guidance, Control and Dynamics, Vol. 15, No. 6, 1992, 1475-1480.

[5] H. Krishnan, N. H. McClamroch, and M. Reyhanoglu, "Attitude Stabilization of a Rigid Spacecraft using Two Control Torques: A Nonlinear Control Approach Based on the Spacecraft Attitude Dynamics," submitted for publication.

[6] H. Krishnan, N. H. McClamroch, and M. Reyhanoglu, "Attitude Stabilization of a Rigid Spacecraft using Two Momentum Wheel Actuators" submitted for publication.

Publication in Edited Book

[7] M. Reyhanoglu and N. H. McClamroch, "Nonlinear Attitude Control of Planar Structures in Space using only Internal Controls," Proceedings of Workshop on Problems in Sensing, Identification and Control of Flexible Structures, The Fields Institute for Research in Mathematical Sciences, Waterloo, Ont., to appear.

Conference Publications

[8] H. Krishnan, N. H. McClamroch and M. Reyhanoglu, "On the Attitude Stabilization of a Rigid Spacecraft using Two Control Torques," American Control Conference, 1992, 1990-1995.

[9] H. Krishnan, N. H. McClamroch and M. Reyhanoglu, "Attitude Stabilization of a Rigid Spacecraft using Momentum Wheel Actuators Operating in a Failure Mode," World Space Congress, August, 1992, IAF-92-0035.

[10] H. Krishnan, M. Reyhanoglu and N. H. McClamroch, "Attitude Stabilization of a Rigid Spacecraft using Gas Jet Actuators in a Failure Mode," Proceedings of IEEE Conference on Decision and Control, December, 1992, Tucson, Az., 1612-1617.

[11] I. Kolmanovsky and N. H. McClamroch, "Planar Reorientation of a Free-Free Beam in Space using Embedded Electromechanical Actuators," SPIE Conference on Smart Structures and Materials '93.

[12] I. Kolmanovsky and N. H. McClamroch, "Efficient Reorientation of a Deformable Body in Space: A Free-Free Beam Example," 1993 American Control Conference.

[13] P. J. McNally and N. H. McClamroch, "Space Station Attitude Disturbances Arising from Internal Motions," 1993 American Control Conference.

Copies of the Ph.D. dissertation abstracts [1,2] are enclosed; copies of the above papers [3-13] are enclosed. One complete copy of each of the Ph.D. dissertations [1,2] has been sent to the NASA Technical Officer, S. Joshi.

Conclusion

Substantial progress has been made in our research on spacecraft with multiple articulated payloads. We have plans to continue our research in this important area The main areas of study will be: formal approaches to design of stabilizing (discontinuous) feedback control laws , extension of our work on attitude control of spacecraft involving flexible components, and extension of our work on non-planar spacecraft attitude control.

The support from NASA has been critical in providing us an opportunity to make major advances in the development of the theory and applications of this new class of space based control problems.

It should be noted that the published papers provide acknowledgement to NASA for financial support of part of this research. In some cases, acknowledgement is also given to the National Science Foundation for their support of our related research on control of nonholonomic dynamic systems.

ABSTRACT

CONTROL AND STABILIZATION OF NONHOLONOMIC DYNAMIC SYSTEMS

by

Mahmut Reyhanoglu

Chairperson: N.H. McClamroch

A theoretical framework is established for the control of nonholonomic dynamic systems, i.e. dynamic systems with nonintegrable constraints. In particular, we emphasize control properties for nonholonomic systems that have no counterpart in holonomic systems. A model for nonholonomic dynamic systems is first presented in terms of differential- algebraic equations defined on a phase space. A reduction procedure is carried out to obtain reduced order state equations. Feedback is then used to obtain a control system in a normal form. The assumptions guarantee that the resulting normal form equations necessarily contain a nontrival drift vector field. Conditions for smooth (C^∞) asymptotic stabilization to an m-dimensional equilibrium manifold are presented; we also demonstrate that a single equilibrium solution cannot be asymptotically stabilized using continuous static or dynamic state feedback. However, any equilibrium is shown to be strongly accessible and small time

locally controllable. An approach using geometric phases is developed as a basis for the control of Caplygin dynamical systems, i.e. nonholonomic systems with certain symmetry properties which can be expressed by the fact that the constraints are cyclic in certain variables. The theoretical development is applied to physical examples of systems that we have studied in detail elsewhere: the control of a knife edge moving on a plane surface and the control of a wheel rolling without slipping on a plane surface. The results are also applied to the reorientation of planar multibody systems using joint torque inputs and to the reorientation of a rigid spacecraft using momentum wheel actuators, since in these examples conservation of angular momentum gives rise to nonintegrable motion invariants.

ABSTRACT

CONTROL OF NONLINEAR SYSTEMS WITH APPLICATIONS TO CONSTRAINED ROBOTS AND SPACECRAFT ATTITUDE STABILIZATION

by

Hariharan Krishnan

Chairman: N. Harris McClamroch

This thesis is organized in two parts. In Part 1, control systems described by a class of nonlinear differential and algebraic equations are introduced. A procedure for local stabilization based on a local state realization is developed. An alternative approach to local stabilization is developed based on a classical linearization of the nonlinear differential-algebraic equations. A theoretical framework is established for solving a tracking problem associated with the differential-algebraic system. First, a simple procedure is developed for the design of a feedback control law which ensures, at least locally, that the tracking error in the closed loop system lies within any given bound if the reference inputs are sufficiently slowly varying. Next, by imposing additional assumptions, a procedure is developed for the design of a feedback control law which ensures that the tracking error in the closed loop system approaches zero exponentially for reference inputs which are not necessarily slowly varying. The control design methodologies are used for simultaneous force and position control in constrained robot systems. The differential-algebraic equations are shown to characterize the slow dynamics of a certain nonlinear control system in nonstandard singularly perturbed form.

In Part 2, the attitude stabilization (reorientation) of a rigid spacecraft using only two control torques is considered. First, the case of momentum wheel actuators is considered. The complete spacecraft dynamics are not controllable. However, the spacecraft dynamics are small time locally controllable in a reduced sense. The reduced spacecraft dynamics cannot be asymptotically stabilized using continuous feedback, but a discontinuous feedback control strategy is constructed. Next, the case of gas jet actuators is considered. If the uncontrolled principal axis is not an axis of symmetry, the complete spacecraft dynamics are small time locally controllable. However, the spacecraft attitude cannot be asymptotically stabilized using continuous feedback, but a discontinuous stabilizing feedback control strategy is constructed. If the uncontrolled principal axis is an axis of symmetry, the complete spacecraft dynamics cannot be stabilized. However, the spacecraft dynamics are small time locally controllable in a reduced sense. The reduced spacecraft dynamics cannot be asymptotically stabilized using continuous feedback, but again a discontinuous feedback control strategy is constructed.

Control and Stabilization of Nonholonomic Dynamic Systems

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Anthony M. Bloch, *Member, IEEE*, Mahmut Reyhanoglu, *Member, IEEE*,
and N. Harris McClamroch, *Fellow, IEEE*

Abstract—A theoretical framework is established for the control of nonholonomic dynamic systems, i.e., dynamic systems with nonintegrable constraints. In particular, we emphasize control properties for nonholonomic systems that have no counterpart in holonomic systems. A model for nonholonomic dynamic systems is first presented in terms of differential-algebraic equations defined on a phase space. A reduction procedure is carried out to obtain reduced-order state equations. Feedback is then used to obtain a nonlinear control system in a normal form. The assumptions guarantee that the resulting normal form equations necessarily contain a nontrival drift vector field. Conditions for smooth (C^∞) asymptotic stabilization to an *m*-dimensional equilibrium manifold are presented; we also demonstrate that a single equilibrium solution cannot be asymptotically stabilized using continuous state feedback. However, any equilibrium is shown to be strongly accessible and small time locally controllable. Finally, an approach using geometric phases is developed as a basis for the control of Caplygin dynamical systems, i.e., nonholonomic systems with certain symmetry properties which can be expressed by the fact that the constraints are cyclic in certain variables. The theoretical development is applied to physical examples of systems that we have studied in detail elsewhere: the control of a knife edge moving on a plane surface and the control of a wheel rolling without slipping on a plane surface. The results of the paper are also applied to the control of a planar multibody system using angular momentum preserving control inputs since the angular momentum may be viewed as a nonholonomic constraint which is an invariant of the motion.

I. INTRODUCTION

NUMEROUS papers have been published in recent years on the control of systems with holonomic constraints. The work of the authors includes McClamroch and Bloch in [17], McClamroch and Wang in [18]. The earliest work on control of nonholonomic systems (that we are aware of) is by Brockett in [6]. Bloch in [2] has examined several control theoretic issues which pertain to both holonomic and nonholonomic systems in a very general form. Related work in robotics [14], [15], [20] and multibody systems [10], [11], [12], [25], [29] has recently

appeared. Our recent work in [3], [4], [22], [23] has also emphasized several classes of physical problems. All of this work has demonstrated that there is a common theoretical framework for a large class of control problems for mechanical systems with nonholonomic constraints. In this paper, we identify that common theoretical framework. Our development is based on the formulation of nonholonomic dynamics by Neimark and Fufaev [21] and the modern formulation of nonlinear geometric control.

II. MODELS OF NONHOLONOMIC SYSTEMS

We consider the class of nonholonomic systems described by the equations

$$M(q)\ddot{q} + F(q,\dot{q}) = J'(q)\lambda + B(q)u \qquad (1)$$

$$J(q)\dot{q} = 0. \qquad (2)$$

Note that a "prime" denotes transpose. We refer to q as an n-vector of generalized configuration variables, $\dot{q}$ as an n-vector of generalized velocity variables, and $\ddot{q}$ as an n-vector of generalized acceleration variables; in addition, λ is an m-vector of constraint multipliers and u is an r-vector of control input variables, where $r \geq n - m$. The $n \times n$ matrix function $M(q)$ is assumed to be symmetric and positive definite, $F(q,\dot{q})$ is an n-vector function, $J(q)$ denotes an $m \times n$ matrix function which is assumed to have full rank and $B(q)$ is a full rank $n \times r$ matrix function. All of these functions are assumed to be smooth (C^∞) and defined on an appropriate open subset of the $(q,\dot{q})$ phase space. The formulation could be given in terms of a system defined on the tangent bundle of a C^∞ manifold; we have not made such a generalization since it is direct. Various assumptions about the control input variables are indicated subsequently.

Differential-algebraic equations of the above form are known to arise for (uncontrolled) nonholonomic systems: see [1] and [21] for many examples. Here, we note that the classical approach for the formulation of constrained dynamics as described in [21] is used. This is in contrast to the variational approach, or "vakonomic" theory (see e.g. [1]). We also note that a Hamiltonian formulation can be developed.

We have assumed that the $m \times n$ matrix $J(q)$ has full rank; hence, there is no loss of generality in assuming that the configuration variables are ordered so that the last n columns of the matrix $J(q)$ constitute an $m \times m$ locally invertible matrix function, i.e., the matrix $J(q)$ can be

expressed as $[J_1(q)J_2(q)]$, where $J_1(q)$ is an $m \times (n - m)$ matrix function and $J_2(q)$ is an $m \times m$ locally nonsingular matrix function. The columns of the $n \times (n - m)$ matrix function

$$C(q) = \begin{bmatrix} I \\ -\bar{J}(q) \end{bmatrix} \quad (3)$$

where I is the $(n - m) \times (n - m)$ identity matrix and $\bar{J}(q) = J_2^{-1}(q)J_1(q)$ is a locally smooth $m \times (n - m)$ matrix function, span the null space of $J(q)$. Formally, the rows of $J(q)$ constitute m linearly independent smooth covector fields defined on the configuration space; these covector fields span a codistribution Ω and the annihilator of the codistribution Ω, denoted $\Omega^{\perp}$, is spanned by $n - m$ linearly independent smooth vector fields

$$\tau_j = \sum_{i=1}^{n} C_{ij}(q) \frac{\partial}{\partial q_i}, \quad j = 1, \cdots, n - m. \quad (4)$$

We present the following definition.

Definition 1 [30]: Consider the following nondecreasing sequence of locally defined distributions

$$N_1 = \Omega^{\perp}$$

$$N_k = N_{k-1} + \text{span} \{[X, Y]|X \in N_1, Y \in N_{k-1}\}.$$

There exists an integer k^* such that

$$N_k = N_{k^*}.$$

for all $k > k^*$. If dim $N_{k^*} = n$ and $k^* > 1$, then the constraints (2) are called completely nonholonomic and the smallest (finite) number k^* is called the degree on nonholonomy.

In this paper, it is assumed that constraint equations (2) are completely nonholonomic with nonholonomy degree k^*. Note that for this to hold $n - m$ must be strictly greater than one. Note also that since the constraints are nonholonomic, there is in fact no explicit restriction on the values of the configuration variables.

We also assume that the matrix product $C'(q)B(q)$ is full rank. As will be seen in Section IV, this assumption guarantees that all $n - m$ degrees of freedom can be (independently) actuated.

The constraints (2) define a $(2n - m)$-dimensional smooth submanifold

$$M = \{(q, \dot{q})|J(q)\dot{q} = 0\} \quad (5)$$

of the phase space. This manifold M plays a critical role in the concept of solutions and the formulation of control and stabilization problems associated with (1) and (2).

We begin by making it clear that (1) and (2) do represent well-posed models in the sense that the associated initial value problem has a unique solution, at least locally.

Definition 2: A pair of vector functions $(q(t), \lambda(t))$ defined on an interval $[0, T)$ is a solution of the initial value problem defined by (1) and (2) and the initial data $(q_0, \dot{q}_0)$ if $q(t)$ is at least twice differentiable, $\lambda(t)$ is integrable,

the vector functions $(q(t), \lambda(t))$ satisfy the differential-algebraic equations (1) and (2) almost everywhere on their domain of definition, and the initial conditions satisfy $(q(0), \dot{q}(0)) = (q_0, \dot{q}_0)$.

The following existence and uniqueness result has been obtained.

Theorem 1 [3]: Assume that the control input function $u: [0, T) \rightarrow R^r$ is a given bounded and measurable function for some $T > 0$. If the initial data satisfy $(q_0, \dot{q}_0) \in M$, then there exists a unique solution (at least locally defined) of the initial value problem corresponding to (1) and (2) which satisfies $(q(t), \dot{q}(t)) \in M$ for each t for which the solution is defined.

Since the differential-algebraic equations (1) and (2) define a smooth vector field on M, a number of other results could be stated, including conditions for continuous dependence of the solution on initial conditions and parameters, conditions for nonexistence of finite escape times, etc. Such results are important, but they are not given here since they are easily obtained. We subsequently use the notation $(Q(t, q_0, \dot{q}_0), \Lambda(t, q_0, \dot{q}_0))$ to denote the solution of (1) and (2) at time $t \geq 0$ corresponding to the initial conditions $(q_0, \dot{q}_0)$. Thus, for each initial condition $(q_0, \dot{q}_0) \in M$ and each bounded, measurable input function $u: [0, T) \rightarrow R^r$, $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \in M$ holds for all $t \geq 0$ where the solution is defined.

A particularly important class of solutions are the equilibrium solutions of (1) and (2). A solution is an equilibrium solution if it is a constant solution; note that if (q^e, λ^e) is an equilibrium solution, we refer to q^e as an equilibrium configuration. The following result should be clear.

Theorem 2: Suppose that $u(t) = 0$, $t \geq 0$. The set of equilibrium configurations of (1) and (2) is given by

$$\{q|F(q, 0) - J'(q)\lambda = 0 \text{ for some } \lambda \in R^m\}.$$

An equivalent expression for the set of equilibrium configurations is

$$\{q|C'(q)F(q, 0) = 0\}.$$

III. CLOSED-LOOP MODELS OF NONHOLONOMIC SYSTEMS

We are interested in feedback control of the form $u = U(q, \dot{q})$ where $U: M \rightarrow R^r$; the corresponding closed loop is described by

$$M(q)\ddot{q} + F(q, \dot{q}) = J'(q)\lambda + B(q)U(q, \dot{q}) \quad (6)$$

$$J(q)\dot{q} = 0. \quad (7)$$

We point out the obvious fact that the closed loop is still defined in terms of the nonholonomic constraint equations.

Suppose $U(q, \dot{q})$ is a smooth function; if the initial conditions satisfy $(q_0, \dot{q}_0) \in M$, then there exists a unique solution $(q(t), \lambda(t))$ (at least locally defined) of the initial value problem corresponding to (6) and (7) which satisfies $(q(t), \dot{q}(t)) \in M$ for each t for which the solution is defined.

The set of equilibrium configurations of (6) and (7) is given by

$$\{q|F(q,0) - J'(q)\lambda = B(q)U(q,0) \text{ for some } \lambda \in R^m\}$$

which is a smooth submanifold of the configuration space. An equivalent expression for the equilibrium submanifold of the configuration space is

$$\{q|C'(q)[F(q,0) - B(q)U(q,0)] = 0\}.$$

We remark that generically the equilibrium manifold has dimension at least m. On the other hand, for certain cases, there may not be even a single equilibrium configuration (e.g., the uncontrolled dynamics of a ball on an inclined plane). However, since we have assumed that $C'(q)B(q)$ is full rank, we can always introduce an equilibrium manifold of dimension at least m by appropriate choice of input.

We now formulate a stabilization problem for nonholonomic systems described by (1) and (2). A suitable stability definition for the closed-loop system described by (6) and (7) is first introduced.

Definition 3: Assume that $u = U(q,\dot{q})$. Let $M_s = \{(q,\dot{q})|\dot{q} = 0\}$ be an embedded submanifold of M. Then M_s is locally stable if for any neighborhood $U \supset M_s$ there is a neighborhood V of M_s with $U \supset V \supset M_s$ such that if $(q_0, \dot{q}_0) \in V \cap M$ then the solution of (6) and (7) satisfies $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \in U \cap M$ for all $t \geq 0$. If, in addition, $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \to (q_s, 0)$ as $t \to \infty$ for some $(q_s, 0) \in M_s$, then we say that M_s is a locally asymptotically stable equilibrium manifold of (6) and (7).

Note that if $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \to (q_s, 0)$ as $t \to \infty$ for some $(q_s, 0) \in M_s$, it follows that there is $\lambda_s \in R^m$ such that $\Lambda(t, q_0, \dot{q}_0) \to \lambda_s$ as $t \to \infty$.

The usual definition of local stability corresponds to the case that M_s is a single equilibrium solution; the more general case is required in the present paper.

The existence of a feedback function so that a certain equilibrium manifold is asymptotically stable is of particular interest; hence, we introduce the following.

Definition 4: The system defined by (1) and (2) is said to be locally asymptotically stabilizable to a smooth equilibrium manifold M_s in M if there exists a feedback function $U: M \to R^r$ such that, for the associated closed-loop equations (6) and (7), M_s is locally asymptotically stable.

If there exists such a feedback function which is smooth on M then we say that (1) and (2) are smoothly asymptotically stabilizable to M_s; of course it is possible (and we subsequently show that it is generic in certain cases) that (1) and (2) might be asymptotically stabilizable to M_s but not smoothly (even not continuously) asymptotically stabilizable to M_s.

IV. NORMAL FORM EQUATIONS FOR NONHOLONOMIC CONTROL SYSTEMS

A number of approaches have been suggested for eliminating the constraint multipliers so that a minimum set of differential equations is obtained: the reduced differential equations characterize the control dependent motion on the constraint manifold.

We first emphasize that the reduced state space is $2n - m$ dimensional. The state of the system can be specified by the n-vector of configuration variables and an $(n - m)$-vector of kinematic variables. Let $q = (q_1, q_2)$ be a partition of the configuration variables corresponding to the partitioning of the matrix function $J(q)$ introduced previously. Then consider the following relation

$$\dot{q} = C(q)\dot{q}_1$$

where $C(q)$ is defined by (3). Taking time derivatives yields

$$\ddot{q} = C(q)\ddot{q}_1 + \dot{C}(q)\dot{q}_1$$

where $\dot{C}(q)$ denotes the time derivative of $C(q)$. Substituting this into (1) and multiplying both sides of the resulting equation by $C'(q)$ gives

$$C'(q)M(q)C(q)\ddot{q}_1$$
$$= C'(q)\left[B(q)u - F(q, C(q)\dot{q}_1) - M(q)\dot{C}(q)\dot{q}_1\right]. \tag{8}$$

Note that $C'(q)M(q)C(q)$ is an $(n - m) \times (n - m)$ symmetric positive definite matrix function.

We also assume that $r = n - m$ (for simplicity). Then the matrix product $C'(q)B(q)$ is locally invertible. Consequently for any $u \in R^r$ there is unique $v \in R^{n-m}$ which satisfies

$$C'(q)\left[B(q)u - F(q, C(q)\dot{q}_1) - M(q)\dot{C}(q)\dot{q}_1\right]$$
$$= C'(q)M(q)C(q)v. \tag{9}$$

(Note that if $r > n - m$ then v can be chosen to depend smoothly on the variables $(q, \dot{q}_1, u)$). This assumption guarantees that the reduced configuration variables satisfy the linear equations

$$\ddot{q}_1 = v.$$

Define the following state variables

$$x_1 = q_1,$$
$$x_2 = q_2,$$
$$x_3 = \dot{q}_1.$$

Then the normal form equations are given by

$$\dot{x}_1 = x_3 \tag{10}$$
$$\dot{x}_2 = -\bar{J}(x_1, x_2)x_3 \tag{11}$$
$$\dot{x}_3 = v. \tag{12}$$

Equations (10)–(12) define a drift vector field $f(x) = (x_3, -\bar{J}(x_1, x_2)x_3, 0)$ and control vector fields $g_i(x) = (0, 0, e_i)$, where e_i is the ith standard basis vector in R^{n-m}, $i = 1, \cdots, n - m$, according to the standard control system form

$$\dot{x} = f(x) + \sum_{i=1}^{n-m} g_i(x)v_i. \tag{13}$$

We consider local properties of (10)–(12), near an equilibrium solution $(x_1^e, x_2^e, 0)$.

Note that the normal form equations (10)–(12) are a special case of the normal form equations in [8]. In particular, the zero dynamics equation of (10) and (12), corresponding to the output x_1, is given by

$$\dot{x}_2 = 0$$

and it is not locally asymptotically stable. The fact that the zero dynamics is a linear system with all zero eigenvalues, means that (10)–(12) are critically minimum phase at the equilibrium; this has important implications in terms of local asymptotic stabilizability of the original equations (1) and (2).

V. STABILIZATION TO AN EQUILIBRIUM MANIFOLD USING SMOOTH FEEDBACK

In this section, we study the problem of stabilization of (1) and (2) to a smooth equilibrium submanifold of M defined by

$$N_e = \{(q, \dot{q}) | \dot{q} = 0, s(q) = 0\}$$

where $s(q)$ is a smooth $n - m$ vector function. We show that with appropriate assumptions, there exists a smooth feedback such that the closed loop is locally asymptotically stable to N_e.

The smooth stabilization problem is the problem of giving conditions so that there exists a smooth feedback function $U: M \rightarrow R^r$ such that N_e is locally asymptotically stable. Of course, we are interested not only in demonstrating that such a smooth feedback exists but also in indicating how such an asymptotically stabilizing smooth feedback can be constructed.

Note that in this section, we consider nonholonomic control systems whose normal form equations satisfy the property that if $q_1(t)$ and $\dot{q}_1(t)$ are exponentially decaying functions, then the solution to

$$\dot{q}_2 = -\bar{J}(q_1(t), q_2)\dot{q}_1(t)$$

is bounded (all the physical examples of nonholonomic systems, of which we are aware, satisfy this assumption).

Note also that the first and second time derivatives of $s(q)$ are given by

$$\dot{s} = \frac{\partial s(q)}{\partial q} C(q)\dot{q}_1,$$

$$\ddot{s} = \frac{\partial}{\partial q}\left(\frac{\partial s(q)}{\partial q}C(q)\dot{q}_1\right)C(q)\dot{q}_1 + \frac{\partial s(q)}{\partial q}C(q)v.$$

Theorem 3: Assume that the above solution property holds. Then the nonholonomic control system, defined by (1) and (2) is locally asymptotically stabilizable to

$$N_e = \{(q, \dot{q}) | \dot{q} = 0, s(q) = 0\} \qquad (14)$$

using smooth feedback, if the transversality condition

$$\det\left(\frac{\partial s(q)}{\partial q_1}\right)\det\left(\frac{\partial s(q)}{\partial q}C(q)\right) \neq 0 \qquad (15)$$

is satisfied.

Proof: It is sufficient to analyze the system in the normal form (10)–(12). By the transversality condition, the change of variables from $(q_1, q_2, \dot{q}_1)$ to $(s, q_2, \dot{s})$ is a diffeomorphism.

Let

$$v = -\left(\frac{\partial s(q)}{\partial q}C(q)\right)^{-1}\left[\frac{\partial}{\partial q}\left(\frac{\partial s(q)}{\partial q}C(q)\dot{q}_1\right)C(q)\dot{q}_1 \right.$$
$$\left. + K_1\frac{\partial s(q)}{\partial q}C(q)\dot{q}_1 + K_2 s(q)\right]$$

where K_1 and K_2 are symmetric positive definite $(n - m) \times (n - m)$ constant matrices. Then, obviously

$$\ddot{s} + K_1\dot{s} + K_2 s = 0$$

is asymptotically stable so that $(s, \dot{s}) \rightarrow 0$ as $t \rightarrow \infty$. The remaining system variables satisfy (11) of the normal form equations (with $x_2 = q_2$), and, by our assumption on the constraint matrix $\bar{J}$, these variables remain bounded for all time. Thus $(q(t), \dot{q}(t)) \rightarrow N_e$ as $t \rightarrow \infty$.

Equations (1) and (2) can be smoothly asymptotically stabilized to the m dimensional equilibrium manifold specified by (14). Condition (15) depends on the specific partitioning of the configuration variables corresponding to the constraint equations (2).

VI. STABILIZATION TO AN EQUILIBRIUM SOLUTION USING PIECEWISE ANALYTIC FEEDBACK

The results in the previous section demonstrate that smooth feedback can be used to asymptotically stabilize certain smooth manifolds N_e in M, where the dimension of N_e is equal to the number m of independent constraints. Consequently, those results do not guarantee smooth asymptotic stabilization to a single equilibrium solution if $m \geq 1$.

In fact, there is no C^1 feedback which can asymptotically stabilize the closed-loop system to a single equilibrium solution. Suppose that there is a C^1 feedback which asymptotically stabilizes, for example, the origin. Then it follows that there is an equilibrium manifold of dimension m containing the origin; that is, the origin is not isolated, which contradicts the assumption that it is asymptotically stable. We state this formally.

Theorem 4: Let $m \geq 1$ and let $(q^e, 0)$ denote an equilibrium solution in M. The nonholonomic control system, defined by (1) and (2), is not asymptotically stabilizable using C^1 state feedback to $(q^e, 0)$.

Proof: A necessary condition for the existence of a C^1 asymptotically stabilizing state feedback law for system (10)–(12) is that the image of the mapping

$$(x_1, x_2, x_3, v) \rightarrow (x_3, -\bar{J}(x_1, x_2)x_3, v)$$

contains some neighborhood of zero (see Brockett [7]). No points of the form

$$\begin{pmatrix} 0 \\ \epsilon \\ \alpha \end{pmatrix}, \qquad \epsilon \neq 0 \text{ and } \alpha \in R^{n-m} \text{ arbitrary}$$

are in its image: it follows that Brockett's necessary condition is not satisfied. Hence, system (10)–(12) cannot be asymptotically stabilized to $(q_1^\epsilon, q_2^\epsilon, 0)$ by a C^1 state feedback law. Consequently, the nonholonomic control system, defined by (1) and (2), is not asymptotically stabilizable to $(q^\epsilon, 0)$ using a C^1 state feedback.

We remark that even C^0 (continuous) state feedback (which results in existence of unique trajectories) is ruled out since Brockett's necessary condition is not satisfied [31].

A corollary of Theorem 4 is that a single equilibrium solution of (1) and (2) cannot by asymptotically stabilized using linear feedback nor can it be asymptotically stabilized using feedback linearization or any other control design approach that uses smooth feedback. Of course, it may be that a single equilibrium solution simply cannot be asymptotically stabilized or it may be that any asymptotically stabilizing state feedback is necessarily not C^0. However, in the subsequent sections, we show that a single equilibrium can be asymptotically stabilized by use of piecewise analytic state feedback.

We first demonstrate that the system of normal form equations (10)–(12), and hence the nonholonomic control system defined by (1) and (2), does indeed satisfy certain strong local controllability properties. In particular, we show that the system is strongly accessible and that the system is small time locally controllable at any equilibrium. These results not only provide a theoretical basis for the use of inherently nonlinear control strategies but they also suggest constructive procedures for the desired control strategies.

Theorem 5: Let $m \geq 1$ and let $(q^\epsilon, 0)$ denote an equilibrium solution in M. The nonholonomic control system defined by (1) and (2) is strongly accessible at $(q^\epsilon, 0)$.

Proof: It suffices to prove that system (10)–(12) is strongly accessible at the origin. Let I denote the set $\{1, \cdots, n - m\}$. The drift and control vector fields can be expressed as

$$f = \sum_{j=1}^{n-m} x_{3,j} \tau_j,$$

$$g_i = \frac{\partial}{\partial x_{3,i}}, \qquad i \in I,$$

where

$$\tau_j = \frac{\partial}{\partial x_{1,j}} - \sum_{i=1}^{n-m} \bar{J}_{ij}(x_1, x_2) \frac{\partial}{\partial x_{2,i}}, \qquad j \in I$$

are considered as vector fields on the (x_1, x_2, x_3) state space. It can be verified that

$$[g_{i_1}, f] = \tau_{i_1}, \qquad i_1 \in I;$$

$$[g_{i_2}, [f, [g_{i_1}, f]]] = [\tau_{i_2}, \tau_{i_1}], \qquad i_1, i_2 \in I;$$

$$\vdots$$

$$[g_{i_{k*}}, [f, \cdots, [g_{i_2}, [f, [g_{i_1}, f]]] \cdots]]$$
$$= [\tau_{i_{k*}}, \cdots, [\tau_{i_2}, \tau_{i_1}] \cdots], \qquad i_k \in I, 1 \leq k \leq k^*$$

hold, where k^* denotes the nonholonomy degree. Let

$$\mathscr{G} = \text{span}\{g_i, i \in I\},$$

$$\mathscr{H} = \text{span}\left\{[g_{i_1}, f], \cdots, \left[g_{i_{k*}}, \left[f, \cdots, [g_{i_1}, [f, [g_{i_1}, f]]] \cdots\right]\right]\right\};$$

$$i_k \in I, 1 \leq k \leq k^*\right\}.$$

Note that $\dim \mathscr{G}(0) = n - m$ and $\dim \mathscr{H}(0) = n$ since the distribution defined by the constraints is completely nonholonomic; moreover $\dim\{\mathscr{G}(0) \cap \mathscr{H}(0)\} = 0$. It follows that the strong accessibility distribution

$$\mathscr{L}_0 = \text{span}\{X: X \in \mathscr{G} \cup \mathscr{H}\}$$

has dimension $2n - m$ at the origin. Hence, the strong accessibility rank condition [28] is satisfied at the origin. Thus system (10)–(12) is strongly accessible at the origin. Consequently, the nonholonomic control system, defined by (1) and (2), is strongly accessible at $(q^\epsilon, 0)$.

Theorem 6: Let $m \geq 1$ and let $(q^\epsilon, 0)$ denote an equilibrium solution in M. The nonholonomic control system, defined by (1) and (2), is small time locally controllable at $(q^\epsilon, 0)$.

Proof: It suffices to prove that system (10)–(12) is small time locally controllable at the origin.

The proof involves the notion of the degree of a bracket. To make this notion well defined we consider, as in [27], a Lie algebra of indeterminates and an associated evaluation map (on vector fields) as follows.

Let $X = (X_0, \cdots, X_{n-m})$ be a finite sequence of indeterminates. Let $A(X)$ denote the free associative algebra over R generated by the X_j, let $L(X)$ denote the Lie subalgebra of $A(X)$ generated by $X_0, \cdots, X_{n-m}$ and let $Br(X)$ be the smallest subset of $L(X)$ that contains $X_0, \cdots, X_{n-m}$ and is closed under bracketing.

Now consider the vector fields $f, g_1, \cdots, g_{n-m}$ on the manifold M. Each $f, g_1, \cdots, g_{n-m}$ is a member of $D(M)$, the algebra of all partial differential operators on $C^\infty(M)$, the space of C^∞ real-valued functions on M. Now let $g_0 = f$, and let $g = (g_0, \cdots, g_{n-m})$ and define the evaluation map

$$Ev(g): A(X) \to D(M)$$

obtained by substituting the g_j for the X_j, i.e.,

$$Ev(g)\left(\sum_I a_I X_I\right) = \sum_I a_I g_I$$

where $g_I = g_{i_1} g_{i_2} \cdots g_{i_k}$, $I = (i_1, \cdots, i_k)$. Note that the kernel of $Ev(g): A(X) \to A(g)$ is the set of all algebraic identities satisfied by the g_i while the kernel of $Ev(g): L(X) \to L(g)$ is the set of Lie algebraic identities satisfied by g_i.

Now, let B be a bracket in $Br(X)$. We define the degree of a bracket to be $\delta(B) = \sum_{i=0}^{n-m}\delta^i(B)$, where $\delta^0(B), \delta^1(B), \cdots, \delta^{n-m}(B)$ denote the number of times $X_0, \cdots, X_{n-m}$, respectively, occur in B. The bracket B is called "bad" if $\delta^0(B)$ is odd and $\delta^i(B)$ is even for each i, $i = 1, \cdots, n - m$. The theorem of Sussmann tells us the system is STLC at the origin if it satisfies the accessibility rank condition; and if B is "bad" there exist brackets $C_1, \cdots, C_k$ of lower degree in $Br(X)$ such that

$$Ev_0(g)(\beta(B)) = \sum_{i=1}^{k} \xi_i Ev_0(g)(C_i)$$

where Ev_0 denotes the evaluation map at the origin and $(\xi_1, \cdots, \xi_k) \in R^k$. Here, $\beta(B)$ is the symmetrization operator, $\beta(B) = \sum_{\pi \in S_{n-m}} \bar{\pi}(B)$, where $\pi \in S_{n-m}$, the group of permutations of $\{1, \cdots, n - m\}$ and for $\pi \in S_{n-m}$, $\bar{\pi}$ is the automorphism of $L(X)$ which fixes X_0 and sends X_i to $X_{\pi(i)}$.

By Theorem 5, the system is accessible at the origin.

The brackets in $\mathcal{G}$ are obviously "good" (not of the type defined as "bad") and $\delta^0(h) = \sum_{j=1}^{n-m}\delta^j(h) \ \forall h \in \mathcal{H}$; thus $\delta(h)$ is even for all h in $\mathcal{H}$, i.e., $\mathcal{H}$ contains "good" brackets only. It follows that the tangent space $T_0 M$ to M at the origin is spanned by the brackets that are all "good." Next we show that the brackets that might be "bad" vanish at the origin. First note that f vanishes at the origin. Let B denote a bracket satisfying $\delta(B) > 1$. If B is a "bad" bracket then, necessarily, $\delta^0(B) \neq \sum_{j=1}^{n-m}\delta^j(B)$, i.e., $\delta(B)$ must be odd. It can be verified that if $\delta^0(B) < \sum_{j=1}^{n-m}\delta^j(B)$ then B is identically zero and if $\delta^0(B) > \sum_{j=1}^{n-m}\delta^j(B)$ the B is of the form $\sum_{i=1}^{n-m} r_i(x_3)Y_i(x_1, x_2)$, for some vector fields $Y_i(x_1, x_2)$, $i \in I$, where $r_i(x_3)$, $i \in I$, are homogeneous functions of degree $(\delta^0(B) - \sum_{j=1}^{n-m}\delta^j(B))$ in x_3; thus B vanishes at the origin. Consequently, the Sussmann condition is satisfied. Hence, system (10)–(12) is small time locally controllable at the origin. It follows that, the nonholonomic control system, defined by (1) and (2), is small time locally controllable at $(q^c, 0)$.

VII. CONSTRUCTION OF PIECEWISE ANALYTIC STABILIZING CONTROLLERS FOR CAPLYGIN SYSTEMS

Our recent work on control of nonholonomic systems in [4], [22], [23] has identified a large class of physical systems, which are referred to as "controlled Caplygin systems." Our subsequent results are developed for this class of systems.

We first describe the class of controlled Caplygin systems. We use the notation introduced previously. If the functions used in defining (1) and (2) do not depend explicitly on the configuration variables q_2, so that the system is locally described by

$$M(q_1)\ddot{q} + F(q_1, \dot{q}) = J'(q_1)\lambda + B(q_1)u \quad (16)$$

$$\bar{J}(q_1)\dot{q}_1 + \dot{q}_2 = 0 \quad (17)$$

where $\bar{J}(q_1)$ is an $m \times (n - m)$ matrix function, then the uncontrolled system is called a "Caplygin system" [21]. In terms of the Lagrangian formalism for the problem this corresponds to the Lagrangian of the free problem being cyclic in (i.e., independent of) the variables q_2, while the constraints are also independent of q_2. The cyclic property is an expression of symmetries in the problem, such symmetries occurring naturally in many physical examples. More generally, if a system can be expressed in the form (16) and (17) using feedback, then we refer to it as a "controlled Caplygin system."

For the Caplygin system described by (16) and (17), (8) becomes

$$C'(q_1)M(q_1)C(q_1)\ddot{q}_1$$
$$= C'(q_1)\left[B(q_1)u - F(q_1, C(q_1)\dot{q}_1) - M(q_1)\dot{C}(q_1)\dot{q}_1 \right]$$
$$(18)$$

which is an equation in the phase variables $(q_1, \dot{q}_1)$ only. As a consequence, q_1 constitutes a reduced configuration space for the system (16) and (17). This reduced configuration space is also referred to as the "base space" (or "shape space") of the system. The term shape space (see [10], [11], [12], [14], [15]) arises from the theory of coupled mechanical systems, where it refers to the internal degrees of freedom of the system. It is possible to consider control theoretic problems which can be expressed solely in the base space, which can be solved using classical methods. However, in our work, we are interested in the more general control problems associated with the complete dynamics defined by (16) and (17), which are reflected in (17) and (18). We remark that the dimension of the base space is unique, equal to the number of degrees of freedom; however the identity of the base space variables is not unique.

As in Section IV, we assume that $r = n - m$ and that the matrix product $C'(q_1)B(q_1)$ is locally invertible; this assumption is not restrictive. Consequently, it can be shown that the normal form equations for the system (16) and (17), following the development in Section IV, are given by

$$\dot{x}_1 = x_3 \quad (19)$$

$$\dot{x}_2 = -\bar{J}(x_1)x_3 \quad (20)$$

$$\dot{x}_3 = v \quad (21)$$

where $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$ and v satisfies

$$C'(q_1)\left[B(q_1)u - F(q_1, C(q_1)\dot{q}_1) - M(q_1)\dot{C}(q_1)\dot{q}_1 \right]$$
$$= C'(q_1)M(q_1)C(q_1)v. \quad (22)$$

Our basic approach is to make use of the normal form equations (19)–(21) to control the Caplygin system (16) and (17). Note that the theoretical results obtained in previous sections certainly apply to the system (16) and (17).

Clearly, there is no continuous state feedback which asymptotically stabilizes a single equilibrium. However, the controllability properties possessed by the system guarantee the existence of a piecewise analytic state feed-

back in the analytic case [26]. We now describe the ideas that are employed to construct such a feedback which does achieve the desired local asymptotic stabilization of a single equilibrium solution. These ideas are based on the use of geometric phase (holonomy) which has proved useful in a variety of kinematics and dynamics problems (see e.g., [10], [11], and [19]). More information concerning geometric phases can be found in the recent book [24] of Shapere and Wilczek, and a review article [16] of Marsden, Montgomery and Ratiu. Our use of geometric phase is, to the best of our knowledge, its first application to nonlinear control systems of the form (19)–(21) which contain nontrivial drift vector fields [5], [22], [23]. The key observation is that the geometric phase, the extent to which a closed path in the base space fails to be closed in the configuration space, depends only on the path traversed in the base space and not on the time history of traversal of the path. Related ideas have been used for a class of path planning problems, based on kinematic relations, in [14], [15], and [12].

For simplicity, we consider control strategies which transfer any initial configuration and velocity (sufficiently close to the origin) to the zero configuration with zero velocity. The proposed control strategy initially transfers the given initial configuration and velocity to the origin of the $(q_1, \dot{q}_1)$ base phase space. The main point then is to determine a closed path in the q_1 base space that achieves the desired geometric phase. We show that, in the analytic case, the indicated assumptions guarantee that this geometric phase construction can be made and that (necessarily piecewise analytic) feedback can be determined which accomplishes the desired control objective.

Let $x^0 = (x_1^0, x_2^0, x_3^0)$ denote an initial state. We now describe two steps involved in construction of a control strategy which transfers the initial state to the origin.

Step 1: Bring the system to the origin of the (x_1, x_3) base phase space, i.e., find a control which transfers the initial state (x_1^0, x_2^0, x_3^0) to $(0, x_2^1, 0)$ in a finite time, form some x_2^1.

Step 2: Traverse a closed path (or a series of closed paths) in the x_1 base space to produce a desired geometric phase in the (x_1, x_2) configuration space, i.e., find a control which transfers $(0, x_2^1, 0)$ to $(0, 0, 0)$.

The desired geometric phase condition is given by

$$x_2^1 = \oint_{\gamma} \bar{J}(x_1)\, dx_1 \qquad (23)$$

where γ denotes a closed path traversed in the base space. The geometric phase is reflected in the fact that traversing a closed path in the base space yields a non-closed path in the full configuration space. Note that here, for notational simplicity in presenting the main idea, we assume that the desired geometric phase can be obtained by a single closed path. In general, more than one loop may be required to produce the desired geometric phase; for such cases γ can be viewed as concatenation of a series of closed paths.

Under the assumptions mentioned previously, explicit procedures can be given for each of the above two steps. Step 1 is classical; it is Step 2, involving the geometric phase, that requires special consideration. Explicit characterization of a closed path γ which satisfies the desired geometric phase condition (23) can be given for several specific examples. In the next section, we present three such examples. However, some problems may require a general computational approach. An algorithm based on Lie algebraic methods as in [13] can be employed to approximately characterize the required closed path. Suppose the closed path γ which satisfies the desired geometric phase condition is chosen. Then a feedback algorithm which realizes the closed path in the base space can be constructed since the base space equations (19), (21) constitute decoupled $n - m$ double integrators on the base space.

This general construction procedure provides a strategy for transferring an arbitrary initial state of (19)–(21) to the origin. Implementation of this control strategy in (necessarily piecewise analytic) feedback form can be accomplished as follows.

Let $a = (a_1, \cdots, a_{n-m})$ and $b = (b_1, \cdots, b_{n-m})$ denote displacement vectors in the x_1 base space and let $\gamma(a, b)$ denote the closed path (in the base space) formed by the line segments from $x_1 = 0$ to $x_1 = a$, from $x_1 = a$ $x_1 = a + b$, from $x_1 = a + b$ to $x_1 = b$, and from $x_1 =$ to $x_1 = 0$. Then the geometric phase of the parameterized family

$$\{\gamma(a, b) \mid a, b \in R^{n-m}\}$$

is determined by the geometric phase function $\gamma(a, b)$ $\alpha(a, b)$ given as

$$\alpha(a, b) = -\oint_{\gamma(a,b)} \bar{J}(x_1)\, dx_1.$$

Now let π_s denote the projection map $\pi_s: (x_1, x_2, \cdot$ $\rightarrow (x_1, x_3)$. In order to construct a feedback control algorithm to accomplish the above two steps, we first define feedback function $V^{x_1^*}(\pi_s x)$ which satisfies: for any $\pi_s x($ there is $t_1 \geq t_0$ such that the unique solution of

$$\dot{x}_1 = x_3,$$
$$\dot{x}_3 = V^{x_1^*}(\pi_s \dot{x}),$$

satisfies $\pi_s x(t_1) = (x_1^*, 0)$. Note that the feedback function is parameterized by the vector x_1^*. Moreover, each x_1^*, there exists such a feedback function. One su feedback function $V^{x_1^*}(\pi_s x) = (V_1^{x_1^*}(\pi_s x), \cdots, V_{n-m}^{x_1^*}(\pi_s x)$ given as

$$V_i^{x_1^*}(\pi_s x) = \begin{cases} -k_i \operatorname{sign}\left(x_{1,i} - x_{1,i}^* + x_{3,i} |x_{3,i}|2k_i\right), \\ \qquad (x_{1,i}, x_{3,i}) \neq (x_{1,i}^*, 0), \\ 0, \\ \qquad (x_{1,i}, x_{3,i}) = (x_{1,i}^*, 0), \end{cases}$$

where k_i, $i = 1, \cdots, n - m$, are arbitrary positive constants.

We specify the control algorithm, with values denoted by v^*, according to the following construction, where x denotes the "current state":

Control Algorithm for v^*:

Step 0: Choose (a^*, b^*) to achieve the desired geometric phase.

Step 1: Set $v^* = V^{a^*}(\pi_s x)$, until $\pi_s x = (a^*, 0)$; then go to Step 2;

Step 2: Set $v^* = V^{a^* - b^*}(\pi_s x)$, until $\pi_s x = (a^* + b^*, 0)$; then go to Step 3;

Step 3: Set $v^* = V^{b^*}(\pi_s x)$, until $\pi_s x = (b^*, 0)$; then go to Step 4;

Step 4: Set $v^* = V^0(\pi_s x)$, until $\pi_s x = (0, 0)$; then go to Step 0.

We assumed here that the desired geometric phase can be obtained by a single closed path. Clearly, the above algorithm can be modified to account for more complex cases.

Note that the control algorithm is constructed by appropriate switchings between members of the parameterized family of feedback functions. On each cycle of the algorithm the particular functions selected depend on the closed path parameters a^*, b^*, computed in Step 0, to correct for errors in x_2.

The control algorithm can be initialized in different ways. The most natural is to begin with Step 4 since v^* in that step does not depend on the closed path parameters; however, many other initializations of the control algorithm are possible. The original control u^* is computed using (22).

Justification that the constructed control algorithm asymptotically stabilizes the origin follows as a consequence of the construction procedure: switching between feedback functions guarantees that the proper closed path (or a sequence of closed paths) is traversed in the base space so that the origin $(0, 0, 0)$ is necessarily reached in a finite time. This construction of a stabilizing feedback algorithm represents an alternative to the approach by Hermes [9], which is based on Lie algebraic properties.

It is important to emphasize that the above construction is based on the *a priori* selection of simply parametrized closed paths in the base space. The above selection simplifies the tracking problem in the base space, but other path selections could be made and they would, of course, lead to a different feedback strategy from that proposed above.

We remark that the technique presented in this section can be generalized to some systems which are not Caplygin. For instance, this generalization is tractible to systems for which (20) takes the form

$$\dot{x}_2 = \rho(x_2)\bar{J}(x_1)$$

where $\rho(x_2)$ denotes a certain Lie group representation (see e.g., [16]). The geometric phase of a closed path for such systems is given as a path ordered exponential rather than a path integral.

VIII. EXAMPLES

Control of Knife Edge Using Steering and Pushing Inputs:
We first consider the control of a knife edge moving in point contact on a plane surface [3]–[5]. Let x and y denote the coordinates of the point of contact of the knife edge on the plane and let ϕ denote the heading angle of the knife edge, measured from the x-axis. Then the equations of motion, with all numerical constants set to unity, are given by

$$\ddot{x} = \lambda \sin \phi + u_1 \cos \phi \qquad (24)$$

$$\ddot{y} = -\lambda \cos \phi + u_1 \sin \phi \qquad (25)$$

$$\dot{\phi} = u_2 \qquad (26)$$

where u_1 denotes the control force in the direction defined by the heading angle, u_2 denotes the control torque about the vertical axis through the point of contact; the components of the force of constraint arise from the scalar nonholonomic constraint

$$\dot{x} \sin \phi - \dot{y} \cos \phi = 0 \qquad (27)$$

which has nonholonomy degree two at any configuration. It is clear that the constraint manifold is a five-dimensional manifold and is defined by

$$M = \left\{ (\phi, x, y, \dot{\phi}, \dot{x}, \dot{y}) \mid \dot{x} \sin \phi - \dot{y} \cos \phi = 0 \right\}$$

and any configuration is an equilibrium if the controls are zero.

Define the variables

$$x_1 = x \cos \phi + y \sin \phi,$$

$$x_2 = \phi,$$

$$x_3 = -x \sin \phi + y \cos \phi,$$

$$x_4 = \dot{x} \cos \phi + \dot{y} \sin \phi - \dot{\phi}(x \sin \phi - y \cos \phi),$$

$$x_5 = \dot{\phi},$$

so that the reduced differential equations are given by

$$\dot{x}_1 = x_4,$$

$$\dot{x}_2 = x_5,$$

$$\dot{x}_3 = -x_1 x_5,$$

$$\dot{x}_4 = u_1 + u_2 x_3 - x_1 x_5^2,$$

$$\dot{x}_5 = u_2.$$

Consequently, (24)–(27) represent a controlled Caplygin system with base space equations which are feedback linearizable. The following conclusions are based on the analysis of the above reduced equations.

Proposition 1: Let $x^e = (x_1^e, x_2^e, x_3^e, 0, 0)$ denote an equilibrium solution of the reduced differential equations corresponding to $u = 0$. The knife edge dynamics described by (24)–(27) have the following properties:

1) There is a smooth feedback which asymptotically stabilizes the closed loop to any smooth one dimensional

equilibrium manifold in M which satisfies the transversality condition.

2) There is no continuous state feedback which asymptotically stabilizes x^e.

3) The system is strongly accessible at x^e since the space spanned by the vectors

$$g_1, g_2, [g_1, f], [g_2, f], [g_2, [f, [g_1, f]]]$$

has dimension 5 at x^e.

4) The system is small time locally controllable at x^e since the brackets satisfy sufficient conditions for small time local controllability.

Note that the base variables are (x_1, x_2). Consider a parameterized rectangular closed path γ in the base space with four corner points

$$(0, 0), (x_1, 0), (x_1, x_2), (0, x_2)$$

i.e., $a = (x_1, 0)$ and $b = (0, x_2)$ following the notation introduced in the general development. By evaluating the integral in (23) in closed form for this case, the desired geometric phase condition is

$$x_3^1 = x_1 x_2.$$

This equation can be explicitly solved to determine a closed path $\gamma^* = \gamma(a^*, b^*)$ which achieves the desired geometric phase. One solution can be given as follows:

$$a^* = \left(\sqrt{|x_3^1|} \operatorname{sign} x_3^1, 0 \right), \qquad b^* = \left(0, \sqrt{|x_3^1|} \right).$$

Note that the previously described feedback algorithm can be used to asymptotically stabilize the knife edge to the origin. A different feedback algorithm for this example is given in [4].

Control of Rolling Wheel Using Steering and Driving Inputs: As a second example, we consider the control of a vertical wheel rolling without slipping on a plane surface [3], [5]. Let x and y denote the coordinates of the point of contact of the wheel on the plane, let ϕ denote the heading angle of the wheel, measured from the x-axis and let θ denote the rotation angle of the wheel due to rolling, measured from a fixed reference. Then the equations of motion, with all numerical constants set to unity, are given by

$$\ddot{x} = \lambda_1 \tag{28}$$

$$\ddot{y} = \lambda_2 \tag{29}$$

$$\ddot{\theta} = -\lambda_1 \cos \phi - \lambda_2 \sin \phi + u_1 \tag{30}$$

$$\ddot{\phi} = u_2 \tag{31}$$

where u_1 denotes the control torque about the rolling axis of the wheel and u_2 denotes the control torque about the vertical axis through the point of contact; the components of the force of constraint arise from the two nonholonomic constraints

$$\dot{x} = \dot{\theta} \cos \phi \tag{32}$$

$$\dot{y} = \dot{\theta} \sin \phi \tag{33}$$

which have nonholonomy degree three at any configuration. The constraint manifold is a six-dimensional manifold and is given by

$$M = \left\{ \left(\theta, \phi, x, y, \dot{\theta}, \dot{\phi}, \dot{x}, \dot{y} \right) \middle| \dot{x} = \dot{\theta} \cos \phi, \dot{y} = \dot{\theta} \sin \phi \right\}$$

and any configuration is an equilibrium if the controls are zero.

Define the variables

$$x_1 = \theta, \quad x_2 = \phi, \quad x_3 = x, \quad x_4 = y, \quad x_5 = \dot{\theta}, \quad x_6 = \dot{\phi}$$

so that the reduced differential equations are given by

$$\dot{x}_1 = x_5,$$

$$\dot{x}_2 = x_6,$$

$$\dot{x}_3 = x_5 \cos x_2,$$

$$\dot{x}_4 = x_5 \sin x_2,$$

$$\dot{x}_5 = \frac{1}{2} u_1,$$

$$\dot{x}_6 = u_2.$$

Consequently, (28)–(33) represent a controlled Caplygin system with base space equations which are feedback linearizable. The following conclusions are based on analysis of the above reduced equations.

Proposition 2: Let $x^e = (x_1^e, x_2^e, x_3^e, x_4^e, 0, 0)$ denote an equilibrium solution of the reduced differential equations corresponding to $u = 0$. The rolling wheel dynamics described by (28)–(33) have the following properties:

1) There is a smooth feedback which asymptotically stabilizes the closed loop to any smooth two-dimensional equilibrium manifold in M which satisfies the transversality condition.

2) There is no continuous state feedback which asymptotically stabilizes x^e.

3) The system is strongly accessible at x^e since the space spanned by the vectors

$$g_1, g_2, [g_1, f], [g_2, f], [g_2, [f, [g_1, f]]],$$
$$[g_2, [f, [g_1, [f, [g_2, f]]]]]$$

has dimension 6 at x^e.

4) The system is small time locally controllable at x^e since the brackets satisfy sufficient conditions for small time local controllability.

Note that the base variables are (x_1, x_2). Consider a parameterized rectangular closed path γ in the base space with four corner points

$$(0, 0), (x_1, 0), (x_1, x_2), (0, x_2).$$

By evaluating the integral in (23) in closed form for this case, the desired geometric phase conditions are

$$x_3^1 = x_1 (\cos x_2 - 1),$$

$$x_4^1 = x_1 \sin x_2.$$

These equations can be explicitly solved to determine a closed path (or a concatenation of closed paths) γ^* which

achieves the desired geometric phase. One solution can be given as follows: if $x_3^1 \neq 0$ then γ^* is the closed path specified by

$$a^* = -\left((x_3^1)^2 + (x_4^1)^2\right)/2x_3^1, 0),$$

$$b^* = \left(0, -\sin^{-1}(2x_3^1 x_4^1/\left((x_3^1)^2 + (x_4^1)^2\right)\right)$$

and if $x_3^1 = 0$ then γ^* is a concatenation of two closed paths specified by

$$a^* = (0.5x_4^1, 0), \qquad b^* = (0, 0.5\pi),$$

$$a^{**} = (-0.5x_4^1, 0), \qquad b^{**} = (0, -0.5\pi).$$

Note that the previously described feedback algorithm can be used (with the modification indicated in the general development) to asymptotically stabilize the rolling wheel to the origin.

Control of Planar Multibody Systems Using Angular Momentum Preserving Inputs: Another interesting class of physical examples is given by the control of a planar multibody system with angular momentum preserving control torques. For more details on the origin of this problem, and references to previous work, see [10] and [25]. Related papers are in [22], [23]. It is assumed that a system of N planar rigid bodies are interconnected by frictionless one degree of freedom joints in the form of an open kinematic chain. The configuration space of the N-body system is T^N, the N-dimensional torus. Define the vector of absolute angles of the N bodies

$$\theta = (\theta_1, \cdots, \theta_N)$$

and the vector of relative angles (or joint angles) corresponding to the $(N - 1)$ joints

$$\psi = (\psi_1, \cdots, \psi_{N-1}).$$

The relationship between the vectors θ and ψ is given by

$$\psi = P\theta$$

where P is a constant $(N - 1) \times N$ matrix. In the absence of potential energy, the equations of motion are given by

$$J(\theta)\ddot{\theta} + F(\theta, \dot{\theta}) = P'u \qquad (34)$$

where the $N \times N$ matrix function $J(\theta)$ is invertible, and

$$F(\theta, \dot{\theta}) = \frac{d}{dt}[J(\theta)]\dot{\theta} - \frac{1}{2}\frac{\partial}{\partial\theta}(\dot{\theta}'J(\theta)\dot{\theta})$$

in an N-vector function, and the control input u is the $N - 1$ vector of joint torques. Assuming that the angular momentum is zero, it follows that

$$1'J(\theta)\dot{\theta} = 0$$

holds, where $1 = (1, \cdots, 1)'$. It can be shown that (35) is nonholonomic for $N \geq 3$. Define the variables

$$x_1 = \psi,$$

$$x_2 = \theta_1$$

$$x_3 = \dot{\psi}_1,$$

so that the reduced differential equations are given by

$$\dot{x}_1 = x_3,$$

$$\dot{x}_2 = -\bar{J}(x_1)x_3,$$

$$\dot{x}_3 = \bar{F}(x_1, x_2) + \bar{B}(x_1)u.$$

The indicated assumptions guarantee that (34) and (35) take the form of a controlled Caplygin system with shape space equations that are feedback linearizable.

The following conclusions are based on analysis of the above reduced equations.

Proposition 3: Let $x^e = (x_1^e, x_2^e, 0)$ denote a regular equilibrium of the reduced differential equations corresponding to $u = 0$, i.e., $(\partial\bar{J}_{i_0}(x_1^e)/\partial x_{1,j_0}^e) - (\partial\bar{J}_{j_0}(x_1^e)/\partial x_{1,i_0}^e) \neq 0$ for some (i_0, j_0). The dynamics of the planar multibody system described by (34) and (35) have the following properties if $N \geq 3$:

1) There is a smooth feedback which asymptotically stabilizes the closed loop to any smooth one dimensional equilibrium manifold in M which satisfies the transversality condition.

2) There is no continuous state feedback which asymptotically stabilizes x^e.

3) The system is strongly accessible at x^e since the space spanned by the vectors

$$g_1, \cdots, g_{N-1}, [g_1, f], \cdots, [g_{N-1}, f], [g_{i_0}, [f, [g_{j_0}, f]]]$$

has dimension $2N - 1$ at x^e.

4) The system is small time locally controllable at x^e since the brackets satisfy sufficient conditions for small time local controllability.

If $N = 1$ or 2, then the system (34) and (35) is neither strongly accessible nor small time locally controllable. If the equilibrium solution x^e is not regular, higher order brackets are required to obtain the same conclusions.

Note that the shape variables are the $N - 1$ joint angles x_1. Following the development in [22], the N bodies can be treated as three interconnected bodies by locking all the joints except the ones labelled (i_0, j_0). Consider a parameterized rectangular closed path γ in the $x_{1,i_0} - x_{1,j_0}$ plane with four corner points

$$(0, 0), (x_{1,i_0}, 0), (x_{1,i_0}, x_{1,j_0}), (0, x_{1,j_0}).$$

In this case, the desired geometric phase condition can be written as

$$x_2^1 = \oint_\gamma \bar{s}_{i_0}(x_{1,i_0}, x_{1,j_0})\, dx_{1,i_0} + \bar{s}_{j_0}(x_{1,i_0}, x_{1,j_0})\, dx_{1,j_0}$$

where $\bar{s}_{i_0}(x_{1,i_0}, x_{1,j_0})$ and $\bar{s}_{j_0}(x_{1,i_0}, x_{1,j_0})$ are obtained by evaluating $\bar{J}_{i_0}(x_1)$ and $\bar{J}_{j_0}(x_1)$ at $x_{1,i} = 0$, for $i = 1, \cdots, N - 1$, $i \neq i_0$, $i \neq j_0$. In this case, the path integral can be computed numerically as a function of the loop parameters x_{1,i_0}, x_{1,j_0} as in [23]. Further, loop parameters x_{1,i_0}^*, x_{1,j_0}^* can be computed numerically, thereby determining a closed path γ^* which achieves the desired geometric phase. Note that the previously described feedback algorithm can be used (with the modification indi-

cated in the general development) to asymptotically stabilize the planar multibody to the origin.

IX. CONCLUSIONS

A class of inherently nonlinear control problems has been identified, the nonlinear features arising directly from physical assumptions about constraints on the motion of a mechanical system. In this paper, we have presented models for mechanical systems with nonholonomic constraints represented both by differential-algebraic equations and by reduced state equations. We have studied control issues for this class of systems and we have derived a number of fundamental results. Although a single equilibrium solution cannot be asymptotically stabilized using continuous state feedback, a general procedure for constructing a piecewise analytic state feedback which achieves the desired result has been suggested. The theoretical issues addressed in the paper have been illustrated through several classes of example problems.

The general approach described in this paper makes substantial use of the geometric approach to nonlinear control. However, the specific nonlinear control strategy suggested is substantially different, both conceptually and in detail, from the smooth nonlinear control strategies most commonly studied in the literature. It is hoped that this paper provides a foundation for future research on this important and challenging class of nonlinear control problems.

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Planar Reorientation Maneuvers of Space Multibody Systems Using Internal Controls

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In this paper a reorientation maneuvering strategy for an interconnection of planar rigid bodies in space is developed. It is assumed that there are no exogeneous torques, and torques generated by joint motors are used as means of control so that the total angular momentum of the multibody system is a constant, assumed to be zero in this paper. The maneuver strategy uses the nonintegrability of the expression for the angular momentum. We demonstrate that large-angle maneuvers can be designed to achieve an arbitrary reorientation of the multibody system with respect to an inertial frame. The theoretical background for carrying out the required maneuvers is briefly summarized. Specifications and computer simulations of a specific reorientation maneuver, and the corresponding control strategies, are described.

I. Introduction

IN this paper we develop a reorientation strategy for a system of N planar rigid bodies in space that are interconnected by ideal frictionless pin joints in the form of an open kinematic chain. Angular momentum preserving controls, e.g., torques generated by joint motors, are considered. The N-body system is assumed to have zero initial angular momentum. Our earlier work[1,2] demonstrated that reorientation of a planar multibody system with three or more interconnected bodies using only joint torque inputs is an inherently nonlinear control problem that is not amenable to classical methods of nonlinear control. The goal of this study is to indicate how control strategies can be explicitly constructed to achieve the desired absolute reorientation of the N-body system.

There are many physical advantages in using internal controls, e.g., joint torque controls, to carry out the desired multibody reorientation maneuvers. First of all, this control approach does not modify the total angular momentum of the multibody system. In addition, internal controls have obvious advantages in terms of energy conservation. Moreover, they can be implemented using standard electrical servo motors, a simple and reliable control actuator technology.

The formal development in this paper is concerned with control of a multibody interconnection in space that has zero angular momentum. Although these results are formulated in a general setting, we have been motivated by several classes of specific problems. Several potential applications of our general results are now described.

Manipulators mounted on space vehicles and space robots have been envisioned to carry out construction, maintenance, and repair tasks in an external space environment. These space systems are essentially multibody systems satisfying the assumptions of this paper. To carry out the desired tasks, they must be capable of performing a variety of reorientation maneuvers. Previous research on maneuvering of such space multibody systems has mainly focused on maneuvers that achieve desired orientation of some of the bodies, e.g., an end effector, whereas the orientation of some of the remaining bodies cannot be specified, at least using the methodologies employed.[3-8] Using the approach suggested in this paper, maneuvers that achieve any desired reorientation for all of the links of the system can be accomplished. Such additional flexibility in performing reorientation maneuvers should have great practical significance for completion of robotic tasks in space.

Another related application is the performance by astronauts of reorientation maneuvers in space. Although it is well known that astronauts in space can perform a variety of complicated reorientation maneuvers, without the use of thrusters, the theoretical basis for such maneuvers is incomplete. Again we note that an astronaut in space can be considered as a multibody system that satisfies all of the assumptions of this paper (except that motion is not restricted to be planar). Consequently, the theory in this paper is applicable in principle to the study of the maneuvering capability of astronauts in space. Previous research in this area[9] has emphasized dynamics issues. Other closely related research has focused on describing the reorientation maneuvers of a falling cat.[10]

Finally, we mention another area of potential application of the results of this paper, namely, the development of deployment maneuvers for multibody antennas connected to a spacecraft. If deployment maneuvers for an antenna, or other deployable structures, are performed using only torque motors at the joints of the antenna segments, then the spacecraft-antenna system is a multibody system that satisfies the assumptions of this paper. Consequently, our results can be used to develop efficient antenna deployment maneuvers. The importance of such deployment maneuvers is that they do not change the final orientation of the spacecraft or the total angular momentum of the spacecraft-antenna system, thereby reducing the requirements of the spacecraft momentum management system. To our knowledge, such control approaches to antenna deployment have not yet been exploited. It is expected that such an approach would have many advantages over the use of existing passive antenna deployment mechanisms.[11]

This paper is organized as follows. In Sec. II, a mathematical model for a planar multibody system in space is derived. We then formulate a control problem associated with planar multibody reorientation. In Sec. III, we first summarize several relevant theoretical results. We then introduce a control strategy to solve this reorientation problem. In Sec. IV, we apply the theoretical results to a three-link system. We present computer simulations illustrating the control strategy. Section V consists of a summary of the main results and concluding remarks about future research. Although a complete treatment of the topics in the paper requires use of differential geometric tools, our presentation avoids these tools and uses only elementary mathematical methods. However, references to relevant literature are provided throughout.

II. Problem Formulation

We consider a system of N planar rigid bodies interconnected by frictionless one-degree-of-freedom joints in the form of an open kinematic chain. The configuration space, for an observer at the center of mass of the system of rigid bodies, is N dimensional. Since we assume an open kinematic chain, there are exactly $N - 1$ joints. We consider controlling the rotational motion of the system using torques at the joints; each joint is assumed to be actuated so as to permit free adjustment of the joint angle. It is assumed that there are no external torques acting on the system. It is clear that the configuration of the N bodies can be described by the absolute angle of any one of the bodies (say body 1) and $N - 1$ joint angles. Denote by θ_1 the absolute angle of body 1 and by the $(N - 1)$ vector $\psi = (\psi_1, \ldots, \psi_{N-1})$ the joint angle vector. Clearly, (θ_1, ψ) is a generalized coordinate vector for the rotational motion. It can be shown that the Lagrangian (which is equal to the rotational kinetic energy under the preceding assumptions), written in terms of these coordinates and their time derivatives, does not contain θ_1 explicitly, i.e., θ_1 is a cyclic or ignorable coordinate. Consequently, the generalized momentum associated with the cyclic coordinate θ_1 is conserved. This conserved quantity is the first integral of the motion corresponding to conservation of angular momentum of the system. In this paper we assume zero initial angular momentum so that angular momentum remains zero throughout a maneuver.

It is clear that Lagrange's equations describe the motion on the joint angle space, and the evolution of θ_1 can be obtained from the expression for conservation of angular momentum. Thus, the motion of a planar multibody system, under the preceding assumptions, can be described by the following reduced-order equations:

$$J_s(\psi)\ddot{\psi} + F_s(\psi, \dot{\psi}) = \tau \qquad (1)$$

$$\dot{\theta}_1 + s'(\psi)\dot{\psi} = 0 \qquad (2)$$

where $\tau = (\tau_1, \ldots, \tau_{N-1})$ denotes the $(N - 1)$ vector of joint torques, $J_s(\psi)$ is a symmetric positive definite $(N - 1) \times (N - 1)$ matrix function, and $s(\psi)$ and $F_s(\psi, \dot{\psi})$ are $(N - 1)$ vector functions. Note that in this paper a prime denotes transpose. The explicit specifications of these functions can be found in the literature.[1,2,12]

State-space equations for Eqs. (1) and (2) are

$$\dot{\theta}_1 = -s(\psi)'\omega \qquad (3)$$

$$\dot{\psi} = \omega \qquad (4)$$

$$\dot{\omega} = -J_s^{-1}(\psi)F_s(\psi, \omega) + J_s^{-1}(\psi)\tau \qquad (5)$$

Note that Eqs. (4) and (5) are expressed in terms of the joint phase variables $(\psi, \dot{\psi})$ only. Hence the joint angle space constitutes a reduced configuration space for the system. This reduced configuration space is also referred to as the "shape space" of the system.[12-16] It is possible to consider control problems expressed solely in terms of the shape space; such problems can be solved using classical methods. However, in our work we are interested in the more general control problems associated with the complete dynamics of the multibody system defined by Eqs. (1) and (2) [or Eqs. (3-5)].

Note that Eqs. (4) and (5) only, which represent the projection of the motion onto the shape phase space, are feedback linearizable using the feedback transformation

$$u = -J_s^{-1}(\psi)F_s(\psi, \omega) + J_s^{-1}(\psi)\tau \qquad (6)$$

where $u \in R^{N-1}$. The previous feedback transformation yields the following normal form equations:

$$\dot{\theta}_1 = -s(\psi)'\omega \qquad (7)$$

$$\dot{\psi} = \omega \qquad (8)$$

$$\dot{\omega} = u \qquad (9)$$

We remark here that it is impossible to completely linearize the system defined by Eqs. (3-5) using static or dynamic feedback combined with any coordinate transformation.

Note that an equilibrium solution of Eqs. (3-5) corresponding to $\tau = 0$ [or equivalently an equilibrium solution of Eqs. (7-9) for $u = 0$] is given by $(\theta_1^e, \psi^e, 0)$, where (θ_1^e, ψ^e) is referred to as an equilibrium configuration. Hence, an equilibrium solution corresponds to a trivial motion of the system for which all of the configuration space variables remain constant.

Note also that Eq. (3) represents conservation of angular momentum. This equation is nonholonomic for $N \geq 3$ (i.e., if the multibody system consists of three or more links), since the differential expression (3) is not integrable for $N \geq 3$. This fact has important implications in terms of controllability properties of the system as will be shown in the subsequent development. As a consequence of the symmetry possessed by the system, θ_1 does not apear explicitly in Eq. (3). Mechanical systems with such symmetry properties are referred to as non holonomic Caplygin systems.[17-21] As a consequence of the nonintegrability for $N \geq 3$, the scalar analytic functions

$$H_{ij}(\psi) = \frac{\partial s_i(\psi)}{\partial \psi_j} - \frac{\partial s_j(\psi)}{\partial \psi_i}, \qquad (i, j) \in I^2 \qquad (10)$$

where $I = \{1, \ldots, N - 1\}$, do not all vanish, except possibly on a set that has measured zero with respect to the shape space.

III. Reorientation Maneuvering Problem

In this section, we address the following control problem associated with planar multibody systems described by Eqs. (1) and (2):

Problem: Given an initial state $(\theta_1^0, \psi^0, \omega^0)$ and a desired equilibrium solution $(\theta_1^e, \psi^e, 0)$, determine a motion

$$[\theta_1(t), \psi(t), \omega(t)], \qquad 0 \leq t \leq t_f$$

such that

$$(\theta_1(0), \psi(0), \omega(0)) = (\theta_1^0, \psi^0, \omega^0), \qquad (\theta_1(t_f)$$

$$\psi(t_f), \qquad \omega(t_f)) = (\theta_1^e, \psi^e, 0)$$

and

$$(\theta_1(t), \psi(t), \omega(t))$$

satisfies Eqs. (1) and (2) for some control function $t \rightarrow \tau(t)$

Note that, in particular, if $\omega^0 = 0$, then this problem cosponds to a rest-to-rest maneuver.

The existence of solutions to this control problem demonstrated in our earlier work.[1,2] In particular, we stud the nonlinear control system described by Eqs. (7-9) and ployed certain results from nonlinear control theory to cha terize controllability properties of planar multibody syst described by Eqs. (1) and (2). These results not only prove existence of solutions of the preceding problem but also vide a theoretical basis for construction of nonlinear cor strategies required to achieve the desired maneuver. We summarize those results.[1,2]

Under the stated assumptions, a planar multibody sy has the following properties if $N \geq 3$, i.e., if it consists of or more links:

1) The system is strongly accessible.

2) The system is small time locally controllable from equilibrium.

3) The system can be transferred from any initial cond to any desired equilibrium in arbitrarily small time.

If $N = 1$ or 2, then the system is not even accessible and is not small time locally controllable, and there exist initial conditions that cannot be transferred to a desired equilibrium.

The proofs[1,19] of the first two results depend on showing that certain Lie algebraic conditions are satisfied if $N \geq 3$. The third result is proved[1,19] constructively.

It should be emphasized that the subsequent development is assumed to be carried out for multibody systems consisting of three or more links ($N \geq 3$); this should be understood even if it is not always explicitly stated. Note that the reorientation problem generally has many solutions. In this paper, we describe one solution approach, outline the theory behind it, and present some data from simulations. The key observation is the following.

Consider Eq. (3). Assume that joint angles are controlled in such a way that $\psi(t)$, $0 \leq t_1 \leq t \leq t_2$, describes a closed path γ in the shape space. Integrating both sides of Eq. (3) from $t = t_1$ to $t = t_2$ and using the fact that $d\psi = \dot{\psi} \, dt$, we obtain

$$\theta_1(t_2) - \theta_1(t_1) = \oint_\gamma s'(\psi) \, d\psi \quad (11)$$

Thus, by proper selection of a path γ in shape space, any desired geometric phase (which is a rotation of link 1) can be obtained. By the nonintegrability property just mentioned, the preceding integral is in fact path dependent, thereby guaranteeing the existence of (many) such paths.

Note that in differential geometry the quantity

$$\alpha(\gamma) = \oint_\gamma s'(\psi) \, d\psi$$

is referred to as the geometric phase (or holonomy) of the closed path γ. This quantity depends only on the geometry of the closed path and is independent of the speed at which the path is traversed.

Note that Stokes' formula can be applied to obtain an equivalent formula for $\alpha(\gamma)$ as a surface integral. For simplicity, assume that $N = 3$, i.e., the shape space is the (ψ_1, ψ_2) plane. Also, let γ be traversed counterclockwise. Then by Stokes' theorem the preceding formula can be written as

$$\alpha(\gamma) = \int_S \left(\frac{\partial s_2}{\partial \psi_1} - \frac{\partial s_1}{\partial \psi_2} \right) d\psi_1 \, d\psi_2$$

where S is the surface within the boundary γ. In the case that the path is traversed clockwise, the surface integral is equal to $-\alpha(\gamma)$.

More information concerning geometric phases can be found in the literature.[15] Geometric phase ideas have proved useful in a variety of inherently nonlinear control problems.[19-21] These ideas have also been used for a class of path planning problems based solely on kinematic relations.[13,14,16]
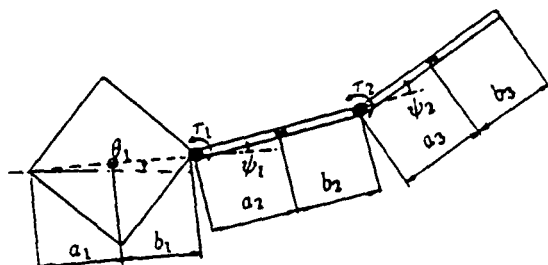

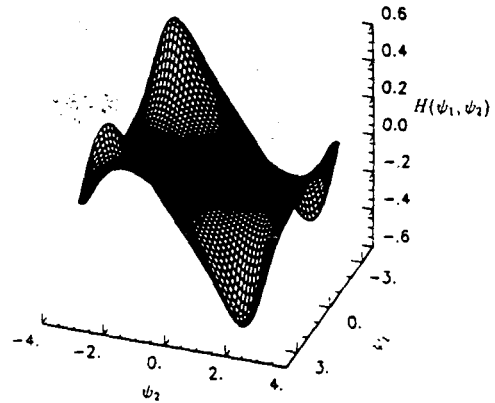Fig. 2 Function $H(\psi_1, \psi_2)$.

We now describe a control strategy, using the preceding geometric phase relation (11), which solves the reorientation problem.

Let $(\theta_1^e, \psi^e, 0)$ denote the desired equilibrium solution. We refer to (θ_1^e, ψ^e) and ψ^e as the desired equilibrium configuration and the desired equilibrium shape, respectively. We describe four steps involved in construction of an open-loop control function $u_{[0,t_f)} = (u_1, \ldots, u_{N-1})'$ that transfers any initial state $(\theta_1^0, \psi^0, \omega^0)$ to $(\theta_1^e, \psi^e, 0)$ in time exactly t_f, where $t_f > 0$ is arbitrary.

Let $0 < t_1 < t_2 < t_3 < t_f$ denote an arbitrary partition of the time interval $[0, t_f)$.

Step 1: Transfer the system to the desired equilibrium shape, i.e., find a control that transfers the initial state $(\theta_1^0, \psi^0, \omega^0)$ to $(\theta_1^1, \psi^e, 0)$ at time t_1, for some θ_1^1.

Since the dynamics on the shape phase space are so simple, namely, decoupled double integrators, step 1 has many solutions that are easily obtained using classical methods. One such control function is

$$u_{[0,t_1)} = \begin{cases} -\dfrac{\pi \omega^0}{t_1} \cos\left(\dfrac{\pi t}{t_1}\right) & t \in [0, 0.5t_1) \\[2ex] \dfrac{8\pi\left[\psi^e - \psi^0 - \omega^0 t_1(0.5 - \pi^{-1})\right]}{t_1^2} \sin\left[\dfrac{2\pi(2t - t_1)}{t_1}\right] & \\[2ex] & t \in [0.5t_1, t_1) \end{cases}$$

$$(12)$$

Next, we select a closed path γ (or a series of closed paths, see remark 1 following) in the shape space that achieves the desired geometric phase. There are many ways to accomplish such a construction; in our work we have found it convenient to use only two joint motions, keeping the other joints locked, and to use a square path in the restricted two-dimensional shape space. It is convenient to select the center of the square path in a region of the shape space that corresponds to a "large" geometric phase change (see remark 2 following).

To make the earlier ideas more concrete, we present a specific construction. Let $(i,j) \in I^2$, $i \neq j$, denote a pair of joints. Assume that for $t \geq t_1$ only this pair of joints are actuated while all of the other joints are kept fixed. This is equivalent to locking all of the joints except the ones labeled i and j and treating the N bodies as three interconnected bodies, for $t \geq t_1$. In this case the desired geometric phase formula can be written as

$$\theta_1(t_f) - \theta_1^1 = \pm\alpha(\gamma)$$

where $+(-)$ corresponds to counterclockwise (clockwise) traversal of the closed path γ. Since we desire to make $\theta_1(t_f) = \theta_1^e$, the closed path γ should be selected to satisfy
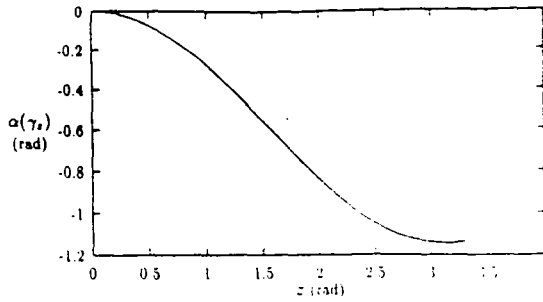
$$\theta_1^e - \theta_1^1 = \pm\alpha(\gamma)$$

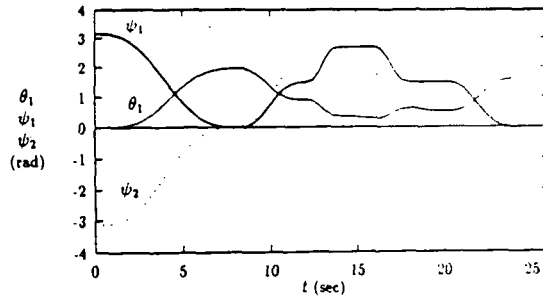Fig. 1 Three-link example.

Fig. 3   Geometric phase curve.

Fig. 4   Time responses for θ_1, ψ_1, and ψ_2.

The path γ lies in the two-dimensional (ψ_i, ψ_j) plane, so that

$$\alpha(\gamma) = \oint_\gamma \bar{s}_i(\psi_i, \psi_j)\, d\psi_i + \bar{s}_j(\psi_i, \psi_j)\, d\psi_j$$

where the scalar functions $\bar{s}_i(\psi_i, \psi_j)$ and $\bar{s}_j(\psi_i, \psi_j)$ are obtained by evaluating $s_i(\psi)$ and $s_j(\psi)$ at $\psi_k = \psi_k^e$, $\forall k \in I$ where $k \neq i, j$.

As mentioned earlier, we choose γ to be a square path in the (ψ_i, ψ_j) plane that is centered at the shape defined by ψ^* and that has side of length z^*, where z^* satisfies

$$\pm \alpha(\gamma_{z^*}) + \theta_1^i - \theta_1^e = 0$$

Here γ_z indicates the dependence of the square path on the size parameter z. In most cases, this equation is easily solved using standard numerical procedures.

Thus the four corner points of this square path are defined by shape vectors

$$p_1^* = \psi^* - 0.5z^*(e_i + e_j)$$

$$p_2^* = \psi^* + 0.5z^*(e_i - e_j)$$

$$p_3^* = \psi^* + 0.5z^*(e_i + e_j)$$

$$p_4^* = \psi^* - 0.5z^*(e_i - e_j)$$

where e_i and e_j are the ith and jth standard basis vectors in R^{N-1}. Thus the specific square path selected depends on the $N-1$ vector ψ^* that is the center of the square and the size of the square z^*.

Remark 1: Note that here, for notational simplicity in presenting the main idea, we assume that the desired geometric phase can be obtained by a single closed path. In general, more than one closed path may be required to produce the desired geometric phase; for such cases γ can be viewed as a concatenation of a series of closed paths.

Remark 2: Selection of the center point ψ^* of the path is rather arbitrary, e.g., one selection is $\psi^* = \psi^e$. However, other choices may provide a greater change in the geometric phase for a given size path. In this regard, the use of Stokes' theo-

rem, as indicated previously, suggests that ψ^* should be chosen where

$$\left| \frac{\partial s_j(\psi)}{\partial \psi_i} - \frac{\partial s_i(\psi)}{\partial \psi_j} \right|$$

is a maximum.

We now describe the remaining three steps as follows.

Step 2: Transfer the system from state $(\theta_1^i, \psi^e, 0)$ to a state corresponding to the corner of γ closest to ψ^e, along an arbitrary path in the shape space, in $t_2 - t_1$ units of time.

As an example, if p_1^* is the corner of γ closest to ψ^e, we propose the following control function for step 2:

$$u_{[t_1, t_2]} = \frac{2\pi(p_1^* - \psi^e)}{(t_2 - t_1)^2} \sin\left[\frac{2\pi(t - t_1)}{(t_2 - t_1)}\right] \tag{13}$$

Step 3: Traverse the selected square path (counterclockwise or clockwise, depending on the sign of the desired geometric phase value), in $t_3 - t_2$ units of time; the resulting change in the angle θ_1 is necessarily $\theta_1^e - \theta_1^i$.

Without loss of generality, we assume that the desired geometric phase value is obtained by counterclockwise traversal of the closed path starting and ending at p_1^*. Then, the following control functions guarantee traversal of the closed path, thereby accomplishing step 3:

$$u_{[t_2, t_2+h]} = \frac{2\pi(p_2^* - p_1^*)}{h^2} \sin\left[\frac{2\pi(t - t_2)}{h}\right] \tag{14}$$

$$u_{[t_2+h, t_2+2h]} = \frac{2\pi(p_3^* - p_2^*)}{h^2} \sin\left[\frac{2\pi(t - t_2 - h)}{h}\right] \tag{15}$$

$$u_{[t_2+2h, t_2+3h]} = \frac{2\pi(p_4^* - p_3^*)}{h^2} \sin\left[\frac{2\pi(t - t_2 - 2h)}{h}\right] \tag{16}$$

$$u_{[t_2+3h, t_3]} = \frac{2\pi(p_1^* - p_4^*)}{h^2} \sin\left[\frac{2\pi(t - t_2 - 3h)}{h}\right] \tag{17}$$

where $h = (t_3 - t_2)/4$.

Step 4: Transfer the system back to the desired equilibrium shape ψ^e following the path used in step 2, in $t_f - t_3$ units of time, thereby guaranteeing that the desired final state $(\theta_1^e, \psi^e, 0)$ is reached at time t_f.

The following control function

$$u_{[t_3, t_f]} = \frac{2\pi(\psi^e - p_1^*)}{(t_f - t_3)^2} \sin\left[\frac{2\pi(t - t_3)}{(t_f - t_3)}\right] \tag{18}$$

accomplishes step 4.

The corresponding control torque τ can be computed using Eq. (6). It is clear that the constructed control torque transfers the initial condition of the system (1) and (2) to the desired equilibrium configuration at time t_f. It is important to emphasize that the preceding construction is based on a priori selection of a square as the closed path in the shape space. Selection of square paths simplifies computation of the controls; how-
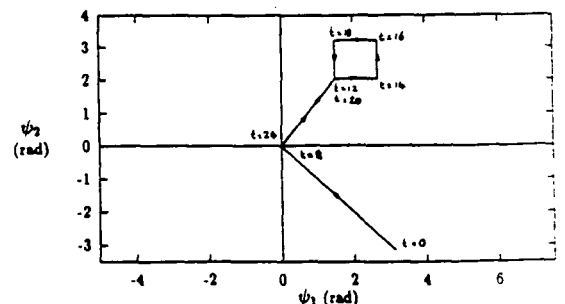

Fig. 5   Motion in shape space.

Table 1 System characteristics

Body i	a, m	b, m	m, kg	I, kg-m^2
1	0.5	0.5	120	10
2	0.5	0.5	12	1
3	0.5	0.5	12	1

ever, other path selections could be made. There are infinitely many choices for control functions that accomplish the preceding four steps, and the total time required is arbitrary.

IV. Example of Maneuvering a Three-Body System

In this section, the theory developed in Sec. III is used to illustrate a specific maneuver for interconnected multibodies using only torque inputs at the joint connections. As discussed, general planar maneuvers cannot be achieved using two or fewer interconnected links. An interconnection of three links provides complete maneuvering capability; consequently, that is the case considered here. Maneuvers of an interconnection of more than three links can always be reduced to a sequence of submaneuvers, each submaneuver involving the motion of only three links.

For illustration purposes we consider a planar three-link system modeled as in Fig. 1. The first link represents a spacecraft, whereas the other two links represent antenna segments; the reorientation maneuver that is studied represents a deployment of the antenna that is to be accomplished while achieving a specified orientation of the spacecraft. The system characteristics are given in Table 1.

Using the notation already introduced with $N = 3$, the following are the reduced-order equations of motion

$$\dot{\theta}_1 = s_1(\psi_1,\psi_2)\omega_1 + s_2(\psi_1,\psi_2)\omega_2 \qquad (19)$$

$$\dot{\psi}_1 = \omega_1 \qquad (20)$$

$$\dot{\psi}_2 = \omega_2 \qquad (21)$$

$$\dot{\omega}_1 = u_1 \qquad (22)$$

$$\dot{\omega}_2 = u_2 \qquad (23)$$

The functions $s_1(\psi_1,\psi_2)$ and $s_2(\psi_1,\psi_2)$, determined from the angular momentum expression, are given as

$$s_i(\psi) = -N_i(\psi)/D(\psi), \qquad i = 1,2$$

where

$$N_1(\psi) = 17.5 + 7.5 \cos \psi_1 + 10.5 \cos \psi_2 + 2.5 \cos(\psi_1 + \psi_2)$$

$$N_2(\psi) = 3.75 + 5.25 \cos \psi_2 + 2.5 \cos(\psi_1 + \psi_2)$$

$$D(\psi) = 32.5 + 15 \cos \psi_1 + 10.5 \cos \psi_2 + 5 \cos(\psi_1 + \psi_2)$$

and the transformed input u is related to the control torque τ by

$$u = -J_s^{-1}(\psi)F_s(\psi,\omega) + J_s^{-1}(\psi)\tau$$

where $J_s(\psi)$ is a 2×2 matrix with entries

$$J_{s_{11}}(\psi) = 17.5 + 10.5 \cos \psi_2 - N_1^2(\psi)/D(\psi)$$

$$J_{s_{12}}(\psi) = 3.75 + 5.25 \cos \psi_2 - N_2^2(\psi)/D(\psi)$$

$$J_{s_{21}}(\psi) = 3.75 + 5.25 \cos \psi_2 - N_2^2(\psi)/D(\psi)$$

$$J_{s_{22}}(\psi) = 3.75 - N_2^2(\psi)/D(\psi)$$

and the vector function $F_s(\psi,\omega)$ can be expressed as

$$F_s(\psi,\omega) = \frac{dJ_s(\psi)}{dt} \omega - \frac{1}{2}\frac{\partial}{\partial \psi}\left[\omega' J_s(\psi)\omega\right]$$

where $\psi = (\psi_1,\psi_2)$ and $\omega = (\omega_1,\omega_2)$. We first compute the function

$$H(\psi) = \frac{\partial s_2(\psi)}{\partial \psi_1} - \frac{\partial s_1(\psi)}{\partial \psi_2}$$

on $[-\pi,\pi] \times [-\pi,\pi]$. This function is shown graphically in Fig. 2. The joint angles (ψ_1,ψ_2), where H takes the largest absolute value, are approximately

$$\left[(2\pi/3, 5\pi/6) , (-5\pi/6, -2\pi/3)\right]$$

Consequently, geometric phases for the square paths centered at $\psi^* = (2\pi/3, 5\pi/6)$ are computed numerically. Figure 3 shows the geometric phase as a function of the size of the square path.

We present a representative rest-to-rest maneuver that deploys the antenna segments from a folded configuration to a deployed configuration while achieving a desired orientation of the spacecraft link. The maneuver is defined by an initial rest configuration $(0, \pi, -\pi)$ and a final rest configuration
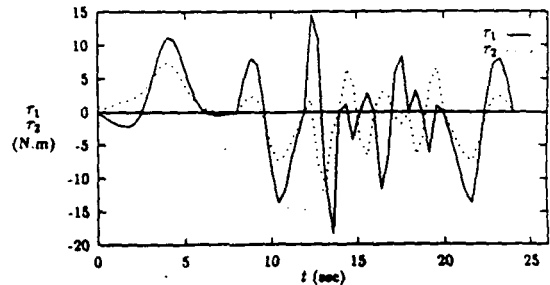

Fig. 6   Control torques τ_1 and τ_2.

Initial Configuration   Step 1   Step 2
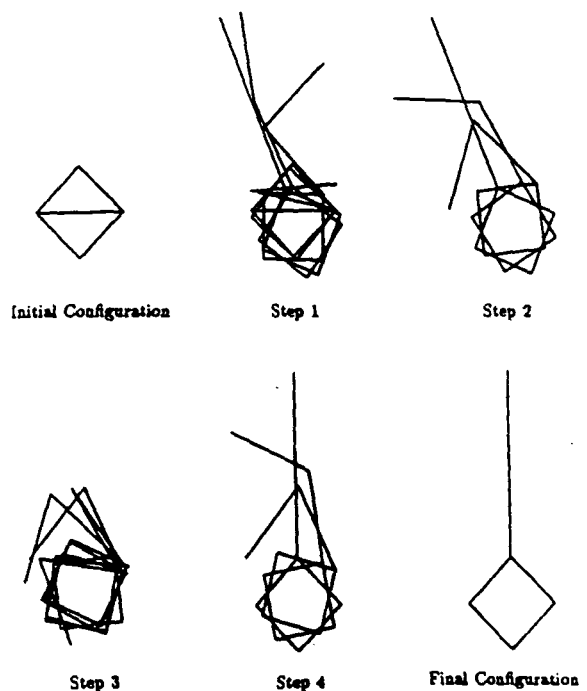
Step 3   Step 4   Final Configuration

Fig. 7   Configuration of links.

$(0.5\pi, 0, 0)$. The specific control functions indicated previously were used in the simulation; the times for each of the indicated steps are $t_1 = 8$, $t_2 = 12$, $t_3 = 20$, and $t_f = 24$. In this particular case, the required geometric phase change $\theta_1^e - \theta_1^i$ was computed to be 0.39 rad, which defined the square path used in the simulation.

The time responses for θ_1, ψ_1, and ψ_2 are shown in Fig. 4. Figure 5 illustrates the motion in the shape space. The control torques τ_1 and τ_2 are shown in Fig. 6. In Fig. 7 the maneuver is demonstrated by showing the configuration of the links for a sequence of uniformly spaced time instants.

V.   Conclusions

In this paper we have developed a reorientation maneuvering strategy for planar rigid bodies interconnected by ideal pin joints in the form of an open kinematic chain. The maneuver strategy uses the nonintegrability of the expression for angular momentum conservation. We have demonstrated that large angle maneuvers can be designed to achieve an arbitrary reorientation of the multibody system with respect to an inertial frame; the maneuvers are performed using internal controls, e.g., servo torque motors located at the joints of the body segments. The theoretical background for carrying out the required maneuvers has been briefly summarized. The results have been applied to a specific space maneuver of a three-body interconnection. We mention two nontrivial extensions of the approach in this paper that are currently being developed. The first extension is to nonplanar reorientation maneuvers of multibody systems; in this case the dynamics issues are much more complicated, but in principle the approach is viable.[22] Another extension is the development of feedback implementations of the controls presented in this paper; some results have been obtained[19] using a (necessarily) discontinuous feedback strategy. These important extensions generally require the use of differential geometric methods for a complete treatment. One motivation of the present paper has been to present the key ideas, in the case of planar reorientation maneuvers, using only elementary methods of analysis.

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Attitude Stabilization of a Rigid Spacecraft Using Two Control Torques:

A Nonlinear Control Approach Based on the Spacecraft Attitude Dynamics

PREV. AND

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by

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Abstract

The attitude stabilization problem of a rigid spacecraft using control torques supplied by gas jet actuators about only two of its principal axes is considered. We first consider the case where the uncontrolled principal axis of the spacecraft is not an axis of symmetry. In this case, the complete spacecraft dynamics are small time locally controllable. However, the spacecraft cannot be asymptotically stabilized to an equilibrium attitude using time-invariant continuous feedback. A discontinuous stabilizing feedback control strategy is constructed which stabilizes the spacecraft to an equilibrium attitude. We next consider the case where the uncontrolled principal axis of the spacecraft is an axis of symmetry. In this case, the complete spacecraft dynamics are not even accessible. However, the spacecraft dynamics are strongly accessible and small time locally controllable in a reduced sense. The reduced spacecraft dynamics cannot be asymptotically stabilized to an equilibrium attitude using time-invariant continuous feedback, but again a discontinuous stabilizing feedback control strategy is constructed. In both cases, the discontinuous feedback controllers are constructed by switching between several feedback functions which are selected to accomplish a sequence of spacecraft maneuvers. The results of the paper show that although standard nonlinear control techniques are not applicable, it is possible to construct a nonlinear discontinuous control law based on the dynamics of the particular physical system.

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1. Introduction

The attitude stabilization problem of a rigid spacecraft using control torques supplied by gas jet actuators about only two of its principal axes is revisited. Although a rigid spacecraft in general is controlled by three independent actuators about its principal axes, the situation considered in this paper may arise due to the failure of one of the actuators of the spacecraft. Since we are considering a space-based system, the problem considered here, namely, the attitude stabilization of a spacecraft operating in an actuator failure mode, is an important control problem. The linearization of the complete spacecraft dynamic equations at any equilibrium attitude has an uncontrollable eigenvalue at the origin. Consequently, controllability and stabilizability properties of the spacecraft cannot be inferred using classical linearization ideas and requires inherently nonlinear analysis. Moreover, a linear feedback control law cannot be used to asymptotically stabilize the spacecraft to an equilibrium attitude. An analysis of the controllability properties of a spacecraft with two independent control torques is made in (Crouch, 1984). Crouch (1984) showed that a necessary and sufficient condition for complete controllability of a spacecraft with control torques supplied by gas jet actuators about only two of its principal axes is that the uncontrolled principal axis must not be an axis of symmetry of the spacecraft. In (Byrnes and Isidori, 1991), it is shown that a rigid spacecraft controlled by two pairs of gas jet actuators about its principal axes cannot be asymptotically stabilized to an equilibrium attitude using a time-invariant continuously differentiable, i.e. C^1, feedback control law. Moreover, using some of the theoretical results in (Sontag, 1989) and (Zabczyk, 1989), it also follows that there does not exist any time-invariant continuous feedback control law which asymptotically stabilizes the spacecraft to an equilibrium attitude. However a smooth C^1 feedback control law is derived in (Byrnes and Isidori, 1991) which locally asymptotically stabilizes the spacecraft to a circular attractor, rather than an isolated equilibrium.

We first consider the case where the uncontrolled principal axis of the spacecraft is not an axis of symmetry. In this case, the complete spacecraft dynamics are small time locally controllable at any equilibrium attitude. However, as stated earlier, the spacecraft cannot be asymptotically stabilized to any equilibrium attitude using a time-invariant continuous feedback control law. Using local controllability results, an algorithm which locally asymptotically stabilizes the spacecraft to an isolated equilibrium is proposed in (Crouch, 1984). That algorithm is extremely complicated and is based on Lie algebraic methods in (Hermes, 1980). The algorithm yields a piecewise constant discontinuous control. Although very complicated, the algorithm is the only one proposed in the literature thus far which locally asymptotically stabilizes the spacecraft attitude to an equilibrium. In this paper a new discontinuous

stabilizing feedback control strategy is constructed which stabilizes the spacecraft to an equilibrium attitude. The control strategy is simple and is based on physical considerations of the problem.

We next consider the case where the uncontrolled principal axis of the spacecraft is an axis of symmetry. In this case, the complete spacecraft dynamics are not even accessible. Under some rather weak assumptions, the spacecraft dynamic equations are strongly accessible and small time locally controllable at any equilibrium attitude in a reduced sense. The reduced spacecraft dynamics cannot be asymptotically stabilized to an equilibrium attitude using time-invariant continuous feedback. Nevertheless, a discontinuous feedback control strategy is constructed which achieves attitude stabilization of the spacecraft.

We conclude this section with a summary of some of the important results on the stabilization of the angular velocity equations (i.e. without considering the attitude equations) of a spacecraft using fewer than three independent control torques. Asymptotic stabilization of the angular velocity equations of a spacecraft using only control torques about two of its principal axes is considered in (Aeyels, 1984) and (Brockett, 1983). It is shown that the angular velocity is asymptotically stabilizable to the origin using smooth C^1 feedback if the uncontrolled principal axis is not an axis of symmetry of the spacecraft. Explicit control laws are derived in (Brockett, 1983) and in (Aeyels, 1984) using center manifold theory. For a spacecraft with no axis of symmetry, asymptotic stabilization using a linear control law is possible using just one control torque about an axis having nonzero components along each principal axis (Aeyels, 1988). The control law, however, is not robust. In the case of an axially symmetric spacecraft controlled using a single control torque about an axis having nonzero components along each principal axis, there exists no linear control law which asymptotically stabilizes the origin; however there exists a nonlinear asymptotically stabilizing control law (Sontag and Sussman, 1988). If there is only one control torque applied about an axis which is a principal axis of the spacecraft, then asymptotic stabilization is not possible (Aeyels, 1985). However, there exist smooth C^1 feedback control laws which make the origin stable in the sense of Lyapunov (Aeyels, 1985). A point to notice is that the resulting closed loop system is robust if the moment of inertia about the control axis is either the maximum or minimum principal moment of inertia. Otherwise, the control law is not robust.

2. Kinematic and Dynamic Equations

Kinematic Equations

The orientation of a rigid spacecraft can be specified using various parametrizations of the special orthogonal group $SO(3)$. Here we use the following Euler angle convention. Consider an inertial $X_1 X_2 X_3$ coordinate frame; let $x_1 x_2 x_3$ be a coordinate frame aligned with the principal axes of the spacecraft with origin at the center of mass of the spacecraft. If the two frames are initially coincident, a series of three rotations about the body axes, performed in the proper sequence, is sufficient to allow the spacecraft to reach any orientation. The three rotations are:

a positive rotation of frame $X_1 X_2 X_3$ by an angle ψ about the X_3 axis; let $x_1' x_2' x_3'$ denote the resulting coordinate frame;

a positive rotation of frame $x_1' x_2' x_3'$ by an angle θ about the x_2' axis; let $x_1'' x_2'' x_3''$ denote the resulting frame;

a positive rotation of frame $x_1'' x_2'' x_3''$ by an angle ϕ about the x_1'' axis; let $x_1 x_2 x_3$ denote the final coordinate frame.

A rotation matrix R relates components of a vector in the inertial frame to components of the same vector in the body frame; in terms of the Euler angles a rotation matrix is of the form

$$R(\psi,\theta,\phi) = \begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix}, \qquad (2.1)$$

where $c\psi = \cos(\psi)$, $s\psi = \sin(\psi)$. We assume that the Euler angles are limited to the ranges $-\pi < \psi < \pi$, $-\pi/2 < \theta < \pi/2$, $-\pi < \phi < \pi$. Suppose ω_1, ω_2, ω_3 are the principal axis components of the absolute angular velocity vector ω of the spacecraft. Then expressions for ω_1, ω_2, ω_3 are given by

$$\omega_1 = \dot{\phi} - \dot{\psi}\sin\theta , \qquad (2.2)$$

$$\omega_2 = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta \sin\phi , \qquad (2.3)$$

$$\omega_3 = -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta \cos\phi . \qquad (2.4)$$

By excluding the case where $\theta = \pm \pi/2$, these equations are invertible. Thus we can solve for $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ in terms of ω_1, ω_2, ω_3 obtaining

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi \ \tan\theta + \omega_3\cos\phi \ \tan\theta \ , \tag{2.5}$$

$$\dot{\theta} = \omega_2\cos\phi - \omega_3\sin\phi \ , \tag{2.6}$$

$$\dot{\psi} = \omega_2\sin\phi \ \sec\theta + \omega_3\cos\phi \ \sec\theta \ . \tag{2.7}$$

Next we consider the dynamic equations which describe the evolution of the angular velocity components of the spacecraft.

Dynamic Equations

Let $J = \text{diag} \ (J_1, J_2, J_3)$, $J_i > 0$, $i = 1, 2, 3$, be the inertia matrix of the spacecraft in a coordinate frame defined by its principal axes. Let H be the angular momentum vector of the spacecraft relative to the inertial frame. Then we have

$$J\omega = R(\psi,\theta,\phi)H \ . \tag{2.8}$$

Differentiating (2.8) we obtain

$$J\dot{\omega} = S(\omega)R(\psi,\theta,\phi)H + R(\psi,\theta,\phi)\dot{H} \ , \tag{2.9}$$

where

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \ . \tag{2.10}$$

We assume that the control torques u'_1 and u'_2 are applied about axes represented by unit vectors b_1 and b_2 respectively. This implies that

$$R(\psi,\theta,\phi)\dot{H} = b_1 u'_1 + b_2 u'_2 \ . \tag{2.11}$$

Without loss of generality, we assume that $b_1 = (1, 0, 0)^T$ and $b_2 = (0, 1, 0)^T$. Thus the equations describing the evolution of the angular velocity of the spacecraft are given by

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u'_1 \ , \tag{2.12}$$

$$J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u'_2 \ , \tag{2.13}$$

$$J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2 \ . \tag{2.14}$$

3. Controllability and Stabilizability Properties of Complete Spacecraft Dynamics with Two Control Torques

As background for our subsequent development, we consider the controllability and stabilizability properties for the complete dynamics of the spacecraft with control torques only about two of its principal axes. Define

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \dfrac{u'_1}{J_1} \\ \dfrac{u'_2}{J_2} \end{bmatrix} .$$

From Section 2 the state equations can be rewritten as

$$\dot{\omega}_1 = a_1\omega_2\omega_3 + u_1 , \tag{3.1}$$

$$\dot{\omega}_2 = a_2\omega_1\omega_3 + u_2 , \tag{3.2}$$

$$\dot{\omega}_3 = a_3\omega_1\omega_2 , \tag{3.3}$$

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi \tan\theta + \omega_3\cos\phi \tan\theta , \tag{3.4}$$

$$\dot{\theta} = \omega_2\cos\phi - \omega_3\sin\phi , \tag{3.5}$$

$$\dot{\psi} = \omega_2\sin\phi \sec\theta + \omega_3\cos\phi \sec\theta , \tag{3.6}$$

where

$$a_1 = \frac{J_2 - J_3}{J_1} , \ a_2 = \frac{J_3 - J_1}{J_2} , \ a_3 = \frac{J_1 - J_2}{J_3} .$$

This is of the form

$$\dot{x} = f(x) + g_1 u_1 + g_2 u_2 , \tag{3.7}$$

where $x = (\omega_1,\omega_2,\omega_3,\phi,\theta,\psi)^T$ and f , g_1, g_2 are vector fields defined appropriately on the open set

$$M = \{x : \omega_i \in R, i = 1, 2, 3, \phi, \psi \in (-\pi , \pi), \theta \in (-0.5\pi , 0.5\pi)\} .$$

It is easily verified that the linearization of the equations about an equilibrium has an uncontrollable eigenvalue at the origin. This implies that an inherently nonlinear analysis is necessary in order to characterize the controllability and stabilizability properties of the complete spacecraft dynamics. Moreover, a linear feedback control law cannot be used to

asymptotically stabilize the spacecraft to an equilibrium attitude.

We now present fundamental results on the controllability and stabilizability properties of the complete spacecraft dynamics described by equations (3.1)-(3.6).

Theorem 3.1: The complete spacecraft dynamics described by state equations (3.1)-(3.6) are strongly accessible $\forall x \in M$ if and only if $J_1 \neq J_2$, i.e. the uncontrolled principal axis is not an axis of symmetry.

Proof: If $J_1 \neq J_2$, the vector fields g_1, g_2, $[g_1 f]$, $[g_2 f]$, $[g_2,[g_1 f]]$, $[[g_2,[g_1 f]]f]$ span a six dimensional space at every $x \in M$. Thus the strong accessibility Lie algebraic rank condition is satisfied and hence the complete spacecraft dynamics are strongly accessible. If $J_1 = J_2$ the complete spacecraft dynamics fails to be accessible since ω_3 is necessarily constant.

Theorem 3.2: The complete spacecraft dynamics described by state equations (3.1)-(3.6) are small time locally controllable at any equilibrium if and only if $J_1 \neq J_2$.

Proof: Suppose $J_1 \neq J_2$. Then the complete spacecraft dynamics are strongly accessible. Following Sussman (1987), let Br(x) denote the smallest Lie algebra of vector fields containing f, g_1, g_2. Let B be any bracket in Br(x). Now denote $\delta^0(B)$, $\delta^1(B)$, $\delta^2(B)$ as the number of occurrences of the vector fields f, g_1, g_2 respectively in the bracket B. The degree of B is equal to the value of $\sum_{i=0}^{2} \delta^i(B)$. The Sussman condition for small time local controllability at an equilibrium is that the so-called *bad* brackets, the brackets with δ^0 odd, and δ^1, δ^2 even, must be a linear combination of brackets of lower degree at that equilibrium. From the proof of Theorem 3.1 it is clear that any bracket of degree greater than four can be expressed as a linear combination of lower order brackets at any equilibrium. Moreover the degree of a bad bracket must necessarily be odd. The bad bracket of degree one is f which vanishes at any equilibrium. The bad brackets of degree three are $[g_1,[g_1 f]]$ and $[g_2,[g_2 f]]$ and both are identically zero vector fields. Thus the complete spacecraft dynamics are small time locally controllable. If $J_1 = J_2$, the complete spacecraft dynamics fails to be accessible at any equilibrium; hence it cannot be small time locally controllable at any equilibrium.

Theorem 3.3: The complete spacecraft dynamics described by state equations (3.1)-(3.6) cannot be locally asymptotically stabilized to an equilibrium by any time-invariant continuous state feedback control law.

This result holds if $J_1 \neq J_2$ and also if $J_1 = J_2$. A weaker version of the above theorem (with "continuous" replaced by "C^1") was proved in (Byrnes and Isidori, 1991). However, Theorem 3.3 follows from (Byrnes and Isidori, 1991) using results in (Sontag, 1989) and (Zabczyk, 1989). This negative result also implies that feedback control approaches based on

linearization, Lyapunov methods, center manifold theory, or zero dynamics cannot be used to asymptotically stabilize the spacecraft to an equilibrium attitude.

Although the full set of equations (3.1)-(3.6) cannot be asymptotically stabilized to an equilibrium via continuous feedback, one may still wish to design a smooth control law which stabilizes at least a particular subset of state variables. Consider the state equations for ω_1, ω_2, ω_3, ϕ and θ given by equations (3.1)-(3.5). These equations are not affected by the Euler angle ψ. Asymptotic stabilization of this subset of the original equations corresponds to stabilization of the motion of the spacecraft about an attractor, which is not an isolated equilibrium. A result from (Byrnes and Isidori, 1991) shows that the closed loop trajectories can be asymptotically stabilized to the manifold

$$\Omega = \{(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) : \omega_1 = \omega_2 = \omega_3 = \phi = \theta = 0\} , \qquad (3.8)$$

using smooth $C^{\hat{k}}$ feedback.

We mention that although the complete spacecraft dynamics described by equations (3.1)-(3.6) cannot be asymptotically stabilized to an equilibrium by continuous feedback, an algorithm generating a piecewise constant discontinuous control has been developed in (Crouch, 1984) which locally asymptotically stabilizes the complete spacecraft dynamics to an equilibrium. The algorithm requires that $J_1 \neq J_2$, i.e. the uncontrolled principal axis must not be an axis of symmetry. The algorithm is based on Lie algebraic methods in (Hermes, 1980). The algorithm is extremely complicated and is not an easily implementable control strategy. However, stabilization of the complete spacecraft dynamic equations (3.1)-(3.6) is an inherently difficult problem and the algorithm in (Crouch, 1984) is the only control strategy proposed in the literature thus far.

4. Attitude Stabilization of a Non-Axially Symmetric Spacecraft with Two Control Torques

In this section, we consider the equations (3.1)-(3.6) describing the motion of a spacecraft controlled by input torques only about two of its principal axes. It is assumed that the uncontrolled principal axis is not an axis of symmetry of the spacecraft; i.e. $J_1 \neq J_2$. As a consequence of the negative result of Theorem 3.3, we restrict our study to the class of discontinuous feedback controllers in order to asymptotically stabilize the complete spacecraft dynamics. However, as shown in the previous section, the complete spacecraft dynamics are small time locally controllable at any equilibrium attitude. This suggests that a piecewise analytic feedback control law can be constructed which asymptotically stabilizes the complete spacecraft dynamics to an equilibrium attitude. Here we present a particular discontinuous

feedback strategy, which is obtained by requiring that the spacecraft undergo a sequence of specified maneuvers. Without loss of generality, we assume that the equilibrium attitude to be stabilized is the origin. We first present a physical interpretation of the sequence of maneuvers that transfers any initial state to the origin.

Maneuvers 1-3. Transfer the initial state of the spacecraft to an equilibrium state in finite time; i.e. bring the spacecraft to rest.

There are control laws based on center manifold theory (Aeyels, 1984) and zero dynamics theory (Byrnes and Isidori, 1991) which accomplish this in an asymptotic sense. Here we use a sequence of three maneuvers, and corresponding feedback control laws, which bring the spacecraft to rest in finite time.

Maneuver 4. Transfer the resulting state to an equilibrium state where $\phi = 0$ in finite time; i.e. so that the spacecraft is at rest with $\phi = 0$. This maneuver is accomplished using the control torque u_1 only.

Maneuver 5. Transfer the resulting state to an equilibrium state where $\phi = 0$, $\theta = 0$ in finite time; i.e. so that the spacecraft is at rest with $\phi = 0$, $\theta = 0$. This maneuver is accomplished using the control torque u_2 only.

In order to complete specification of the sequence of maneuvers, the Euler angle ψ must be brought to zero. This cannot be accomplished directly since a control torque cannot be applied about the third principal axis of the spacecraft. However, the resulting state can be transferred to the origin indirectly using three maneuvers. The three maneuvers correspond to three consecutive rotations about the two controlled principal axes of the spacecraft, the first and the third being around the first principal axis. This produces a net change in the orientation of the spacecraft (see Figure 9 in Marsden et. al, 1991) so that the state of the spacecraft is transferred to the origin in finite time. The three maneuvers are described as follows.

Maneuver 6. Transfer the resulting state to an equilibrium state where $\phi = \dfrac{\pi}{2}$, $\theta = 0$ in finite time; i.e. so that the spacecraft is at rest with $\phi = \dfrac{\pi}{2}$, $\theta = 0$. This maneuver is accomplished using the control torque u_1 only.

Maneuver 7. Transfer the resulting state to the equilibrium state $(0,0,0,\dfrac{\pi}{2},0,0)^T$ in finite time. This maneuver is accomplished using the control torque u_2 only.

Maneuver 8. Transfer the equilibrium state $(0,0,0,\dfrac{\pi}{2},0,0)^T$ to the equilibrium state $(0,0,0,0,0,0)^T$ in finite time. This maneuver is accomplished using the control torque u_1 only.

Note that, excluding the first three maneuvers where the spacecraft is brought to rest, all subsequent maneuvers are such that the angular velocity component ω_3 is maintained identically zero. This is accomplished by carrying out maneuvers which require use of only a single control torque at a time. It is convenient to introduce some notation. Throughout, assume $k > 0$, and define

$$
G(x_1, x_2) = \begin{cases} k \;\; \text{if} \;\; \{x_1 + \dfrac{x_2|x_2|}{2k} > 0\} \; \text{or} \; \{x_1 + \dfrac{x_2|x_2|}{2k} = 0 \; \text{and} \; x_2 > 0\} \\[3mm] -k \;\; \text{if} \;\; \{x_1 + \dfrac{x_2|x_2|}{2k} < 0\} \; \text{or} \; \{x_1 + \dfrac{x_2|x_2|}{2k} = 0 \; \text{and} \; x_2 < 0\} \\[3mm] 0 \;\; \text{if} \;\; \{x_1 = 0 \; \text{and} \; x_2 = 0\} \end{cases}
$$

We use the well-known property that the feedback control

$$
u = - G(x_1 - \bar{x}_1, x_2)
$$

for the system

$$
\dot{x}_1 = x_2
$$

$$
\dot{x}_2 = u
$$

transfers any initial state to the final state $(\bar{x}_1, 0)$ in a finite time. We also use the standard notation that

$$
\text{sign}(x_1) = \begin{cases} 1 \;\; \text{if} \;\; x_1 > 0 \\ -1 \;\; \text{if} \;\; x_1 < 0 \; . \\ 0 \;\; \text{if} \;\; x_1 = 0 \end{cases}
$$

Our mathematical construction of a control strategy which transfers an arbitrary initial state of the spacecraft to the origin is based on a sequence of equilibrium subsets and a sequence of control functions which transfers a state in one subset to another. Consider the following equilibrium subsets of M

$$
M_1 = \{x = (0,0,0,\phi,\theta,\psi)^T \mid \phi,\theta,\psi \; \text{arbitrary}\} \,,
$$

$$
M_2 = \{x = (0,0,0,0,\theta,\psi)^T \mid \theta,\psi \; \text{arbitrary}\} \,,
$$

$$
M_3 = \{x = (0,0,0,0,0,\psi)^T \mid \psi \; \text{arbitrary}\} \,,
$$

$$M_4 = \{x = (0,0,0,\frac{\pi}{2},0,\psi)^T \mid \psi \text{ arbitrary}\} .$$

We now present the feedback control laws that accomplish the sequential maneuvers described above; for each case we show that a desired terminal state which defines the maneuver is reached.

Transferring any initial state to a state in M_1

In order to transfer the arbitrary initial state to a final state which satisfies $\omega_1 = \omega_2 = \omega_3 = 0$ three sequential maneuvers are required. The first maneuver results in $\omega_1 = \omega_2 = 0$ while $\omega_3 \neq 0$ in general; the second maneuver results in $\omega_1 = \omega_1^*$ and $\omega_2 = \omega_2^*$, where ω_1^*, ω_2^* are chosen to guarantee that at the end of the third maneuver $\omega_1 = \omega_2 = \omega_3 = 0$. These three maneuvers are described in detail as follows.

Maneuver 1. Let $(\omega_1^0, \omega_2^0, \omega_3^0, \phi^0, \theta^0, \psi^0)^T \in M$ denote an initial state for the complete spacecraft dynamics described by equations (3.1)-(3.6). Define

$$v_1 = a_1 \omega_2 \omega_3 + u_1 ,$$

$$v_2 = a_2 \omega_3 \omega_1 + u_2 .$$

Equations (3.1)-(3.3) can now be rewritten as

$$\dot{\omega}_1 = v_1 , \tag{4.1}$$

$$\dot{\omega}_2 = v_2 , \tag{4.2}$$

$$\dot{\omega}_3 = a_3 \omega_1 \omega_2 . \tag{4.3}$$

Apply the feedback control functions

$$v_1 = - k \operatorname{sign} \omega_1 ,$$

$$v_2 = - k \operatorname{sign} \omega_2 .$$

It is easy to see that after a finite time given by $\max(\dfrac{|\omega_1^0|}{k}, \dfrac{|\omega_2^0|}{k})$, $\omega_1 = \omega_2 = 0$; at this instant let $\omega_3 = \bar{\omega}_3$ where the constant value $\bar{\omega}_3$ can be evaluated.

Maneuver 2. Apply the feedback control functions

$$v_1 = - k \operatorname{sign}(\omega_1 - \omega_1^*) ,$$

$$v_2 = -k\,\mathrm{sign}(\omega_2 - \omega_2^*)\,,$$

where

$$\omega_1^* = \left[\frac{3k\,|\overline{\omega}_3|}{2|a_3|}\right]^{\frac{1}{3}}\,, \quad \omega_2^* = -\omega_1^*\,\mathrm{sign}\overline{\omega}_3\,\mathrm{sign}a_3\,.$$

It is again easy to see that after a finite time given by $\dfrac{\omega_1^*}{k}$, $\omega_1 = \omega_1^*$, $\omega_2 = \omega_2^*$, and in addition it can be shown that $\omega_3 = \dfrac{\overline{\omega}_3}{2}$.

Maneuver 3. Apply the feedback control functions

$$v_1 = -k\,\mathrm{sign}\omega_1\,,$$

$$v_2 = -k\,\mathrm{sign}\omega_2\,.$$

It can be seen that after a finite time given by $\dfrac{\omega_1^*}{k}$, $\omega_1 = 0$, $\omega_2 = 0$ and it can be shown that $\omega_3 = 0$.

Consequently, the resulting state after these three sequential maneuvers is $(0,0,0,\phi^1,\theta^1,\psi^1)^T \in M_1$ for some ϕ^1, θ^1, ψ^1.

Transferring a state in M_1 to a state in M_2 (Maneuver 4)

Let $(0,0,0,\phi^1,\theta^1,\psi^1)^T \in M_1$ denote a state of the spacecraft. Apply the feedback control functions

$$u_1 = -G(\phi,\,\omega_1)\,,$$

$$u_2 = 0\,.$$

It follows that

$$\omega_2 = 0\,,\ \omega_3 = 0\,,$$

$$\theta = \theta^1\,,\ \psi = \psi^1\,,$$

satisfy equations (3.2), (3.3), (3.5), (3.6) while equations (3.1), (3.4) become

$$\dot{\omega}_1 = -G(\phi,\,\omega_1)\,,$$

$$\dot{\phi} = \omega_1 .$$

Consequently, after a finite time $\omega_1 = 0$, $\phi = 0$; and thus the maneuver transfers a state $(0,0,0,\phi^1,\theta^1,\psi^1)^T \in M_1$ to the state $(0,0,0,0,\theta^1,\psi^1)^T \in M_2$ in finite time.

Transferring a state in M_2 to a state in M_3 (Maneuver 5)

Let $(0,0,0,0,\theta^1,\psi^1)^T \in M_2$ denote a state of the spacecraft. Apply the feedback control functions

$$u_1 = 0 ,$$

$$u_2 = - G(\theta, \omega_2) .$$

It follows that

$$\omega_1 = 0 , \ \omega_3 = 0 ,$$

$$\phi = 0 , \psi = \psi^1 ,$$

satisfy equations (3.1), (3.3), (3.4), (3.6) while equations (3.2), (3.5) become

$$\dot{\omega}_2 = - G(\theta, \omega_2) ,$$

$$\dot{\theta} = \omega_2 .$$

Consequently, after a finite time $\omega_2 = 0$, $\theta = 0$; and thus the maneuver transfers a state $(0,0,0,0,\theta^1,\psi^1)^T \in M_2$ to the state $(0,0,0,0,0,\psi^1)^T \in M_3$ in finite time.

Transferring a state in M_3 to a state in M_4 (Maneuver 6)

Let $(0,0,0,0,0,\psi^1)^T \in M_3$ denote a state of the spacecraft. Apply the feedback control functions

$$u_1 = - G(\phi - \frac{\pi}{2}, \omega_1) ,$$

$$u_2 = 0 .$$

It follows that

$$\omega_2 = 0 , \ \omega_3 = 0 ,$$

$$\theta = 0 \ , \ \psi = \psi^1 \ ,$$

satisfy equations (3.2), (3.3), (3.5), (3.6) while equations (3.1), (3.4) become

$$\dot{\omega}_1 = - G \left(\phi - \frac{\pi}{2}, \ \omega_1 \right) ,$$

$$\dot{\phi} = \omega_1 \ .$$

Consequently, after a finite time $\omega_1 = 0$, $\phi = \dfrac{\pi}{2}$; and thus the maneuver transfers a state $(0,0,0,0,0,\psi^1)^T \in M_3$ to the state $\left(0,0,0,\dfrac{\pi}{2},0,\psi^1\right)^T \in M_4$ in finite time.

Transferring a state in M_4 to $\left(0,0,0,\dfrac{\pi}{2},0,0\right)^T$ (Maneuver 7)

Let $\left(0,0,0,\dfrac{\pi}{2},0,\psi^1\right)^T \in M_4$ denote a state of the spacecraft. Apply the feedback control functions

$$u_1 = 0 \ ,$$

$$u_2 = - G (\psi, \ \omega_2) \ .$$

It follows that

$$\omega_1 = 0 \ , \ \omega_3 = 0 \ ,$$

$$\phi = \frac{\pi}{2} \ , \theta = 0 \ ,$$

satisfy equations (3.1), (3.3), (3.4), (3.5) while equations (3.2), (3.6) become

$$\dot{\omega}_2 = - G (\psi, \ \omega_2) \ ,$$

$$\dot{\psi} = \omega_2 \ .$$

Consequently, after a finite time $\omega_2 = 0$, $\psi = 0$; and thus the maneuver transfers a state $\left(0,0,0,\dfrac{\pi}{2},0,\psi^1\right)^T \in M_4$ to the state $\left(0,0,0,\dfrac{\pi}{2},0,0\right)^T$ in finite time.

Transferring $(0,0,0,\frac{\pi}{2},0,0)^T$ **to** $(0,0,0,0,0,0)^T$ **(Maneuver 8)**

Let $(0,0,0,\frac{\pi}{2},0,0)^T$ denote the state of the spacecraft. Apply the feedback control functions

$$u_1 = - G(\phi, \omega_1),$$

$$u_2 = 0.$$

It follows that

$$\omega_2 = 0, \quad \omega_3 = 0,$$

$$\theta = 0, \quad \psi = 0,$$

satisfy equations (3.2), (3.3), (3.5), (3.6) while equations (3.1), (3.4) become

$$\dot{\omega}_1 = - G(\phi, \omega_1),$$

$$\dot{\phi} = \omega_1.$$

Consequently, after a finite time $\omega_1 = 0$, $\phi = 0$; and thus the maneuver transfers $(0,0,0,\frac{\pi}{2},0,0)^T$ to the state $(0,0,0,0,0,0)^T$ in finite time.

In summary, the feedback control strategy outlined above can be implemented by sequential switching between the following feedback functions.

Maneuver 1. Apply

$$u_1^1(x) = - a_1\omega_2\omega_3 - k\,\text{sign}\,\omega_1,$$

$$u_2^1(x) = - a_2\omega_3\omega_1 - k\,\text{sign}\,\omega_2,$$

until $(\omega_1,\omega_2,\omega_3) = (0,0,\varpi_3)$ for some value ϖ_3; then go to Manuever 2.

Maneuver 2. Compute

$$\omega_1^* = \left[\frac{3k\,|\varpi_3|}{2|a_3|}\right]^{\frac{1}{3}}, \quad \omega_2^* = - \left[\frac{3k\,|\varpi_3|}{2|a_3|}\right]^{\frac{1}{3}}\text{sign}\,\varpi_3\,\text{sign}\,a_3;$$

apply

$$u_1^2(x) = -a_1\omega_2\omega_3 - k\,\mathrm{sign}(\omega_1 - \omega_1^*) \,,$$

$$u_2^2(x) = -a_1\omega_3\omega_1 - k\,\mathrm{sign}(\omega_2 - \omega_2^*) \,,$$

until $(\omega_1,\omega_2,\omega_3) = (\omega_1^*,\omega_2^*,\dfrac{\omega_3}{2})$; then go to Maneuver 3.

Maneuver 3. Apply

$$u_1^3(x) = -a_1\omega_2\omega_3 - k\,\mathrm{sign}\omega_1 \,,$$

$$u_2^3(x) = -a_2\omega_3\omega_1 - k\,\mathrm{sign}\omega_2 \,,$$

until $(\omega_1,\omega_2,\omega_3) = (0,0,0)$, i.e. $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi)^T \in M_1$; then go to Maneuver 4.

Maneuver 4: Apply

$$u_1^4(x) = -G(\phi,\,\omega_1) \,,$$

$$u_2^4(x) = 0 \,,$$

until $(\omega_1,\omega_2,\omega_3,\phi) = (0,0,0,0)$, i.e. $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi)^T \in M_2$; then go to Maneuver 5.

Maneuver 5: Apply

$$u_1^5(x) = 0 \,,$$

$$u_2^5(x) = -G(\theta,\,\omega_2) \,,$$

until $(\omega_1,\omega_2,\omega_3,\phi,\theta) = (0,0,0,0,0)$, i.e. $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi)^T \in M_3$; then go to Maneuver 6.

Maneuver 6: Apply

$$u_1^6(x) = -G(\phi - \frac{\pi}{2},\,\omega_1) \,,$$

$$u_2^6(x) = 0 \,,$$

until $(\omega_1,\omega_2,\omega_3,\phi,\theta) = (0,0,0,\frac{\pi}{2},0)$, i.e. $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi)^T \in M_4$; then go to Maneuver 7.

Maneuver 7: Apply

$$u_1^7(x) = 0 \,,$$

$$u_2^7(x) = -G(\psi,\,\omega_2) \,,$$

until $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) = (0,0,0,\frac{\pi}{2},0,0)$; then go to Maneuver 8.

Maneuver 8: Apply

$$u_1^8(x) = -G(\phi, \omega_1),$$

$$u_2^8(x) = 0,$$

until $(\omega_1, \omega_2, \omega_3, \phi, \theta, \psi) = (0,0,0,0,0,0)$.

This feedback control strategy achieves attitude stabilization of the spacecraft by executing a sequence of maneuvers. This strategy is discontinuous and nonclassical in nature. Justification that it stabilizes the complete spacecraft dynamics to an equilibrium attitude in finite time, under the ideal model assumptions, follows as a consequence of the construction procedure. A computer implementation of the feedback control strategy can be easily carried out.

5. Attitude Stabilization of an Axially Symmetric Spacecraft with Two Control Torques

From the analysis made in Section 3, we find that the complete dynamics of a spacecraft controlled by two control torques supplied by gas jet actuators, as described by equations (3.1)-(3.6), fail to be controllable or even accessible if the uncontrolled principal axis is an axis of symmetry of the spacecraft, i.e. if $J_1 = J_2$. Due to the lack of controllability, the control algorithm proposed in (Crouch, 1984) is not applicable to this case. In this section we concentrate on the case where the uncontrolled principal axis of the spacecraft is an axis of symmetry, i.e. $J_1 = J_2$. In particular we ask the question: what restricted control and stabilization properties of the spacecraft can be demonstrated in this case? Our analysis begins by demonstrating that, under appropriate restrictions of interest, the spacecraft equations can be expressed in a reduced form. Controllability and stabilizability properties for this case follow from an analysis of the reduced equations.

Consider the equations (3.1)-(3.6) describing the motion of a spacecraft controlled by input torques supplied by gas jet actuators about only two of its principal axes. It is assumed that the uncontrolled principal axis is an axis of symmetry of the spacecraft. From equations (3.1)-(3.6) and $J_1 = J_2$ we have

$$\dot{\omega}_1 = a_1 \omega_2 \omega_3 + u_1, \tag{5.1}$$

$$\dot{\omega}_2 = a_2 \omega_1 \omega_3 + u_2, \tag{5.2}$$

$$\dot{\omega_3} = 0 , \tag{5.3}$$

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi \, \tan\theta + \omega_3\cos\phi \, \tan\theta , \tag{5.4}$$

$$\dot{\theta} = \omega_2\cos\phi - \omega_3\sin\phi , \tag{5.5}$$

$$\dot{\psi} = \omega_2\sin\phi \, \sec\theta + \omega_3\cos\phi \, \sec\theta . \tag{5.6}$$

If $\omega_3(0) \neq 0$ then ω_3 cannot be transferred to zero using any control function. If we assume that $\omega_3(0) = 0$, then $\omega_3 \equiv 0$. Under the restriction $\omega_3(0) = 0$, the reduced spacecraft dynamics for this case are described by

$$\dot{\omega_1} = u_1 , \tag{5.7}$$

$$\dot{\omega_2} = u_2 , \tag{5.8}$$

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi \, \tan\theta , \tag{5.9}$$

$$\dot{\theta} = \omega_2\cos\phi , \tag{5.10}$$

$$\dot{\psi} = \omega_2\sin\phi \, \sec\theta . \tag{5.11}$$

The following results can now be easily shown. The proofs of Theorem 5.1 and Theorem 5.2 are similar to the proofs of Theorem 3.1 and Theorem 3.2 respectively. Theorem 5.3 follows from the results in (Brockett, 1983), (Sontag, 1989) and (Zabczyk, 1989).

Theorem 5.1: The reduced dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (5.7)-(5.11) are strongly accessible.

Theorem 5.2: The reduced dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (5.7)-(5.11) are small time locally controllable at any equilibrium.

Theorem 5.3: The reduced dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (5.7)-(5.11) cannot be asymptotically stabilized to an equilibrium using a time-invariant continuous feedback control law.

The implications of the properties stated above are as follows. For all initial conditions that satisfy $\omega_3(0) = 0$, the axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (5.1)-(5.6) can be controlled to any equilibrium attitude. However, any time-invariant feedback control law that asymptotically stabilizes the spacecraft to an isolated equilibrium attitude must necessarily be discontinuous. Thus arbitrary reorientation of the spacecraft can be achieved if $\omega_3(0) = 0$; if $\omega_3(0) \neq 0$, reorientation of the spacecraft to an equilibrium attitude cannot be achieved.

Conveniently, it turns out that sequential execution of the maneuvers defined as Manuevers 3 through 8 in the previous section transfers any initial state of the reduced spacecraft dynamics (5.7)-(5.11) to the origin in finite time. The physical interpretation of the manuevers is the same as described previously; the overall feedback control strategy is as follows.

Maneuver 1. Apply

$$u_1^1(x) = -k\,\text{sign}\,\omega_1 ,$$

$$u_2^1(x) = -k\,\text{sign}\,\omega_2 ,$$

until $(\omega_1,\omega_2) = (0,0)$; then go to Maneuver 2.

Maneuver 2: Apply

$$u_1^2(x) = -G(\phi,\ \omega_1) ,$$

$$u_2^2(x) = 0 ,$$

until $(\omega_1,\omega_2,\phi) = (0,0,0)$; then go to Maneuver 3.

Manuever 3: Apply

$$u_1^3(x) = 0 ,$$

$$u_2^3(x) = -G(\theta,\ \omega_2) ,$$

until $(\omega_1,\omega_2,\phi,\theta) = (0,0,0,0)$; then go to Maneuver 4.

Maneuver 4: Apply

$$u_1^4(x) = -G(\phi - \frac{\pi}{2},\ \omega_1) ,$$

$$u_2^4(x) = 0 ,$$

until $(\omega_1,\omega_2,\phi,\theta) = (0,0,\frac{\pi}{2},0)$, then go to Maneuver 5.

Maneuver 5: Apply

$$u_1^5(x) = 0 ,$$

$$u_2^5(x) = -G(\psi,\ \omega_2) ,$$

until $(\omega_1,\omega_2,\phi,\theta,\psi) = (0,0,\frac{\pi}{2},0,0)$; then go to Maneuver 6.

Maneuver 3, the control torques u_1 and u_2 are both applied to bring the spacecraft to rest. But once the spacecraft is brought to rest, the subsequent maneuvers are such that only one of the control torques is nonzero in any interval of time. Thus ω_3 remains zero at all time beyond 1.73 seconds, and ω_1 and ω_2 vary so that only one is nonzero at any time interval beyond 1.73 seconds. Three dimensional visualization schemes have been developed using a Silicon Graphics Iris work station in order to display the reorientation maneuvers of the spacecraft.

7. Conclusion

The attitude stabilization problem of a spacecraft using control torques supplied by gas jet actuators about only two of its principal axes has been considered. If the uncontrolled principal axis is not an axis of symmetry of the spacecraft, the complete spacecraft dynamics cannot be asymptotically stabilized to an equilibrium attitude using continuous feedback. A discontinuous feedback control strategy was constructed which stabilizes the spacecraft to an equilbrium attitude in finite time. If the uncontrolled principal axis is an axis of symmetry of the spacecraft, the complete spacecraft dynamics cannot be stabilized. The reduced spacecraft dynamics cannot be asymptotically stabilized using continuous feedback, but again a discontinuous feedback control strategy was constructed which stabilizes the spacecraft (in the reduced sense) to an equilibrium attitude in finite time. The results of the paper show that although standard nonlinear control techniques do not apply, it is possible to construct a stabilizing control law by performing a sequence of maneuvers.

One of the advantages of the development in this paper is that feedback control strategies are constructed which guarantee attitude stabilization in a finite time. The total time required to complete the spacecraft reorientation is the sum of the times required to complete the sequence of maneuvers described. From the analysis provided, it should be clear that the time required to complete each maneuver depends on the single positive parameter k in the corresponding control law. There is a trade off between the required control levels, determined by the selection of k, and the resulting times to complete each of the maneuvers and hence the total time required to reorient the spacecraft. In particular, the time to reorient the spacecraft from a given initial state to the origin can be expressed as a function of the value of the parameter k and of the initial state.

For each of the two attitude stabilization problems considered, we have presented one example of a sequence of maneuvers which achieves the desired spacecraft attitude stabilization. There are many other maneuver sequences, and corresponding feedback control strategies, which will also achieve the desired attitude stabilization of the spacecraft. But each such strategy is necessarily discontinuous.

We have demonstrated the closed loop properties for the special feedback control strategies presented. Our analysis was based on a number of assumptions which are required to justify the mathematical models studied. Further robustness analysis is required to determine effects of model uncertainities and external disturbances. Unfortunately, such robustness analysis is quite difficult since the closed loop vector fields are necessarily discontinuous. Perhaps, feedback control strategies which stabilize the spacecraft attitude, different from ones presented in this paper, would provide improved closed loop robustness.

Acknowledgements

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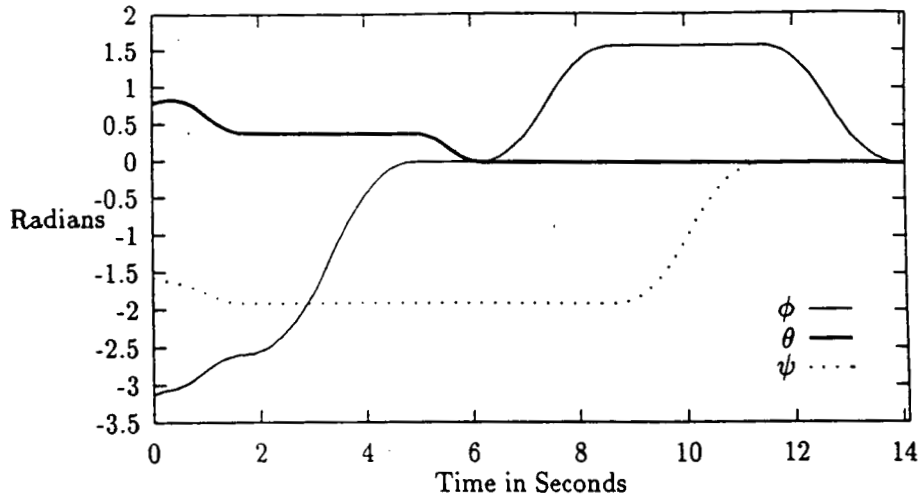
Figure 1: Plot of Euler Angles
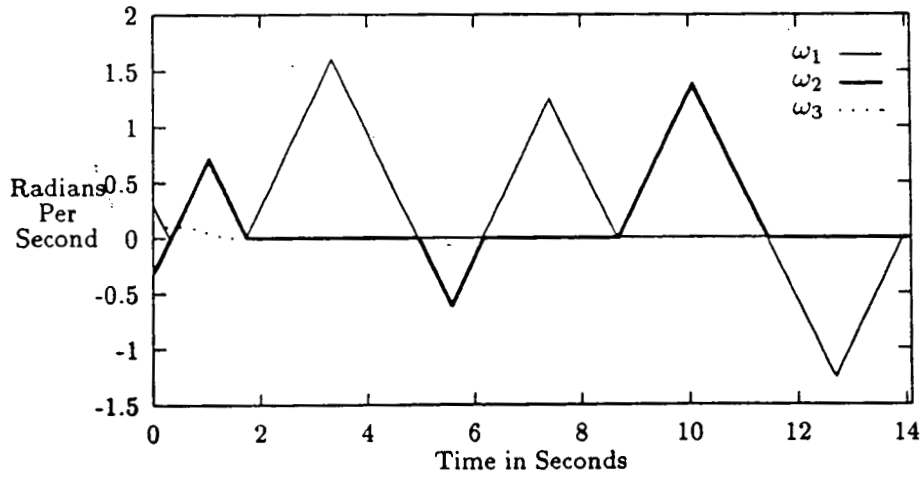
Figure 2: Plot of Angular Velocities
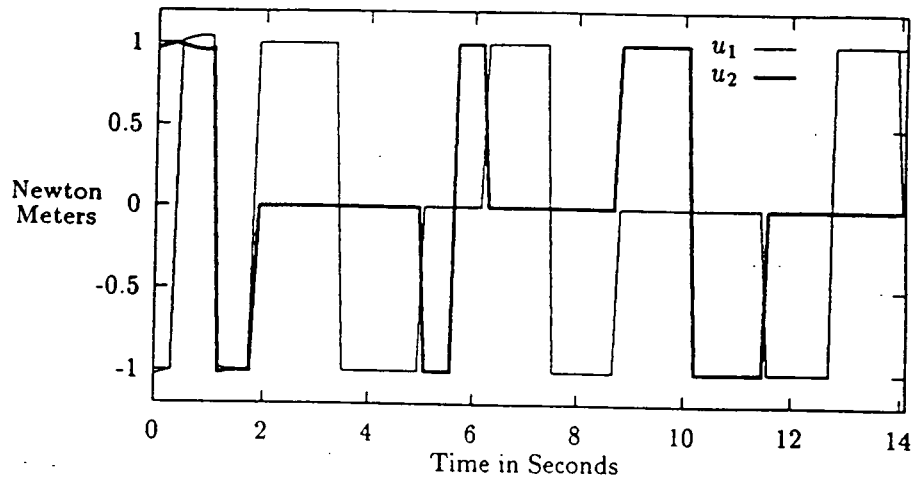
Figure 3: Plot of u_1 and u_2

ATTITUDE STABILIZATION OF A RIGID SPACECRAFT USING TWO MOMENTUM WHEEL ACTUATORS

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Abstract

It is well known that three momentum wheel actuators can be used to control the attitude of a rigid spacecraft and that arbitrary reorientation maneuvers of the spacecraft can be accomplished using smooth feedback. If failure of one of the momentum wheel actuators occurs, we demonstrate that two momentum wheel actuators can be used to control the attitude of a rigid spacecraft and that arbitrary reorientation maneuvers of the spacecraft can be accomplished. Although the complete spacecraft equations are not controllable, the spacecraft equations are small time locally controllable in a reduced nonlinear sense. The reduced spacecraft dynamics cannot be asymptotically stabilized to any equilibrium attitude using a time-invariant continuous feedback control law, but discontinuous feedback control strategies are constructed which stabilize any equilibrium attitude of the spacecraft in finite time. Consequently, reorientation of the spacecraft can be accomplished using discontinuous feedback control.

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1. Introduction

We consider the attitude control of a spacecraft modeled as a rigid body. It is well known that three actuators, either gas jets or momentum wheels, can be used to control the attitude of a rigid spacecraft and that arbitrary reorientation maneuvers of the spacecraft can be accomplished using smooth feedback[1-7]. If failure of one of the actuators occurs, then one is left with only two actuators. In this paper, the attitude stabilization problem of a rigid spacecraft using only two control torques supplied by momentum wheel actuators is considered. Since we are considering a space-based system, the problem considered here, namely, the attitude stabilization of a spacecraft operating in an actuator failure mode, is an important control problem. It is assumed that the center of mass of the system consisting of the spacecraft and the momentum wheel actuators is fixed in space.

Attitude stabilization of a rigid spacecraft using two momentum wheel actuators is not a mature subject in the literature. Controllability results for a rigid spacecraft controlled by momentum wheel actuators are presented in Ref. 8. We mention that most of the previous researchers have considered the problem of controlling a rigid spacecraft using two gas jet actuators[8-22]. Attitude stabilization of a rigid spacecraft using two gas jet actuators is considered in Refs. 8-13. Refs. 14-22 consider only the stabilization of the angular velocity equations of a rigid spacecraft using two gas jet actuators.

We consider the attitude stabilization of a spacecraft using control torques supplied by two momentum wheel actuators about axes spanning a two dimensional plane orthogonal to a principal axis of the spacecraft. The linearization of the complete spacecraft dynamic equations at any equilibrium attitude has an uncontrollable eigenvalue at the origin. Consequently, controllability and stabilizability properties of the spacecraft cannot be inferred using classical linearization ideas. The complete spacecraft dynamics is, in fact, not controllable. Under a rather weak assumption, the spacecraft dynamics is small time locally controllable at any equilibrium attitude in a reduced nonlinear sense. The reduced spacecraft dynamics cannot be asymptotically stabilized to any equilibrium attitude using time-invariant continuous feedback. Nevertheless, two different discontinuous feedback control strategies are constructed which achieves reorientation of the spacecraft in finite time. Using the concept of geometric phase[23], a discontinuous feedback control strategy is presented based on the nonholonomic control theory in Ref. 24. An alternate discontinuous feedback control strategy, based on the fact that rigid body rotations do not commute, is also presented.

This paper is based on our ealier work presented in Ref. 10 and is a companion to Ref. 11 and Ref. 12, which treat the attitude stabilization of a rigid spacecraft using two gas jet actuators.

2. Kinematic and Dynamic Equations

The orientation of a rigid spacecraft can be specified using various parametrizations of the special orthogonal group $SO(3)$. Here we use the Z-Y-X Euler angle convention for parametrizing the orientation of the rigid spacecraft[25]. The corresponding rotation matrix is denoted as $R(\psi, \theta, \phi)$, where ψ, θ, ϕ are the Euler angles. We assume that the Euler angles are limited to the ranges $-\pi < \psi < \pi$, $-\pi/2 < \theta < \pi/2$, $-\pi < \phi < \pi$. Suppose ω_1, ω_2, ω_3 are the principal axis components of the absolute angular velocity vector ω of the spacecraft. Then we have[25]

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \tan\theta + \omega_3 \cos\phi \tan\theta, \tag{2.1}$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi, \tag{2.2}$$

$$\dot{\psi} = \omega_2 \sin\phi \sec\theta + \omega_3 \cos\phi \sec\theta. \tag{2.3}$$

Next we consider the dynamic equations which describe the evolution of the angular velocity components of the spacecraft. Consider two momentum wheel actuators spinning about axes defined by unit vectors b_1, b_2 fixed in the spacecraft such that the center of mass of the i-th wheel lies on the axis defined by b_i, and a control torque $-\bar{u}_i$ is supplied to the i-th wheel about the axis defined by b_i by a motor fixed in the spacecraft. Consequently, an equal and opposite torque $\bar{u}_i$ is exerted by the wheel on the spacecraft. We assume that b_i defines a principal axis for the i-th wheel which is symmetric about b_i. Further b_1 and b_2 span a two dimensional plane which is orthogonal to a principal axis of the spacecraft and, without loss of generality, b_i are assumed to be of the form

$$b_i = (b_{i1}, b_{i2}, 0)^T, i = 1, 2. \tag{2.4}$$

The mass of spacecraft, wheel 1 and wheel 2 are denoted as m_1, m_2 and m_3 respectively, and ρ_1, ρ_2, ρ_3 denote the position vectors of the center of mass of the spacecraft, wheel 1 and wheel 2 respectively with respect to the center of mass of the whole system. Thus from the location of the wheels

$$\rho_2 = \rho_1 + d_1 b_1, \tag{2.5}$$

$$\rho_3 = \rho_1^1 + d_2 b_2, \tag{2.6}$$

where d_1, d_2 are constants. Since, by the definition of center of mass,

$$\sum_{i=1}^{3} m_i \rho_i = 0, \tag{2.7}$$

further manipulation of equations (2.5)-(2.7) gives expressions for ρ_1, ρ_2 and ρ_3 which we denote as $\rho_i = (c_{i1}, c_{i2}, 0)^T$, $i = 1, 2, 3$. The total angular momentum vector of the system is given, in the spacecraft body frame, by

$$R(\psi, \theta, \phi)H = J\omega + v, \tag{2.8}$$

where

$$J = [I_1 + \sum_{i=1}^{3} \overline{I}_i + \sum_{i=2}^{3} (I_i - \underline{I}_i)], \tag{2.9}$$

$$\overline{I}_i = m_i \begin{bmatrix} c_{i2}^2 & -c_{i1}c_{i2} & 0 \\ -c_{i1}c_{i2} & c_{i1}^2 & 0 \\ 0 & 0 & c_{i1}^2 + c_{i2}^2 \end{bmatrix}, \quad i = 1, 2, 3, \tag{2.10}$$

$$\underline{I}_2 = b_1 b_1^T j_1, \tag{2.11}$$

$$\underline{I}_3 = b_2 b_2^T j_2, \tag{2.12}$$

$$v = \underline{I}_2(\omega + b_1\dot{\theta}_1) + \underline{I}_3(\omega + b_2\dot{\theta}_2), \tag{2.13}$$

where I_1, I_2, and I_3 denote the inertia tensors of the spacecraft, wheel 1 and wheel 2 respectively, j_1 is the moment of inertia of wheel 1 about the axis defined by b_1, j_2 is the moment of inertia of wheel 2 about the axis defined by b_2, and θ_1, θ_2 are the angles of rotation of wheel 1 and wheel 2 about the axes defined by b_1 and b_2 respectively. Here H denotes the angular momentum vector of the system expressed in the inertial coordinate frame. The angular momentum vector H is a constant since there is no external moment about the center of mass of the system. Suppose $\overline{u}_1$ and $\overline{u}_2$ are the control torques; then

$$\dot{v} = -(b_1\overline{u}_1 + b_2\overline{u}_2). \tag{2.14}$$

Differentiating (2.8) with respect to time we obtain

$$J\dot{\omega} = S(\omega)R(\psi, \theta, \phi)H + b_1\overline{u}_1 + b_2\overline{u}_2, \tag{2.15}$$

where

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

Note that

$$I_1 = \text{diag}(I_{11}, I_{12}, I_{13}),$$

$$I_2 = \text{block diag}(I_{21}, I_{22}),$$

$$I_3 = \text{block diag}(I_{31}, I_{32}),$$

where I_{21}, I_{31} are invertible 2×2 matrices, $I_{11}, I_{12}, I_{13}, I_{22}, I_{32}$ are nonzero real numbers and therefore J is a positive definite matrix of the form

$$J = \text{block diag } (J_1, J_2), \tag{2.16}$$

where J_1 is an invertible 2×2 matrix and J_2 is a nonzero real number.

3. Controllability and Stabilizability Properties

In this section we consider the controllability and stabilizablity properties of the spacecraft dynamics controlled by two momentum wheel actuators. Define

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = J_1^{-1} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}.$$

From Section 2 the complete spacecraft dynamics can be rewritten as

$$\dot{\omega} = \begin{bmatrix} J_1^{-1} & 0_{(2 \times 1)} \\ 0_{(1 \times 2)} & J_2^{-1} \end{bmatrix} S(\omega)R(\psi, \theta, \phi)H + \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}, \tag{3.1}$$

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi \, \tan\theta + \omega_3\cos\phi \, \tan\theta, \tag{3.2}$$

$$\dot{\theta} = \omega_2\cos\phi - \omega_3\sin\phi, \tag{3.3}$$

$$\dot{\psi} = \omega_2\sin\phi \, \sec\theta + \omega_3\cos\phi \, \sec\theta, \tag{3.4}$$

where H is a constant vector.

The linearization of the complete spacecraft dynamic equations (3.1)-(3.4) at any equilibrium attitude has an uncontrollable eigenvalue at the origin. Consequently, the controllability

and stabilizability properties of the complete spacecraft dynamics cannot be inferred using classical linearization ideas. However, from equations (2.4), (2.11)-(2.13) and the definition

$$c = (0, 0, 1)^T,$$ (3.5)

we have $c^T v = 0$. Therefore from equation (2.8) we have

$$c^T R (\psi, \theta, \phi) H = c^T J \omega.$$ (3.6)

Since H is a constant vector, this equation represents a constraint on the motion of the spacecraft irrespective of the controls applied. Thus the complete spacecraft dynamics is not completely controllable. Therefore we ask the following question: what restricted control and stabilization properties of the spacecraft can be demonstrated in this case? Our analysis begins by demonstrating that, under an appropriate restriction of interest, the spacecraft equations have restricted controllability and stabilizability properties.

Consider equations (3.1)-(3.4) and suppose the angular momentum vector H of the system is zero. From equations (2.16), (3.5) and (3.6) it follows that the angular velocity component of the spacecraft about the uncontrolled principal axis is identically zero, i.e., $\omega_3 \equiv 0$. Under such a restriction, the reduced spacecraft dynamics are described by

$$\dot{\omega}_1 = u_1,$$ (3.7)

$$\dot{\omega}_2 = u_2,$$ (3.8)

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \tan\theta,$$ (3.9)

$$\dot{\theta} = \omega_2 \cos\phi,$$ (3.10)

$$\dot{\psi} = \omega_2 \sin\phi \sec\theta.$$ (3.11)

Notice that the linearization of the equations (3.7)-(3.11) at any equilibrium has an uncontrollable eigenvalue at the origin. Therefore analysis of the controllability and stabilizability properties of the reduced spacecraft dynamics requires inherently nonlinear techniques. The following results follow directly based on an analysis similar to that in Ref. 24.

Theorem 3.1: The reduced dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (3.7)-(3.11) are small time locally controllable at any equilibrium.

Theorem 3.2: The reduced dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (3.7)-(3.11) cannot be asymptotically stabilized to any

equilibrium using a time-invariant continuous feedback control law, but the reduced dynamics can be asymptotically stabilized to any equilibrium using a piecewise continuous feedback control law.

Theorem 3.1 follows from the fact that a sufficient condition for small time local controllability given in Ref. 26 is satisfied by the equations (3.7)-(3.11). The first part of Theorem 3.2 follows from the fact that a necessary condition for the existence of a time-invariant continuous feedback control law given in Ref. 17 is not satisfied by equations (3.7)-(3.11); the second part is a consequence of small time local controllability[26]. The implications of the properties stated above are as follows. Suppose the angular momentum vector H is zero. Then the spacecraft controlled by two momentum wheel actuators can be controlled to any equilibrium attitude but the feedback control law must necessarily be discontinuous. Thus arbitrary reorientation of the spacecraft can be achieved under the restriction $H = 0$; If $H \neq 0$, equation (3.6) implies that reorientation of the spacecraft to an equilibrium attitude cannot be achieved.

4. Feedback Stabilization Algorithms

We restrict our study to the class of discontinuous feedback controllers in order to asymptotically stabilize the reduced spacecraft dynamics described by state equations (3.7)-(3.11). Clearly, traditional nonlinear control design methods are of no use since there is no general procedure for the design of a discontinuous feedback control. However, an algorithm generating a discontinuous feedback control which asymptotically stabilizes an equilibrium can be constructed, as suggested by the controllability properties of the system. Without loss of generality, we assume that the equilibrium to be stabilized is the origin. We present two different discontinuous control strategies which stabilize the origin of equations (3.7)-(3.11) in finite time.

4.1. Feedback stabilization based on nonholonomic control theory

Consider a diffeomorphism defined by

$$y_1 = \cos\phi \, ln(\sec\theta + \tan\theta) + \psi\sin\phi, \tag{4.1}$$

$$y_2 = \omega_2\sec\theta - y_4 y_5, \tag{4.2}$$

$$y_3 = \phi, \tag{4.3}$$

$$y_4 = \omega_1 + \omega_2 \sin\phi \, \tan\theta, \tag{4.4}$$

$$y_5 = \sin\phi \, \ln(\sec\theta + \tan\theta) - \psi\cos\phi, \tag{4.5}$$

If we now define the feedback relations

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\phi\sin\theta & (1 - y_5\sin\phi\sin\theta) \\ \cos\theta & y_5\cos\theta \end{bmatrix} \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} \cos\phi y_5\sin\phi\sec^2\theta\omega_2^2 \\ 0 \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} -y_4^2 y_1 + \cos\phi(\sec\theta\tan\theta\omega_2^2 - y_5 y_4\tan\theta\omega_2) \\ \cos\phi(y_4\tan\theta\omega_2 + \sin\phi\sec^2\theta\omega_2^2) \end{bmatrix} \right\}, \tag{4.6}$$

then the reduced spacecraft dynamics (3.7)-(3.11) are described in the new variables by the normal form equations

$$\dot{y}_1 = y_2, \tag{4.7}$$

$$\dot{y}_2 = v_1, \tag{4.8}$$

$$\dot{y}_3 = y_4, \tag{4.9}$$

$$\dot{y}_4 = v_2, \tag{4.10}$$

$$\dot{y}_5 = y_4 y_1. \tag{4.11}$$

From equations (4.1)-(4.5), notice that $\omega_1 = \omega_2 = \phi = \theta = \psi = 0$ implies that $y_1 = y_2 = y_3 = y_4 = y_5 = 0$. Hence asymptotic stabilization of equations (3.7)-(3.11) to the origin is equivalent to asymptotic stabilization of the normal form equations (4.7)-(4.11) to the origin; hence we consider asymptotic stabilization of the normal form equations. The normal form equations (4.7)-(4.11) are in a familiar form which has been studied in Ref. 24 and therefore can be stabilized by the following discontinuous control strategy.

- First, transfer the initial state of the normal form equations (4.7)-(4.11) to the equilibrium state $(0, 0, 0, 0, y_5^1)$, for some y_5^1, in finite time.

- Next, traverse a closed path γ in the (y_1, y_3) space in finite time, where the path γ is selected to satisfy

$$-y_5^1 = \int_\gamma y_1 dy_3; \tag{4.12}$$

this transfers the state $(0, 0, 0, 0, y_5^1)$ to the origin in finite time.

Here we consider a rectangular path γ in the (y_1, y_3) space formed by line segments from $(0, 0)$ to $(y_1^*, 0)$, from $(y_1^*, 0)$ to (y_1^*, y_3^*), from (y_1^*, y_3^*) to $(0, y_3^*)$, and from $(0, y_3^*)$ to $(0, 0)$. For such a path, the line integral in equation (4.12) can be explicitly evaluated as $y_1^* y_3^*$ so that equation (4.12) becomes

$$- y_5^1 = y_1^* y_3^*,$$
(4.13)

and the parameters y_1^* and y_3^* specifying the particular rectangular path are chosen to satisfy the above equation.

Throughout, assume $k > 0$, and define

$$G(x_1, x_2) = \begin{cases} k & \text{if } \{x_1 + \dfrac{x_2 |x_2|}{2k} > 0\} \text{ or} \\ & \quad \{x_1 + \dfrac{x_2|x_2|}{2k} = 0 \text{ and } x_2 > 0\} \\ -k & \text{if } \{x_1 + \dfrac{x_2|x_2|}{2k} < 0\} \text{ or} \\ & \quad \{x_1 + \dfrac{x_2|x_2|}{2k} = 0 \text{ and } x_2 < 0\} \\ 0 & \text{if } \{x_1 = 0 \text{ and } x_2 = 0\} \end{cases}$$

We use the well-known property that any initial state of the system

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = - G(x_1 - \bar{x}_1, x_2),$$

is transferred to the final state $(\bar{x}_1, 0)$ in a finite time.

We now present a specific feedback control algorithm which stabilizes the spacecraft to the origin in finite time; this feedback control algorithm implements the approach just described.

Maneuver 1: Apply

$$v_1 = - G(y_1, y_2),$$

$$v_2 = - G(y_3, y_4),$$

until $(y_1, y_2, y_3, y_4, y_5) = (0, 0, 0, 0, y_5^1)$ where y_5^1 is arbitrary; then go to Maneuver 2.

Maneuver 2: If $y_5^1 \geq 0$, choose $y_1^* = -y_3^* = \sqrt{(y_5^1)}$; else choose $y_1^* = y_3^* = \sqrt{(-y_5^1)}$; Apply

$$v_1 = -G(y_1 - y_1^*, y_2),$$

$$v_2 = -G(y_3, y_4),$$

until $(y_1, y_2, y_3, y_4, y_5) = (y_1^*, 0, 0, 0, y_5^1)$; then go to Maneuver 3.

Maneuver 3: Apply

$$v_1 = -G(y_1 - y_1^*, y_2),$$

$$v_2 = -G(y_3 - y_3^*, y_4),$$

until $(y_1, y_2, y_3, y_4, y_5) = (y_1^*, 0, y_3^*, 0, 0)$; then go to Maneuver 4.

Maneuver 4: Apply

$$v_1 = -G(y_1, y_2),$$

$$v_2 = -G(y_3 - y_3^*, y_4),$$

until $(y_1, y_2, y_3, y_4, y_5) = (0, 0, y_3^*, 0, 0)$; then go to Maneuver 5.

Maneuver 5: Apply

$$v_1 = -G(y_1, y_2),$$

$$v_2 = -G(y_3, y_4),$$

until $(y_1, y_2, y_3, y_4, y_5) = (0, 0, 0, 0, 0)$; then go to Maneuver 2.

It can be verified that the execution of Maneuver 1 transfers the initial state of the normal form equations to the equilibrium state $(0, 0, 0, 0, y_5^1)$, for some y_5^1, in finite time. Subsequent execution of Maneuvers 2 through 5 then transfers the state $(0, 0, 0, 0, y_5^1)$ to the origin in finite time. This control algorithm is nonclassical and involves switching between various feedback functions. Justification that it stabilizes the origin of the normal form equations (4.7)-(4.11) in finite time follows as a consequence of the construction procedure. Since stabilization of the normal form equations to the origin is equivalent to stabilization of the state equations (3.7)-(3.11) to the origin, we conclude that the control inputs u_1 and u_2 given by equation (4.6) with v_1 and v_2 defined by the above control algorithm stabilizes the reduced spacecraft dynamics described by equations (3.7)-(3.11) to the equilibrium $(\omega_1, \omega_2, \phi, \theta, \psi) = (0, 0, 0, 0, 0)$ in finite time. A computer implementation of the feedback control strategy can be easily carried out.

4.2. Feedback stabilization based on rigid body rotational characteristics

We now present an alternate discontinuous feedback control strategy for stabilizing the origin of equations (3.7)-(3.11) in finite time. This strategy requires that the spacecraft undergo a sequence of specified maneuvers and is based on the fact that rigid body rotations do not commute. The physical interpretation of the sequence of maneuvers that transfers any initial state of equation (3.7)-(3.11) to the origin is as follows.

- Transfer the initial state of equations (3.7)-(3.11) to any equilibrium state in finite time; i.e. bring the spacecraft to rest.

- Transfer the resulting state to an equilibrium state where $\phi = 0$ in finite time; i.e. so that the spacecraft is at rest with $\phi = 0$.

- Transfer the resulting state to an equilibrium state where $\phi = 0$, $\theta = 0$ in finite time; i.e. so that the spacecraft is at rest with $\phi = 0$, $\theta = 0$.

- Transfer the resulting state to an equilibrium state where $\phi = \dfrac{\pi}{2}$, $\theta = 0$ in finite time; i.e. so that the spacecraft is at rest with $\phi = \dfrac{\pi}{2}$, $\theta = 0$.

- Transfer the resulting state to the equilibrium state $(0, 0, \dfrac{\pi}{2}, 0, 0)$ in finite time.

- Transfer the equilibrium state $(0, 0, \dfrac{\pi}{2}, 0, 0)$ to the equilibrium state $(0, 0, 0, 0, 0)$ in finite time.

We now present a feedback control algorithm which stabilizes the spacecraft to the origin in finite time; this feedback control algorithm implements the approach just described.

Maneuver 1. Apply

$$u_1 = -k \operatorname{sign} \omega_1 \,,$$

$$u_2 = -k \operatorname{sign} \omega_2 \,,$$

until $(\omega_1, \omega_2) = (0, 0)$; then go to Maneuver 2.

Maneuver 2: Apply

$$u_1 = -G(\phi, \omega_1) \,,$$

$$u_2 = 0 \,,$$

until $(\omega_1, \omega_2, \phi) = (0, 0, 0)$; then go to Maneuver 3.

Manuever 3: Apply

$$u_1 = 0 \,,$$

$$u_2 = - G(\theta, \omega_2) \,,$$

until $(\omega_1, \omega_2, \phi, \theta) = (0, 0, 0, 0)$; then go to Maneuver 4.

Maneuver 4: Apply

$$u_1 = - G(\phi - \frac{\pi}{2}, \omega_1) \,,$$

$$u_2 = 0 \,,$$

until $(\omega_1, \omega_2, \phi, \theta) = (0, 0, \frac{\pi}{2}, 0)$, then go to Maneuver 5.

Maneuver 5: Apply

$$u_1 = 0 \,,$$

$$u_2 = - G(\psi, \omega_2) \,,$$

until $(\omega_1, \omega_2, \phi, \theta, \psi) = (0, 0, \frac{\pi}{2}, 0, 0)$; then go to Maneuver 6.

Maneuver 6: Apply

$$u_1 = - G(\phi, \omega_1) \,,$$

$$u_2 = 0 \,,$$

until $(\omega_1, \omega_2, \phi, \theta, \psi) = (0, 0, 0, 0, 0, 0)$; then go to Maneuver 1.

It can be verified that the execution of Maneuver 1 transfers the initial state of equations (3.7)-(3.11) to the equilibrium state $(0, 0, \phi^1, \theta^1, \psi^1)$, for some ϕ^1, θ^1, ψ^1, in finite time. Execution of Manuever 2 then transfers the state $(0, 0, \phi^1, \theta^1, \psi^1)$ to the state $(0, 0, 0, \theta^1, \psi^1)$; execution of Manuever 3 then transfers the state $(0, 0, 0, \theta^1, \psi^1)$ to the state $(0, 0, 0, 0, \psi^1)$; execution of Maneuver 4 then transfers the state $(0, 0, 0, 0, \psi^1)$ to the state $(0, 0, \frac{\pi}{2}, 0, \psi^1)$; execution of Maneuver 5 then transfers the state $(0, 0, \frac{\pi}{2}, 0, \psi^1)$ to the state $(0, 0, \frac{\pi}{2}, 0, 0)$; finally, execution of Manuever 6 transfers the state $(0, 0, \frac{\pi}{2}, 0, 0)$ to the state $(0, 0, 0, 0, 0)$. This strategy is discontinuous and nonclassical in nature. A computer implementation of the

feedback control strategy can be easily carried out.

4.3 Comments

We have introduced two different control laws which transfer any initial state of equations (3.7)-(3.11) to the origin in finite time. Each of these control laws is in feedback form, since the control values depend on the current state; and each control law is discontinuous. The first construction procedure makes use of the nonholonomic features of the reduced spacecraft dynamics, while the second construction procedure uses physical insight about rigid body rotations. The first control law constructed makes use of both control actuators simultaneously, while the second control law (after Maneuver 1) uses only a single actuator at a time. The two discontinuous feedback control laws exhibited are illustrations of the class of control laws which asymptotically stabilize equations (3.7)-(3.11) to the origin. There are other maneuver sequences, and corresponding feedback control laws, which will also achieve the desired attitude stabilization of the spacecraft. But each such strategy is necessarily discontinuous.

One of the advantages of the development in Sections 4.1 and 4.2 is that feedback control strategies are constructed which guarantee attitude stabilization in a finite time. The total time required to complete the spacecraft reorientation is the sum of the times required to complete the sequence of maneuvers described. It should be clear that the time required to complete each maneuver depends on the single positive parameter k in the corresponding control law. There is a trade off between the required control levels, determined by the selection of k, and the resulting times to complete each of the maneuvers and hence the total time required to reorient the spacecraft. In particular, the time to reorient the spacecraft from a given initial state to the origin can be expressed as a function of the value of the parameter k and of the initial state.

We have demonstrated, by construction, the closed loop properties for the special feedback control strategies presented. Our analysis was based on an ideal model assumption. Further robustness analysis is required to determine effects of model uncertainities and external disturbances. Unfortunately, such robustness analysis is quite difficult since the closed loop vector fields are necessarily discontinuous. Perhaps, feedback control strategies which stabilize the spacecraft attitude, different from ones presented in this paper, would provide improved closed loop robustness. These issues are to be studied in future research.

5. Simulation

We illustrate the results of the paper using an example. Consider a rigid spacecraft with no control torque about the third principal axis and two control torques, generated by momentum wheel actuators, are applied about the other two principal axes. Therefore the vectors b_1 and b_2 are given by $b_1 = (1, 0, 0)^T$, $b_2 = (0, 1, 0)^T$. For our simulation, we use the spacecraft parameters used in Ref. 2. The mass of the spacecraft, m_1, is 500 Kg, and the masses of the momentum wheels, m_2 and m_3, are each 5 Kg. The center of mass of the momentum wheels are located at a distance 0.2 m from the center of mass of the spacecraft, i.e., $d_1 = d_2 = 0.2$ m. The moment of inertia of the wheels about its axis of rotation is 0.5 Kg.m^2, i.e., $j_1 = j_2 = 0.5$. The inertia tensor of the spacecraft and the two momentum wheels are

$$I_1 = \text{diag } (86.215, 85.07, 113.565) \text{ Kg.m}^2 ,$$

$$I_2 = \text{diag } (0.5, 0.25, 0.25) \text{ Kg.m}^2 ,$$

$$I_3 = \text{diag } (0.25, 0.5, 0.25) \text{ Kg.m}^2 .$$

Using these parameters, the inertia matrix J can be calculated which equals

$$J = \text{diag}(86.7, 85.5, 114.5) \text{ Kg.m}^2 ,$$

approximately. The complete dynamics of the spacecraft system defined by equations (3.1)-(3.4) is not controllable, but we consider the restriction that the angular momentum vector $H = 0$. Consequently, we are interested in stabilizing the reduced spacecraft dynamics described by equations (3.7)-(3.11) to the equilibrium $(\omega_1, \omega_2, \phi, \theta, \psi) = (0, 0, 0, 0, 0)$. The spacecraft is initially at rest (i.e., $\omega_1^0 = \omega_2^0 = 0$) with an initial orientation given by the Euler angles $\phi^0 = \pi$, $\theta^0 = 0.25\pi$ and $\psi^0 = -0.5\pi$.

First, a computer implementation of the feedback control algorithm specified in Section 4.1 was used to stabilize the spacecraft to the origin. The value of the gain k was chosen as 1. The time responses of the Euler angles, angular velocities and the control torques are shown in Fig. 1, Fig. 2 and Fig. 3 respectively. After a total maneuver time of 11.77 seconds, $\omega_1 = \omega_2 = \phi = \theta = \psi = 0$. Next, a computer implementation of the feedback control algorithm specified in Section 4.2 was used to stabilize the spacecraft to the origin. The value of the gain k was chosen as 1. The time responses of the Euler angles, angular velocities and the control torques are shown in Fig. 4, Fig. 5 and Fig. 6 respectively. After a total maneuver time of 13 seconds, $\omega_1 = \omega_2 = \phi = \theta = \psi = 0$.

6. Conclusion

The attitude stabilization problem of a spacecraft using control torques supplied by two momentum wheel actuators about axes spanning a two dimensional plane orthogonal to a principal axis has been considered. The complete spacecraft dynamics are not controllable. However, the spacecraft dynamics are small time locally controllable in a reduced sense. The reduced spacecraft dynamics cannot be asymptotically stabilized using time-invariant continuous feedback, but discontinuous feedback control strategies have been constructed which stabilizes the spacecraft (in the reduced sense) to an equilibrium attitude in finite time. The results of the paper show that although classical nonlinear control techniques do not apply, it is possible to construct control laws based on the particular spacecraft dynamics.

Acknowledgements

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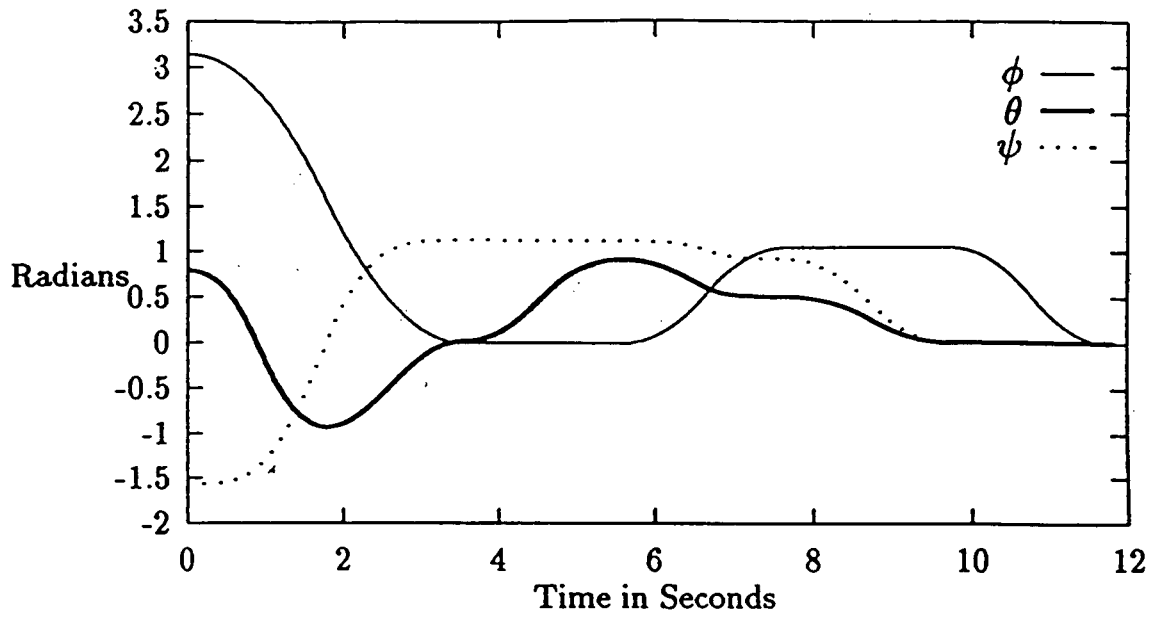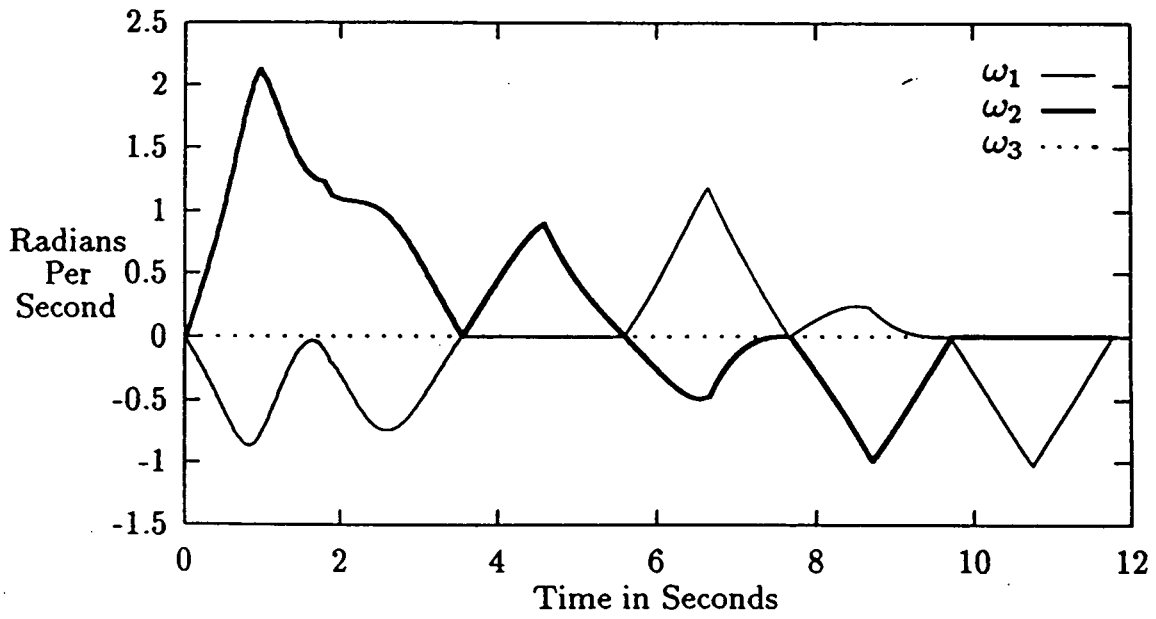
Fig. 1: Euler Angles

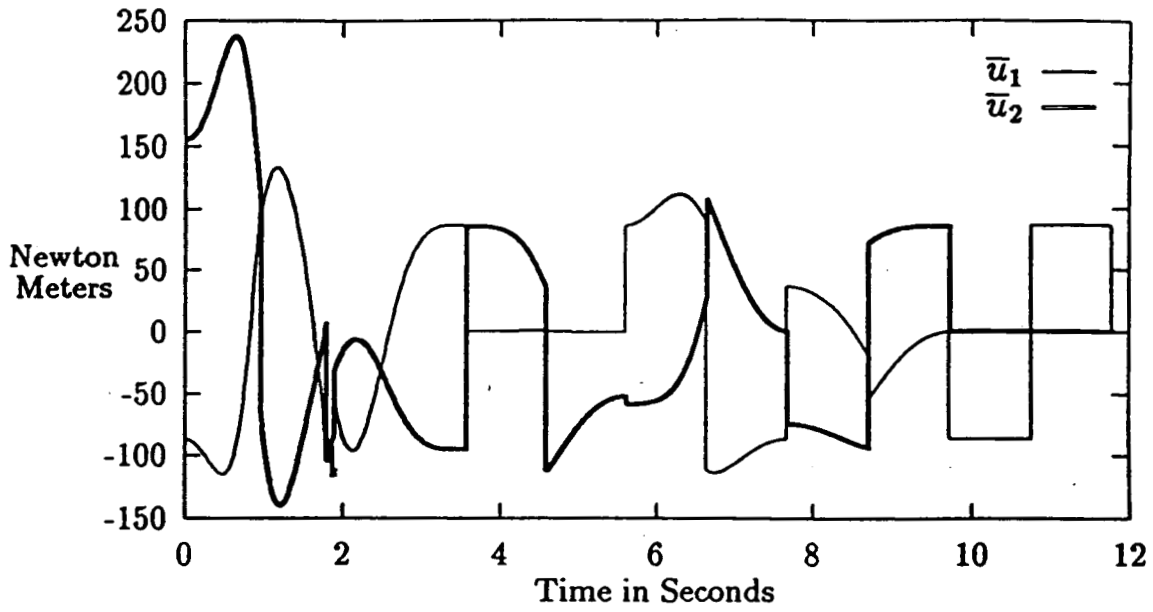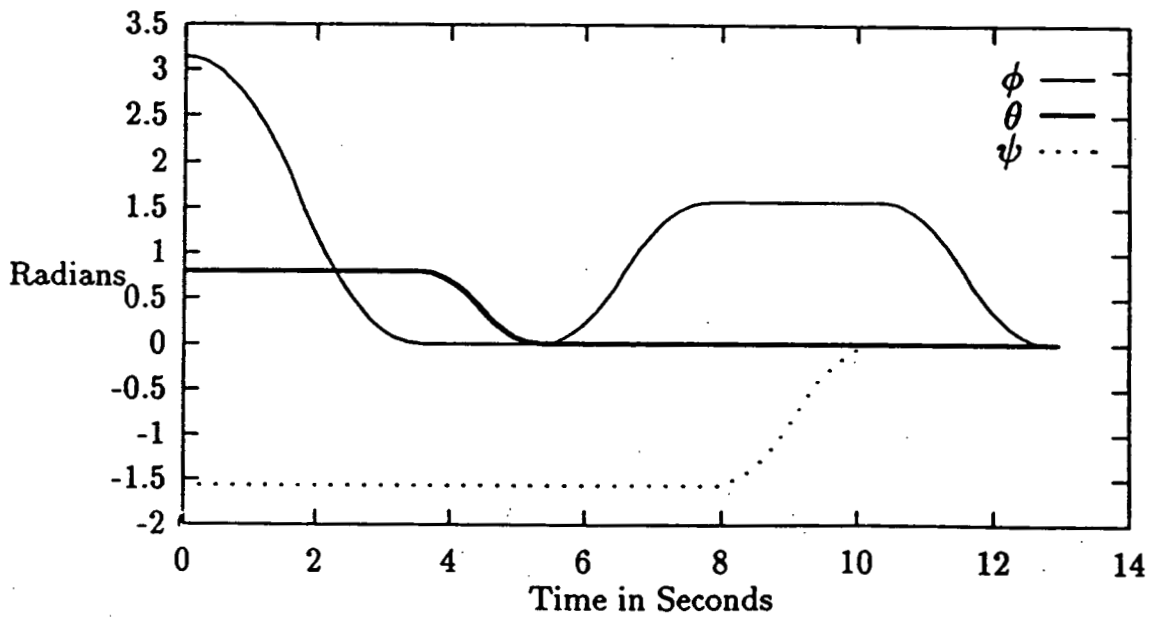Fig. 2: Anglular Velocities
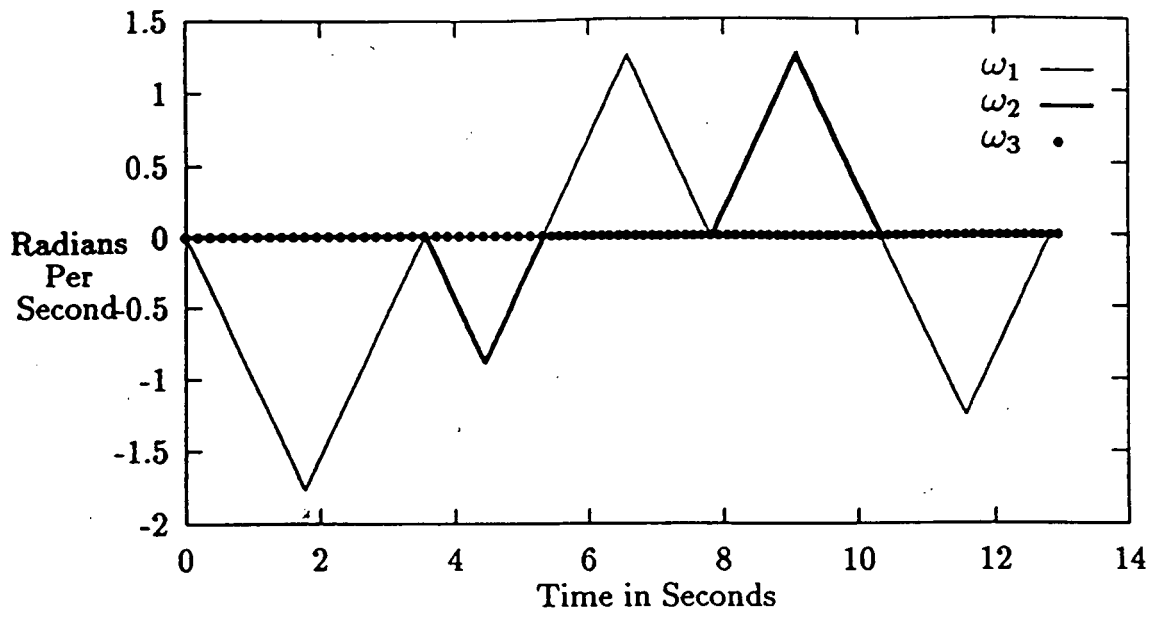
Fig. 3: Control Torques

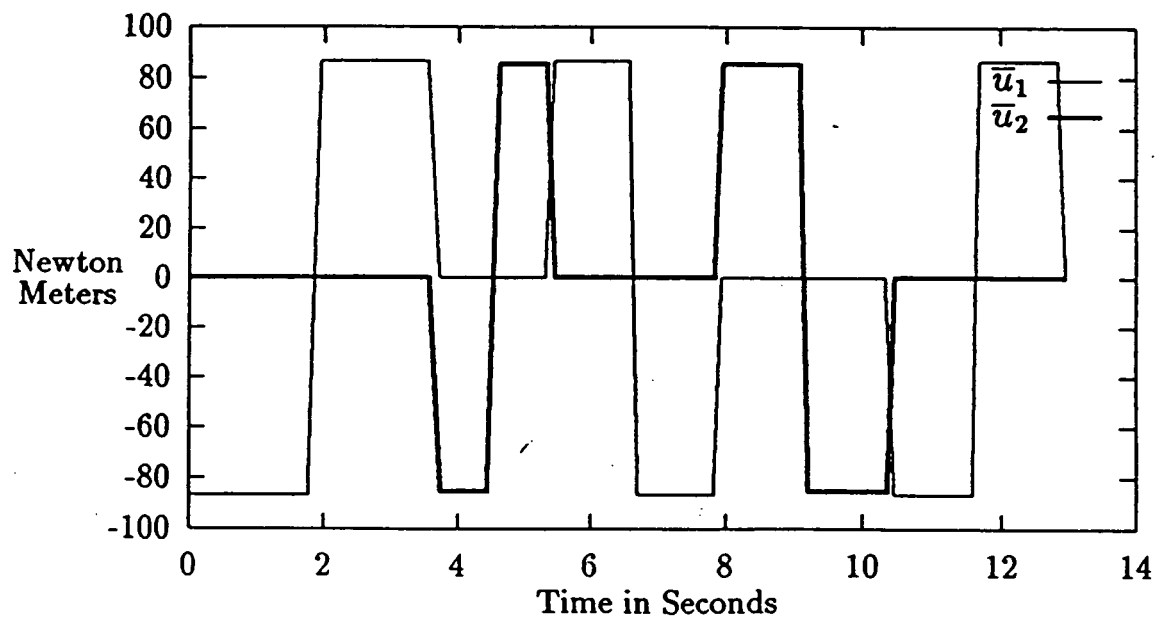Fig. 4: Euler Angles

Fig. 5: Anglular Velocities

Fig. 6: Control Torques

NONLINEAR ATTITUDE CONTROL OF PLANAR STRUCTURES IN SPACE USING ONLY INTERNAL CONTROLS

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Abstract

An attitude control strategy for maneuvers of an interconnection of planar bodies in space is developed. It is assumed that there are no exogeneous torques and that torques generated by joint motors are used as means of control so that the total angular momentum of the multibody system is a constant, assumed to be zero in this paper. The control strategy utilizes the nonintegrability of the expression for the angular momentum. Large angle maneuvers can be designed to achieve an arbitrary reorientation of the multibody system with respect to an inertial frame. The theoretical background for carrying out the required maneuvers is summarized.

1. Introduction

In this paper we develop an attitude control strategy for a system of N planar rigid bodies in space which are interconnected by ideal frictionless pin joints in the form of an open kinematic chain. Angular momentum preserving controls, e.g. torques generated by joint motors, are considered. The N-body system is assumed to have zero initial angular momentum. Our earlier work[1,2] demonstrated that reorientation of a planar multibody system with three or more interconnected bodies using only joint torque inputs is an inherently nonlinear control problem which is not amenable to classical methods of nonlinear control. The goal of this study is to indicate how control strategies can be explicitly constructed to achieve the desired absolute reorientation of the N-body system. The key is to excite certain oscillatory motions in the shape of the structure, thereby providing a capability for reorientation of the structure with respect to an inertial frame.

There are many physical advantages in using internal controls, e.g. joint torque controls, to carry out the desired multibody reorientation maneuvers. First of all, this control approach does not modify the total angular momentum of the multibody system. In addition, internal controls have obvious advantages in terms of energy conservation. Moreover, they can be implemented using standard electrical servo motors, a simple and reliable control actuator technology.

The formal development in this paper is concerned with control of a multibody interconnection in space which has zero angular momentum. Although these results are formulated in a general setting, we have been motivated by several classes of specific problems. Several potential applications of our general results are now described.

Manipulators mounted on space vehicles and space robots have been envisioned to carry out construction, maintenance and repair tasks in an external space environment. Previous research on maneuvering of such space multibody systems has mainly focused on maneuvers which achieve desired orientation of some of the

bodies, e.g. an end effector, while the orientation of some of the remaining bodies cannot be specified, at least using the methodologies employed[3-8]. Another related application is the performance by astronauts of reorientation maneuvers in space. Previous research in this area[9] has emphasized dynamics issues. Closely related research has focused on describing the reorientation maneuvers of a falling cat[10]. Finally, we mention another area of potential application of the results of this paper, namely the development of deployment maneuvers for multibody antennas connected to a spacecraft. It is expected that such an approach would have many advantages over the use of existing passive antenna deployment mechanisms[11].

This paper is organized as follows. In Section 2, a mathematical model for a planar multibody system in space is derived. We then formulate an attitude control problem associated with the planar multibody system. In Section 3, we first summarize several relevant theoretical results. We then introduce an attitude control strategy to solve this reorientation problem. Section 4 consists of a summary of the main results and concluding remarks about our continuing research. Although a complete treatment of the topics in the paper requires use of differential geometric tools, our presentation avoids these tools and uses only elementary mathematical methods. However, references to relevant literature are provided throughout.

2. Mathematical Model for Planar Multibody System

We consider a system of N planar rigid bodies interconnected by frictionless one degree of freedom joints in the form of an open kinematic chain. The configuration space, for an observer at the center of mass of the system of rigid bodies, is N dimensional. Since we assume an open kinematic chain there are exactly $N-1$ joints. We consider controlling the rotational motion of the system using torques at the joints; each joint is assumed to be actuated so as to permit free adjustment of the joint angle. It is assumed that there are no external torques acting on the system. It is clear that the configuration of the N bodies can be described by the absolute angle of any one of the bodies (say body 1) and $N-1$ joint angles. Denote by θ_1 the absolute angle of body 1, and by the $(N-1)$-vector $\psi = (\psi_1, \cdots, \psi_{N-1})$

the joint angle vector. Clearly, (θ_1, ψ) is a generalized coordinate vector for the rotational motion. It can be shown that the Lagrangian (which is equal to the rotational kinetic energy under the above assumptions), written in terms of these coordinates and their time derivatives, does not contain θ_1 explicitly, i.e. θ_1 is a cyclic or ignorable coordinate. Consequently, the generalized momentum associated with the cyclic coordinate θ_1 is conserved. This conserved quantity is the first integral of the motion corresponding to conservation of angular momentum of the system. In this paper we assume zero initial angular momentum so that angular momentum remains zero throughout a maneuver.

It is clear that Lagrange's equations describe the motion on the joint angle space, and the evolution of θ_1 can be obtained from the expression for conservation of angular momentum. Thus, the motion of a planar multibody system, under the above assumptions, can be described by the following reduced order equations

$$J_s(\psi)\ddot{\psi} + F_s(\psi, \dot{\psi}) = \tau \ , \tag{1}$$

$$\dot{\theta}_1 + s'(\psi)\dot{\psi} = 0 \tag{2}$$

where $\tau = (\tau_1, \cdots, \tau_{N-1})$ denotes the $(N-1)$-vector of joint torques, $J_s(\psi)$ is a symmetric positive definite $(N-1) \times (N-1)$ matrix function; and $s(\psi)$, $F_s(\psi, \dot{\psi})$ are $(N-1)$-vector functions. Note that in this paper a "prime" denotes tranpose. The explicit specifications of these functions can be found in the literature[1,2,12].

State space equations for (1) and (2) are

$$\dot{\theta}_1 = -s(\psi)'\omega \ , \tag{3}$$

$$\dot{\psi} = \omega \ , \tag{4}$$

$$\dot{\omega} = -J_s^{-1}(\psi)F_s(\psi, \omega) + J_s^{-1}(\psi)\tau \ . \tag{5}$$

Note that equations (4),(5) are expressed in terms of the joint phase variables $(\psi, \dot{\psi})$ only. Hence the joint angle space constitutes a reduced configuration space

for the system. This reduced configuration space is also referred to as the "shape space" of the system[12-16]. It is possible to consider control problems expressed solely in terms of the shape space; such problems can be solved using classical methods. However, in our work we are interested in the more general control problems associated with the complete dynamics of the multibody system defined by equations (1)-(2) (or (3)-(5)).

Note that equations (4)-(5) only, which represent the projection of the motion onto the shape phase space, are feedback linearizable using the feedback transformation

$$u = -J_s^{-1}(\psi)F_s(\psi,\omega) + J_s^{-1}(\psi)\tau \tag{6}$$

where $u \in R^{N-1}$. The above feedback transformation yields the following normal form equations

$$\dot{\theta}_1 = -s(\psi)'\omega \ , \tag{7}$$

$$\dot{\psi} = \omega \ , \tag{8}$$

$$\dot{\omega} = u \ . \tag{9}$$

We remark here that it is impossible to completely linearize the system defined by equations (3)-(5) using static or dynamic feedback combined with any coordinate transformation.

Note that an equilibrium solution of equations (3)-(5) corresponding to $\tau = 0$ (or equivalently an equilibrium solution of equations (7)-(9) for $u = 0$) is given by $(\theta_1^e, \ \psi^e, \ 0)$, where (θ_1^e, ψ^e) is referred to as an equilibrium configuration. Hence an equilibrium solution corresponds to a trivial motion of the system for which all the configuration space variables remain constant.

Note also that equation (3) represents conservation of angular momentum. This equation is not integrable for $N \geq 3$ (i.e. if the multibody system consists of three

or more links). This fact has important implications in terms of controllability properties of the system as will be shown in the subsequent development. As a consequence of the symmetry possessed by the system, θ_1 does not appear explicitly in equation (3). Mechanical systems with such symmetry properties are referred to as Caplygin systems[17-21]. As a consequence of the nonintegrability for $N \geq 3$, the scalar analytic functions

$$H_{ij}(\psi) = \frac{\partial s_i(\psi)}{\partial \psi_j} - \frac{\partial s_j(\psi)}{\partial \psi_i} \;,\; (i,j) \in I^2 \;, \tag{10}$$

where $I = \{1, \cdots, N-1\}$, do not all vanish, except possibly on a set which has measure zero with respect to the shape space.

3. Attitude Control Problem

In this section, we address the following control problem associated with planar multibody systems described by equations (1)-(2) :

Problem : *Given an initial state $(\theta_1^0, \psi^0, \omega^0)$ and a desired equilibrium solution $(\theta_1^e, \psi^e, 0)$, determine a motion $(\theta_1(t), \psi(t), \omega(t))$, $0 \leq t \leq t_f$, such that $(\theta_1(0), \psi(0), \omega(0)) = (\theta_1^0, \psi^0, \omega^0)$, $(\theta_1(t_f), \psi(t_f), \omega(t_f)) = (\theta_1^e, \psi^e, 0)$ and $(\theta_1(t), \psi(t), \omega(t))$ satisfies equations (1)-(2) for some control function $t \longmapsto \tau(t)$.*

Note that, in particular, if $\omega^0 = 0$ then the above problem corresponds to a rest-to-rest maneuver.

The existence of solutions to the above control problem was demonstrated in our earlier work[1,2]. In particular, we studied the nonlinear control system described by equations (7)-(9) and employed certain results from nonlinear control theory to characterize controllability properties of planar multibody systems described by equations (1)-(2). These results not only prove the existence of solutions of the above problem but they also provide a theoretical basis for construction of nonlinear control strategies required to achieve the desired maneuver. We next

summarize those results[1,2].

Under the stated assumptions, a planar multibody system has the following properties if $N \geq 3$, i.e. if it consists of three or more links:

1. *The system is strongly accessible.*
2. *The system is small time locally controllable from any equilibrium.*
3. *The system can be transferred from any initial condition to any desired equilibrium in arbitrarily small time.*

If $N = 1$ or $N = 2$, then the system is not even accessible, not small time locally controllable and there exist initial conditions which cannot be transferred to a desired equilibrium.

The proofs[1,19] of the first two results depend on showing that certain Lie algebraic conditions are satisfied if $N \geq 3$. The third result is proved[1,19] constructively.

It should be emphasized that the subsequent development is assumed to be carried out for multibody systems consisting of three or more links ($N \geq 3$) Note that the reorientation or attitude control problem generally has many solutions. In this paper, we describe one solution approach, outline the theory behind it, and present some data from simulations. The key observation is that there is nonlinear coupling between changes in the shape of the structure and the rotational motion of the structure as a whole; this coupling is used to achieve reorientation of the structure.

Consider equation (3). Assume that joint angles are controlled in such a way that $\psi(t)$, $0 \leq t_1 \leq t \leq t_2$, describes a closed path γ in the shape space. Integrating both sides of equation (3) from $t = t_1$ to $t = t_2$ and using the fact that $d\psi = \dot{\psi}dt$, we obtain

$$\theta_1(t_2) - \theta_1(t_1) = \oint_\gamma s'(\psi)d\psi . \tag{11}$$

Thus by proper selection of a path γ in shape space, any desired geometric phase

(which is a rotation of link 1) can be obtained. By the nonintegrability property mentioned previously, the above integral is in fact path dependent thereby guaranteeing the existence of (many) such paths.

Note that in differential geometry the quantity

$$\alpha(\gamma) = \oint_\gamma s'(\psi)d\psi$$

is referred to as the geometric phase (or holonomy) of the closed path γ. This quantity depends only on the geometry of the closed path and is independent of the speed at which the path is traversed.

Note that Stokes' formula can be applied to obtain an equivalent formula for $\alpha(\gamma)$ as a surface integral. For simplicity, assume that $N = 3$, i.e. the shape space is the (ψ_1, ψ_2) plane. Also, let γ be traversed counterclockwise. Then by Stokes' theorem the above formula can be written as

$$\alpha(\gamma) = \int_S (\frac{\partial s_2}{\partial \psi_1} - \frac{\partial s_1}{\partial \psi_2})d\psi_1 d\psi_2$$

where S is the surface within the boundary γ. In the case that the path is traversed clockwise, the surface integral is equal to $-\alpha(\gamma)$.

More information concerning geometric phases can be found in the literature[15]. Geometric phase ideas have proved useful in a variety of inherently nonlinear control problems[19-21]. These ideas have also been used for a class of path planning problems based solely on kinematic relations[13,14,16].

We now describe a control strategy, using the above geometric phase relation (11), which solves the reorientation problem.

Let $(\theta_1^e, \psi^e, 0)$ denote the desired equilibrium solution. We refer to (θ_1^e, ψ^e) and ψ^e as the desired equilibrium configuration and the desired equilibrium shape, respectively. We describe four steps involved in construction of an open loop control function $u_{[0,t_f)} = (u_1, \cdots, u_{N-1})'$ which transfers any initial state $(\theta_1^0, \psi^0, \omega^0)$ to

$(\theta_1^e, \psi^e, 0)$ in time exactly t_f, where $t_f > 0$ is arbitrary.

Let $0 < t_1 < t_2 < t_3 < t_f$ denote an arbitrary partition of the time interval $[0, t_f)$.

Step 1 : *Transfer the system to the desired equilibrium shape, i.e. find a control which transfers the initial state $(\theta_1^0, \psi^0, \omega^0)$ to $(\theta_1^1, \psi^e, 0)$ at time t_1, for some θ_1^1.*

Since the dynamics on the shape phase space are so simple, namely decoupled double integrators, Step 1 has many solutions which are easily obtained using classical methods. One such control function is

$$u_{[0,t_1)} = \begin{cases} -\frac{\pi \omega^0}{t_1} \cos(\frac{\pi t}{t_1}) & t \in [0, 0.5t_1) \\ \frac{8\pi(\psi^e - \psi^0 - \omega^0 t_1(0.5 - \pi^{-1}))}{t_1^2} \sin(\frac{2\pi(2t - t_1)}{t_1}) & t \in [0.5t_1, t_1) \ . \end{cases} \tag{12}$$

Next, we select a closed path γ (or a series of closed paths - see Remark 1 below) in the shape space which achieves the desired geometric phase. There are many ways to accomplish such a construction; in our work we have found it convenient to use only two joint motions, keeping the other joints locked, and to use a square path in the restricted two dimensional shape space. It is convenient to select the center of the square path in a region of the shape space which corresponds to a "large" geometric phase change (see Remark 2 below).

To make the above ideas more concrete, we present a specific construction. Let $(i, j) \in I^2$, $i \neq j$, denote a pair of joints. Assume that for $t \geq t_1$ only this pair of joints are actuated while all the other joints are kept fixed. This is equivalent to locking all the joints except the ones labelled i and j and treating the N bodies as three interconnected bodies, for $t \geq t_1$. In this case the desired geometric phase formula can be written as

$$\theta_1(t_f) - \theta_1^1 = \pm \alpha(\gamma)$$

where $+(-)$ corresponds to counterclockwise (clockwise) traversal of the closed path γ. Since we desire to make $\theta_1(t_f) = \theta_1^e$, the closed path γ should be selected to

satisfy

$$\theta_1^e - \theta_1^1 = \pm\alpha(\gamma) \ .$$

The path γ lies in the two dimensional (ψ_i, ψ_j) plane, so that

$$\alpha(\gamma) = \oint_\gamma \tilde{s}_i(\psi_i, \psi_j) d\psi_i + \tilde{s}_j(\psi_i, \psi_j) d\psi_j$$

where the scalar functions $\tilde{s}_i(\psi_i, \psi_j)$ and $\tilde{s}_j(\psi_i, \psi_j)$ are obtained by evaluating $s_i(\psi)$ and $s_j(\psi)$ at $\psi_k = \psi_k^e$, $\forall k \in I$ where $k \neq i, j$.

As mentioned above we choose γ to be a square path in the (ψ_i, ψ_j) plane which is centered at the shape defined by ψ^* and which has side of length z^*, where z^* satisfies

$$\pm\alpha(\gamma_{z^*}) + \theta_1^1 - \theta_1^e = 0 \ .$$

Here γ_z indicates the dependence of the square path on the size parameter z. In most cases, this equation is easily solved using standard numerical procedures.

Remark 1 : Note that here, for notational simplicity in presenting the main idea, we assume that the desired geometric phase can be obtained by a single closed path. In general, more than one closed path may be required to produce the desired geometric phase; for such cases γ can be viewed as a concatenation of a series of closed paths. In any event, the motion along such a closed path defines a periodic motion corresponding to a change in the shape of the structure.

Remark 2 : Selection of the center point ψ^* of the path is rather arbitrary, e.g. one selection is $\psi^* = \psi^e$. However, other choices may provide a greater change in the geometric phase for a given size path. In this regard, the use of Stoke's theorem, as indicated previously, suggests that ψ^* should be chosen where

$$|\frac{\partial s_j(\psi)}{\partial \psi_i} - \frac{\partial s_i(\psi)}{\partial \psi_j}|$$

is a maximum.

We now describe the remaining three steps as follows.

Step 2 : *Transfer the system from state $(\theta_1^1, \psi^e, 0)$ to a state corresponding to the corner of γ closest to ψ^e, along an arbitrary path in the shape space, in $t_2 - t_1$ units of time.*

As an example, if p_1^* is the corner of γ closest to ψ^e we propose the following control function for Step 2.

$$u_{[t_1,t_2)} = \frac{2\pi(p_1^* - \psi^e)}{(t_2 - t_1)^2} \sin\left(\frac{2\pi(t - t_1)}{(t_2 - t_1)}\right) . \tag{13}$$

Step 3 : *Traverse the selected square path (counterclockwise or clockwise, depending on the sign of the desired geometric phase value), in $t_3 - t_2$ units of time; the resulting change in the angle θ_1 is necessarily $\theta_1^e - \theta_1^1$.*

Without loss of generality, we assume that the desired geometric phase value is obtained by counterclockwise traversal of the closed path starting and ending at p_1^*. Then, the following control functions guarantee traversal of the closed path, thereby accomplishing Step 3.

$$u_{[t_2,t_2+h)} = \frac{2\pi(p_2^* - p_1^*)}{h^2} \sin\left(\frac{2\pi(t - t_2)}{h}\right) , \tag{14}$$

$$u_{[t_2+h,t_2+2h)} = \frac{2\pi(p_3^* - p_2^*)}{h^2} \sin\left(\frac{2\pi(t - t_2 - h)}{h}\right) , \tag{15}$$

$$u_{[t_2+2h,t_2+3h)} = \frac{2\pi(p_4^* - p_3^*)}{h^2} \sin\left(\frac{2\pi(t - t_2 - 2h)}{h}\right) , \tag{16}$$

$$u_{[t_2+3h,t_3)} = \frac{2\pi(p_1^* - p_4^*)}{h^2} \sin\left(\frac{2\pi(t - t_2 - 3h)}{h}\right) \tag{17}$$

where $h = (t_3 - t_2)/4$.

Step 4 : *Transfer the system back to the desired equilibrium shape ψ^e following the path used in Step 2, in $t_f - t_3$ units of time; thereby guaranteeing that the desired final state $(\theta_1^e, \psi^e, 0)$ is reached at time t_f.*

The following control function

$$u_{[t_3,t_f)} = \frac{2\pi(\psi^e - p_1^*)}{(t_f - t_3)^2} \sin\left(\frac{2\pi(t - t_3)}{(t_f - t_3)}\right) \qquad (18)$$

accomplishes Step 4.

The corresponding control torque τ can be computed using equation (6). It is clear that the constructed control torque transfers the initial condition of the system (1)-(2) to the desired equilibrium configuration at time t_f. It is important to emphasize that the above construction is based on a priori selection of a square as the closed path in the shape space. Selection of square paths simplifies computation of the controls; however other path selections, e.g. corresponding to sinusoidal changes in the shape of the structure, could be made. There are infinitely many choices for control functions which accomplish the above four steps, and the total time required is arbitrary.

4. Conclusions

In this paper we have developed an attitude control strategy for planar rigid bodies interconnected by ideal pin joints in the form of an open kinematic chain. The control strategy utilizes the nonintegrability of the expression for angular momentum. We have demonstrated that large angle maneuvers can be designed to achieve an arbitrary reorientation of the multibody system with respect to an inertial frame; the maneuvers are performed using internal controls, e.g. servo torque motors located at the joints of the body segments. The theoretical background for carrying out the required maneuvers has been briefly summarized. We mention two nontrivial extensions of the approach in this paper which are currently being developed. The first extension is to non-planar reorientation maneuvers of multibody systems consisting of rigid and flexible links; in this case the dynamics issues are much more complicated but in principle the approach is viable[22]. Another extension is the development of feedback implementations of the controls presented in this paper; some results have been obtained[19] using a (necessarily)

discontinuous feedback strategy. These important extensions generally require the use of differential geometric methods for a complete treatment.

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On the Attitude Stabilization of a Rigid Spacecraft Using Two Control Torques

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Abstract

The attitude stabilization problem of a rigid spacecraft using only two control torques is considered. The control torques are assumed to be generated by either gas jet actuators or momentum wheel actuators. In particular, we focus on the development of a control strategy for the following two problems which have not been considered elsewhere: the attitude stabilization of an axially symmetric spacecraft using control torques supplied by two pairs of gas jet actuators about axes spanning a two dimensional plane orthogonal to the axis of symmetry; the attitude stabilization of a spacecraft using control torques supplied by two momentum wheel actuators about axes spanning a two dimensional plane orthogonal to a principal axis. The complete dynamics of the spacecraft system fails to be controllable or even accessible in these cases. However, the spacecraft dynamics are strongly accessible and small time locally controllable in a restricted sense; but the restricted spacecraft dynamics cannot be asymptotically stabilized using any smooth C^1 feedback. A nonsmooth feedback control strategy is developed for the restricted spacecraft dynamics which achieves an arbitrary reorientation of the spacecraft.

1. Introduction

The attitude stabilization problem of a rigid spacecraft using only two control torques is revisited. This may represent a spacecraft controlled by three control torques operating in a failure mode. The linearization of the dynamic equations at an equilibrium of such a system has an uncontrollable eigenvalue at the origin. Thus controllability and stabilizability properties of the system cannot be inferred using classical linearization ideas and requires inherently nonlinear analysis. An analysis of the controllability properties of a spacecraft with two independent control torques is made in [7]. In [7] it is shown that a necessary and sufficient condition for complete controllability of a spacecraft with control torques supplied by two pairs of gas jet actuators about axes spanning a two dimensional plane is that the axis orthogonal to this plane must not be a principal axis of symmetry of the spacecraft. For such a system, it is further shown that controllability is equivalent to local controllability at any equilibrium. When a spacecraft is controlled by less than three independent momentum wheel actuators, the system is not controllable or even accessible at any equilibrium [7]. Hence, all the results in the literature on the stabilization of a spacecraft with two control torques assume that the control torques are generated by gas jet actuators. In what follows in this section, it is assumed that the control torques are generated by gas jet actuators unless stated otherwise.

In [6], it is shown that a rigid spacecraft controlled by two pairs of gas jet actuators cannot be asymptotically stabilized to an equilibrium using a continuously differentiable, i.e. C^1, feedback control law. However a smooth C^1 feedback control law is derived which locally asymptotically stabilizes the spacecraft to a circular attractor, rather than an isolated equilibrium. Using local controllability results, an algorithm which locally asymptotically stabilizes the system to an isolated equilibrium is proposed in [7]. The algorithm is extremely complicated and is based on Lie algebraic methods as proposed in [8]. The algorithm yields a piecewise constant discontinuous control. Although very complicated, the algorithm is the only one proposed in the literature which locally asymptotically stabilizes the system to an equilibrium.

In this paper, we focus on two important control problems which have not been considered elsewhere:

the attitude stabilization of an axially symmetric spacecraft using control torques supplied by two pairs of gas jet actuators about axes spanning a two dimensional plane orthogonal to the axis of symmetry;

the attitude stabilization of a spacecraft using control torques supplied by two momentum wheel actuators about axes spanning a two dimensional plane orthogonal to a principal axis.

The focus of this paper is on the development of a control strategy for attitude stabilization, in a restricted sense, of the above mentioned spacecraft systems. The control results in [7] and [6] are not applicable to these spacecraft systems since the complete dynamic

equations fail to be controllable or even accessible at any equilibrium. Under some rather weak assumptions, it is shown that the dynamic equations of both the spacecraft systems being considered reduce to an identical set of equations which are identified as a nonholonomic control system. A coordinate transformation is made and feedback is then used to obtain a nonlinear control model in a normal form. The linearization of the normal form equations at an equilibrium has an uncontrollable eigenvalue at the origin. Based on analysis of the normal form equations, the spacecraft systems are strongly accessible and small time locally controllable at any equilibrium in a restricted sense. The spacecraft systems cannot be asymptotically stabilized to an equilibrium using smooth C^1 feedback. Nevertheless, a nonsmooth feedback control strategy is developed which achieves reorientation of the spacecraft. The feedback control strategy is based on holonomy or geometric phase, which is due to the presence of a nonintegrable constraint on the spacecraft motion in the cases studied.

We conclude this section with a summary of some of the important results on the stabilization of the angular velocity equations (i.e. without considering the attitude equations) of a spacecraft using fewer than three independent control torques. Asymptotic stabilization of the angular velocity equations of a spacecraft using only control torques about two of its principal axes is considered in [1,5]. It is shown that the angular velocity is asymptotically stabilizable to the origin using smooth C^1 feedback if the uncontrolled principal axis is not an axis of symmetry of the spacecraft. Explicit control laws are derived in [5] and [1] using center manifold theory. For a spacecraft with no axis of symmetry, asymptotic stabilization using a linear control law is possible using just one control torque about an axis having nonzero components along each principal axis [3]. The control law, however, is not robust. In the case of an axially symmetric spacecraft controlled using a single control torque about an axis having nonzero components along each principal axis, there exists no linear control law which asymptotically stabilizes the origin; however there exists a nonlinear asymptotically stabilizing control law [10]. If there is only one control torque applied about an axis which is a principal axis of the spacecraft, then asymptotic stabilization is not possible [2]. However, there exist smooth C^1 feedback control laws which make the origin stable in the sense of Lyapunov [2]. A point to notice is that the resulting closed loop system is robust if the moment of inertia about the control axis is either the maximum or minimum principal moment of inertia. Otherwise, the control law is nonrobust.

2. Kinematic and Dynamic Equations

Kinematic Equations

The orientation of a rigid spacecraft can be specified using various parametrizations of $SO(3)$. Here we use the following Euler angle convention. Consider an inertial $X_1X_2X_3$ coordinate frame; let $x_1x_2x_3$ be a coordinate frame aligned with the principal axes of the spacecraft with origin at the center of mass of the spacecraft. If the two frames are initially coincident, a series of three rotations about the body axes, performed in the proper sequence, is sufficient to allow the spacecraft to reach any orientation. The three rotations are:

a positive rotation of frame $X_1X_2X_3$ by angle ψ about the X_3 axis; let $x_1'x_2'x_3'$ denote the resulting coordinate frame;

a positive rotation of frame $x_1'x_2'x_3'$ by angle θ about the x_2' axis; let $x_1''x_2''x_3''$ denote the resulting frame;

a positive rotation of frame $x_1''x_2''x_3''$ by angle ϕ about the x_1'' axis; let $x_1x_2x_3$ denote the final coordinate frame.

A rotation matrix R relates components of a vector in the inertial frame to components of the same vector in the body frame; in terms of the Euler angles a rotation matrix is

$$R(\psi,\theta,\phi) = \begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix} \quad (2.1)$$

where $c\psi = \cos(\psi)$, $s\psi = \sin(\psi)$. We assume that the Euler angles are limited to the ranges $-\pi < \psi < \pi$, $-\pi/2 < \theta < \pi/2$, $-\pi < \phi < \pi$. Suppose ω_1, ω_2, ω_3 are the principal axis components of the absolute angular velocity vector ω of the spacecraft. Then expressions for ω_1, ω_2, ω_3 are given by

$$\omega_1 = \dot\phi - \dot\psi \sin\theta , \quad (2.2)$$

$$\omega_2 = \dot\theta\cos\phi + \dot\psi\cos\theta\,\sin\phi , \quad (2.3)$$

$$\omega_3 = -\dot\theta\sin\phi + \dot\psi\cos\theta\,\cos\phi . \quad (2.4)$$

By excluding the case where $\theta = \pm \pi/2$, these equations are invertible. Thus we can solve for ϕ, θ, ψ in terms of ω_1, ω_2, ω_3 obtaining

$$\dot\phi = \omega_1 + \omega_2\sin\phi\,\tan\theta + \omega_3\cos\phi\,\tan\theta , \quad (2.5)$$

$$\dot\theta = \omega_2\cos\phi - \omega_3\sin\phi , \quad (2.6)$$

$$\dot\psi = \omega_2\sin\phi\,\sec\theta + \omega_3\cos\phi\,\sec\theta . \quad (2.7)$$

Next we consider the dynamic equations which describe the evolution of the angular velocity components of the spacecraft.

Dynamic Equations: Gas Jet Actuators

Let $J = \mathrm{diag}(J_1, J_2, J_3)$ be the inertia matrix of the spacecraft in a coordinate frame defined by its principal axes. Let H be the angular momentum vector of the spacecraft relative to the inertial frame. Then we have

$$J\omega = R(\psi,\theta,\phi)H . \quad (2.8)$$

Differentiating (2.8) we get

$$J\dot\omega = S(\omega)R(\psi,\theta,\phi)H + R(\psi,\theta,\phi)\dot H , \quad (2.9)$$

where

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} . \quad (2.10)$$

We assume that the control torques u'_1 and u'_2 are applied about the axes represented by unit vectors b_1 and b_2 respectively. This implies that

$$R(\psi,\theta,\phi)\dot H = b_1 u'_1 + b_2 u'_2 . \quad (2.11)$$

Further the vectors b_1 and b_2 are assumed to span a two dimensional plane orthogonal to a principal axis of the spacecraft. Without loss of generality, b_i are assumed to be of the form $(b_{i1}, b_{i2}, 0)^T$. Thus the equations describing the evolution of the angular velocity of the spacecraft are given by

$$J_1\dot\omega_1 = (J_2 - J_3)\omega_2\omega_3 + b_{11}u'_1 + b_{21}u'_2 , \quad (2.12)$$

$$J_2\dot\omega_2 = (J_3 - J_1)\omega_3\omega_1 + b_{12}u'_1 + b_{22}u'_2 , \quad (2.13)$$

$$J_3\dot\omega_3 = (J_1 - J_2)\omega_1\omega_2 . \quad (2.14)$$

Dynamic Equations: Momentum Wheel Actuators

Consider a rigid spacecraft with two momentum wheel actuators spinning about axes defined by unit vectors b_1, b_2 fixed in the spacecraft such that the center of mass of the i-th wheel lies on the axis defined by b_i, and a control torque $-u'_i$ is supplied to the i-th wheel about the axis defined by b_i by a motor fixed in the spacecraft. Consequently, an equal and opposite torque u'_i is exerted by the wheel on the spacecraft. We refer to the spacecraft and the two wheels as body 1, body 2 and body 3 respectively. Let C_i denote a coordinate frame aligned with the principal axes at the center of mass of body i. We assume that b_i defines a principal axis for the i-th wheel which is symmetric about b_i. Further b_1 and b_2 span a two dimensional plane which is orthogonal to a principal axis of the spacecraft. Without loss of generality b_i are assumed to be of the form $(b_{i1}, b_{i2}, 0)^T$.

Let I_i denote the inertia tensor of body i in the coordinate frame C_1. The mass of body i is denoted as m_i and ρ_i^1 denotes the position vector, expressed in the C_1 frame, of the center of mass of body i with respect to the center of mass of the whole system. Let ω denote the absolute angular velocity of the spacecraft expressed

in the spacecraft body frame. By the definition of the center of mass, we have

$$\sum_{i=1}^{3} m_i \rho_i^1 = 0 , \quad (2.15)$$

and from the location of the wheels

$$\rho_2^1 = \rho_1^1 + (d_1, d_2, 0)^T , \quad (2.16)$$

$$\rho_3^1 = \rho_1^1 + (d_3, d_4, 0)^T , \quad (2.17)$$

where $(d_1, d_2, 0)^T$ and $(d_3, d_4, 0)^T$ are position vectors of the center of mass of body 2 and body 3 respectively, relative to the frame C_1. Further manipulation of equations (2.15)-(2.17) gives

$$\rho_1^1 = (-\frac{m_2}{m}d_1 - \frac{m_3}{m}d_3, -\frac{m_2}{m}d_2 - \frac{m_3}{m}d_4, 0)^T , \quad (2.18)$$

$$\rho_2^1 = (\frac{m_1+m_3}{m}d_1 - \frac{m_3}{m}d_3, \frac{m_1+m_3}{m}d_2 - \frac{m_3}{m}d_4, 0)^T , \quad (2.19)$$

$$\rho_3^1 = (-\frac{m_2}{m}d_1 + \frac{m_1+m_2}{m}d_3, -\frac{m_2}{m}d_2 + \frac{m_1+m_2}{m}d_4, 0)^T , \quad (2.20)$$

where $m = m_1+m_2+m_3$. We denote ρ_i^1 as $\rho_i^1 = (c_{i1}, c_{i2}, 0)^T$. The total angular momentum vector of the system is given, in the spacecraft body frame, by

$$R(\psi,\theta,\phi)H = J\omega + v , \quad (2.21)$$

where

$$J = [I_1 + \sum_{i=1}^{3}\bar I_i + \sum_{i=2}^{3}(I_i - \underline I_i)] , \quad (2.22)$$

$$\bar I_i = m_i \begin{bmatrix} c_{i2}^2 & -c_{i1}c_{i2} & 0 \\ -c_{i1}c_{i2} & c_{i1}^2 & 0 \\ 0 & 0 & c_{i1}^2+c_{i2}^2 \end{bmatrix} , \quad (2.23)$$

$$v = (v_1, v_2, 0)^T = \underline I_2(\omega + b_1\dot\theta_1) + \underline I_3(\omega + b_2\dot\theta_2) , \quad (2.24)$$

$$\underline I_2 = b_1 b_1^T j_1 , \quad \underline I_3 = b_2 b_2^T j_2 , \quad (2.25)$$

$$I_1 = \mathrm{diag}(I_{11}, I_{12}, I_{13}), \quad (2.26)$$

$$I_2 = \text{block diag}(I_{21}, I_{22}), \quad I_3 = \text{block diag}(I_{31}, I_{32}), \quad (2.27)$$

where I_{21}, I_{31} are invertible 2×2 matrices, I_{11}, I_{12}, I_{13}, I_{22}, I_{32} are nonzero real numbers, j_1 is the moment of inertia of body 2 about the axis defined by b_1, j_2 is the moment of inertia of body 3 about the axis defined by b_2, and θ_1, θ_2 are the angles of rotation of the wheels about the axes defined by b_1 and b_2 respectively. Here H denotes the angular momentum vector of the system expressed in the inertial coordinate frame. The angular momentum vector H is a constant since there is no external moment about the center of mass of the system. Suppose u'_1 and u'_2 are the control torques; then

$$\dot v = -(b_1 u'_1 + b_2 u'_2) . \quad (2.28)$$

Differentiating (2.21) we obtain

$$J\dot\omega = S(\omega)R(\psi,\theta,\phi)H + b_1 u'_1 + b_2 u'_2 , \quad (2.29)$$

where H is a constant vector. Note that $I_1 + \sum_{i=1}^{3}\bar I_i$ represents the moment of inertia of the system with the momentum wheels replaced by point masses and is hence positive definite. The matrices $I_i - \underline I_i$, $i = 2, 3$ are positive semidefinite and block diagonal. Therefore J is a positive definite matrix and hence invertible. In fact J is of the form

$$J = \text{block diag}(J_1, J_2) , \quad (2.30)$$

where J_1 is an invertible 2×2 matrix and J_2 is a nonzero real number.

3. Controllability and Stabilizability Properties of Complete Spacecraft Dynamics with Two Control Torques

As background for our subsequent development, we present controllability and stabilizability properties for the complete dynamics of the spacecraft systems described in the previous section. The case for gas jet actuators is shown to depend significantly on the condition $J_1 \ne J_2$. The case for momentum wheel actuators is straightforward.

Results for Gas Jet Actuators

We first consider the equations describing the motion of a spacecraft controlled by two pairs of gas jet actuators. Define

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{J_1} & 0 \\ 0 & \dfrac{1}{J_2} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix}.$$

From Section 2 the state equations can be rewritten as

$$\dot{\omega}_1 = a_1 \omega_2 \omega_3 + u_1 , \tag{3.1}$$

$$\dot{\omega}_2 = a_2 \omega_1 \omega_3 + u_2 , \tag{3.2}$$

$$\dot{\omega}_3 = a_3 \omega_1 \omega_2 , \tag{3.3}$$

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \, \tan\theta + \omega_3 \cos\phi \, \tan\theta , \tag{3.4}$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi , \tag{3.5}$$

$$\dot{\psi} = \omega_2 \sin\phi \, \sec\theta + \omega_3 \cos\phi \, \sec\theta , \tag{3.6}$$

where

$$a_1 = \frac{J_2 - J_3}{J_1} , \quad a_2 = \frac{J_3 - J_1}{J_2} , \quad a_3 = \frac{J_1 - J_2}{J_3} .$$

This is of the form

$$\dot{x} = f(x) + g_1 u_1 + g_2 u_2 , \tag{3.7}$$

where $x = (\omega_1, \omega_2, \omega_3, \phi, \theta, \psi)^T$ and f, g_1, g_2 are vector fields defined appropriately. Let M denote the open set

$$M = \{ x : \omega_i \in R, \, i = 1,2,3, \, \phi, \, \psi \in (-\pi, \pi), \, \theta \in (-0.5\pi, 0.5\pi) \} .$$

It is easily verified that the linearization of the equations about an equilibrium has an uncontrollable eigenvalue at the origin. This implies that an inherently nonlinear analysis is necessary in order to characterize the controllability and stabilizability properties of the system.

We present important results on the controllability and stabilizability properties of the complete spacecraft dynamics described by (3.1)-(3.6).

Theorem 3.1: The complete spacecraft dynamics described by state equations (3.1)-(3.6) are strongly accessible $\forall x \in M$ if and only if $J_1 \neq J_2$, i.e. the uncontrolled principal axis is not an axis of symmetry.

Proof: If $J_1 \neq J_2$, the vector fields $g_1, g_2, [g_1, f], [g_2, f], [g_2, [g_1, f]], [[g_2, [g_1, f]], f]$ span a six dimensional space at every $x \in M$. Thus the strong accessibility Lie algebraic rank condition is satisfied and hence the complete spacecraft dynamics are strongly accessible. If $J_1 = J_2$ the complete spacecraft dynamics fails to be accessible since ω_3 is necessarily constant.

Theorem 3.2: The complete spacecraft dynamics described by state equations (3.1)-(3.6) are small time locally controllable at any equilibrium if and only if $J_1 \neq J_2$.

Proof: Suppose $J_1 \neq J_2$. Then the complete spacecraft dynamics are strongly accessible. Following Sussman [11], let $Br(x)$ denote the smallest Lie algebra of vector fields containing f, g_1, g_2. Let B be any bracket in $Br(x)$. Now denote $\delta^0(B), \delta^1(B), \delta^2(B)$ as the number of occurrences of the vector fields f, g_1, g_2 respectively in the bracket B. The degree of B is equal to the value of $\sum_{i=0}^{2} \delta^i(B)$.

The Sussman condition for small time local controllability at an equilibrium is that the so-called *bad* brackets, the brackets with δ^0 odd, and δ^1, δ^2 even, must be a linear combination of brackets of lower degree at that equilibrium. From the proof of Theorem 3.1 it is clear that any bracket of degree greater than four can be expressed as a linear combination of lower order brackets at any equilibrium. Moreover the degree of a bad bracket must necessarily be odd. The bad bracket of degree one is f which vanishes at any equilibrium. The bad brackets of degree three are $[g_1, [g_1, f]]$ and $[g_2, [g_2, f]]$ and both are identically zero vector fields. Thus the complete spacecraft dynamics are small time locally controllable. If $J_1 = J_2$, the complete spacecraft dynamics fails to be accessible at any equilibrium; hence it cannot be small time locally controllable at any equilibrium.

Theorem 3.3: The complete spacecraft dynamics described by state equations (3.1)-(3.6) cannot be locally asymptotically stabilized to

an equilibrium by any C^1 static or dynamic state feedback control law.

The above theorem was proved in [6] by appealing to Krasnoselski's theorem. Although the full set of equations (3.1)-(3.6) cannot be asymptotically stabilized to an equilibrium via C^1 feedback, one may still wish to design a smooth control law which stabilizes at least a particular subset of state variables. Consider the state equations for $\omega_1, \omega_2, \omega_3, \phi$ and θ given by (3.1)-(3.5). These equations are not affected by the Euler angle variable ψ. Asymptotic stabilization of this subset of the original equations corresponds to stabilization of the motion of the spacecraft about an attractor, which is not an isolated equilibrium. The following theorem from [6] shows that the closed loop trajectories can be asymptotically stabilized to the manifold

$$\Omega = \{ (\omega_1, \omega_2, \omega_3, \phi, \theta, \psi) : \omega_1 = \omega_2 = \omega_3 = \phi = \theta = 0 \} , \tag{3.8}$$

using smooth C^1 feedback.

Theorem 3.4: Suppose $J_1 \neq J_2$. The feedback control law

$$u_1 = -\omega_1 - \phi - A_1 \omega_3 - B_1 \omega_3^2 - (A_1 + 2B_1 \omega_3)(\frac{J_1 - J_2}{J_3})\omega_1 \omega_2$$

$$\quad - (\frac{J_2 - J_3}{J_1})\omega_2 \omega_3 - \omega_1 - \omega_2 \sin\phi \, \tan\theta - \omega_3 \cos\phi \, \tan\theta ,$$

$$u_2 = -\omega_2 - \theta - A_2 \omega_3 - B_2 \omega_3^2 - (A_2 + 2B_2 \omega_3)(\frac{J_1 - J_2}{J_3})\omega_1 \omega_2$$

$$\quad - (\frac{J_3 - J_1}{J_2})\omega_1 \omega_3 - \omega_2 \cos\phi + \omega_3 \sin\phi ,$$

where A_1, A_2, B_1, B_2 are gain parameters which satisfy $A_1 A_2 = 0$ and

$$(\frac{J_1 - J_2}{J_3})(A_1(A_1 - B_2) - A_2(A_2 + B_1)) < 0 , \tag{3.9}$$

locally asymptotically stabilizes the rigid spacecraft to the one dimensional manifold Ω defined by (3.8).

We mention that although the complete spacecraft dynamics described by equations (3.1)-(3.6) cannot be asymptotically stabilized to an equilibrium by C^1 feedback, an algorithm generating a piecewise constant discontinuous control is developed in [7] which locally asymptotically stabilizes the complete spacecraft dynamics to an equilibrium. The algorithm requires that $J_1 \neq J_2$, i.e. the uncontrolled principal axis must not be an axis of symmetry. The algorithm is based on Lie algebraic methods as proposed in [8]. The algorithm is extremely complicated and is not an easily implementable control strategy. However, stabilization of the complete spacecraft dynamic equations (3.1)-(3.6) is an inherently difficult problem and the algorithm in [7] is the only control strategy proposed in the literature thus far.

Results for Momentum Wheel Actuators

Now let us consider the case of a rigid spacecraft controlled by two momentum wheel actuators. Define

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = J_1^{-1} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix}.$$

From Section 2 the state equations can be rewritten as

$$\dot{\omega} = \begin{bmatrix} J_1^{-1} & 0_{(2 \times 1)} \\ 0_{(1 \times 2)} & J_2^{-1} \end{bmatrix} S(\omega) R(\psi, \theta, \phi) H + (u_1, u_2, 0)^T , \tag{3.10}$$

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \, \tan\theta + \omega_3 \cos\phi \, \tan\theta , \tag{3.11}$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi , \tag{3.12}$$

$$\dot{\psi} = \omega_2 \sin\phi \, \sec\theta + \omega_3 \cos\phi \, \sec\theta , \tag{3.13}$$

where H is a constant vector. In [7] it is shown that the complete dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (3.10)-(3.13) are not controllable or even accessible at any equilibrium. As a consequence of this negative result, the complete spacecraft dynamics cannot be asymptotically stabilized using two momentum wheel actuators.

4. Controllability and Stabilizability Properties of Restricted Spacecraft Dynamics with Two Control Torques

From the analysis made in the previous section, we find that

the complete dynamics of a spacecraft system controlled by two control torques supplied by gas jet actuators as described by equations (3.1)-(3.6) fail to be controllable or even accessible if the uncontrolled principal axis is an axis of symmetry of the spacecraft, i.e. if $J_1 = J_2$. Due to the lack of controllability, the stabilizing control algorithm proposed in [7] is not applicable to this case. Moreover, the control law given by Theorem 3.4 which asymptotically stabilizes the spacecraft to a one dimensional manifold will not work in this case since condition (3.9) is violated. Also, the complete dynamics of a spacecraft system controlled by two momentum wheel actuators as described by equations (3.10)-(3.13) fail to be controllable or even accessible. Note that in this case it is not necessary that the uncontrolled principal axis be an axis of symmetry of the spacecraft. In this section we concentrate on these important cases. In particular we ask the question: what restricted control and stabilization properties of the spacecraft can be demonstrated in the cases considered? Our analysis begins by demonstrating that, under appropriate restrictions of interest, the spacecraft equations can be expressed in terms of normal form equations. Restricted controllability and stabilizability properties for each case follow as a consequence of previous work.

Normal Form Equations

We first consider the equations (3.1)-(3.6) describing the motion of a spacecraft controlled by input torques supplied by two pairs of gas jet actuators about axes spanning a two dimensional plane orthogonal to a principal axis of the spacecraft. It is assumed that the uncontrolled principal axis is an axis of symmetry of the spacecraft. From equations (3.1)-(3.6) and $J_1 = J_2$ we have

$$\dot{\omega}_1 = a_1 \omega_2 \omega_3 + u_1 , \tag{4.1}$$

$$\dot{\omega}_2 = a_2 \omega_1 \omega_3 + u_2 , \tag{4.2}$$

$$\dot{\omega}_3 = 0 , \tag{4.3}$$

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \, \tan\theta + \omega_3 \cos\phi \, \tan\theta , \tag{4.4}$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi , \tag{4.5}$$

$$\dot{\psi} = \omega_2 \sin\phi \, \sec\theta + \omega_3 \cos\phi \, \sec\theta . \tag{4.6}$$

If we assume that the initial angular velocity component of the spacecraft about the axis of symmetry is zero, i.e. $\omega_3(0) = 0$, then $\omega_3 \equiv 0$. Under such a restriction, the restricted spacecraft dynamics for this case are described by

$$\dot{\omega}_1 = u_1 , \tag{4.7}$$

$$\dot{\omega}_2 = u_2 , \tag{4.8}$$

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \, \tan\theta , \tag{4.9}$$

$$\dot{\theta} = \omega_2 \cos\phi , \tag{4.10}$$

$$\dot{\psi} = \omega_2 \sin\phi \, \sec\theta . \tag{4.11}$$

We next consider the equations (3.10)-(3.13) describing the motion of a spacecraft controlled by input torques supplied by two momentum wheel actuators about axes spanning a two dimensional plane orthogonal to a principal axis. Suppose the angular momentum vector H of the system is zero. From equations (2.21), (2.24) and (2.30) it follows that the angular velocity component of the spacecraft about the uncontrolled principal axis is identically zero, i.e. $\omega_3 \equiv 0$. The restricted spacecraft dynamics for this case are described by

$$\dot{\omega}_1 = u_1 , \tag{4.12}$$

$$\dot{\omega}_2 = u_2 , \tag{4.13}$$

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \, \tan\theta , \tag{4.14}$$

$$\dot{\theta} = \omega_2 \cos\phi , \tag{4.15}$$

$$\dot{\psi} = \omega_2 \sin\phi \, \sec\theta . \tag{4.16}$$

The equations describing the motion of the spacecraft, under the restrictions specified, reduce to an identical set of equations in both the cases considered; we say that equations (4.7)-(4.11) (or (4.12)-(4.16)) describe restricted spacecraft dynamics since, in each case, assumptions have been made which a priori guarantee that the component of the spacecraft angular velocity $\omega_3 = 0$. According to equation (2.4), the condition that $\omega_3 = 0$ implies that

$$-(\sin\phi)d\theta + (\cos\theta\cos\phi)d\psi = 0 ; \tag{4.17}$$

this represents a nonintegrable constraint on the spacecraft motion. Therefore the dynamic equations in each case define a nonholonomic control system of the form studied in [4,9].

Now consider a diffeomorphism defined by

$$y_1 = \cos\phi \, ln(\sec\theta + \tan\theta) + \psi\sin\phi , \tag{4.18}$$

$$y_2 = \omega_2 \sec\theta - y_4 y_5 , \tag{4.19}$$

$$y_3 = \phi , \tag{4.20}$$

$$y_4 = \omega_1 + \omega_2 \sin\phi \, \tan\theta , \tag{4.21}$$

$$y_5 = \sin\phi \, ln(\sec\theta + \tan\theta) - \psi\cos\phi . \tag{4.22}$$

The state equations (4.7)-(4.11) (or (4.12)-(4.16)) in the new variables are given by

$$\dot{y}_1 = y_2 , \tag{4.23}$$

$$\dot{y}_2 = -y_5 u_1 + (\sec\theta - y_5\sin\phi\tan\theta)u_2 - y_4^2 y_1 + \cos\phi(\sec\theta\tan\theta\omega_2^2 - y_5 y_4\tan\theta\omega_2 - y_5\sin\phi\sec^2\theta\omega_2^2) , \tag{4.24}$$

$$\dot{y}_3 = y_4 , \tag{4.25}$$

$$\dot{y}_4 = u_1 + \sin\phi\tan\theta u_2 + \cos\phi(y_4\tan\theta\omega_2 + \sin\phi\sec^2\theta\omega_2^2) , \tag{4.26}$$

$$\dot{y}_5 = y_4 y_1 . \tag{4.27}$$

If we now define the feedback relations

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\phi\sin\theta & (1-y_5\sin\phi\sin\theta) \\ \cos\theta & y_5\cos\theta \end{bmatrix} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right. \tag{4.28}$$

$$\left. - \begin{bmatrix} -y_4^2 y_1 + \cos\phi(\sec\theta\tan\theta\omega_2^2 - y_5 y_4\tan\theta\omega_2 - y_5\sin\phi\sec^2\theta\omega_2^2) \\ \cos\phi(y_4\tan\theta\omega_2 + \sin\phi\sec^2\theta\omega_2^2) \end{bmatrix} \right)$$

then the restricted spacecraft dynamics are described by normal form equations

$$\dot{y}_1 = y_2 , \tag{4.29}$$

$$\dot{y}_2 = v_1 , \tag{4.30}$$

$$\dot{y}_3 = y_4 , \tag{4.31}$$

$$\dot{y}_4 = v_2 , \tag{4.32}$$

$$\dot{y}_5 = y_4 y_1 . \tag{4.33}$$

Note that the origin of equations (4.7)-(4.11) (or (4.12)-(4.16)) corresponds to the origin of the normal form equations (4.29)-(4.33).

The above normal form equations thus represent the spacecraft control system for each of the cases considered in the restricted sense; namely, for the gas jet actuator case $\omega_3(0) = 0$ is a priori assumed, and for the momentum wheel actuator case the angular momentum vector of the system H is a priori assumed to be zero. The following results stated for each of the spacecraft systems, are based on the normal form equations above and follow directly from general results in [4].

Results for Gas Jet Actuators

As indicated previously, the complete dynamics of an axially symmetric spacecraft ($J_1 = J_2$) controlled by two pairs of gas jet actuators as described by equations (4.1)-(4.6) is not strongly accessible, it is not small time locally controllable, and it cannot by asymptotically stabililized to an equilibrium by a C^1 feedback control law. On the other hand, if we a priori add the restriction that $\omega_3(0) = 0$ then the resulting restricted spacecraft dynamics are described by equations (4.7)-(4.11), and hence by the normal form equations (4.29)-(4.33). We now indicate that this restricted control system satisfies certain controllability and stabilizability properties.

Theorem 4.1: The restricted dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (4.7)-(4.11) are strongly accessible.

Theorem 4.2: The restricted dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (4.7)-(4.11) are small time locally controllable at any equilibrium.

Theorem 4.3: The restricted dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (4.7)-(4.11) cannot be asymptotically stabilized to an equilibrium using a C^1 feedback control law.

Theorem 4.4: The restricted dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (4.7)-(4.11) can be asymptotically stabilized to the one dimensional equilibrium manifold

$$\Omega = \{(\omega_1,\omega_2,\phi,\theta,\psi) : \omega_1 = \omega_2 = \phi = \theta = 0\} , \qquad (4.34)$$

using a smooth feedback control law given by (4.28) with v_1 and v_2 given by

$$v_1 = -k_{11}y_2 - k_{12}y_1 , \qquad (4.35)$$

$$v_2 = -k_{21}y_4 - k_{22}y_3 , \qquad (4.36)$$

where $k_{11}, k_{12}, k_{21}, k_{22} > 0$ and y_1, y_2, y_3, y_4 are defined by (4.18)-(4.22).

The implications of the properties stated above are as follows. For all initial conditions that satisfy $\omega_3(0) = 0$, the axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (4.1)-(4.6) can be controlled to any isolated equilibrium. However, any feedback control law that asymptotically stabilizes the spacecraft to an isolated equilibrium must necessarily be nonsmooth. Thus arbitrary reorientation of the spacecraft can be achieved if $\omega_3(0) = 0$; if $\omega_3(0) \neq 0$, reorientation of the spacecraft cannot be achieved.

Results for Momentum Wheel Actuators

From Section 3, the complete dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (3.10)-(3.13) is not controllable or even accessible at any equilibrium. On the other hand, if we a priori add the restriction that the angular momentum vector $H = 0$ then the resulting restricted spacecraft dynamics are described by equations (4.12)-(4.16), and hence by the normal form equations (4.29)-(4.33). We now indicate that this restricted control system satisfies certain controllability and stabilizability properties.

Theorem 4.5: The restricted dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (4.12)-(4.16) are strongly accessible.

Theorem 4.6: The restricted dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (4.12)-(4.16) are small time locally controllable at any equilibrium.

Theorem 4.7: The restricted dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (4.12)-(4.16) cannot be asymptotically stabilized to an equilibrium using a C^1 feedback control law.

Theorem 4.8: The restricted dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (4.12)-(4.16) can be asymptotically stabilized to the one dimensional equilibrium manifold

$$\Omega = \{(\omega_1,\omega_2,\phi,\theta,\psi) : \omega_1 = \omega_2 = \phi = \theta = 0\} , \qquad (4.37)$$

using a smooth feedback control law given by (4.28) with v_1 and v_2 given by

$$v_1 = -k_{11}y_2 - k_{12}y_1 , \qquad (4.38)$$

$$v_2 = -k_{21}y_4 - k_{22}y_3 , \qquad (4.39)$$

where $k_{11}, k_{12}, k_{21}, k_{22} > 0$ and y_1, y_2, y_3, y_4 are defined by (4.18)-(4.22).

The implications of the properties stated above are as follows. Suppose the angular momentum vector H is zero. Then the spacecraft controlled by two momentum wheel actuators as described by equations (3.10)-(3.13) can be controlled to any isolated equilibrium. However, any feedback control law that asymptotically stabilizes the spacecraft to an isolated equilibrium must necessarily be nonsmooth. This implies that arbitrary reorientation of the spacecraft can be achieved under the restriction $H = 0$; if $H \neq 0$ reorientation of the spacecraft cannot be achieved.

5. Feedback Stabilization Algorithm for Restricted Spacecraft Dynamics with Two Control Torques

We must restrict our study to the class of non-smooth feedback controllers in order to asymptotically stabilize the restricted spacecraft dynamics described by state equations (4.7)-(4.11) (or (4.12)-(4.16)). Clearly, traditional nonlinear control design methods are of no use. However, a control algorithm generating a non-smooth feedback control which asymptotically stabilizes an equilibrium can be constructed - as suggested by the controllability properties of the system. Without loss of generality, we assume that the equilibrium to be stabilized is the origin. Asymptotic stabilization of equations (4.7)-(4.11) (or (4.12)-(4.16)) to the origin is equivalent to asymptotic stabilization of the normal form equations (4.29)-(4.33) to the origin; hence we consider asymptotic stabilization of the normal form equations.

From equation (4.33) we find that if the spacecraft motion defines a closed path γ in the (y_1,y_3) space then

$$\Delta y_5 = \int_\gamma y_1 dy_3 , \qquad (5.1)$$

where Δy_5 is the lift in the variable y_5. This is the *holonomy* or *geometric phase*. This holonomy can be used to control the system to the origin using a two step procedure; the procedure is subsequently implemented as (nonsmooth) feedback.

Let $(\omega_1^0,\omega_2^0,\phi^0,\theta^0,\psi^0)$ denote an initial state for the restricted spacecraft dynamics described by equations (4.7)-(4.11) (or (4.12)-(4.16)). This corresponds to an initial state $(y_1^0 y_2^0 y_3^0 y_4^0 y_5^0)$ for the normal form equations (4.29)-(4.33).

Step 1: Transfer the initial state $(y_1^0 y_2^0 y_3^0 y_4^0 y_5^0)$ of the normal form equations to the state $(0,0,0,0,y_5^f)$, for some y_5^f, in finite time.

Step 2: Traverse a closed path γ in the (y_1,y_3) space in finite time where the path γ is selected to produce the desired holonomy

$$-y_5^f = \int_\gamma y_1 dy_3 . \qquad (5.2)$$

Note that the execution of step 1 is classical. Execution of step 2 requires explicit characterization of a closed path γ which produces the desired holonomy. In general there may be infinitely many closed paths which are candidates for step 2. Here we consider a rectangular path in the (y_1,y_3) space formed by line segments from $(0,0)$ to $(y_1^*,0)$, from $(y_1^*,0)$ to (y_1^*,y_3^*), from (y_1^*,y_3^*) to $(0,y_3^*)$, and from $(0,y_3^*)$ to $(0,0)$. The desired holonomy given by equation (5.2) now becomes

$$-y_5^f = y_1^* y_3^* , \qquad (5.3)$$

and y_1^* and y_3^* are chosen to satisfy the above equation. This guarantees, by construction, that the origin is reached. We now present a specific feedback control algorithm which asymptotically stabilizes the spacecraft to the origin; this feedback control algorithm implements the previous control approach. For additional details on this feedback implementation see [4,9].

Feedback Control Algorithm

Step 0: If $y_5^f \geq 0$, choose $y_1^* = -y_3^* = \sqrt{(y_5^f)}$; else choose $y_1^* = y_3^* = \sqrt{(-y_5^f)}$.

Step 1: Set

$$v_1 = \begin{cases} -\text{sign}(y_1 - y_1^* + \dfrac{y_2|y_2|}{2}) ; & (y_1,y_2) \neq (y_1^*,0) \\ 0 ; & (y_1,y_2) = (y_1^*,0) \end{cases}$$

$$v_2 = \begin{cases} -\text{sign}(y_3 + \dfrac{y_4|y_4|}{2}) ; & (y_3,y_4) \neq (0,0) \\ 0 ; & (y_3,y_4) = (0,0) \end{cases}$$

until $(y_1,y_2,y_3,y_4) = (y_1^*,0,0,0)$; then go to step 2.

Step 2: Set

$$v_1 = \begin{cases} -\text{sign}(y_1 - y_1^* + \dfrac{y_2|y_2|}{2}) ; & (y_1,y_2) \neq (y_1^*,0) \\ 0 ; & (y_1,y_2) = (y_1^*,0) \end{cases}$$

$$v_2 = \begin{cases} -\text{sign}(y_3 - y_3^* + \dfrac{y_4|y_4|}{2}) ; & (y_3,y_4) \neq (y_3^*,0) \\ 0 ; & (y_3,y_4) = (y_3^*,0) \end{cases}$$

until $(y_1,y_2,y_3,y_4) = (y_1^*,0,y_3^*,0)$; then go to step 3.

Step 3: Set

$$v_1 = \begin{cases} -\text{sign}(y_1 + \dfrac{y_2|y_2|}{2}) ; & (y_1,y_2) \neq (0,0) \\ 0 ; & (y_1,y_2) = (0,0) \end{cases}$$

$$v_2 = \begin{cases} -\text{sign}(y_3 - y_3^* + \dfrac{y_4|y_4|}{2}) ; & (y_3,y_4) \neq (y_3^*,0) \\ 0 ; & (y_3,y_4) = (y_3^*,0) \end{cases}$$

until $(y_1, y_2, y_3, y_4) = (0,0,y_3,0)$; then go to step 4.

Step 4: Set

$$v_1 = \begin{cases} -\text{sign}(y_1 + \dfrac{y_2|y_2|}{2}) \, ; & (y_1, y_2) \neq (0,0) \\ \qquad 0 \, ; & (y_1, y_2) = (0,0) \end{cases}$$

$$v_2 = \begin{cases} -\text{sign}(y_3 + \dfrac{y_4|y_4|}{2}) \, ; & (y_3, y_4) \neq (0,0) \\ \qquad 0 \, ; & (y_3, y_4) = (0,0) \end{cases}$$

until $(y_1, y_2, y_3, y_4) = (0,0,0,0)$; then go to step 0.

The most natural way to initialize the control algorithm is to begin with step 4 since the control inputs do not depend on the values of y_1 and y_3 in that step. This control algorithm is nonclassical and involves cyclic switching between various feedback functions. Justification that the constructed control algorithm globally asymptotically stabilizes the origin of the normal form equations (4.29)-(4.33) follows as a consequence of the construction procedure. Since stabilization of the normal form equations to the origin is equivalent to stabilization of the state equations (4.7)-(4.11) (or (4.12)-(4.16)) to its origin, we conclude that the control inputs defined by (4.28) with v_1 and v_2 defined by the above control algorithm asymptotically stabilizes the restricted spacecraft dynamics described by equations (4.7)-(4.11) (or (4.12)-(4.16)) to the equilibrium $(\omega_1, \omega_2, \phi, \theta, \psi) = (0,0,0,0,0)$.

6. Simulation

We illustrate the results of the previous sections using an example. We consider an axially symmetric rigid spacecraft in the form of a cylinder. It is assumed that there is no control torque about the axis of symmetry and two control torques, generated by gas jet actuators, are applied about the other principal axes. The complete dynamics of the spacecraft system cannot be asymptotically stabilized, but we consider the restriction that $\omega_3(0) = 0$. Consequently, we are interested in asymptotically stabilizing the restricted spacecraft dynamics described by equations (4.7)-(4.11) to the equilibrium $(\omega_1, \omega_2, \phi, \theta, \psi) = (0,0,0,0,0)$. The spacecraft is initially at rest (i.e. $\omega_1^0 = \omega_2^0 = 0$) with an initial orientation given by the Euler angles $\phi^0 = \pi$, $\theta^0 = 0.25\pi$ and $\psi^0 = -0.5\pi$. The initial state of the system corresponds to an initial state $(y_1^0, y_2^0, y_3^0, y_4^0, y_5^0) = (-0.881,0,\pi,0,-0.5\pi)$ for the normal form equations (4.29)-(4.33). A computer implementation of the feedback control algorithm specified in Section 5 was used to asymptotically stabilize the equilibrium. The algorithm was initialized at step 4. At the end of step 4, $y_1^1 = y_2^1 = y_3^1 = y_4^1 = 0$ and $y_5^1 = -1.118$. The desired holonomy is produced by traversing a square path in the (y_1, y_3) space with $y_1^* = y_3^* = 1.057$. The time responses of the Euler angle variables ϕ, θ and ψ are shown in figure 1. Three dimensional visualization schemes have been developed using a Silicon Graphics Iris work station in order to display the reorientation maneuver of the spacecraft.
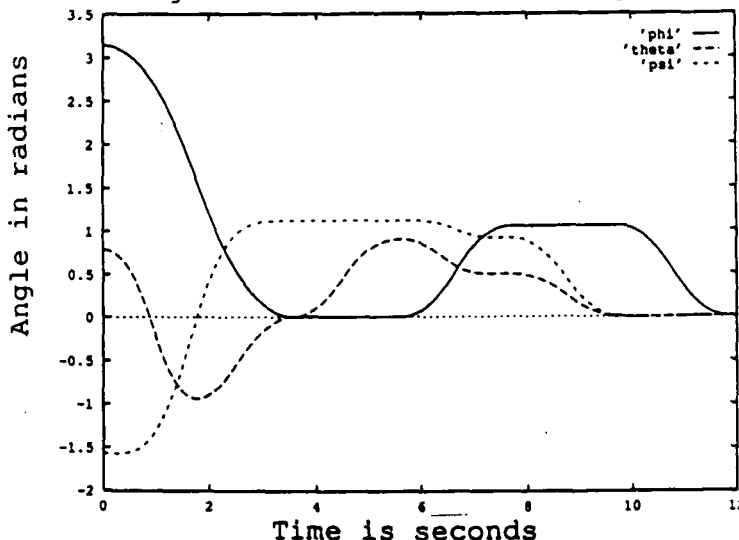
Since the restricted dynamics for a spacecraft controlled by two momentum wheel actuators about two of its principal axes have the same reduced form, we do not consider a separate example to illustrate this case.

7. Conclusion

The attitude stabilization problem of a spacecraft using only two control inputs has been considered. Particular emphasis has been given to the development of a control strategy for two important problems which have not been considered elsewhere: the attitude stabilization of an axially symmetric spacecraft using control torques supplied by gas jet actuators about axes spanning a two dimensional plane orthogonal to the axis of symmetry; the attitude stabilization of a spacecraft using control torques supplied by momentum wheel actuators about axes spanning a two dimensional plane orthogonal to a principal axis. The complete dynamics of the spacecraft system fails to be controllable or even accessible about any equilibrium. Under some rather weak assumptions, it has been shown that the restricted spacecraft dynamic equations in both cases reduce to an identical set of equations which represents a nonholonomic control system. A feedback control strategy based on holonomy has been presented which achieves arbitrary reorientation of the spacecraft.

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Figure 1: Plot of Euler Angles

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ATTITUDE STABILIZATION OF A RIGID SPACECRAFT USING GAS JET ACTUATORS OPERATING IN A FAILURE MODE*

N93-29992

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Abstract

We consider the attitude stabilization of a rigid spacecraft using control torques supplied by gas jet actuators about only two of its principal axes. First, the case where the uncontrolled principal axis of the spacecraft is not an axis of symmetry is considered. In this case, the complete spacecraft dynamics are small time locally controllable. However, the spacecraft cannot be asymptotically stabilized to an equilibrium attitude using time-invariant continuous feedback. A discontinuous stabilizing feedback control strategy is constructed which stabilizes the spacecraft to an equilibrium attitude. Next, the case where the uncontrolled principal axis of the spacecraft is an axis of symmetry is considered. In this case, the complete spacecraft dynamics are not even accessible. However, the spacecraft dynamics are strongly accessible and small time locally controllable in a reduced sense. The reduced spacecraft dynamics cannot be asymptotically stabilized to an equilibrium attitude using time-invariant continuous feedback, but again a discontinuous stabilizing feedback control strategy is constructed. In both cases, the discontinuous feedback controllers are constructed by switching between one of several feedback functions.

1. Introduction

We consider the attitude stabilization of a rigid spacecraft using control torques supplied by gas jet actuators about only two of its principal axes. A rigid spacecraft in general is controlled by three independent actuators about its principal axes. The situation considered here may arise due to the failure of one of the actuators. The linearization of the complete spacecraft dynamic equations at any equilibrium attitude has an uncontrollable eigenvalue at the origin. Consequently, controllability and stabilizability properties of the spacecraft cannot be inferred using classical linearization ideas and requires inherently nonlinear analysis. Moreover, a linear feedback control law cannot be used to asymptotically stabilize the spacecraft to an equilibrium attitude. An analysis of the controllability properties of a spacecraft with two independent control torques is made in [7]. In [7] it is shown that a necessary and sufficient condition for complete controllability of a spacecraft with control torques supplied by gas jet actuators about only two of its principal axes is that the uncontrolled principal axis must not be an axis of symmetry of the spacecraft. In [6], it is shown that a rigid spacecraft controlled by two pairs of gas jet actuators about its principal axes cannot be asymptotically stabilized to an equilibrium attitude using a time-invariant continuously differentiable, i.e., C^1, feedback control law. Moreover, using some of the theoretical results in [9] and [12], it also follows that there does not exist any time-invariant continuous feedback control law which asymptotically stabilizes the spacecraft to an equilibrium attitude. However a smooth C^1 feedback control law is derived in [6] which locally asymptotically stabilizes the spacecraft to a circular attractor, rather than an isolated equilibrium.

We first consider the case where the uncontrolled principal axis of the spacecraft is not an axis of symmetry. In this case, the complete spacecraft dynamics are small time locally controllable at any equilibrium attitude. However, as stated earlier, the spacecraft cannot be asymptotically stabilized to any equilibrium attitude using a time-invariant continuous feedback control law. Using local controllability results, an algorithm which locally asymptotically stabilizes the spacecraft to an isolated equilibrium is proposed in [7]. That algorithm is extremely complicated and is based on Lie algebraic methods in [8]. The algorithm yields a piecewise constant discontinuous control. Although very complicated, the algorithm is the only one proposed in the literature thus far which locally asymptotically stabilizes the spacecraft attitude to an equilibrium. In this paper a new discontinuous stabilizing feedback control strategy is constructed which stabilizes the spacecraft to an equilibrium attitude. The control strategy is simple and is based on physical considerations of the problem.

We next consider the case where the uncontrolled principal axis of the spacecraft is an axis of symmetry. In this case, the complete spacecraft dynamics are not even accessible. Under some rather weak assumptions, the spacecraft dynamic equations are strongly accessible and small time locally controllable at any equilibrium attitude in a reduced sense. The reduced spacecraft dynamics cannot be asymptotically stabilized to an equilibrium attitude using time-invariant continuous feedback. Nevertheless, a discontinuous feedback control strategy is constructed which achieves attitude stabilization of the spacecraft.

2. Kinematic and Dynamic Equations

The orientation of a rigid spacecraft can be specified using various parametrizations of the special orthogonal group $SO(3)$. Here we use the following Euler angle convention. Consider an inertial $X_1 X_2 X_3$ coordinate frame; let $x_1 x_2 x_3$ be a coordinate frame aligned with the principal axes of the spacecraft with origin at the center of mass of the spacecraft. If the two frames are initially coincident, a series of three rotations about the body axes, performed in the proper sequence, is sufficient to allow the spacecraft to reach any orientation. The three rotations are [14]:

a positive rotation of frame $X_1 X_2 X_3$ by an angle ψ about the X_3 axis; let $x_1' x_2' x_3'$ denote the resulting coordinate frame;

a positive rotation of frame $x_1' x_2' x_3'$ by an angle θ about the x_2' axis; let $x_1'' x_2'' x_3''$ denote the resulting frame;

a positive rotation of frame $x_1'' x_2'' x_3''$ by an angle ϕ about the x_1'' axis; let $x_1 x_2 x_3$ denote the final coordinate frame.

A rotation matrix $R(\psi,\theta,\phi)$ relates components of a vector in the inertial frame to components of the same vector in the body frame [14]. We assume that the Euler angles are limited to the ranges

$$-\pi < \psi, \phi < \pi, -\pi/2 < \theta < \pi/2 . \tag{2.1}$$

Suppose ω_1, ω_2, ω_3 are the principal axis components of the absolute angular velocity vector ω of the spacecraft. Then expressions for ω_1, ω_2, ω_3 are given by

$$\omega_1 = \dot{\phi} - \dot{\psi}\sin\theta , \tag{2.2}$$

$$\omega_2 = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta \sin\phi , \tag{2.3}$$

$$\omega_3 = -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta \cos\phi . \tag{2.4}$$

Since these equations are invertible, we can solve for $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ in terms of ω_1, ω_2, ω_3 obtaining

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi \tan\theta + \omega_3\cos\phi \tan\theta , \tag{2.5}$$

$$\dot{\theta} = \omega_2\cos\phi - \omega_3\sin\phi , \tag{2.6}$$

$$\dot{\psi} = \omega_2\sin\phi \sec\theta + \omega_3\cos\phi \sec\theta . \tag{2.7}$$

Next we consider the dynamic equations which describe the evolution of the angular velocity components of the spacecraft. Let $J = \text{diag}(J_1, J_2, J_3)$, $J_i > 0$, $i = 1, 2, 3$, be the inertia matrix of the spacecraft in a coordinate frame defined by its principal axes. Let H be the angular momentum vector of the spacecraft relative to the inertial frame. Then we have

$$J\omega = R(\psi,\theta,\phi)H . \tag{2.8}$$

Differentiating (2.8) we obtain

$$J\dot{\omega} = S(\omega)R(\psi,\theta,\phi)H + R(\psi,\theta,\phi)\dot{H} , \tag{2.9}$$

where

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}. \tag{2.10}$$

We assume that the control torques u'_1 and u'_2 are applied about axes represented by unit vectors b_1 and b_2 respectively. This implies that

$$R(\psi,\theta,\phi)\dot{H} = b_1 u'_1 + b_2 u'_2. \tag{2.11}$$

Without loss of generality, we assume that $b_1 = (1, 0, 0)^T$ and $b_2 = (0, 1, 0)^T$. Thus the equations describing the evolution of the angular velocity of the spacecraft are given by

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3 + u'_1, \tag{2.12}$$

$$J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1 + u'_2, \tag{2.13}$$

$$J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2. \tag{2.14}$$

3. Controllability and Stabilizability Properties of Complete Spacecraft Dynamics

As background for our subsequent development, we consider the controllability and stabilizability properties for the complete dynamics of the spacecraft with control torques only about two principal axes. Define

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \dfrac{u'_1}{J_1} \\ \dfrac{u'_2}{J_2} \end{bmatrix}.$$

From Section 2 the state equations can be rewritten as

$$\dot{\omega}_1 = a_1\omega_2\omega_3 + u_1, \tag{3.1}$$

$$\dot{\omega}_2 = a_2\omega_1\omega_3 + u_2, \tag{3.2}$$

$$\dot{\omega}_3 = a_3\omega_1\omega_2, \tag{3.3}$$

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi\,\tan\theta + \omega_3\cos\phi\,\tan\theta, \tag{3.4}$$

$$\dot{\theta} = \omega_2\cos\phi - \omega_3\sin\phi, \tag{3.5}$$

$$\dot{\psi} = \omega_2\sin\phi\,\sec\theta + \omega_3\cos\phi\,\sec\theta, \tag{3.6}$$

where

$$a_1 = \frac{J_2 - J_3}{J_1}, \ a_2 = \frac{J_3 - J_1}{J_2}, \ a_3 = \frac{J_1 - J_2}{J_3}.$$

It is easily verified that the linearization of the equations about an equilibrium has an uncontrollable eigenvalue at the origin. This implies that an inherently nonlinear analysis is necessary in order to characterize the controllability and stabilizability properties of the complete spacecraft dynamics. Moreover, a linear feedback control law cannot be used to asymptotically stabilize the spacecraft to an equilibrium attitude.

We now present fundamental results on the controllability and stabilizability properties of the complete spacecraft dynamics described by equations (3.1)-(3.6). The reader in referred to [13] for additional details.

Theorem 3.1: The complete spacecraft dynamics described by state equations (3.1)-(3.6) are strongly accessible if and only if $J_1 \ne J_2$, i.e., the uncontrolled principal axis is not an axis of symmetry.

Theorem 3.2: The complete spacecraft dynamics described by state equations (3.1)-(3.6) are small time locally controllable at any equilibrium if and only if $J_1 \ne J_2$.

Theorem 3.3: The complete spacecraft dynamics described by state equations (3.1)-(3.6) cannot be locally asymptotically stabilized to an equilibrium by any time-invariant continuous state feedback control law.

Theorem 3.3 holds if $J_1 \ne J_2$ and also if $J_1 = J_2$. A weaker version (with "continuous" replaced by "C^1") was proved in [6]. However, Theorem 3.3 follows from [6] using results in [9] and [12]. This negative result also implies that feedback control approaches based on linearization, Lyapunov methods, center manifold theory, or zero dynamics cannot be used to asymptotically stabilize the spacecraft to an equilibrium attitude.

Although the full set of equations (3.1)-(3.6) cannot be asymptotically stabilized to an equilibrium via continuous feedback, one may still wish to design a smooth control law which stabilizes at least a particular subset of state variables. Consider the state equations for ω_1, ω_2, ω_3, ϕ and θ given by equations (3.1)-(3.5). These equations are invariant with respect to the Euler angle ψ. Asymptotic stabilization of this subset of the original equations corresponds to stabilization of the motion of the spacecraft about an attractor, which is not an isolated equilibrium. A result from [6] shows that the closed loop trajectories can be asymptotically stabilized to the manifold

$$\Omega = \{(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) : \omega_1 = \omega_2 = \omega_3 = \phi = \theta = 0\} \tag{3.8}$$

using smooth C^1 feedback.

We mention that although the complete spacecraft dynamics described by equations (3.1)-(3.6) cannot be asymptotically stabilized to an equilibrium by continuous feedback, an algorithm generating a piecewise constant discontinuous control has been developed in [7] which locally asymptotically stabilizes the complete spacecraft dynamics to an equilibrium. The algorithm requires that $J_1 \ne J_2$, i.e., the uncontrolled principal axis must not be an axis of symmetry. The algorithm is based on Lie algebraic methods in [8]. The algorithm is extremely complicated and is not an easily implementable control strategy. However, stabilization of the complete spacecraft dynamic equations (3.1)-(3.6) is an inherently difficult problem and the algorithm in [7] is the only control strategy proposed in the literature thus far.

4. Attitude Stabilization of a Non-Axially Symmetric Spacecraft with Two Control Torques

In this section, we consider the equations (3.1)-(3.6) describing the motion of a spacecraft controlled by input torques only about two of its principal axes. It is assumed that the uncontrolled principal axis is not an axis of symmetry; i.e., $J_1 \ne J_2$. As a consequence of the negative result of Theorem 3.3, we restrict our study to the class of discontinuous feedback controllers in order to asymptotically stabilize the complete spacecraft dynamics. However, as shown in the previous section, the complete spacecraft dynamics are small time locally controllable at any equilibrium attitude. This suggests that a piecewise analytic feedback control law can be constructed which asymptotically stabilizes the complete spacecraft dynamics to an equilibrium attitude. Here we present a particular discontinuous feedback strategy, which is obtained by requiring that the spacecraft undergo a sequence of specified maneuvers. Without loss of generality, we assume that the equilibrium attitude to be stabilized is the origin. We first present a physical interpretation of the sequence of maneuvers that transfers any initial state to the origin.

Maneuvers 1-3. Transfer the initial state of the spacecraft to an equilibrium state in finite time; i.e., bring the spacecraft to rest.

There are control laws based on center manifold theory [1] and zero dynamics theory [6] which accomplish this in an asymptotic sense. Here we use a sequence of three maneuvers, and corresponding feedback control laws, which bring the spacecraft to rest in finite time.

Maneuver 4. Transfer the resulting state to an equilibrium state where $\phi = 0$ in finite time; i.e., so that the spacecraft is at rest with $\phi = 0$. This maneuver is accomplished using the control torque u_1 only.

Maneuver 5. Transfer the resulting state to an equilibrium state where $\phi = 0$, $\theta = 0$ in finite time; i.e., so that the spacecraft is at rest with $\phi = 0$, $\theta = 0$. This maneuver is accomplished using the control torque u_2 only.

In order to complete specification of the sequence of maneuvers, the Euler angle ψ must be brought to zero. This cannot be accomplished directly since a control torque cannot be applied about the third principal axis of the spacecraft. However, the resulting state can be transferred to the origin indirectly using three maneuvers. The three maneuvers correspond to three consecutive rotations about the two controlled principal axes of the spacecraft, the first and the third being around the first principal axis. This produces a net change in the orientation of the spacecraft so that the state of the spacecraft is transferred to the origin in finite time. The three maneuvers are described as follows.

Maneuver 6. Transfer the resulting state to an equilibrium state where $\phi = 0.5\pi$, $\theta = 0$ in finite time; i.e., so that the spacecraft is at rest with $\phi = 0.5\pi$, $\theta = 0$. This maneuver is accomplished using the control torque u_1 only.

Maneuver 7. Transfer the resulting state to the equilibrium state $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) = (0,0,0,0.5\pi,0,0)$ in finite time. This maneuver is accomplished using the control torque u_2 only.

Maneuver 8. Transfer the equilibrium state $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) = (0,0,0,0.5\pi,0,0)$ to the equilibrium state $(0,0,0,0,0,0)$ in finite time. This maneuver is accomplished using the control torque u_1 only.

Note that, excluding the first three maneuvers where the spacecraft is brought to rest, all subsequent maneuvers are such that the angular velocity component ω_3 is maintained identically zero. This is accomplished by carrying out maneuvers which require use of only a single control torque at a time.

It is convenient to introduce some notation. Throughout, assume $k > 0$, and define

$$G(x_1, x_2) = \begin{cases} k & \text{if } \{x_1 + \dfrac{x_2|x_2|}{2k} > 0\} \text{ or} \\ & \{x_1 + \dfrac{x_2|x_2|}{2k} = 0 \text{ and } x_2 > 0\} \\ -k & \text{if } \{x_1 + \dfrac{x_2|x_2|}{2k} < 0\} \text{ or} \\ & \{x_1 + \dfrac{x_2|x_2|}{2k} = 0 \text{ and } x_2 < 0\} \\ 0 & \text{if } \{x_1 = 0 \text{ and } x_2 = 0\} \end{cases}$$

We use the well-known property that the feedback control

$$u = - G(x_1 - \mathcal{X}_1, x_2)$$

for the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$

transfers any initial state to the final state $(\mathcal{X}_1, 0)$ in a finite time. We also use the standard notation that

$$\text{sign}(x_1) = \begin{cases} 1 & \text{if } x_1 > 0 \\ -1 & \text{if } x_1 < 0 \\ 0 & \text{if } x_1 = 0 \end{cases}.$$

Our mathematical construction of a control strategy which transfers an arbitrary initial state of the spacecraft to the origin is based on a sequence of equilibrium subsets and a sequence of control functions which transfers a state in one subset to another. Consider the following equilibrium subsets

$$M_1 = \{(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) = (0,0,0,\phi,\theta,\psi), \ \phi,\theta,\psi \text{ arbitrary}\},$$
$$M_2 = \{(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) = (0,0,0,0,\theta,\psi), \ \theta,\psi \text{ arbitrary}\},$$
$$M_3 = \{(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) = (0,0,0,0,0,\psi), \ \psi \text{ arbitrary}\},$$
$$M_4 = \{(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) = (0,0,0,0.5\pi,0,\psi), \ \psi \text{ arbitrary}\}.$$

We now present the feedback control laws that accomplish the sequential maneuvers described above; for each case we show that a desired terminal state which defines the maneuver is reached.

Transferring any initial state to a state in M_1

In order to transfer the arbitrary initial state to a final state which satisfies $\omega_1 = \omega_2 = \omega_3 = 0$ three sequential maneuvers are required. The first maneuver results in $\omega_1 = \omega_2 = 0$ while $\omega_3 \neq 0$ in general; the second maneuver results in $\omega_1 = \omega_1^*$ and $\omega_2 = \omega_2^*$, where ω_1^*, ω_2^* are chosen to guarantee that at the end of the third maneuver $\omega_1 = \omega_2 = \omega_3 = 0$. These three maneuvers are described in detail as follows.

Maneuver 1. Let $(\omega_1^0,\omega_2^0,\omega_3^0,\phi^0,\theta^0,\psi^0)$ denote an initial state for the complete spacecraft dynamics described by equations (3.1)-(3.6). Define

$$v_1 = a_1\omega_2\omega_3 + u_1,$$
$$v_2 = a_2\omega_3\omega_1 + u_2.$$

Equations (3.1)-(3.3) can now be rewritten as

$$\dot{\omega}_1 = v_1, \tag{4.1}$$
$$\dot{\omega}_2 = v_2, \tag{4.2}$$
$$\dot{\omega}_3 = a_3\omega_1\omega_2. \tag{4.3}$$

Apply the feedback control functions

$$v_1 = - k \, \text{sign}\omega_1,$$
$$v_2 = - k \, \text{sign}\omega_2.$$

It is easy to see that after a finite time given by $\max(\dfrac{|\omega_1^0|}{k}, \dfrac{|\omega_2^0|}{k})$, $\omega_1 = \omega_2 = 0$; at this instant let $\omega_3 = \bar{\omega}_3$ where the constant value $\bar{\omega}_3$ can be evaluated.

Maneuver 2. Apply the feedback control functions

$$v_1 = - k \, \text{sign}(\omega_1 - \omega_1^*),$$
$$v_2 = - k \, \text{sign}(\omega_2 - \omega_2^*),$$

where

$$\omega_1^* = \left[\frac{3k|\bar{\omega}_3|}{2|a_3|} \right]^{\frac{1}{3}}, \quad \omega_2^* = - \omega_1^* \, \text{sign}\bar{\omega}_3 \, \text{sign}a_3.$$

It is again easy to see that after a finite time given by $\dfrac{\omega_1^*}{k}$, $\omega_1 = \omega_1^*$, $\omega_2 = \omega_2^*$, and in addition it can be shown that $\omega_3 = \dfrac{\bar{\omega}_3}{2}$.

Maneuver 3. Apply the feedback control functions

$$v_1 = - k \, \text{sign}\omega_1,$$
$$v_2 = - k \, \text{sign}\omega_2.$$

It can be seen that after a finite time given by $\dfrac{\omega_1^*}{k}$, $\omega_1 = 0$, $\omega_2 = 0$ and it can be shown that $\omega_3 = 0$.

Consequently, the resulting state after these three sequential maneuvers is $(0,0,0,\phi^1,\theta^1,\psi^1) \in M_1$ for some ϕ^1, θ^1, ψ^1.

Transferring a state in M_1 to a state in M_2 (Maneuver 4)

Let $(0,0,0,\phi^1,\theta^1,\psi^1) \in M_1$ denote a state of the spacecraft. Apply the feedback control functions

$$u_1 = - G(\phi, \omega_1),$$
$$u_2 = 0.$$

It follows that

$$\omega_2 = 0, \ \omega_3 = 0,$$
$$\theta = \theta^1, \ \psi = \psi^1,$$

satisfy equations (3.2), (3.3), (3.5), (3.6) while equations (3.1), (3.4) become

$$\dot{\omega}_1 = - G(\phi, \omega_1),$$
$$\dot{\phi} = \omega_1.$$

Consequently, after a finite time $\omega_1 = 0$, $\phi = 0$; and thus a state $(0,0,0,\phi^1,\theta^1,\psi^1) \in M_1$ is transferred to the state $(0,0,0,0,\theta^1,\psi^1) \in M_2$ in finite time.

Transferring a state in M_2 to a state in M_3 (Maneuver 5)

Let $(0,0,0,0,\theta^1,\psi^1) \in M_2$ denote a state of the spacecraft. Apply the feedback control functions

$$u_1 = 0,$$
$$u_2 = - G(\theta, \omega_2).$$

It follows that

$$\omega_1 = 0, \ \omega_3 = 0,$$
$$\phi = 0, \ \psi = \psi^1,$$

satisfy equations (3.1), (3.3), (3.4), (3.6) while equations (3.2), (3.5) become

$$\dot{\omega}_2 = - G(\theta, \omega_2),$$
$$\dot{\theta} = \omega_2.$$

Consequently, after a finite time $\omega_2 = 0$, $\theta = 0$; and thus a state $(0,0,0,0,\theta^1,\psi^1) \in M_2$ is transferred to the state $(0,0,0,0,0,\psi^1) \in M_3$ in finite time.

Transferring a state in M_3 to a state in M_4 (Maneuver 6)

Let $(0,0,0,0,0,\psi^1) \in M_3$ denote a state of the spacecraft. Apply the feedback control functions

$$u_1 = -G(\phi - 0.5\pi, \omega_1),$$

$$u_2 = 0.$$

It follows that

$$\omega_2 = 0, \quad \omega_3 = 0,$$

$$\theta = 0, \quad \psi = \psi^1,$$

satisfy equations (3.2), (3.3), (3.5), (3.6) while equations (3.1), (3.4) become

$$\dot\omega_1 = -G(\phi - 0.5\pi, \omega_1),$$

$$\dot\phi = \omega_1.$$

Consequently, after a finite time $\omega_1 = 0$, $\phi = 0.5\pi$; and thus a state $(0,0,0,0,0,\psi^1) \in M_3$ is transferred to the state $(0,0,0,0.5\pi,0,\psi^1) \in M_4$ in finite time.

Transferring a state in M_4 to $(0,0,0,0.5\pi,0,0)$ (Maneuver 7)

Let $(0,0,0,0.5\pi,0,\psi^1) \in M_4$ denote a state of the spacecraft. Apply the feedback control functions

$$u_1 = 0,$$

$$u_2 = -G(\psi, \omega_2).$$

It follows that

$$\omega_1 = 0, \quad \omega_3 = 0,$$

$$\phi = 0.5\pi, \quad \theta = 0,$$

satisfy equations (3.1), (3.3), (3.4), (3.5) while equations (3.2), (3.6) become

$$\dot\omega_2 = -G(\psi, \omega_2),$$

$$\dot\psi = \omega_2.$$

Consequently, after a finite time $\omega_2 = 0$, $\psi = 0$; and thus a state $(0,0,0,0.5\pi,0,\psi^1) \in M_4$ is transferred to the state $(0,0,0,0.5\pi,0,0)$ in finite time.

Transferring $(0,0,0,0.5\pi,0,0)^T$ to $(0,0,0,0,0,0)$ (Maneuver 8)

Let $(0,0,0,0.5\pi,0,0)$ denote the state of the spacecraft. Apply the feedback control functions

$$u_1 = -G(\phi, \omega_1),$$

$$u_2 = 0.$$

It follows that

$$\omega_2 = 0, \quad \omega_3 = 0,$$

$$\theta = 0, \quad \psi = 0,$$

satisfy equations (3.2), (3.3), (3.5), (3.6) while equations (3.1), (3.4) become

$$\dot\omega_1 = -G(\phi, \omega_1),$$

$$\dot\phi = \omega_1.$$

Consequently, after a finite time $\omega_1 = 0$, $\phi = 0$; and thus the state $(0,0,0,0.5\pi,0,0)$ is transferred to the state $(0,0,0,0,0,0)$ in finite time.

In summary, the feedback control strategy outlined above can be implemented by sequential switching between the following feedback functions.

Maneuver 1. Apply

$$u_1^1(x) = -a_1\omega_2\omega_3 - k\,\mathrm{sign}\omega_1,$$

$$u_2^1(x) = -a_2\omega_3\omega_1 - k\,\mathrm{sign}\omega_2,$$

until $(\omega_1,\omega_2,\omega_3) = (0,0,\bar\omega_3)$ for some value $\bar\omega_3$; then go to Maneuver 2.

Maneuver 2. Compute

$$\omega_1^* = \left[\frac{3k|\bar\omega_3|}{2|a_3|}\right]^{\frac13}, \quad \omega_2^* = -\left[\frac{3k|\bar\omega_3|}{2|a_3|}\right]^{\frac13}\mathrm{sign}\bar\omega_3\,\mathrm{sign}a_3;$$

apply

$$u_1^2(x) = -a_1\omega_2\omega_3 - k\,\mathrm{sign}(\omega_1 - \omega_1^*),$$

$$u_2^2(x) = -a_1\omega_3\omega_1 - k\,\mathrm{sign}(\omega_2 - \omega_2^*),$$

until $(\omega_1,\omega_2,\omega_3) = (\omega_1^*,\omega_2^*,\dfrac{\bar\omega_3}{2})$; then go to Maneuver 3.

Maneuver 3. Apply

$$u_1^3(x) = -a_1\omega_2\omega_3 - k\,\mathrm{sign}\omega_1,$$

$$u_2^3(x) = -a_2\omega_3\omega_1 - k\,\mathrm{sign}\omega_2,$$

until $(\omega_1,\omega_2,\omega_3) = (0,0,0)$, i.e., $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) \in M_1$; then go to Maneuver 4.

Maneuver 4. Apply

$$u_1^4(x) = -G(\phi, \omega_1),$$

$$u_2^4(x) = 0,$$

until $(\omega_1,\omega_2,\omega_3,\phi) = (0,0,0,0)$, i.e., $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) \in M_2$; then go to Maneuver 5.

Maneuver 5. Apply

$$u_1^5(x) = 0,$$

$$u_2^5(x) = -G(\theta, \omega_2),$$

until $(\omega_1,\omega_2,\omega_3,\phi,\theta) = (0,0,0,0,0)$, i.e., $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) \in M_3$; then go to Maneuver 6.

Maneuver 6. Apply

$$u_1^6(x) = -G(\phi - 0.5\pi, \omega_1),$$

$$u_2^6(x) = 0,$$

until $(\omega_1,\omega_2,\omega_3,\phi,\theta) = (0,0,0,0.5\pi,0)$, i.e., $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) \in M_4$; then go to Maneuver 7.

Maneuver 7. Apply

$$u_1^7(x) = 0,$$

$$u_2^7(x) = -G(\psi, \omega_2),$$

until $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) = (0,0,0,0.5\pi,0,0)$; then go to Maneuver 8.

Maneuver 8. Apply

$$u_1^8(x) = -G(\phi, \omega_1),$$

$$u_2^8(x) = 0,$$

until $(\omega_1,\omega_2,\omega_3,\phi,\theta,\psi) = (0,0,0,0,0,0)$.

This feedback control strategy achieves attitude stabilization of the spacecraft by executing a sequence of maneuvers. This strategy is discontinuous and nonclassical in nature. Justification that it stabilizes the complete spacecraft dynamics to the equilibrium attitude (at the origin) in finite time, under the ideal model assumptions, follows as a consequence of the construction procedure. A computer implementation of the feedback control strategy can be easily carried out.

5. Attitude Stabilization of an Axially Symmetric Spacecraft with Two Control Torques

From the analysis made in Section 3, we find that the complete dynamics of a spacecraft controlled by two control torques supplied by gas jet actuators, as described by equations (3.1)-(3.6), fail to be controllable or even accessible if the uncontrolled principal axis is an axis of symmetry of the spacecraft, i.e., if $J_1 = J_2$. Due to the lack of controllability, the control algorithm proposed in [7] is not applicable to this case. In this section we concentrate on the case where the uncontrolled principal axis of the spacecraft is an axis of symmetry, i.e., $J_1 = J_2$. In particular we ask the question: what restricted control and stabilization properties of the spacecraft can be demonstrated in this case? Our analysis begins by demonstrating that, under appropriate restrictions of interest, the spacecraft equations can be expressed in a reduced form. Controllability and stabilizability properties for this case follow from an analysis of the reduced equations.

Consider the equations (3.1)-(3.6) describing the motion of a spacecraft controlled by input torques supplied by gas jet actuators about only two of its principal axes. It is assumed that the uncontrolled principal axis is an axis of symmetry of the spacecraft. From equations (3.1)-(3.6) and $J_1 = J_2$ we have

$$\dot{\omega}_1 = a_1\omega_2\omega_3 + u_1 , \qquad (5.1)$$

$$\dot{\omega}_2 = a_2\omega_1\omega_3 + u_2 , \qquad (5.2)$$

$$\dot{\omega}_3 = 0 , \qquad (5.3)$$

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi\,\tan\theta + \omega_3\cos\phi\,\tan\theta , \qquad (5.4)$$

$$\dot{\theta} = \omega_2\cos\phi - \omega_3\sin\phi , \qquad (5.5)$$

$$\dot{\psi} = \omega_2\sin\phi\,\sec\theta + \omega_3\cos\phi\,\sec\theta . \qquad (5.6)$$

If $\omega_3(0) \neq 0$ then ω_3 cannot be transferred to zero using any control function. If we assume that $\omega_3(0) = 0$, then $\omega_3 \equiv 0$. Under the restriction $\omega_3(0) = 0$, the reduced spacecraft dynamics for this case are described by

$$\dot{\omega}_1 = u_1 , \qquad (5.7)$$

$$\dot{\omega}_2 = u_2 , \qquad (5.8)$$

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi\,\tan\theta , \qquad (5.9)$$

$$\dot{\theta} = \omega_2\cos\phi , \qquad (5.10)$$

$$\dot{\psi} = \omega_2\sin\phi\,\sec\theta . \qquad (5.11)$$

The following results can be easily shown. The proofs of Theorem 5.1 and Theorem 5.2 are similar to the proofs of Theorem 3.1 and Theorem 3.2 respectively in [13]. Theorem 5.3 follows from the results in [5], [9] and [12].

Theorem 5.1: The reduced dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (5.7)-(5.11) are strongly accessible.

Theorem 5.2: The reduced dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (5.7)-(5.11) are small time locally controllable at any equilibrium.

Theorem 5.3: The reduced dynamics of an axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (5.7)-(5.11) cannot be asymptotically stabilized to an equilibrium using a time-invariant continuous feedback control law.

The implications of the properties stated above are as follows. For all initial conditions that satisfy $\omega_3(0) = 0$, the axially symmetric spacecraft controlled by two pairs of gas jet actuators as described by equations (5.1)-(5.6) can be controlled to any equilibrium attitude. However, any time-invariant feedback control law that asymptotically stabilizes the spacecraft to an isolated equilibrium attitude must necessarily be discontinuous. Thus arbitrary reorientation of the spacecraft can be achieved if $\omega_3(0) = 0$; if $\omega_3(0) \neq 0$, reorientation of the spacecraft to an equilibrium attitude cannot be achieved.

Conveniently, it turns out that sequential execution of the maneuvers defined as Maneuvers 3 through 8 in the previous section transfers any initial state of the reduced spacecraft dynamics (5.7)-(5.11) to the origin in finite time. The physical interpretation of the maneuvers is the same as described previously; the overall feedback control strategy is as follows.

Maneuver 1. Apply

$$u_1^1(x) = -k\,\mathrm{sign}\,\omega_1 ,$$

$$u_2^1(x) = -k\,\mathrm{sign}\,\omega_2 ,$$

until $(\omega_1,\omega_2) = (0,0)$; then go to Maneuver 2.

Maneuver 2: Apply

$$u_1^2(x) = -G(\phi, \omega_1) ,$$

$$u_2^2(x) = 0 ,$$

until $(\omega_1,\omega_2,\phi) = (0,0,0)$; then go to Maneuver 3.

Manuever 3: Apply

$$u_1^3(x) = 0 ,$$

$$u_2^3(x) = -G(\theta, \omega_2) ,$$

until $(\omega_1,\omega_2,\phi,\theta) = (0,0,0,0)$; then go to Maneuver 4.

Maneuver 4: Apply

$$u_1^4(x) = -G(\phi - 0.5\pi, \omega_1) ,$$

$$u_2^4(x) = 0 ,$$

until $(\omega_1,\omega_2,\phi,\theta) = (0,0,0.5\pi,0)$, then go to Maneuver 5.

Maneuver 5: Apply

$$u_1^5(x) = 0 ,$$

$$u_2^5(x) = -G(\psi, \omega_2) ,$$

until $(\omega_1,\omega_2,\phi,\theta,\psi) = (0,0,0.5\pi,0,0)$; then go to Maneuver 6.

Maneuver 6: Apply

$$u_1^6(x) = -G(\phi, \omega_1) ,$$

$$u_2^6(x) = 0 ,$$

until $(\omega_1,\omega_2,\phi,\theta,\psi) = (0,0,0,0,0,0)$.

This feedback control strategy achieves attitude stabilization of the spacecraft, in the sense described previously, by executing a sequence of maneuvers. This strategy is discontinuous and nonclassical in nature. A computer implementation of the feedback control strategy can be easily carried out.

Notice that according to equation (2.4), the condition that $\omega_3 = 0$ implies that

$$-(\sin\phi)d\,\theta + (\cos\theta\cos\phi)d\,\psi = 0 ;$$

this represents a nonintegrable invariant of the spacecraft motion. Therefore the reduced spacecraft dynamic equations define a nonlinear control system of the form studied in [4]. An alternate discontinuous control strategy which achieves attitude stabilization of the spacecraft is presented in [13].

6. Simulation

We illustrate the results of the paper with an example of a nonaxially symmetric spacecraft with principal moments of inertia $J_1 = 100\ Kg.\ M^2$, $J_2 = 250\ Kg.\ M^2$, and $J_3 = 350\ Kg.\ M^2$. There is no control torque about the third principal axis and two control torques, generated by gas jet actuators, are applied about the other two principal axes. The spacecraft has an initial orientation described by the Euler angles $\phi^0 = -\pi$, $\theta^0 = 0.25\pi$, and $\psi^0 = -0.5\pi$ radians, and an initial angular velocity given by $\omega_1^0 = 0.3$, $\omega_2^0 = -0.3$, and $\omega_3^0 = 0.1$ radians per second. A computer implementation of the feedback control strategy described in Section 4 was used to asymptotically stabilize the spacecraft to the origin. The value of k is chosen to be 1. Fig. 1, Fig. 2 and Fig. 3 show the time responses on the Euler angles, angular velocities and the control torques respectively. At $t = 0.3$ seconds, which is the end of Maneuver 1 of the algorithm, ω_1 and ω_2 are both zero while $\omega_3 = \overline{\omega}_3 = 0.1039$ radians per second. At $t = 1.73$ seconds, which is the end of Maneuver 3 of the algorithm, $\omega_1 = \omega_2 = \omega_3 = 0$, and $\phi = -2.59$, $\theta = 0.37$ and $\psi = -1.913$ radians. The subsequent maneuvers described by Steps 4 through 8 results in $\phi = \theta = \psi = \omega_1 = \omega_2 = \omega_3 = 0$ as shown in Fig. 1 and Fig. 2. It might be observed from Fig. 3 that until 1.73 seconds, which is the end of Maneuver 3, the control torques u_1 and u_2 are both applied to bring the spacecraft to rest. But once the spacecraft is brought to rest, the subsequent maneuvers are such that only one of the control torques is nonzero in any interval of time. Thus ω_3 remains zero at all time beyond 1.73 seconds, and ω_1 and ω_2 vary so that only one is nonzero at any time interval beyond 1.73 seconds. Since the feedback control strategy for the reorientation of an axially symmetric spacecraft is similar to the feedback control strategy for the reorientation of a non-axially symmetric spacecraft, we do not consider a seperate example to illustrate this case.

7. Conclusion

The attitude stabilization problem of a spacecraft using control torques supplied by gas jet actuators about only two of its principal axes has been considered. If the uncontrolled principal axis is not an axis of symmetry of the spacecraft, the complete spacecraft dynamics cannot be asymptotically stabilized to an equilibrium attitude using continuous feedback. A discontinuous feedback control strategy was constructed which stabilizes the spacecraft to an equilibrium attitude in finite time. If the uncontrolled principal axis is an axis of symmetry of the spacecraft, the

complete spacecraft dynamics cannot be stabilized. The reduced spacecraft dynamics cannot be asymptotically stabilized using continuous feedback, but again a discontinuous feedback control strategy was constructed which stabilizes the spacecraft (in the reduced sense) to an equilibrium attitude in finite time. The results of the paper show that although standard nonlinear control techniques do not apply, it is possible to construct a stabilizing control law by performing a sequence of maneuvers.

One of the advantages of the development in this paper is that feedback control strategies are constructed which guarantee attitude stabilization in a finite time. The total time required to complete the spacecraft reorientation is the sum of the times required to complete the sequence of maneuvers described. From the analysis provided, it should be clear that the time required to complete each maneuver depends on the single positive parameter k in the corresponding control law. There is a trade off between the required control levels, determined by the selection of k, and the resulting times to complete each of the maneuvers and hence the total time required to reorient the spacecraft. In particular, the time to reorient the spacecraft from a given initial state to the origin can be expressed as a function of the value of the parameter k and of the initial state.

For each of the two attitude stabilization problems considered, we have presented one example of a sequence of maneuvers which achieves the desired spacecraft attitude stabilization. There are many other maneuver sequences, and corresponding feedback control strategies, which will also achieve the desired attitude stabilization of the spacecraft. But each such strategy is necessarily discontinuous.

We have demonstrated the closed loop properties for the special feedback control strategies presented. Our analysis was based on a number of assumptions which are required to justify the mathematical models studied. Further robustness analysis is required to determine effects of model uncertainities and external disturbances. Unfortunately, such robustness analysis is quite difficult since the closed loop vector fields are necessarily discontinuous. Perhaps, feedback control strategies which stabilize the spacecraft attitude, different from ones presented in this paper, would provide improved closed loop robustness.

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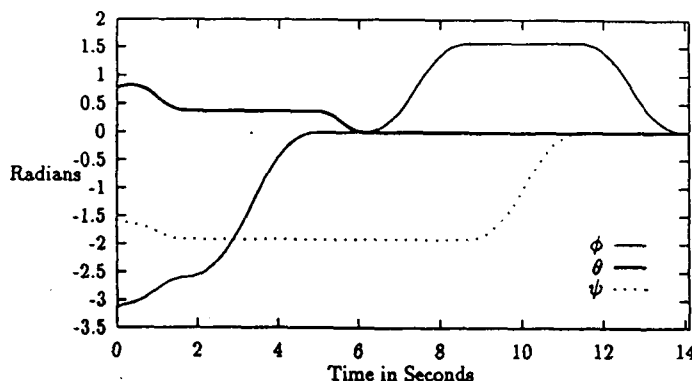[14] D. Greenwood, "Principles of Dynamics", Prentice Hall, New Jersey, Second Edition, 1988.
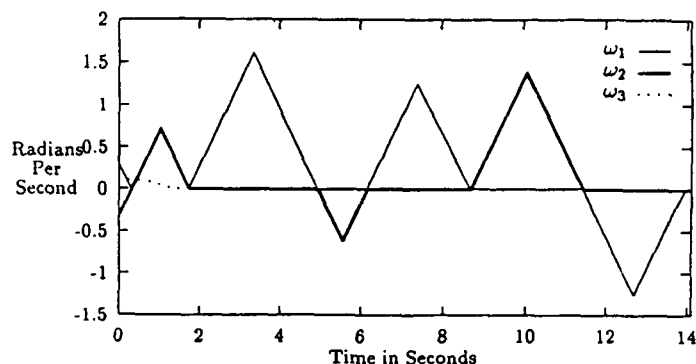
Figure 2: Plot of Angular Velocities.

Figure 1: Plot of Euler Angles.

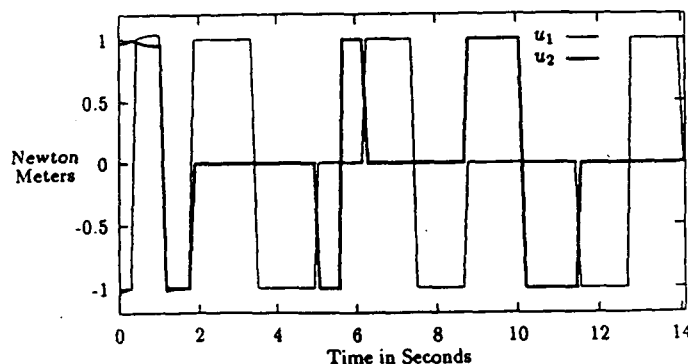Figure 3: Plot of u_1 and u_2.

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Planar reorientation of a free-free beam in space using embedded electromechanical actuators

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ABSTRACT

It is demonstrated that the planar reorientation of a free-free beam in zero gravity space can be accomplished by periodically changing the shape of the beam using embedded electromechanical actuators. The dynamics which determine the shape of the free-free beam is assumed to be characterized by the Euler-Bernoulli equation, including material damping, with apppropriate boundary conditions. The coupling between the rigid body motion and the flexible motion is explained using the angular momentum expression which includes rotatory inertia and kinematically exact effects. A control scheme is proposed where the embedded actuators excite the flexible motion of the beam so that it rotates in the desired sense with respect to a fixed inertial reference. Relations are derived which relate the average rotation rate to the amplitudes and the frequencies of the periodic actuation signal and the properties of the beam. These reorientation maneuvers can be implemented by using feedback control.

1. INTRODUCTION

Classical models of uniform free-free flexible beams in zero-gravity space result in complete decoupling of rigid body motion and flexible motion. However, conservation of the angular momentum of the beam indicates that the classical models are incomplete in the sense that there is in fact higher order nonlinear coupling between the rigid body motion and the flexible motion, if rotatory inertia and kinematically exact modeling effects are included. Assuming that the angular momentum of the beam is always zero, oscillations in the shape of the flexible beam can actually cause a rotation of the beam with respect to a fixed inertial reference. The rotation of the beam over one period depends only on the shape of the beam over the period and does not depend on the length of the period; hence this phenomenon is referred to as a geometric phase change.

These observations lead to a scheme for carrying out asymptotic reorientation of a free-free flexible beam in space using only electromechanical actuators embedded in the beam. These embedded electromechanical actuators, e.g. piezoelectric actuators, do not change the angular momentum of the free-free beam but they can be used to change the shape of the beam in a periodic way thereby resulting in a rotation of the beam in space. This reorientation scheme, based on the use of embedded actuators, does not require use of momentum wheels or gas jets and thus requires a minimal use of fuel to achieve a given beam reorientation.

In this paper, the basic modeling issues are addressed. The dynamics which characterize the shape of the free-free beam is assumed to be characterized by the Euler-Bernoulli equation, including material damping, with apppropriate boundary conditions. The coupling between the rigid body motion and the flexible motion is explained using the angular momentum expression. A control scheme is proposed where the embedded actuators excite the flexible motion of the beam so that it rotates in the desired sense. Relations are derived which relate the average rotation rate to the amplitudes and the frequencies of the periodic actuation signal and the properties of the beam. These reorientation maneuvers can be implemented by using feedback control. Important features of the approach are indicated.

2. A PLANAR FREE–FREE BEAM MODEL

Consider a uniform free–free beam of undeformed length $2L$ in space with zero angular momentum and zero linear momentum. Referring to Fig. 1 the motion of the beam is constrained to a plane defined by vectors $(\bar{e}_1, \bar{e}_3)$ where $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ is an orthonormal basis for an inertial frame whose origin is at the center of mass of the beam. Let $(\bar{i}, \bar{j}, \bar{k})$ be a rotating frame with its origin fixed at the origin of the inertial frame such that the vectors $(\bar{i}, \bar{k})$ lie in the plane $(\bar{e}_1, \bar{e}_3)$ and $\bar{j} = \bar{e}_2$. The straight line passing through the origin in the direction of vector $\bar{k}$ is called the reference line. Let the beam initially be at rest in a straight line configuration aligned with the reference line. Then, the location of each point on the line of mass centroids of the beam can be described in terms of the parameter $s \in [-L, L]$. This parameter s can be viewed as a label for each of the crossections. We assume that as the beam deforms the shape and the area of the crossections remain invariant. Following other researchers[1,2,3] we introduce three functions $u(s, t), y(s, t) : [-L, L] \times \Re \to \Re$ and $\psi(s, t) : [-L, L] \times \Re \to T^1$ such that $(u(s, t) + s, y(s, t))$ define the coordinates of the line of centroids in the deformed configuration with respect to the moving frame $(\bar{i}, \bar{k})$ at time t. The angle $\psi(s, t)$ between the normal to the crossection at s and $\bar{e}_3$ specifies the orientation of the crossection. The normal to the crossection at s is denoted by $\bar{t}_3$. We define the material basis $(\bar{t}_1, \bar{t}_2, \bar{t}_3)$ to be orthonormal so that $\bar{t}_1$ lies in the plane $(\bar{e}_1, \bar{e}_3)$. The crossection itself can be associated with the set of points (ξ_1, ξ_2) in a compact set $A \subset \Re^2$ such that $\xi_1 \bar{t}_1 + \xi_2 \bar{t}_2 + (u(s, t) + s)\bar{k} + (y(s, t))\bar{i}$ gives the location of any point on the beam as ξ_1 and ξ_2 vary through A and s varies from $-L$ to L.

Since the origin of the inertial frame is fixed at the center of mass of the beam we obtain

$$\int_{-L}^{L} y(s, t)ds = 0, \tag{1}$$

$$\int_{-L}^{L} u(s, t)ds = 0. \tag{2}$$

Let ρ denote the constant mass density per unit volume of the beam. We assume that the beam has a symmetric crossection so that the first moment of inertia of the crossection about the line of centroids is

$$\int_{A} \rho \xi_1 d\xi_1 d\xi_2 = 0. \tag{3}$$

The second moment of inertia of the crossection about the line of centroids is

$$I_2 = \int_{A} \rho \xi_1{}^2 d\xi_1 d\xi_2. \tag{4}$$

The mass per unit length of the crossection is given by

$$m_0 = \int_{A} \rho d\xi_1 d\xi_2. \tag{5}$$

We define the angle $\theta(t)$ between $\bar{e}_3$ and $\bar{k}$ so that $y(s, t)$ measured from the reference line satisfies the following orthogonality condition

$$\int_{-L}^{L} sy(s, t)ds = 0. \tag{6}$$

The existence of the angle $\theta(t)$ follows from the geometry indicated in Fig. 1. This definition provides a separation between the motion which determines the shape of the beam, given by $y(s, t)$, $-L \le s \le L$, and the rotation of the beam as a whole, given by $\theta(t)$.

3. EQUATIONS OF MOTION

We first develop a kinematically exact expression for the angular momentum of the free-free beam. Let $\bar{\varphi}(s, \xi_1, \xi_2, \theta, t)$ be the vector from the origin of the inertial frame to a point (s, ξ_1, ξ_2) on the beam at time t; then

$$\bar{\varphi} = (s\sin\theta + y\cos\theta + \xi_1\cos\psi + u\sin\theta)\bar{e}_1 + (\xi_2)\bar{e}_2 + (s\cos\theta - \xi_1\sin\psi - y\sin\theta + u\cos\theta)\bar{e}_3 \qquad (7)$$

where $\theta = \theta(t), y = y(s,t)$ and $\psi = \psi(s,t)$. The angular momentum about the origin of the inertial frame at time t is zero so that

$$\int_{-L}^{L}\int_A \rho\bar{\varphi} \times \frac{d\bar{\varphi}}{dt}d\xi_1 d\xi_2 ds = 0. \qquad (8)$$

Substituting equation (7) into equation (8) and using equations (4) and (5) we can express $\dot{\theta}$ in terms of y, u and α as

$$\dot{\theta} = \frac{\int_{-L}^{L}\{m_0 s\frac{\partial y}{\partial t} + I_2\dot{\alpha} + m_0(\frac{\partial y}{\partial t}u - \frac{\partial u}{\partial t}y)\}ds}{\int_{-L}^{L}\{-m_0 s^2 - m_0 y^2 - I_2\}ds} \qquad (9)$$

where $\alpha = \psi - \theta$ is the angle between the normal $\bar{t}_3$ to the crossection at s and the reference line.

Assume that the beam is unshearable and inextensible and that the deformations are small. This implies, using equation (2), that

$$u(s,t) = 0. \qquad (10)$$

and that

$$\alpha \approx y_s. \qquad (11)$$

We use the Euler-Bernoulli beam model to characterize the shape of the beam.[4,5] Thus $y(s,t)$ satisfies the Euler-Bernoulli equation of the form

$$m_0 y_{tt} + \gamma y_{tssss} + EI y_{ssss} = -\sum_{j=1}^{m} v_j(t)\delta'(s - s_j) \qquad (12)$$

with the boundary conditions

$$y_{ss}(-L) = y_{ss}(L) = 0, \qquad (13)$$

$$y_{sss}(-L) = y_{sss}(L) = 0 \qquad (14)$$

where $I = I_2/\rho$, E is Young's elasticity modulus, δ' is the distributional derivative of the delta function and where for simplicity we assume Kelvin-Voigt damping with a positive damping coefficient γ. In addition. $y(s,t)$ must satisfy conditions (1) and (6). Internal bending torques $v_j(t), j = 1,\ldots,m$ are produced by m point actuators located at $s = s_j$ on the beam where $s_j \in [-L, L]$. These embedded electromechanical actuators change the shape of the beam but at the same time preserve the angular momentum. Although such actuators are capable of inducing relatively small displacements one can excite the beam periodically at a frequency near one of the lower resonant frequencies of the beam to obtain relatively large periodic shape change.

Using expressions (6), (10) and (11) in equation (9) we obtain

$$\dot{\theta} = \frac{-\int_{-L}^{L} I_2 y_{ts}ds}{\tau + \int_{-L}^{L} m_0 y^2 ds} \qquad (15)$$

where $\tau = \frac{2}{3}m_0 L^3 + 2I_2 L$. This expression demonstrates the nonlinear coupling between the beam's shape and its rigid body motion. Expression (15) is non-integrable in the sense that if $y(s,t)$ is a periodic function of time, the integral of $\dot{\theta}$ over one period is, in general, non-zero.

Remark 3.1 If in the above derivation we had not used the kinematically exact expression for the angular momentum but had used the linearized strain assumptions we would have obtained the expression

$$\dot{\theta} = -\frac{1}{\tau} \int_{-L}^{L} I_2 y_{st} ds. \tag{16}$$

As can be seen expression (16) leads to the incorrect conclusion that a periodic change in the shape of the beam does not result in rotation of the beam. Note that inclusion of rotatory inertia effects and the use of the kinematically exact expression for the angular momentum is necessary in order to demonstrate that the beam can rotate in space due to periodic shape change.

We expand the solution $y(s,t)$ to equation (12) in the series

$$y(s,t) = \sum_{i=1}^{\infty} w_i(s) q_i(t) \tag{17}$$

where $w_i(s), i = 1, 2, \ldots$ are the orthonormal elastic mode shapes of the Euler-Bernoulli model. The elastic mode shapes are given by

$$w_i(s) = \begin{cases} \cos(\beta_i s) - \frac{\sin(\beta_i L)}{\sinh(\beta_i L)} \cosh(\beta_i s) & \text{if } i = 1,3,5\ldots \\ \sin(\beta_i s) + \frac{\cos(\beta_i L)}{\cosh(\beta_i L)} \sinh(\beta_i s) & \text{if } i = 2,4,6\ldots \end{cases}$$

where β_i are the positive roots of the equation

$$\cos(2\beta L) \cosh(2\beta L) = 1$$

ordered according to their magnitude.

Expansion (17) provides the modal decomposition

$$\ddot{q}_i + c_i \dot{q} + \omega_i{}^2 q = \sum_{j=1}^{m} b_{ij} v_j(t), i = 1,2\ldots \tag{18}$$

where $\omega_i{}^2 = \frac{EI\beta_i^4}{m_0}$, $c_i = \frac{\gamma \omega_i{}^2}{EI}$ and $b_{ij} = \frac{\partial w_i}{\partial s}|_{s=s_j}$. Equation (12), or equivalently equation (18), determines the shape of the beam and is called the shape space equation. Substituting equation (17) into equation (15) we obtain

$$\dot{\theta} = \frac{-I_2 \sum_{i=1}^{\infty}(J_i \dot{q}_i)}{\tau + \sum_{i=1}^{\infty} q_i{}^2} \tag{19}$$

where $J_i = w_i(L) - w_i(-L)$. We note that (19) is non-integrable for any truncation of the infinite series in (17).

4. ASYMPTOTIC REORIENTATION MANEUVERS

The goal is to accomplish asymptotic maneuvers, i.e. starting with $\theta(t_0) = \theta_0$, $y(s, t_0) = y_t(s, t_0) = 0$ we want to rotate the beam so that $\theta(t) \to \theta_d$, $y(s, t) \to 0$ and $y_t(s, t) \to 0$ as $t \to \infty$ for some desired angle θ_d.

Consider the periodic excitation of the beam at a single frequency ω as

$$v_j(t) = v_j^0 + v_j^\omega \cos(\omega t), j = 1, 2, \ldots, m \tag{20}$$

Since the shape space dynamics of the free-free beam is asymptotically stable, the steady-state motion of the beam is given by

$$q_i(t) = l_i + a_i \cos(\omega t + \phi_i) \tag{21}$$

where

$$l_i = \frac{1}{\omega_i^2} \sum_{j=1}^{m} b_{ij} v_j^0, \tag{22}$$

$$a_i = \frac{1}{\sqrt{(\omega_i - \omega^2)^2 + c_i^2 \omega_i^2}} \sum_{j=1}^{m} b_{ij} v_j^\omega, \tag{23}$$

and

$$\phi_i = -arctg\left(\frac{c_i \omega_i}{\omega_i^2 - \omega^2}\right). \tag{24}$$

The excitation function (20) should be sufficiently small so that the Euler–Bernoulli model for the shape space dynamics remains valid[6].

If $\sum_{i=1}^{\infty} q_i^2$ is small comparing with τ we can approximate

$$\frac{1}{\tau + \sum_{i=1}^{\infty} q_i^2} \approx \frac{1}{\tau}(1 - \frac{\sum_{i=1}^{\infty} q_i^2}{\tau})$$

and thus

$$\dot{\theta} \approx -\frac{1}{\tau} \sum_{i=1}^{\infty} J_i \dot{q}_i + \frac{1}{\tau^2} \left[\sum_{i=1}^{\infty} J_i \dot{q}_i\right] \left[\sum_{j=1}^{\infty} q_j^2\right].$$

Integrating over one period and using equation (21) we obtain

$$\theta(\frac{2\pi}{\omega}) - \theta(0) = \int_0^{\frac{2\pi}{\omega}} \frac{1}{\tau^2} \left[\sum_{i=1}^{\infty} J_i \dot{q}_i\right] \left[\sum_{j=1}^{\infty} q_j^2\right] dt$$

$$= \int_0^{\frac{2\pi}{\omega}} \frac{1}{\tau^2} \left[\sum_{i=1}^{\infty} -a_i J_i \omega \sin(\omega t + \phi_i)\right] \left[\sum_{j=1}^{\infty} (l_j + a_j \cos(\omega t + \phi_j))^2\right] dt$$

$$= \frac{2\pi}{\tau^2} \sum_{i=1}^{\infty} \sum_{j=1, j \neq i}^{\infty} a_i J_i l_j a_j \sin(\phi_j - \phi_i). \tag{25}$$

Expression (25) implies that, in general, the change in angle θ in steady-state over one period is non-zero thereby proving that a periodic change in shape of the beam results in a rotation of the beam; the steady-state difference $\theta(\frac{2\pi}{\omega}) - \theta(0)$ is referred to as the geometric phase. There are cases, however, when the

geometric phase turns out to be zero.

Proposition 4.1 Assume that the steady–state motion of the beam is described by equation (21). Then, $\theta(\frac{2\pi}{\omega}) - \theta(0) = 0$ if any of the following conditions hold:

1. $a_i = 0$ for all i

2. $l_i = 0$ for all i

3. $\phi_i = \phi_j$ for all i, j

The second statement of the proposition is the most important. It implies that for a non-zero geometric phase the beam should necessarily vibrate about a non-straight line reference configuration. It follows from expression (19) that following the motion $\bar{q}_i(t) = -l_i - a_i\cos(\omega t + \phi_i)$ yields a steady–state geometric phase change negative to that of (21). Therefore, in order to rotate the beam in the opposite direction it is sufficient to reverse the signs of v_j^ω and v_j^0.

Remark 4.1 Expression (25) can be used in order to predict the sign and the value of the geometric phase. Consider a beam which has a square crossection with side size R. Assume that two actuators at $s_1 = -rL$ and $s_2 = rL$ where $0 \le r \le 1$ produce torques according to equation (20). Using two first modes in the series (25) yields

$$\theta(\frac{2\pi}{\omega}) - \theta(0) \approx \frac{81.5661((v_1^\omega)^2 - (v_2^\omega)^2)(v_1^0 - v_2^0)\Phi(r)\sin(\phi_2 - \phi_1)}{\rho^2 L^3 E R^7(\frac{2}{3}(\frac{L}{R})^2 + \frac{1}{6})^2\sqrt{(\omega_1^2 - \omega^2)^2 + c_1^2\omega^2}\sqrt{(\omega_2^2 - \omega^2)^2 + c_2^2\omega^2}} \tag{26}$$

where

$$\Phi(r) = (\sin(2.36502r) + 0.1329\sinh(2.36502r))^2(\cos(3.9266r) - 0.0279\cosh(3.9266r)).$$

We are now in a position to formulate a specific control strategy to accomplish the asymptotic maneuver. Starting at rest with $\theta(t_0) = \theta_0$ application of control law (20) results in a nonzero rotation over a period. By repetition of cycles of motion (21) as many times as necessary the beam can be caused to rotate closer and closer to θ_d. As $\theta(t)$ approach θ_d we can reduce the amplitude of the oscillations to zero in a way so that $\theta(t) \to \theta_d$ as $t \to \infty$.

The proposed control law is of the form

$$v_j(t) = \varepsilon_k\left[\bar{v}_j^0 + \bar{v}_j^\omega\cos(\omega t)\right], j = 1,\ldots, m, \tag{27}$$

where $\frac{2(k-1)\pi p}{\omega} \le t - t_0 < \frac{2k\pi p}{\omega}$, $k = 1, 2, \ldots$; that is, the control excitation is an amplitude modulated function, where $\bar{v}_j^0$, $\bar{v}_j^\omega$ are constants and ε_k denotes the scalar amplitude modulation sequence that defines the control excitation on the k-th cycle. Each cycle is exactly p periods.

The constants ω, $\bar{v}_j^0$, $\bar{v}_j^\omega$ can be chosen nearly arbitrary, although one approach is to choose $\bar{v}_j^0$, $\bar{v}_j^\omega$ to maximize the geometric phase expression

$$\sum_{i=1}^\infty \sum_{j=1, j\ne i}^\infty a_i J_i l_j a_j \sin(\phi_j - \phi_i)$$

Since $|\varepsilon_k| \to 0$ then $q_i(t) \to 0$ and $\dot{q}_i \to 0$ as $t \to \infty$. By continuity $\theta(t) \to \theta^{con}$ for some constant θ^{con} as $t \to \infty$. We want to show that $\theta^{con} = \theta_d$.

By contradiction, assume that $\theta^{con} > \theta_d$. Let $\delta_3 > 0$ be sufficiently small so that $\theta^{con} - \delta_3 > \theta_d$. Choose ξ_3 so that

$$\left(\frac{\theta^{con} - \theta_d - \delta_3}{\Delta\theta^*} \right) > \xi_3^3 > 0.$$

Then, there exists an integer N_3 such that for any $k > N_3$ it follows that $|\varepsilon_k| < \xi_3$ and $|\theta_k^{ave} - \theta^{con}| < \delta_3$. Note that for any $k > N_3 + 1$ and $l > N_3 + 1$

$$r_k = \left(\frac{\theta_d - \theta_{k-1}^{ave}}{\Delta\theta^*} \right)^{\frac{1}{3}} < 0$$

and

$$|r_k| \geq \left(\frac{\theta^{con} - \delta_3 - \theta_d}{\Delta\theta^*} \right)^{\frac{1}{3}} > \xi_3 > |\varepsilon_l|.$$

Thus, we conclude from (A2) that for any $k, l > N_3 + 1$ it follows that $\varepsilon_k = \varepsilon_l \neq 0$. Hence, we obtain a contradiction to the convergence of the sequence ε_k to zero as $k \to \infty$. Similar arguments lead to a contradiction in case $\theta^{con} < \theta_d$. □

Finally, it follows from equations (28) and (23) that

$$\lim_{t \to \infty} \theta(t) = \theta_d, \lim_{t \to \infty} \left(\begin{array}{c} y(s,t) \\ y_t(s,t) \end{array} \right) = 0, -L \leq s \leq L$$

The controller which we have constructed has two functions. Its main function is to excite the oscillations of the beam in such a way so that the beam rotates in the desired sense. Subsequently, the controller serves to suppress the vibrations previously excited so that the free–free beam comes to rest with a desired orientation. Note that control law (27) is a non–smooth feedback control law.

5. NUMERICAL EXAMPLE

Space structures can often be modeled as light and flexible beams. Consider a beam with half-length $L = 1[m]$, density per unit volume $\rho = 1400[kg/m^3]$ and square crossection with the side size $R = 0.1[m]$. Young's modulus of the beam is $E = 3.0 \times 10^6[N/m^2]$ and the damping coefficient of Kelvin-Voigt damping is $\gamma = 0.2$. Two actuators are installed near both ends of the beam at $r = 0.9$. The maximal torque each of the actuators can produce is equal $100[Nm]$. The excitation frequency $\omega = 13[Hz]$ is selected to lie between the first $10.6[Hz]$ and the second $29[Hz]$ resonant frequencies of the beam; $\bar{v}_j^0$ and $\bar{v}_j^w$, $j = 1.2$ are chosen using expression (26) to maximize the geometric phase change over one period. For this example we choose $p = 5$ and $\gamma_1 = \gamma_2 = 0.9$. We want to rotate the beam from $\theta_0 = 0.1[rad]$ at $t = 0$ [sec] to $\theta_d = 0[rad]$. The dependence of the angle $\theta(t)$ [rad] on time $t[sec]$ is shown for a part of the maneuver in Fig. 2. In this case the geometric phase change over one period in steady–state predicted by expression (26) is equal to -2.7465×10^{-4} [rad] whereas its actual value is equal to -3.0411×10^{-4} [rad]. The dependence of the control parameter ε on time is shown in Figure 3.

where $a_i, l_i, \phi_i, i = 1, \ldots,$ are related to $\bar{v}_j^0, \bar{v}_j^\omega, j = 1, \ldots, m$ according to expressions (22)–(24), and $\bar{v}_j^0, \bar{v}_j^\omega$ are constrained by

$$\sum_{j=1}^m \left(\bar{v}_j^0\right)^2 \leq \alpha_c, \sum_{j=1}^m \left(\bar{v}_j^\omega\right)^2 \leq \beta_c.$$

In terms of $\bar{v}_j^0, \bar{v}_j^\omega, j = 1, \ldots, m$ this is a constrained mathematical programming problem which is linear in $\bar{v}_j^0$ (for fixed $\bar{v}_j^\omega$) and quadratic in $\bar{v}_j^\omega$ (for fixed $\bar{v}_j^0$). We will subsequently denote the maximum value of this constrained optimization problem as $\Delta\theta^*$.

The modulation sequence ε_{k+1} is defined in terms of an "average" of $\theta(t)$, over the k-th cycle, that is

$$\theta_k^{ave} = \frac{1}{2}\left(\max \theta(t) + \min \theta(t)\right) \tag{28}$$

where the maximum and minimum are over $\frac{2(k-1)\pi p}{\omega} \leq t - t_0 \leq \frac{2k\pi p}{\omega}$. We also introduce two auxilary variables $\theta_0^{ave} = \theta_0$ and $\varepsilon_0 = \text{sign}\left(\frac{\theta_d - \theta_0}{\Delta\theta^*}\right)$. We express ε_k in terms of θ_{k-1}^{ave} and ε_{k-1} as indicated below:

(A1) Compute

$$r_k = \left(\frac{\theta_d - \theta_{k-1}^{ave}}{\Delta\theta^*}\right)^{\frac{1}{3}}.$$

(A2) In case $|r_k| \geq |\varepsilon_{k-1}|$, if r_k and ε_{k-1} have the same signs then $\varepsilon_k = |\varepsilon_{k-1}|\text{sign}(r_k)$; if r_k and ε_{k-1} have opposite signs then $\varepsilon_k = \gamma_1|\varepsilon_{k-1}|\text{sign}(r_k)$, where $0 < \gamma_1 < 1$.

(A3) If $0 < |r_k| < |\varepsilon_{k-1}|$ then $\varepsilon_k = \gamma_2 r_k$, where $0 < \gamma_2 < 1$.

(A4) If $r_k = 0$ then $\varepsilon_k = \varepsilon_{k-1}$.

Proposition 4.2 If the proposed control law is of the form (27) where ε_k is selected according to steps **(A1)–(A4)**, then

$$\lim_{k \to \infty} \theta_k^{ave} = \theta_d, \lim_{k \to \infty} \varepsilon_k = 0.$$

Sketch of the Proof. The sequence $|\varepsilon_k|$ is non–increasing and bounded on $[0, 1]$. Therefore, there exists $b \in [0, 1]$ such that $b = \inf_k |\varepsilon_k|$. We want to show that $b = 0$.

By contradiction, assume $b \neq 0$. Then, for $\xi = \frac{b(1 - \max(\gamma_1, \gamma_2))}{2\max(\gamma_1, \gamma_2)}$ we can find an integer N_1 such that for all $k > N_1$ $|\varepsilon_k| - b < \xi$. From **(A2)** and **(A3)** we conclude that only two cases are possible: $\varepsilon_k = b$ for all $k > N_1$ or $\varepsilon_k = -b$ for all $k > N_1$.

Assume that the former case is true. Since the transient decays to zero and using continuity of θ with respect to q_i and $\dot{q}_i$ we assert that for $\xi_1 = \frac{1}{2}b^3\Delta\theta^*$ there exists an integer N_2 such that for any $k > N_2$

$$b^3\Delta\theta^* - \xi_1 < \theta_{k+1}^{ave} - \theta_k^{ave} < b^3\Delta\theta^* + \xi_1,$$

where $\Delta\theta^* > 0$. Note that $\frac{\theta_d - \theta_{N_2}^{ave}}{\Delta\theta^*} > b^3 > 0$. Choosing an integer l so that $l > 2\frac{\theta_d - \theta_{N_2}^{ave}}{b^3\Delta\theta^*} + 1$ we conclude that

$$0 > \frac{-(b^3\Delta\theta^* - \xi_1)l + \theta_d - \theta_{N_2}^{ave}}{\Delta\theta^*} > \frac{\theta_d - \theta_{N_2+l}^{ave}}{\Delta\theta^*} > b^3 > 0.$$

Therefore, the former case can never occur. Similarly, we can verify that the latter case also leads to a contradiction. Hence, $b = 0$.

6. CONCLUSION

In this paper the angular momentum expression for a planar free–free beam in space is derived. It is shown how the general motion of the beam can be separated into rigid and elastic motions. The change of shape of the beam is described by the Euler–Bernoulli equation with free-free boundary conditions. Angular momentum conservation leads to the nonlinear dependence of the rigid motion on the shape of the beam. As shown this dependence is non–integrable in the sense that a periodic change in shape of the beam results in a non-zero rotation of the beam over one period. Approximate relationships expressing the average rate of rotation of the beam in terms of the amplitudes and phases of periodic excitation of the beam by internal actuators are derived. Finally, a control strategy for a planar asymptotic reorientation maneuver is developed.

A general treatment of the interplay between deformations and rotations of deformable bodies is given by Shapere and Wilczek.[7] Reyhanoglu and McClamroch[8] have developed a framework for reorientation of multibody systems in space. In this paper, we have used the framework developed by Shapere and Wilczek for the specific problem of reorientation of a free-free beam in space; our results represent, in a certain sense, the limiting case of the multibody results obtained by Reyhanoglu and McClamroch when the number of bodies increases without limit.

Although our study in this paper has been concerned with the ideal case of reorientation of a free–free beam in space, we note that the same ideas are applicable to reorientation of a wide class of deformable space structures, using only actuators embedded into the structure. In this sense, smart structures technology can be used to accomplish a variety of efficient reorientation maneuvers for space structures.

7. ACKNOWLEDGMENTS

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Fig 1. Inertial, Moving and Material coordinate frames.

$\bar{t}_3$

$\bar{t}_1$

ψ

$\bar{e}_1$

$y(s)$

$u(s)+s$

$\bar{i}$

$\bar{k}$

$\bar{e}_3$

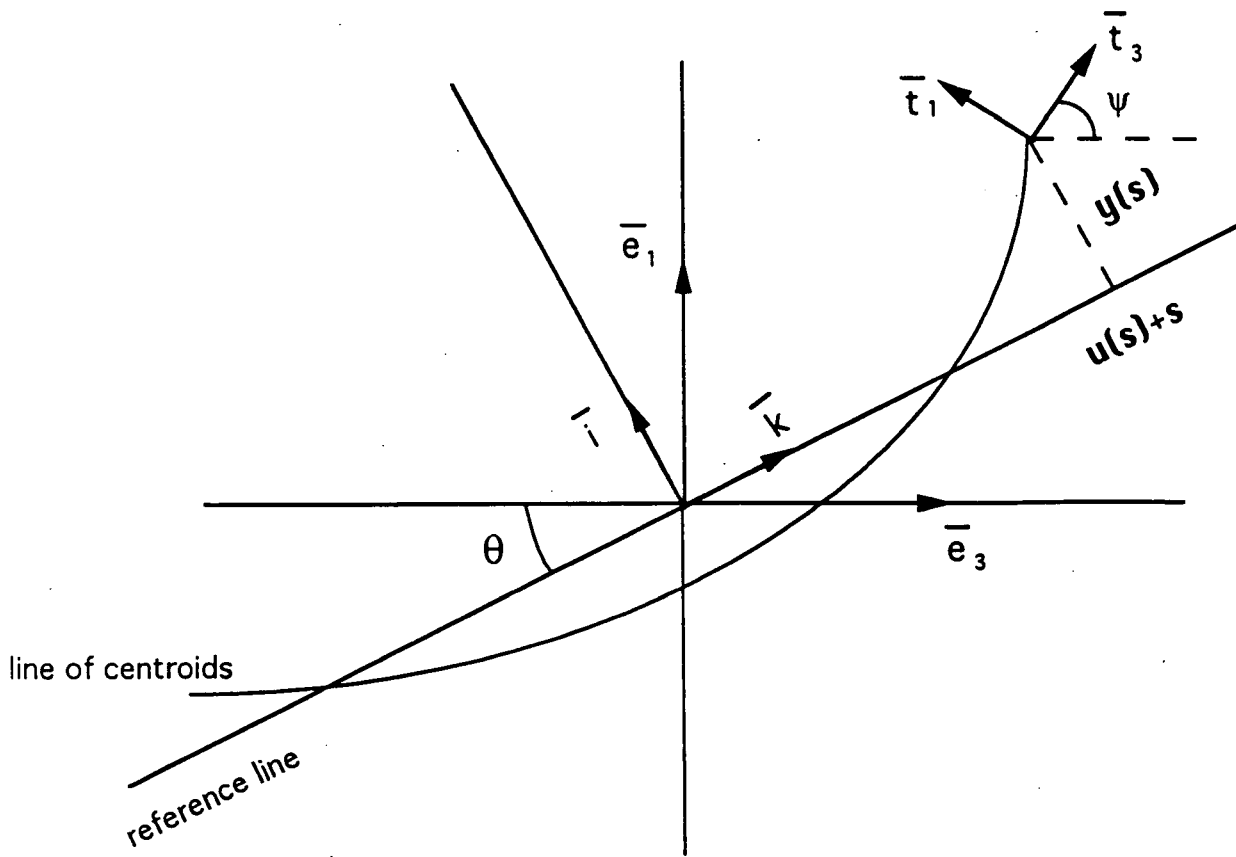θ

line of centroids

reference line

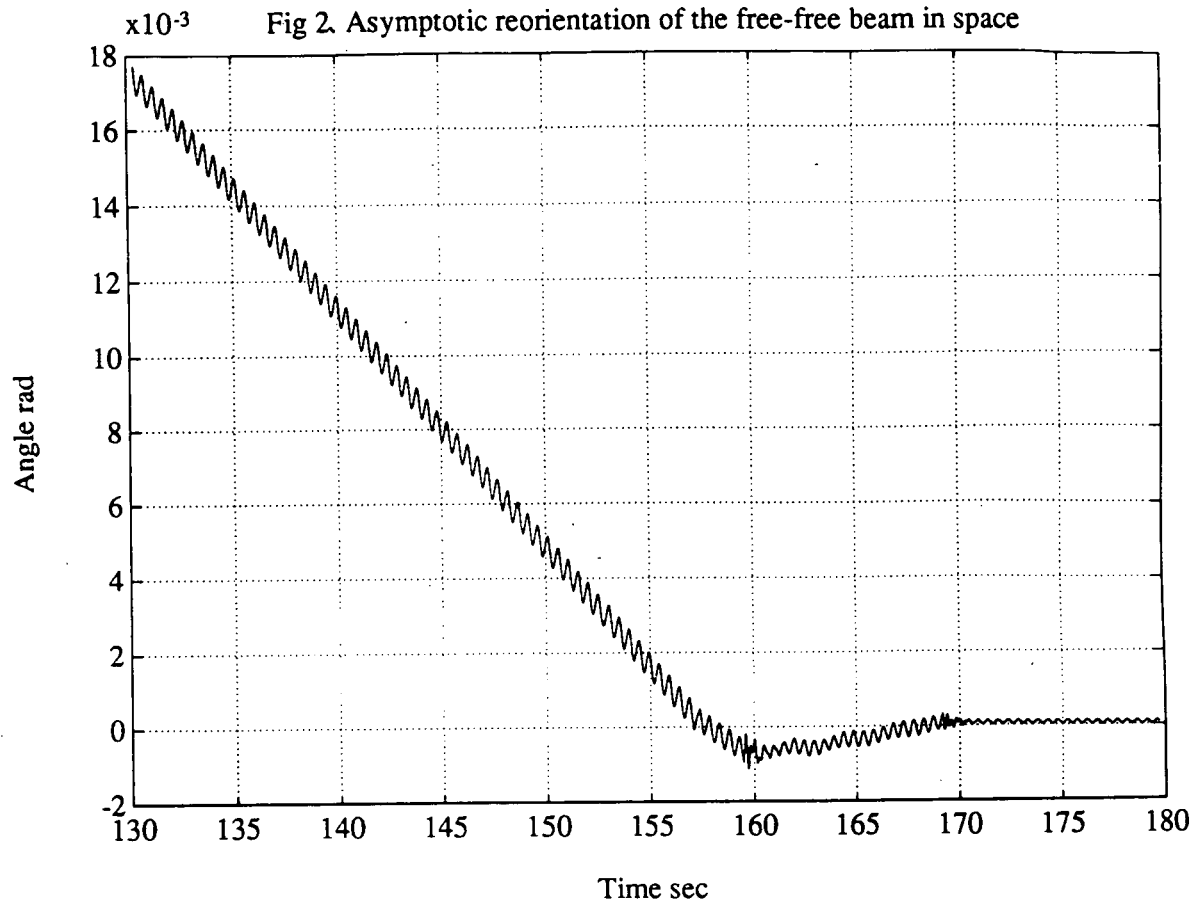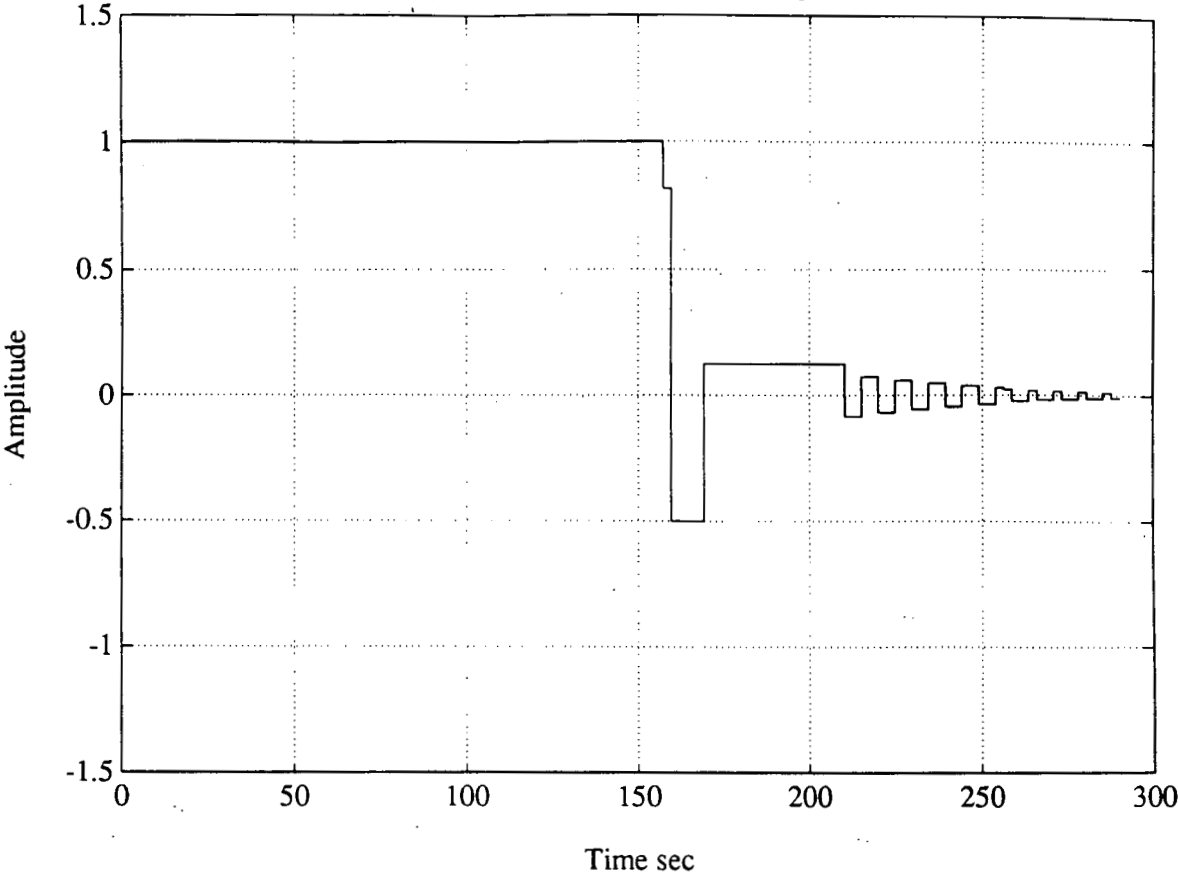Fig 2. Asymptotic reorientation of the free-free beam in space

Fig 3. Amplitude modulation sequence

EFFICIENT REORIENTATION OF A DEFORMABLE BODY IN SPACE: A FREE-FREE BEAM EXAMPLE

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Abstract

It is demonstrated that the planar reorientation of a free-free beam in zero gravity space can be accomplished by periodically changing the shape of the beam using internal actuators. A control scheme is proposed in which electromechanical actuators excite the flexible motion of the beam so that it rotates in the desired manner with respect to a fixed inertial reference. The results can be viewed as an extension of previous work to a distributed parameter case.

1. Introduction

Following [8], we introduce the concept of a deformable body, for which distances between the points of the body can change during the motion. Examples of deformable bodies include both lumped and distributed parameter systems such as multilink rigid body interconnections and structures with distributed flexibility. The orientation of a deformable body with respect to a fixed inertial reference can be specified by a choice of body frame. In general, there are many ways to choose a body frame. For example, in the case of planar motion a body frame can be identified with any two distinct points in the body. The shape of a deformable body can be specified in terms of the position of the body relative to the body frame. Thus, an arbitrary motion of a deformable body can be separated into rigid body motion and shape change.

Assume that both linear and angular momenta about the center of mass of the body are conserved and equal to zero. These conditions hold if the body is in a circular orbit around the Earth or is in a free fall. As a consequence of angular momentum conservation, shape change and the rigid body motion are coupled. This coupling is inherently nonlinear. In particular, one may be interested in inducing a rotation of a deformable body with respect to a fixed inertial reference by periodically changing the shape of the body with internal (momenta preserving) actuators. Reorientation strategies for lumped parameter mechanical systems have been extensively studied in the literature [4, 5, 7, 8]. Reorientation schemes based on the use of internal actuators require a minimal use of fuel to achieve the desired reorientation maneuver.

In this paper we extend the aforementioned reorientation strategies to the case of flexible bodies. In particular, we are interested in a planar reorientation of a free-free beam in space using only electromechanical actuators. These electromechanical actuators, e.g. piezoelectric or shape memory actuators, do not change the angular momentum of the free-free beam but can be used to change the shape of the beam in a periodic way. Assuming that the angular momentum of the beam is always zero, oscillations in the shape of the beam can cause a rotation of the beam with respect to a fixed inertial reference. The rotation of the beam over one period depends only on the shape of the beam over one period and does not depend on the length of the period; hence this phenomenon is referred to as a geometric phase change.

The extension of existing strategies to the free-free beam case is not straightforward for several reasons. Classical models of uniform free-free flexible beams in zero gravity space result in complete decoupling of rigid body motion and flexible motion. Higher order nonlinear coupling between rigid body motion and flexible motion is captured in geometrically exact beam theories [9]. The resulting models, however, are complicated. The free-free beam is an infinite dimensional superarticulated system. Thus, an arbitrary shape change cannot be produced by a finite number of actuators. In addition, the body frame of the beam needs to be chosen so that the shape change is independent of the rigid body motion. Such a choice of body frame is natural for lumped parameter systems since variables specifying orientation are ignorable.

In this paper, we first address basic modeling issues. The dynamics which determine the shape of the free-free beam are assumed to be characterized by the Euler-Bernoulli equation, including material damping, with appropriate boundary conditions. The higher order coupling between the rigid body motion and the flexible motion is captured using the angular momentum expression which includes rotatory inertia and kinematically exact effects. A control scheme is proposed in which the actuators excite the flexible motion of the beam so that the beam rotates in the desired sense.

2. A Planar Free-Free Beam Model

Consider a uniform free-free beam of undeformed length $2L$ in space with zero angular momentum and zero linear momentum. Referring to Fig. 1 the motion of the beam is constrained to a plane defined by vectors $(\bar{e}_1, \bar{e}_3)$ where $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ is an orthonormal basis for an inertial frame whose origin is at the center of mass of the beam. Let $(\bar{i}, \bar{j}, \bar{k})$ be a rotating

frame with its origin fixed at the origin of the inertial frame such that the vectors $(\bar{i}, \bar{k})$ lie in the plane $(\bar{e}_1, \bar{e}_3)$ and $\bar{j} = \bar{e}_2$. The straight line passing through the origin in the direction of vector $\bar{k}$ is called the reference line. Let the beam initially be at rest in a straight line configuration aligned with the reference line. Then, the location of each point on the line of mass centroids of the beam can be described in terms of the parameter $s \in [-L, L]$. This parameter s can be viewed as a label for each of the crossections. We assume that as the beam deforms the shape and the area of the crossections remain invariant. Following other researchers [1, 6, 9] we introduce three functions $u(s, t), y(s, t) : [-L, L] \times \Re \longrightarrow \Re$ and $\psi(s, t) : [-L, L] \times \Re \longrightarrow T^1$ such that $(u(s, t) + s, y(s, t))$ define the coordinates of the line of centroids in the deformed configuration with respect to the moving frame $(\bar{i}, \bar{k})$ at time t. The angle $\psi(s, t)$ between the normal to the crossection at s and $\bar{e}_3$ specifies the orientation of the crossection. The normal to the crossection at s is denoted by $\bar{t}_3$. We define the material basis $(\bar{t}_1, \bar{t}_2, \bar{t}_3)$ to be orthonormal so that $\bar{t}_1$ lies in the plane $(\bar{e}_1, \bar{e}_3)$. The crossection itself can be associated with the set of points (ξ_1, ξ_2) in a compact set $A \subset \Re^2$ such that $\xi_1 \bar{t}_1 + \xi_2 \bar{t}_2 + (u(s, t) + s)\bar{k} + (y(s, t))\bar{i}$ gives the location of any point on the beam as ξ_1 and ξ_2 vary through A and s varies from $-L$ to L.

Since the origin of the inertial frame is fixed at the center of mass of the beam we obtain

$$\int_{-L}^{L} y(s, t)ds = 0, \qquad (1)$$

$$\int_{-L}^{L} u(s, t)ds = 0. \qquad (2)$$

Let ρ denote the constant mass density per unit volume of the beam. We assume that the beam has a symmetric crossection so that the first moment of inertia of the crossection about the line of centroids is

$$\int_{A} \rho \xi_1 d\xi_1 d\xi_2 = 0. \qquad (3)$$

The second moment of inertia of the crossection about the line of centroids, referred to as the rotatory inertia, is

$$I_2 = \int_{A} \rho \xi_1^2 d\xi_1 d\xi_2. \qquad (4)$$

and assumed to be positive. The mass per unit length of the crossection is given by

$$m_0 = \int_{A} \rho d\xi_1 d\xi_2. \qquad (5)$$

We define the angle $\theta(t)$ between $\bar{e}_3$ and $\bar{k}$ so that $y(s, t)$ measured from the reference line satisfies the following orthogonality condition

$$\int_{-L}^{L} s y(s, t)ds = 0. \qquad (6)$$

The existence of the angle $\theta(t)$ follows from the geometry indicated in Fig. 1. This definition provides a

separation between the motion which determines the shape of the beam, given by $y(s, t)$, $-L \le s \le L$, and the rotation of the beam as a whole, given by $\theta(t)$.
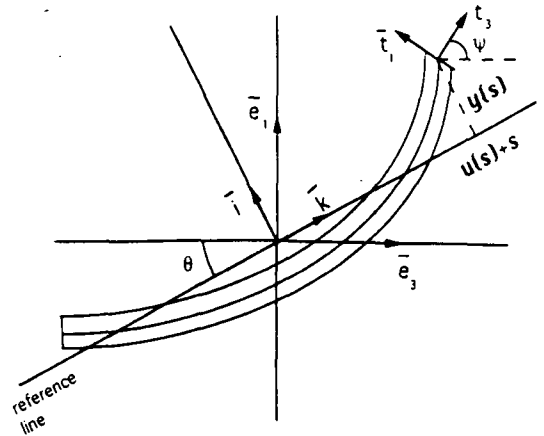

Fig 1. Planar Beam Model

We next develop a kinematically exact expression for the angular momentum of the free–free beam. Let $\bar{\varphi}(s, \xi_1, \xi_2, \theta, t)$ be the vector from the origin of the inertial frame to a point (s, ξ_1, ξ_2) on the beam at time t; then

$$\bar{\varphi} = (s \sin\theta + y\cos\theta + \xi_1\cos\psi + u\sin\theta)\bar{e}_1 +$$
$$(\xi_2)\bar{e}_2 + (s\cos\theta - \xi_1\sin\psi - y\sin\theta + u\cos\theta)\bar{e}_3 \quad (7)$$

where $\theta = \theta(t), y = y(s, t)$ and $\psi = \psi(s, t)$. The angular momentum about the origin of the inertial frame at time t is zero so that

$$\int_{-L}^{L} \int_{A} \rho \bar{\varphi} \times \frac{d\bar{\varphi}}{dt} d\xi_1 d\xi_2 ds = 0. \qquad (8)$$

Substituting equation (7) into equation (8) and using equations (4) and (5) we can express θ in terms of y, u and α as

$$\dot{\theta} = \frac{\int_{-L}^{L} \{m_0 s \frac{\partial y}{\partial t} + I_2 \dot{\alpha} + m_0(\frac{\partial y}{\partial t}u - \frac{\partial u}{\partial t}y)\}ds}{\int_{-L}^{L} \{-m_0 s^2 - m_0 y^2 - I_2\}ds} \qquad (9)$$

where $\alpha = \psi - \theta$ is the angle between the normal $\bar{t}_3$ to the crossection at s and the reference line.

Assume that the beam is unshearable and inextensible and that the deformations are small. This implies, using equation (2), that

$$u(s, t) = 0. \qquad (10)$$

and that

$$\alpha \approx y_s. \qquad (11)$$

We use the Euler–Bernoulli beam model to characterize the shape of the beam [3]. Thus $y(s, t)$ satisfies the Euler-Bernoulli equation of the form

$$m_0 y_{tt} + \gamma y_{tssss} + EI y_{ssss} = -\sum_{j=1}^{m} v_j(t)\delta'(s - s_j) \quad (12)$$

with the boundary conditions ·

$$y_{ss}(-L) = y_{ss}(L) = 0, \qquad (13)$$

$$y_{sss}(-L) = y_{sss}(L) = 0 \qquad (14)$$

where $I = I_2/\rho$, E is Young's elasticity modulus. δ' is the distributional derivative of the delta function and where for simplicity we assume Kelvin-Voigt damping with a positive damping coefficient γ. In addition, $y(s,t)$ must satisfy conditions (1) and (6). Internal bending torques $v_j(t)$, $j = 1, \ldots, m$ are produced by m point actuators located at $s = s_j$ on the beam where $s_j \in [-L, L]$. These actuators change the shape of the beam but at the same time preserve the angular momentum. Although such actuators are capable of inducing relatively small displacements one can excite the beam periodically at a frequency near one of the lower resonant frequencies of the beam to obtain relatively large periodic shape change.

Using expressions (6), (10) and (11) in equation (9) we obtain

$$\dot\theta = \frac{-\int_{-L}^{L} I_2 y_{ts}\, ds}{\tau + \int_{-L}^{L} m_0 y^2\, ds} \qquad (15)$$

where $\tau = \frac{2}{3} m_0 L^3 + 2I_2 L$. This expression demonstrates the nonlinear coupling between the beam's shape and its rigid body motion. Expression (15) is non-integrable in the sense that if $y(s,t)$ is a periodic function of time, the integral of $\dot\theta$ over one period is, in general, non-zero.

We can expand the solution $y(s,t)$ to equation (12) in the series

$$y(s,t) = \sum_{i=1}^{\infty} w_i(s) q_i(t) \qquad (16)$$

where $w_i(s)$, $i = 1, 2, \ldots$ are the orthonormal *elastic* mode shapes of the Euler-Bernoulli model. The solution $y(s,t)$ satisfies equations (1) and (6), which can be viewed as orthogonality conditions for the rigid body modes and elastic modes. Expansion (16) provides the modal description

$$\ddot{q}_i + c_i \dot{q} + \omega_i^2 q = \sum_{j=1}^{m} b_{ij} v_j(t), i = 1, 2 \ldots \qquad (17)$$

Equation (12), or equivalently equation (17), determines the shape of the beam and is called the shape space equation. Substituting equation (16) into equation (15) we obtain

$$\dot\theta = \frac{-I_2 \sum_{i=1}^{\infty} (J_i \dot{q}_i)}{\tau + \sum_{i=1}^{\infty} q_i^2} \qquad (18)$$

where $J_i = w_i(L) - w_i(-L)$. We note that (18) is, in general, non-integrable for any truncation of the infinite series in (16).

3. Asymptotic Reorientation Maneuvers

The goal is to accomplish asymptotic maneuvers, i.e. starting with $\theta(t_0) = \theta_0$, $y(s, t_0) = y_t(s, t_0) = 0$ we want to rotate the beam so that $\theta(t) \to \theta_d$, $y(s,t) \to 0$ and $y_t(s,t) \to 0$ as $t \to \infty$ for some desired angle θ_d.

Consider the periodic excitation of the beam at a single frequency ω as

$$v_j(t) = v_j^0 + v_j^\omega \cos(\omega t), j = 1, 2, \ldots, m \qquad (19)$$

Since the shape space dynamics of the free–free beam is asymptotically stable, the steady-state motion of the beam is given by

$$q_i(t) = l_i + a_i \cos(\omega t + \phi_i) \qquad (20)$$

where the parameters l_i, a_i and ϕ_i can be expressed in terms of v_j^ω and v_j^0 according to equation (17). The excitation function (19) should be sufficiently small so that the Euler–Bernoulli model for the shape space dynamics remains valid. Substituting equation (20) into equation (18) and integrating over one period we obtain the steady–state change in angle θ over one period is given by

$$\int_0^{\frac{2\pi}{\omega}} \frac{\varsigma_0 \cos(\omega t + \chi_0) dt}{1 + \varsigma_1 (\cos(\omega t + \chi_1) + \varsigma_2 \cos(2\omega t + \chi_2)} \qquad (21)$$

for constants ς_0, ς_1, ς_2, χ_0, χ_1 and χ_2. Expression (21) implies that, in general, the change in angle θ in steady–state over one period is non-zero, thereby proving that a periodic change in shape of the beam results in a rotation of the beam. The steady-state difference $\theta(\frac{2\pi}{\omega}) - \theta(0)$ is referred to as the geometric phase. If $\sum_{i=1}^{\infty} q_i^2$ is small as compared with τ, we can approximate

$$\frac{1}{\tau + \sum_{i=1}^{\infty} q_i^2} \approx \frac{1}{\tau}(1 - \frac{\sum_{i=1}^{\infty} q_i^2}{\tau})$$

and thus using equation (20) we obtain

$$\theta(\frac{2\pi}{\omega}) - \theta(0) = \frac{2\pi}{\tau^2} \sum_{i=1}^{\infty} \sum_{j=1, j \neq i}^{\infty} a_i J_i l_j a_j \sin(\phi_j - \phi_i).$$

$$(22)$$

Although the geometric phase is generally non-zero, there are cases when the geometric phase is zero.

Proposition 4.1 Assume that the steady–state motion of the beam is described by equation (20). Then $\theta(\frac{2\pi}{\omega}) - \theta(0) = 0$ if any of the following conditions hold:

1. $a_i = 0$ for all i

2. $l_i = 0$ for all i

3. $\phi_i = \phi_j$ for all i, j

The second statement of the proposition is the most important. It implies that for a non-zero geometric phase the beam should necessarily vibrate about a

non-straight line reference configuration. It follows from expression (18) that in order to rotate the beam in the opposite direction it is sufficient to reverse the signs of v_j^ω and v_j^0.

We are now in a position to formulate a specific control strategy to accomplish the desired asymptotic maneuver. Starting at rest with $\theta(t_0) = \theta_0$ application of control law (19) results in a nonzero geometric phase change over one period. By repetition of cycles of motion as many times as necessary the beam can be caused to rotate closer and closer to θ_d. As $\theta(t)$ approach θ_d we can reduce the amplitude of the oscillations to zero in a way so that $\theta(t) \to \theta_d$ as $t \to \infty$.

The proposed control law is of the form

$$v_j(t) = \varepsilon_k \left[\bar{v}_j^0 + \bar{v}_j^\omega \cos(\omega t) \right], j = 1, \ldots, m. \quad (23)$$

where $\frac{2(k-1)\pi p}{\omega} \le t - t_0 < \frac{2k\pi p}{\omega}$, $k = 1, 2, \ldots$; that is, the control excitation is an amplitude modulated function, where $\bar{v}_j^0$, $\bar{v}_j^\omega$ are constants and ε_k denotes the scalar amplitude modulation sequence that defines the control excitation on the k-th cycle. Each cycle is exactly p periods. The constants ω, $\bar{v}_j^0$, $\bar{v}_j^\omega$ can be chosen nearly arbitrary, although one approach is to choose $\bar{v}_j^0$, $\bar{v}_j^\omega$ to maximize geometric phase expression (22) where $a_i, l_i, \phi_i, i = 1, \ldots,$ are related to $\bar{v}_j^0$, $\bar{v}_j^\omega, j = 1, \ldots, m$ according to expressions (20) and (17), and $\bar{v}_j^0$, $\bar{v}_j^\omega$ are constrained in norm. In terms of $\bar{v}_j^0$, $\bar{v}_j^\omega, j = 1, \ldots, m$ this is a constrained mathematical programming problem which is linear in $\bar{v}_j^0$ (for fixed $\bar{v}_j^\omega$) and quadratic in $\bar{v}_j^\omega$ (for fixed $\bar{v}_j^0$). We will subsequently denote the maximum value of this constrained optimization problem as $\Delta\theta^*$.

The modulation sequence ε_{k+1} is defined in terms of an average of $\theta(t)$, over the k-th cycle, that is

$$\theta_k^{ave} = \frac{1}{2} \left(\max \theta(t) + \min \theta(t) \right) \quad (24)$$

where the maximum and minimum are over $\frac{2(k-1)\pi p}{\omega} \le t - t_0 \le \frac{2k\pi p}{\omega}$. We also introduce two auxilary variables $\theta_0^{ave} = \theta_0$ and $\varepsilon_0 = \text{sign}\left(\frac{\theta_d - \theta_0}{\Delta\theta^*}\right)$. We express ε_k in terms of θ_{k-1}^{ave} and ε_{k-1} as indicated below:

(A1) Compute

$$r_k = \left(\frac{\theta_d - \theta_{k-1}^{ave}}{\Delta\theta^*} \right)^{\frac{1}{3}}.$$

(A2) In case $|r_k| \ge |\varepsilon_{k-1}|$, if r_k and ε_{k-1} have the same signs then $\varepsilon_k = |\varepsilon_{k-1}|\text{sign}(r_k)$; if r_k and ε_{k-1} have opposite signs then $\varepsilon_k = \gamma_1|\varepsilon_{k-1}|\text{sign}(r_k)$, where $0 < \gamma_1 < 1$.

(A3) If $0 < |r_k| < |\varepsilon_{k-1}|$ then $\varepsilon_k = \gamma_2 r_k$, where $0 < \gamma_2 < 1$.

(A4) If $r_k = 0$ then $\varepsilon_k = \varepsilon_{k-1}$.

Proposition 4.2 If the proposed control law is of the form (23) where ε_k is selected according to steps (A1)–(A4), then

$$\lim_{k \to \infty} \theta_k^{ave} = \theta_d, \lim_{k \to \infty} \varepsilon_k = 0.$$

Sketch of the Proof. The sequence $|\varepsilon_k|$ is non-increasing and bounded on $[0, 1]$. Therefore, there exists $b \in [0, 1]$ such that $b = \inf_k |\varepsilon_k|$. It can be shown that by construction of the sequence b must be zero.

Since $|\varepsilon_k| \to 0$ then $q_i(t) \to 0$ and $\dot{q}_i \to 0$ as $t \to \infty$. By continuity $\theta(t) \to \theta^{con}$ for some constant θ^{con} as $t \to \infty$. It can be shown that $\theta^{con} = \theta_d$. □

Finally, it follows from equations (24) and (20) that

$$\lim_{t \to \infty} \theta(t) = \theta_d, \lim_{t \to \infty} \begin{pmatrix} y(s, t) \\ y_t(s, t) \end{pmatrix} = 0, -L \le s \le L$$

The controller which we have constructed has two functions. Its main function is to excite the oscillations of the beam in such a way that the beam rotates in the desired sense. Subsequently, the controller serves to suppress the vibrations previously excited so that the free–free beam comes to rest with a desired orientation. Note that control law (23) is a non–smooth feedback control law [2].

4. Numerical Example

Consider a beam with half–length $L = 1[m]$, density per unit volume $\rho = 1400[kg/m^3]$ and square crossection with the side size $R = 0.1[m]$. Young's modulus of the beam is $E = 3.0 \times 10^6[N/m^2]$ and the Kelvin–Voigt damping coefficient is $\gamma = 0.2$. Two actuators are installed near both ends of the beam at $s_1 = -0.9[m]$ and $s_2 = 0.9[m]$. The maximal torque each of the actuators can produce is equal to $100[Nm]$. The excitation frequency $\omega = 13[Hz]$ is selected to lie between the first $10.6[Hz]$ and the second $29[Hz]$ resonant frequencies of the beam; $\bar{v}_j^0$ and $\bar{v}_j^\omega$, $j = 1, 2$ are chosen using expression (22) to maximize the geometric phase change over one period. For this example we choose $p = 5$ and $\gamma_1 = \gamma_2 = 0.9$. The first four elastic modes of the beam are used in our simulation.

We want to rotate the beam from $\theta_0 = 0.1[rad]$ at $t = 0[sec]$ to $\theta_d = 0[rad]$. The dependence of the angle $\theta(t)[rad]$ on time $t[sec]$ is shown for a part of the maneuver in Fig. 2. In this case the geometric phase change over one period in steady–state predicted by expression (22) is equal to -2.7465×10^{-4} $[rad]$ whereas its actual simulation value is equal to -3.0411×10^{-4} $[rad]$. The dependence of the modulation parameter ε on time is shown in Figure 3.
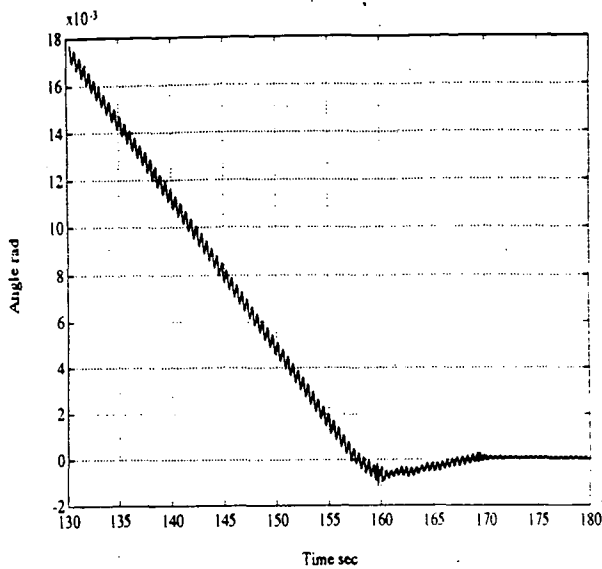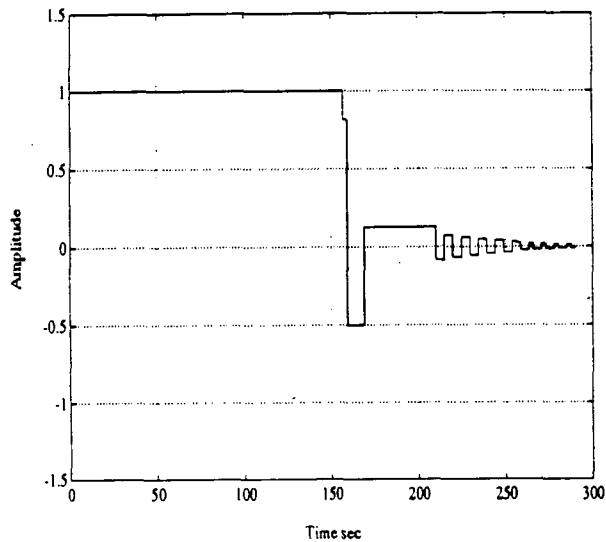
Fig 2. Asymptotic Reorientation Maneuver

Fig 3. Amplitude Modulation Sequence

5. Conclusion

In this paper the angular momentum expression for a planar free-free beam in space is derived. It is shown how the general motion of the beam can be separated into rigid and elastic motions. The change of shape of the beam is described by the Euler–Bernoulli equation with free-free boundary conditions. Angular momentum conservation leads to the nonlinear dependence of the rigid motion on the shape of the beam. As shown this dependence is non-integrable in the sense that a periodic change in shape of the beam results in a non-zero rotation of the beam over one period. Approximate relationships expressing the average rate of rotation of the beam in terms of the amplitudes and phases of periodic excitation of the

beam by internal actuators are derived. Finally, a control strategy for a planar asymptotic reorientation maneuver is developed.

A general treatment of the interplay between deformations and rotations of deformable bodies is given by Shapere and Wilczek [8]. Reyhanoglu and McClamroch [7] have developed a framework for reorientation of multibody systems in space. In this paper, we have used the framework developed by Shapere and Wilczek for the specific problem of reorientation of a free-free beam in space; our results represent, in a certain sense, the limiting case of the multibody results obtained by Reyhanoglu and McClamroch when the number of bodies increases without limit.

Although our study in this paper has been concerned with the ideal case of reorientation of a free-free beam in space, we note that the same ideas are applicable to reorientation of a wide class of deformable space structures, using only actuators embedded into the structure. In this sense, smart structures technology can be used to accomplish a variety of efficient reorientation maneuvers for space structures.

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ATTITUDE STABILIZATION OF A RIGID SPACECRAFT USING MOMENTUM WHEEL ACTUATORS OPERATING IN A FAILURE MODE

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ATTITUDE STABILIZATION OF A RIGID SPACECRAFT USING MOMENTUM WHEEL ACTUATORS OPERATING IN A FAILURE MODE

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Abstract

It is well known that three momentum wheel actuators can be used to control the attitude of a rigid spacecraft and that arbitrary reorientation maneuvers of the spacecraft can be accomplished. If failure of one of the momentum wheel actuators occurs, it is commonly thought that attitude control is not possible using only two momentum wheel actuators. In this paper, we demonstrate that, in fact, two momentum wheel actuators can be used to control the attitude of a rigid spacecraft and that arbitrary reorientation maneuvers of the spacecraft can be accomplished in a specific sense. The complete spacecraft dynamics cannot be stabilized to an equilibrium attitude. However, the spacecraft equations are small time locally controllable in a reduced nonlinear sense. The reduced spacecraft dynamics cannot be asymptotically stabilized to an equilibrium attitude using a time-invariant continuous feedback control law, but a discontinuous feedback control strategy is constructed which asymptotically stabilizes an equilibrium attitude of the spacecraft. Consequently, arbitrary reorientation of the spacecraft can be accomplished using this discontinuous feedback control strategy.

1. Introduction

The attitude stabilization problem of a rigid spacecraft using only two control torques supplied by momentum wheel actuators is revisited. Although a rigid spacecraft in general is controlled by three independent actuators, the situation considered in this paper may arise due to the failure of one of the actuators of the spacecraft. Since we are considering a space-based system, the problem considered here, namely, the attitude stabilization of a spacecraft operating in an actuator failure mode, is an important control problem.

In this paper, we consider the attitude stabilization of a spacecraft using control torques supplied by two momentum wheel actuators about axes spanning a two dimensional plane orthogonal to a principal axis. The linearization of the complete spacecraft dynamic equations at any equilibrium attitude has an uncontrollable eigenvalue at the origin. Consequently, controllability and stabilizability properties of the spacecraft cannot be inferred using classical linearization ideas. Moreover, a linear feedback control law cannot be used to asymptotically stabilize the spacecraft to an equilibrium attitude. It is shown that the complete spacecraft dynamics controlled by two momentum wheel actuators is not controllable at any equilibrium attitude. Thus any equilibrium attitude of the complete spacecraft dynamics is not stabilizable. Under some rather weak assumptions, the spacecraft dynamic equations are shown to have a nonintegrable motion invariant, so that they fall within the class of nonlinear control systems previously studied[4]. A coordinate transformation is made and feedback is then used to obtain a nonlinear control model in a normal form. The linearization of the normal form equations at an equilibrium has an uncontrollable eigenvalue at the origin. Based on analysis of the normal form equations, the spacecraft dynamics are small time locally controllable at any equilibrium attitude in a reduced sense. The reduced spacecraft dynamics cannot be asymptotically stabilized to an equilibrium attitude using time-invariant continuous feedback. Nevertheless, a discontinuous feedback control strategy is constructed which achieves reorientation of the spacecraft. The feedback control strategy is based on geometric phase, which is due to the presence of a nonintegrable invariant of the spacecraft motion.

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2. Kinematic and Dynamic Equations

Kinematic Equations

The orientation of a rigid spacecraft can be specified using various parametrizations of the special orthogonal group $SO(3)$. Here we use the following Euler angle convention. Consider an inertial $X_1 X_2 X_3$ coordinate frame; let $x_1 x_2 x_3$ be a coordinate frame aligned with the principal axes of the spacecraft with origin at the center of mass of the spacecraft. If the two frames are initially coincident, a series of three rotations about the body axes, performed in the proper sequence, is sufficient to allow the spacecraft to reach any orientation. The three rotations are:

a positive rotation of frame $X_1 X_2 X_3$ by an angle ψ about the X_3 axis; let $x_1' x_2' x_3'$ denote the resulting coordinate frame;

a positive rotation of frame $x_1' x_2' x_3'$ by an angle θ about the x_2' axis; let $x_1'' x_2'' x_3''$ denote the resulting frame;

a positive rotation of frame $x_1'' x_2'' x_3''$ by an angle ϕ about the x_1'' axis; let $x_1 x_2 x_3$ denote the final coordinate frame.

A rotation matrix R relates components of a vector in the inertial frame to components of the same vector in the body frame; in terms of the Euler angles a rotation matrix is of the form

$$
R(\psi,\theta,\phi) = \begin{bmatrix} c\psi c\theta & & \\ -s\psi c\phi + c\psi s\theta s\phi & & \\ s\psi s\phi + c\psi s\theta c\phi & & \\ & s\psi c\theta & -s\theta \\ & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix}
$$

where $c\psi = \cos(\psi)$, $s\psi = \sin(\psi)$. We assume that the Euler angles are limited to the ranges $-\pi < \psi < \pi$, $-\pi/2 < \theta < \pi/2$, $-\pi < \phi < \pi$. Suppose $\omega_1, \omega_2, \omega_3$ are the principal axis components of the absolute angular velocity vector ω of the spacecraft. Then expressions for $\omega_1, \omega_2, \omega_3$ are given by

$$\omega_1 = \dot{\phi} - \dot{\psi}\sin\theta, \qquad (2.1)$$

$$\omega_2 = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\,\sin\phi, \qquad (2.2)$$

$$\omega_3 = -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\,\cos\phi. \qquad (2.3)$$

By excluding the case where $\theta = \pm\pi/2$, these equations are invertible. Thus we can solve for ϕ, θ, ψ in terms of $\omega_1, \omega_2, \omega_3$ obtaining

$$\dot{\phi} = \omega_1 + \omega_2\sin\phi\,\tan\theta + \omega_3\cos\phi\,\tan\theta, \quad (2.4)$$

$$\dot{\theta} = \omega_2\cos\phi - \omega_3\sin\phi, \qquad (2.5)$$

$$\dot{\psi} = \omega_2\sin\phi\,\sec\theta + \omega_3\cos\phi\,\sec\theta. \qquad (2.6)$$

Next we consider the dynamic equations which describe the evolution of the angular velocity components of the spacecraft.

Dynamic Equations

Consider a rigid spacecraft with two momentum wheel actuators spinning about axes defined by unit vectors b_1, b_2 fixed in the spacecraft such that the center of mass of the i-th wheel lies on the axis defined by b_i, and a control torque $-u'_i$ is supplied to the i-th wheel about the axis defined by b_i by a motor fixed in the spacecraft. Consequently, an equal and opposite torque u'_i is exerted by the wheel on the spacecraft. We refer to the spacecraft and the two wheels as body 1, body 2 and body 3 respectively. Let C_i denote a coordinate frame aligned with the principal axes at the center of mass of body i. We assume that b_i defines a principal axis for the i-th wheel which is symmetric about b_i. Further b_1 and b_2 span a two dimensional plane which is orthogonal to a principal axis of the spacecraft. Without loss of generality b_i are assumed to be of the form

$$b_i = (b_{i1}, b_{i2}, 0)^T, \ i = 1, 2. \qquad (2.7)$$

Let I_i denote the inertia tensor of body i in the coordinate frame C_1. The mass of body i is denoted as m_i and ρ_i^1 denotes the position vector, expressed in the C_1 frame, of the center of mass of body i with respect to the center of mass of the whole system. Let ω denote the absolute angular velocity of the spacecraft expressed in the spacecraft body frame. By the definition of the center of mass, we have

$$\sum_{i=1}^{3} m_i \rho_i^1 = 0, \qquad (2.8)$$

and from the location of the wheels

$$\rho_2^1 = \rho_1^1 + (d_1, d_2, 0)^T, \qquad (2.9)$$

$$\rho_3^1 = \rho_1^1 + (d_3, d_4, 0)^T, \qquad (2.10)$$

where $(d_1, d_2, 0)^T$ and $(d_3, d_4, 0)^T$ are position vectors of the center of mass of body 2 and body 3 respectively, relative to the frame C_1. Further manipulation of equations (2.8)-(2.10) gives

$$\rho_1^1 = (-\frac{m_2}{m}d_1 - \frac{m_3}{m}d_3, \ -\frac{m_2}{m}d_2 - \frac{m_3}{m}d_4, \ 0)^T$$

2

$$\rho_2^1 = (\frac{m_1 + m_3}{m}d_1 - \frac{m_3}{m}d_3,$$

$$\frac{m_1 + m_3}{m}d_2 - \frac{m_3}{m}d_4, 0)^T$$

$$\rho_3^1 = (-\frac{m_2}{m}d_1 + \frac{m_1 + m_2}{m}d_3,$$

$$-\frac{m_2}{m}d_2 + \frac{m_1 + m_2}{m}d_4, 0)^T$$

where $m = m_1 + m_2 + m_3$. We denote ρ_i^1 as $\rho_i^1 = (c_{i1}, c_{i2}, 0)^T$. The total angular momentum vector of the system is given, in the spacecraft body frame, by

$$R(\psi, \theta, \phi)H = J\omega + v, \qquad (2.11)$$

where

$$J = [I_1 + \sum_{i=1}^{3}\overline{I}_i + \sum_{i=2}^{3}(I_i - \underline{I}_j)], \qquad (2.12)$$

$$\overline{I}_i = m_i \begin{bmatrix} c_{i2}^2 & -c_{i1}c_{i2} & 0 \\ -c_{i1}c_{i2} & c_{i1}^2 & 0 \\ 0 & 0 & c_{i1}^2 + c_{i2}^2 \end{bmatrix}, \qquad (2.13)$$

$$v = (v_1, v_2, 0)^T \qquad (2.14)$$

$$= \underline{I}_2(\omega + b_1\dot{\theta}_1) + \underline{I}_3(\omega + b_2\dot{\theta}_2), \qquad (2.15)$$

$$\underline{I}_2 = b_1 b_1^T j_1, \qquad (2.16)$$

$$\underline{I}_3 = b_2 b_2^T j_2, \qquad (2.17)$$

$$I_1 = \text{diag}(I_{11}, I_{12}, I_{13}), \qquad (2.18)$$

$$I_2 = \text{block diag}(I_{21}, I_{22}), \qquad (2.19)$$

$$I_3 = \text{block diag}(I_{31}, I_{32}), \qquad (2.20)$$

where I_{21}, I_{31} are invertible 2×2 matrices, $I_{11}, I_{12}, I_{13}, I_{22}, I_{32}$ are nonzero real numbers, j_1 is the moment of inertia of body 2 about the axis defined by b_1, j_2 is the moment of inertia of body 3 about the axis defined by b_2, and θ_1, θ_2 are the angles of rotation of the wheels about the axes defined by b_1 and b_2 respectively. Here H denotes the angular momentum vector of the system expressed in the inertial coordinate frame. The angular momentum vector H is a constant since there is no external moment about the center of mass of the system. Suppose u'_1 and u'_2 are the control torques; then

$$\dot{v} = -(b_1 u'_1 + b_2 u'_2). \qquad (2.21)$$

Differentiating (2.11) we obtain

$$J\dot{\omega} = S(\omega)R(\psi, \theta, \phi)H$$

$$+ b_1 u'_1 + b_2 u'_2 \qquad (2.22)$$

where H is a constant vector. Note that $I_1 + \sum_{i=1}^{3}\overline{I}_i$ represents the moment of inertia of the system with the momentum wheels replaced by point masses and is hence positive definite. The matrices $I_i - \underline{I}_j$, $i = 2, 3$ are positive semidefinite and block diagonal. Therefore J is a positive definite matrix and hence invertible. In fact J is of the form

$$J = \text{block diag}(J_1, J_2), \qquad (2.23)$$

where J_1 is an invertible 2×2 matrix and J_2 is a nonzero real number.

3. Controllability and Stabilizability of Complete Spacecraft Dynamics

In this section we consider the controllability and stabilizablity properties of the complete spacecraft spacecraft dynamics controlled by two momentum wheel actuators. Define

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = J_1^{-1} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix}.$$

From Section 2 the complete spacecraft dynamics can be rewritten as

$$\dot{\omega} = \begin{bmatrix} J_1^{-1} & 0_{(2 \times 1)} \\ 0_{(1 \times 2)} & J_2^{-1} \end{bmatrix} S(\omega)R(\psi, \theta, \phi)H$$

$$+ (u_1, u_2, 0)^T \qquad (3.1)$$

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \tan\theta + \omega_3 \cos\phi \tan\theta \qquad (3.2)$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi \qquad (3.3)$$

$$\dot{\psi} = \omega_2 \sin\phi \sec\theta + \omega_3 \cos\phi \sec\theta \qquad (3.4)$$

where H is a constant vector. From equation (2.21) and

$$c = (0, 0, 1)^T, \qquad (3.5)$$

we have $c^T \dot{v} = 0$; hence $c^T v = \alpha$ where α is a constant. Therefore from equation (2.11) we have

$$c^T R(\psi, \theta, \phi)H = c^T J\omega + \alpha. \qquad (3.6)$$

Since H is also a constant, this equation represents a constraint on the motion of the spacecraft irrespective of the controls applied. Thus we conclude that the complete spacecraft dynamics is not controllable. Moreover this implies that the complete spacecraft dynamics cannot be stabilized to an equilibrium attitude.

4. Controllability and Stabilizability of Restricted Spacecraft Dynamics

From the analysis made in Section 3, we find that the complete dynamics of a spacecraft controlled by two control torques supplied by momentum wheel actuators fail to be controllable. In this section we ask the following question: what restricted control and stabilization properties of the spacecraft can be demonstrated in this case? Our analysis begins by demonstrating that, under appropriate restrictions of interest, the spacecraft equations can be expressed in a special form that we previously studied. Restricted controllability and stabilizability properties follow as a consequence of previous work.

Normal Form Equations

We consider the equations (3.1)-(3.4) describing the motion of a spacecraft controlled by input torques supplied by two momentum wheel actuators about axes spanning a two dimensional plane orthogonal to a principal axis. Suppose the angular momentum vector H of the system is zero. From equations (2.11), (2.14) and (2.23) it follows that the angular velocity component of the spacecraft about the uncontrolled principal axis is identically zero, i.e. $\omega_3 \equiv 0$. Under such a restriction, the reduced spacecraft dynamics are described by

$$\dot{\omega}_1 = u_1, \tag{4.1}$$

$$\dot{\omega}_2 = u_2, \tag{4.2}$$

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \, \tan\theta, \tag{4.3}$$

$$\dot{\theta} = \omega_2 \cos\phi, \tag{4.4}$$

$$\dot{\psi} = \omega_2 \sin\phi \, \sec\theta. \tag{4.5}$$

According to equation (2.3), the condition that $\omega_3 = 0$ implies that

$$-(\sin\phi)d\theta + (\cos\theta\cos\phi)d\psi = 0; \tag{4.6}$$

this represents a nonintegrable invariant of the spacecraft motion. Therefore the reduced spacecraft dynamic equations define a nonlinear control system of the form studied earlier[4]. Now consider a diffeomorphism defined by

$$y_1 = \cos\phi \, ln(\sec\theta + \tan\theta) + \psi\sin\phi \tag{4.7}$$

$$y_2 = \omega_2 \sec\theta - y_4 y_5 \tag{4.8}$$

$$y_3 = \phi \tag{4.9}$$

$$y_4 = \omega_1 + \omega_2 \sin\phi \, \tan\theta \tag{4.10}$$

$$y_5 = \sin\phi \, ln(\sec\theta + \tan\theta) - \psi\cos\phi \tag{4.11}$$

The state equations (4.1)-(4.5) in the new variables are given by

$$\dot{y}_1 = y_2 \tag{4.12}$$

$$\dot{y}_2 = -y_5 u_1 + (\sec\theta - y_5\sin\phi\tan\theta)u_2$$
$$- y_4^2 y_1 + \cos\phi(\sec\theta\tan\theta\omega_2^2$$
$$- y_5 y_4\tan\theta\omega_2 - y_5\sin\phi\sec^2\theta\omega_2^2) \tag{4.13}$$

$$\dot{y}_3 = y_4 \tag{4.14}$$

$$\dot{y}_4 = u_1 + \sin\phi\tan\theta u_2 + \cos\phi(y_4\tan\theta\omega_2$$
$$+ \sin\phi\sec^2\theta\omega_2^2) \tag{4.15}$$

$$\dot{y}_5 = y_4 y_1 \tag{4.16}$$

If we now define the feedback relations

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\sin\phi\sin\theta & (1-y_5\sin\phi\sin\theta) \\ \cos\theta & y_5\cos\theta \end{bmatrix} \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right.$$
$$- \begin{bmatrix} -y_4^2 y_1 + \cos\phi(\sec\theta\tan\theta\omega_2^2 - y_5 y_4\tan\theta\omega_2) \\ \cos\phi(y_4\tan\theta\omega_2 + \sin\phi\sec^2\theta\omega_2^2) \end{bmatrix}$$
$$\left. + \begin{bmatrix} \cos\phi y_5\sin\phi\sec^2\theta\omega_2^2 \\ 0 \end{bmatrix} \right\} \tag{4.17}$$

then the reduced spacecraft dynamics are described by normal form equations

$$\dot{y}_1 = y_2, \tag{4.18}$$

$$\dot{y}_2 = v_1, \tag{4.19}$$

$$\dot{y}_3 = y_4, \tag{4.20}$$

$$\dot{y}_4 = v_2, \tag{4.21}$$

$$\dot{y}_5 = y_4 y_1. \tag{4.22}$$

Note that the origin of equations (4.1)-(4.5) corresponds to the origin of the normal form equations (4.18)-(4.22). The following results are based on the normal form equations above and follow directly from general results in the work of Bloch, et. al.[4].

Theorem 4.1: The reduced dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (4.1)-(4.5) are small time locally controllable at any equilibrium.

Theorem 4.2: The reduced dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (4.1)-(4.5) cannot be asymptotically stabilized to an equilibrium using a time-invariant continuous feedback control law.

Theorem 4.3: The reduced dynamics of a spacecraft controlled by two momentum wheel actuators as described by equations (4.1)-(4.5) can be asymptotically stabilized to the one dimensional equilibrium manifold

$$\Omega = \{(\omega_1, \omega_2, \phi, \theta, \psi) : \omega_1 = \omega_2$$

$$= \phi = \theta = 0\}, \quad (4.23)$$

using a smooth feedback control law given by (4.17) with v_1 and v_2 given by

$$v_1 = -k_{11}y_2 - k_{12}y_1, \quad (4.24)$$

$$v_2 = -k_{21}y_4 - k_{22}y_3, \quad (4.25)$$

where $k_{11} > 0$, $k_{12} > 0$, $k_{21} > 0$, $k_{22} > 0$ and y_1, y_2, y_3, y_4 are defined by (4.7)-(4.11).

The implications of the properties stated above are as follows. Suppose the angular momentum vector H is zero. Then the spacecraft controlled by two momentum wheel actuators as described by equations (3.1)-(3.4) can be controlled to any isolated equilibrium attitude. However, any time-invariant feedback control law that asymptotically stabilizes the spacecraft to an isolated equilibrium attitude must necessarily be discontinuous. Thus arbitrary reorientation of the spacecraft can be achieved under the restriction $H = 0$; if $H \neq 0$, reorientation of the spacecraft to an equilibrium attitude cannot be achieved.

5. Feedback Stabilization Algorithm

We restrict our study to the class of discontinuous feedback controllers in order to asymptotically stabilize the reduced spacecraft dynamics described by state equations (4.1)-(4.5). Clearly, traditional nonlinear control design methods are of no use. However, an algorithm generating a discontinuous feedback control which asymptotically stabilizes an equilibrium can be constructed, as suggested by the controllability properties of the system. Without loss of generality, we assume that the equilibrium to be stabilized is the origin. Asymptotic stabilization of equations (4.1)-(4.5) to the origin is equivalent to asymptotic stabilization of the normal form equations (4.18)-(4.22) to the origin; hence we consider asymptotic stabilization of the normal form equations.

From equation (4.22) we find that if the spacecraft motion defines a closed path γ in the (y_1, y_3) space then

$$\Delta y_5 = \int_\gamma y_1 dy_3, \quad (5.1)$$

where Δy_5 is the change in the variable y_5 or the geometric phase[9] corresponding to the path γ. This relationship is the basis for control of the system to the origin using (discontinuous) feedback.

First, transfer the initial state of the normal form equations (4.18)-(4.22) to the state $(0,0,0,0,y_5^1)$, for some y_5^1, in finite time.

Next, traverse a closed path γ in the (y_1, y_3) space in finite time, where the path γ is selected to satisfy

$$-y_5^1 = \int_\gamma y_1 dy_3. \quad (5.2)$$

Note that the execution of the first maneuver is classical. Execution of the second maneuver requires explicit characterization of a closed path γ which produces the desired path integral. Since $y_1 dy_3$ is not exact, such a path necessarily exists; and there are many such closed paths. Here we consider a rectangular path γ in the (y_1, y_3) space formed by line segments from $(0,0)$ to $(y_1^*, 0)$, from $(y_1^*, 0)$ to (y_1^*, y_3^*), from (y_1^*, y_3^*) to $(0, y_3^*)$, and from $(0, y_3^*)$ to $(0,0)$. For such a path, the line integral in equation (5.2) can be explicitly evaluated as $y_1^* y_3^*$ so that equation (5.2) becomes

$$-y_5^1 = y_1^* y_3^*, \quad (5.3)$$

and the parameters y_1^* and y_3^* specifying the particular rectangular path are chosen to satisfy the above equation. Note that this selection guarantees that, at the end of the maneuver, $y_5 = 0$. Since the path γ is closed, $y_1 = 0$, $y_3 = 0$ at the end of the maneuver. The sequential implementation of the two maneuvers guarantees, by construction, that any initial state of the normal form equations is tranferred to the origin in finite time.

We now present a specific feedback control algorithm which asymptotically stabilizes the spacecraft to the origin; this feedback control algorithm implements the approach just described. Throughout, assume $k > 0$, and define

$$G(x_1, x_2) = \begin{cases} k & \text{if } \{x_1 + \dfrac{x_2|x_2|}{2k} > 0\} \text{ or} \\ & \{x_1 + \dfrac{x_2|x_2|}{2k} = 0 \text{ and } x_2 > 0\} \\ -k & \text{if } \{x_1 + \dfrac{x_2|x_2|}{2k} < 0\} \text{ or} \\ & \{x_1 + \dfrac{x_2|x_2|}{2k} = 0 \text{ and } x_2 < 0\} \\ 0 & \text{if } \{x_1 = 0 \text{ and } x_2 = 0\} \end{cases}$$

We use the well-known property that the feedback control

$$u = -G(x_1 - \bar{x}_1, x_2)$$

for the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

transfers any initial state to the final state $(\bar{x}_1, 0)$ in a

finite time.

Step 0: If $y_5^1 \geq 0$, choose $y_1^* = -y_3^* = \sqrt{(y_5^1)}$; else choose $y_1^* = y_3^* = \sqrt{(-y_5^1)}$; then go to Step 1.

Step 1: Set

$$v_1 = -G(y_1 - y_1^*, y_2),$$

$$v_2 = -G(y_3, y_4),$$

until $(y_1, y_2, y_3, y_4) = (y_1^*, 0, 0, 0)$; then go to Step 2.

Step 2: Set

$$v_1 = -G(y_1 - y_1^*, y_2),$$

$$v_2 = -G(y_3 - y_3^*, y_4),$$

until $(y_1, y_2, y_3, y_4) = (y_1^*, 0, y_3^*, 0)$; then go to Step 3.

Step 3: Set

$$v_1 = -G(y_1, y_2),$$

$$v_2 = -G(y_3 - y_3^*, y_4),$$

until $(y_1, y_2, y_3, y_4) = (0, 0, y_3^*, 0)$; then go to Step 4.

Step 4: Set

$$v_1 = -G(y_1, y_2),$$

$$v_2 = -G(y_3, y_4),$$

until $(y_1, y_2, y_3, y_4) = (0, 0, 0, 0)$; then go to Step 0.

The most natural way to initialize the control algorithm is to begin with Step 4 since the control inputs do not depend on the values of y_1^* and y_3^* in that step. It can be verified that the execution of Step 4 transfers the initial state of the normal form equations to the state $(0, 0, 0, 0, y_5^1)$, for some y_5^1, in finite time. Execution of Steps 0 through 4 then transfers the state $(0, 0, 0, 0, y_5^1)$ to the origin in finite time. This control algorithm is nonclassical and involves switching between various feedback functions. Justification that the constructed control algorithm globally asymptotically stabilizes the origin of the normal form equations (4.18)-(4.22) follows as a consequence of the construction procedure. Since stabilization of the normal form equations to the origin is equivalent to stabilization of the state equations (4.1)-(4.5) to the origin, we conclude that the control inputs u_1 and u_2 given by equation (4.17) with v_1 and v_2 defined by the above control algorithm asymptotically stabilizes the reduced spacecraft dynamics described by equations (4.1)-(4.5) to the equilibrium $(\omega_1, \omega_2, \phi, \theta, \psi) = (0, 0, 0, 0, 0)$ in finite time.

6. Simulation

We illustrate the results of the paper using an example. Consider a rigid spacecraft with no control torque about the third principal axis and two control torques, generated by momentum wheel actuators, are applied about the other two principal axes. The complete dynamics of the spacecraft system defined by equations (3.1)-(3.4) cannot be asymptotically stabilized, but we consider the restriction that the angular momentum vector $H = 0$. Consequently, we are interested in asymptotically stabilizing the restricted spacecraft dynamics described by equations (4.1)-(4.5) to the equilibrium $(\omega_1, \omega_2, \phi, \theta, \psi) = (0, 0, 0, 0, 0)$. Here we present a simulation of equations (4.1)-(4.5). The spacecraft is initially at rest (i.e. $\omega_1^0 = \omega_2^0 = 0$) with an initial orientation given by the Euler angles $\phi^0 = \pi$, $\theta^0 = 0.25\pi$ and $\psi^0 = -0.5\pi$. The initial state of the system corresponds to an initial state $(y_1^0, y_2^0, y_3^0, y_4^0, y_5^0) = (-0.881, 0, \pi, 0, -0.5\pi)$ for the normal form equations (4.18)-(4.22). A computer implementation of the feedback control algorithm specified in Section 5 was used to asymptotically stabilize the equilibrium. The value of the gain k was chosen as 1. The algorithm was initialized at Step 4. The simulations are shown in Fig. 1 through Fig. 5. At $t = 3.55$ seconds, which is the end of Step 4, $y_1^1 = y_2^2 = y_3^1 = y_4^1 = 0$ and $y_5^1 = -1.118$. The desired geometric phase is produced by traversing a square path in the (y_1, y_3) space with $y_1^* = y_3^* = 1.057$, which is calculated from Step 0. At $t = 5.6$ seconds, which is the end of Step 1, $y_1 = 1.057$, $y_2 = y_3 = y_4 = 0$, and y_5 remains at -1.118. At $t = 7.66$ seconds, which is the end of Step 2, y_1 and y_2 remain the same while $y_3 = 1.057$, $y_4 = 0$ and $y_5 = 0$. At $t = 9.71$ seconds, which is the end of Step 3, $y_1 = y_2 = 0$, $y_3 = 1.057$, $y_4 = 0$ and $y_5 = 0$. Finally at $t = 11.77$ seconds, which is the end of Step 4, $y_1 = y_2 = y_3 = y_4 = y_5 = 0$. Thus $\omega_1 = \omega_2 = \phi = \theta = \psi = 0$ after a total maneuver time of 11.77 seconds. Three dimensional visualization schemes have been developed using a Silicon Graphics Iris work station in order to display the reorientation maneuvers of the spacecraft.

7. Conclusion

The attitude stabilization problem of a spacecraft using control torques supplied by two momentum wheel actuators about axes spanning a two dimensional plane orthogonal to a principal axis has been considered. The complete spacecraft dynamics are not controllable. However, the spacecraft dynamics are small time locally controllable in a reduced sense. The reduced spacecraft dynamics cannot be asymptotically stabilized using continuous feedback, but a discontinuous feedback control strategy has

been constructed which stabilizes the spacecraft (in the reduced sense) to the equilibrium attitude in finite time. The results of the paper show that although standard control techniques do not apply, it is possible to construct a control law based on the particular spacecraft dynamics.

Acknowledgements

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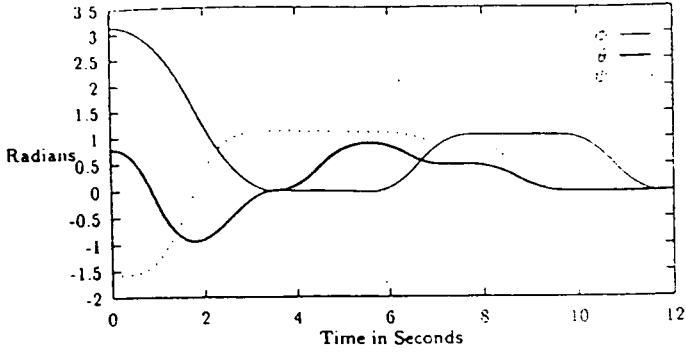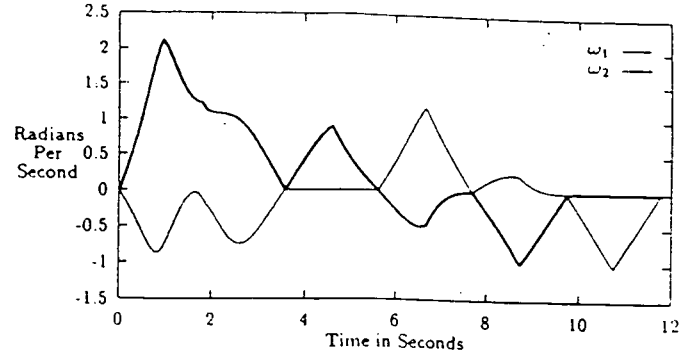
Figure 1: Plot of Euler Angles
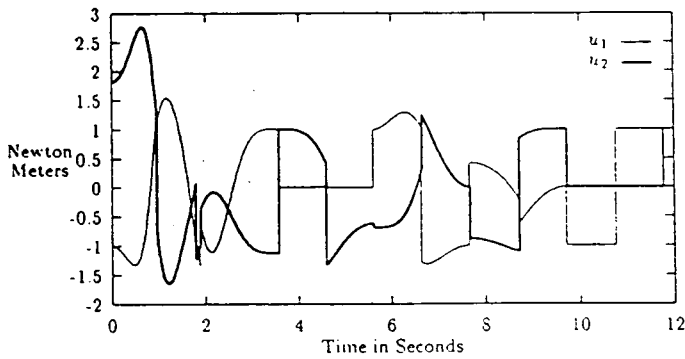

Figure 2: Plot of Angular Velocities

Figure 3: Plot of u_1 and u_2

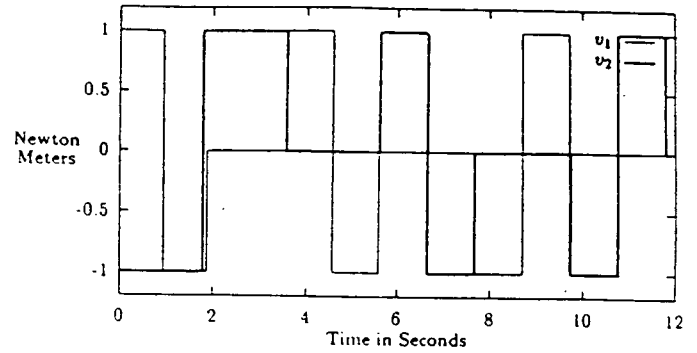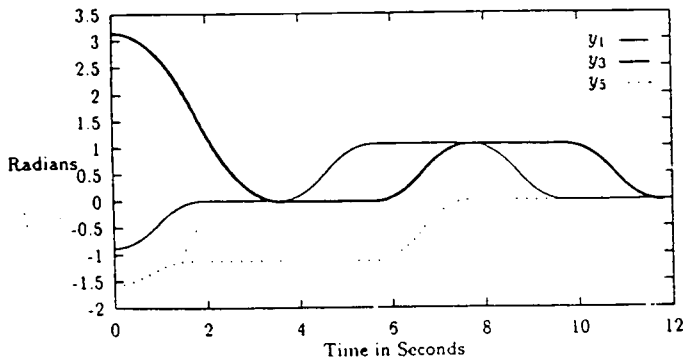Figure 4: Plot of v_1 and v_2
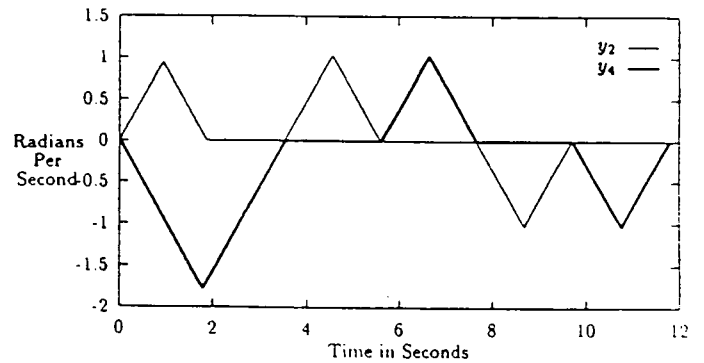

Figure 5: Plot of y_1, y_3 and y_5

Figure 6: Plot of y_2 and y_4

Space Station Attitude Disturbance Arising from Internal Motions*

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Abstract

A source of space station attitude disturbances is identified. The attitude disturbance is driven by internal space station motions and is a direct result of conservation of angular momentum. Three examples are used to illustrate the effect: a planar three link system, a rigid carrier body with two moveable masses, and a nonplanar five link system. Simulation results are given to show the magnitude of the attitude change in each example. Factors which accentuate or attenuate this disturbance effect are discussed.

Introduction

A variety of nonclassical, inherently nonlinear dynamic modelling and control problems have been investigated recently. These investigations have revealed new possibilities for controlling certain systems and new explanations for certain sources of disturbances. These ideas are directly related to control and disturbance analyses for the space station.

For example, a planar multilink system can be reoriented to an arbitrary attitude using only internal motions [1], [2]. Internal motions are executed in the shape space, defined by the relative angles of the links, to achieve a desired change in the absolute orientation. This effect can be extended to nonplanar multibody systems (as shown in an example given later) to allow arbitrary reorientation.

An example [3] of a rigid body with point mass oscillators also illustrates this effect. In this example, point masses move in slots at controlled rates. The model was motivated by the attitude drift of the Hubble Space Telescope due to thermally excited solar panel vibrations.

These examples and other space and non-space related examples [4], [5] illustrate the basic phenomena: that internal motions for a multibody system for which angular momentum is conserved can give rise to absolute orientation changes of the multibody system. In our case, we are interested in exploiting our insight into this phenomena to study potential attitude disturbances to the space station due to internal motions. Internal motions are the relative motions of the system sub-structures, payloads, and modules with respect to each other.

This paper illustrates these effects for the space station through several examples. These examples serve to illustrate the magnitude of this disturbance effect, as well as to distinguish this disturbance from other disturbances such as atmospheric drag and solar wind. The emphasis in this paper is on internal motions of the space station, driven externally or internally, which can result in an attitude change of the space station.

Space Station and Large Space Structures

Design of the space station (or other large spacecraft structures) is presented with competing requirements. Of particular focus here is the requirement to maintain stable pointing of the overall structure in the presence of additional requirements to point antennas and payloads, stabilize appendages, and conduct internal operations. For instance, the momentum management and attitude control system for the space station must provide space station attitude control within 5 deg of the local vertical and local horizontal lines, with an attitude rate boundary of 0.02 deg/sec. The design goal for nominal operation is to maintain the station attitude excursion to less than 0.2 deg from the average equilibrium attitude and the total attitude within 5 deg of the local vertical and local horizontal lines. The attitude excursion is relaxed to 1 deg during attitude seeking [6]. Nominal operations, however, include astronaut activities, solar panel actuation, antenna actuation, and many other potential disturbances.

We are interested in exploring a particular class of disturbances that can modify the attitude of the space station. Some elements of the space station that may produce such attitude disturbance effects include:

1) Motions of flexible bodies, such as solar arrays, connecting beam structures, and laboratory modules, excited by external or internal forces. These motions can change over time due to thermal effects and flexibility effects.

2) Manipulated elements such as antennas, robot arms, solar panels, solar dynamic power concentrators, attached pointing payloads, and new station segments added through construction. The space station design includes several elements which are manipulated independently of one another, through a dedicated local control system. The overall effect of these independent manipulations will cause the system shape, as described by the relative orientations of manipulated elements, to change with time. Also, during construction large elements are manipulated

into place to define new overall configurations.

3) Internal motions of astronauts, servicing robots, centrifuges, and circulating pumps. For the space station, a servicing robot has been discussed that would traverse the beam sections of the space station.

Effects of the above three classes of configuration changes are illustrated by three examples.

Example 1: Planar Three Link Model

Consider a planar model of a space station with central body and two rigid appendages (figure 1). Appendages could represent mechanical links, such as the space station beta joints, or they could represent a lumped parameter approximation of a large flexible structure. The model is characterized by (1) the distance between each link center of mass and the connecting link hinge
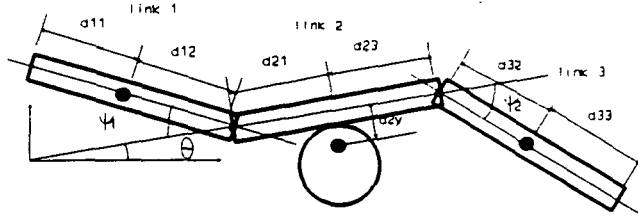
Figure 1: Planar Three Link System

points, (2) the mass and inertia of each link. The appendages are restricted to move as rotational links only. The configuration space is given by the two hinge angles (ψ_1, ψ_2) and the overall orientation of one of the links, θ. The shape space is given by the two hinge angles. This type of dynamic system has received much attention in the literature [1], [2], [7], [8]. We are interested here in a modification of the special kinematic case presented in [7]. Our modification includes an offset of the middle link center of mass from the line connecting the two hinge points. The model can also be extended to include additional links; however, three links are sufficient (and necessary) to illustrate the attitude change effect.

The primary relation of importance for our discussion is the angular momentum expression for the system. Since we are considering zero external torque on the system, angular momentum is constant throughout the motion of the appendages. The angular momentum μ is written as:

$$\mu = J(\psi_1, \psi_2)\dot{\theta} + N_1(\psi_1, \psi_2)\dot{\psi}_1 + N_2(\psi_1, \psi_2)\dot{\psi}_2 \quad (1)$$

where

$$J(\psi_1, \psi_2) = k_1 + k_2\cos(\psi_1) + k_3\cos(\psi_2) + k_4\cos(\psi_1 + \psi_2) + k_5\sin(\psi_1) + k_6\sin(\psi_2)$$

$$N_1(\psi_1, \psi_2) = k_7 + k_8\cos(\psi_1) + k_9\cos(\psi_2) + k_{10}\cos(\psi_1 + \psi_2) + k_{11}\sin(\psi_2)$$

$$N_2(\psi_1, \psi_2) = k_{12} + k_{13}\cos(\psi_1) + k_{14}\cos(\psi_2) + k_{15}\cos(\psi_1 + \psi_2) + k_{16}\sin(\psi_2)$$

The constants k_1 through k_{16} are functions of the link kinematic parameters only [8]. Note that the angular momentum is not a function of the orientation angle. The Lagrangian function constructed for this system would show that θ is ignorable. We assume that the appendages are excited according to:
Further, this excitation is persistent for a long period of time (several orbital periods). The excitation is characterized by (1)

$$\psi_1(t) = \alpha\sin\left(\frac{2\pi t}{T} + \phi_1\right) + \psi_{10}$$

$$\psi_2(t) = \alpha\sin\left(\frac{2\pi t}{T} + \phi_2\right) + \psi_{20} \quad (2)$$

a phase difference between the two appendages $(\phi_1 \neq \phi_2)$, and (2) a nonzero mean value $(\psi_{10} \neq 0$ and $\psi_{20} \neq 0)$. The importance of these two assumptions is explained later.

In order to make our results concrete, a set of parameters is selected for this example, representing an approximation of a large space structure with two flexible appendages (see table 1). For this example, $(\phi_1, \phi_2) = (0.0, \pi/2)$, $(\psi_{10}, \psi_{20}) = (\pi/8, \pi/8)$, and $\alpha = \pi/8$ rad. For this system, simulation results clearly indicate that there is a small but steady drift in the orientation angle of the base link (figure 2).

Table 1: Parameters for 3 Link System

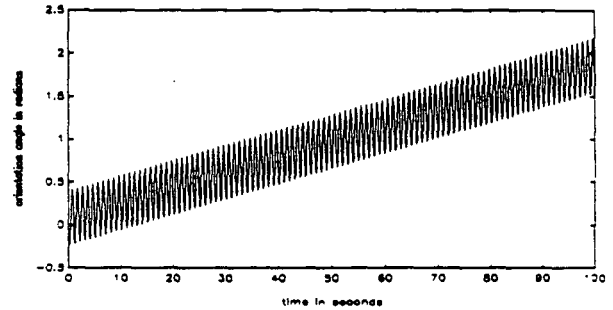| Link 1	Link 2	Link 3
d11 = 20 m	d21 = 20 m	d32 = 20 m
d12 = 20 m	d23 = 20 m	d33 = 20 m
m1 = 3000 kg	d2y = 10 m	m3 = 3000 kg
I1 = 2.3 e-5 kg-m^2	m2 = 6000 kg	I3 = 2.3 e-5 kg-m^2
	I2 = 2.0 e-6 kg-m^2	

Figure 2: Orientation Change for 3 Link System

Example 2: Rigid Body with Moveable Point Masses

Consider a model of a rigid space station module with two internal moveable masses, for instance representing astronaut motions, mobile robot motions, or a centrifuge facility (figure 3). This model is an adaptation of a model originally presented in [3]. The model is characterized by (1) the path along which the masses move, and (2) the carrier body inertia matrix and the masses of each element. For this model, $R \in SO(3)$ represents the orientation of the carrier body with respect to the inertial frame and q_1 and q_2 are the position vectors of the oscillators with respect to the carrier-fixed frame. Also, Ω is the angular velocity of the body in the carrier frame, I_c is the inertia matrix of the carrier body, and $(\hat{\ })$ represents the skew symmetric matrix formed by the components of $(\ )$ under the standard isomorphism $\hat{\ }: R^3 \mapsto so(3)$ given by:

$$(x_1, x_2, x_3) \mapsto \begin{vmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{vmatrix} \quad (3)$$

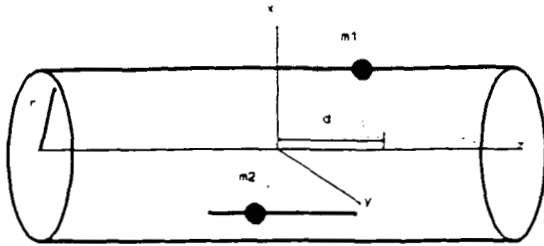The important relation here is the angular momentum expression.

Figure 3: Rigid Central Body with 2 Moveable Masses

$$\mu = R(I_0 + \Delta I_0)\,\Omega + D_1\dot{q}_1 + D_2\dot{q}_2 \qquad (4)$$

where

$$\Delta I_0 = -m[e_1\dot{q}_1^2 + e_2\dot{q}_2^2 - (e_1\dot{q}_1 + e_2\dot{q}_2)^2]$$

$$D_1 = m[(1-e_1)e_2\dot{q}_1 - e_1e_2\dot{q}_2]$$

$$D_2 = m[-e_1e_2\dot{q}_1 + e_1(1-e_2)\dot{q}_2]$$

$$e_i = \frac{m_i}{m}$$

$$m = m_0 + m_1 + m_2$$

Consider again zero initial angular momentum. The body angular velocity vector is given by:

$$\Omega = -I_{lock}^{-1}(D_1\dot{q}_1 + D_2\dot{q}_2) \qquad (5)$$

where

$$I_{lock} = I_0 + \Delta I_0$$

For illustration purposes, these two point masses are assumed to move relative to the rigid body with the following motions:

$$q_1(t) = [r \quad 0 \quad d(1+\cos(\tfrac{2\pi t}{T_1} + \Phi_1))]^T$$

$$q_2(t) = [0 \quad r \quad -d(1+\cos(\tfrac{2\pi t}{T_2} + \Phi_2))]^T \qquad (6)$$

Properties of this motion include (1) the masses are offset from each other, and (2) their velocity vectors are orthogonal. Other motions could be chosen; these were chosen to illustrate general motions of the base body. (In particular, circular motions of either particle will directly lead to an attitude drift).

The angular momentum equation can be integrated numerically for the given motions to obtain body rates over time. In order to illustrate how these body rates effect the overall attitude of the base body, consider an Euler 3-2-1 system represented by (ψ, θ, ϕ) attached to the base body, initially at $(0,0,0)$ and integrate the following transformation equations from the body rates $\Omega = (\omega_x, \omega_y, \omega_z)$ to the orientation rates, to obtain the base body attitude as a function of time, expressed in orientation angles:

$$\dot{\psi} = (\omega_y\sin\phi + \omega_z\cos\phi)\sec\theta$$

$$\dot{\theta} = (\omega_y\cos\phi - \omega_z\sin\phi) \qquad (7)$$

$$\dot{\phi} = \omega_x + (\omega_y\sin\phi + \omega_z\cos\phi)\tan\theta$$

Again, to make this example concrete the set of parameters in table 2 were used to define a simulation. The body rates for this simulation are shown in figure 4. The orientation angles for this simulation as a function of time are given in figure 5.

Table 2: Parameters for Cylinder with Moveable Masses

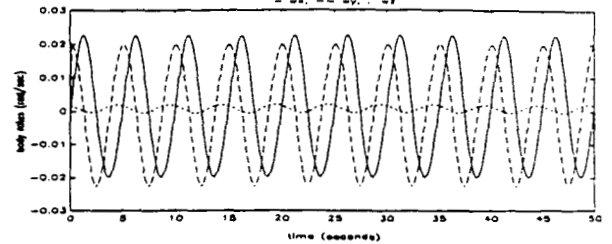| Cylinder	Mass 1	Mass 2
height = 33 m	m1 = 100 kg	m2 = 100 kg
radius = 10 m	T1 = 5 0	T2 = 5 0
mass = 2000 kg	phi 1 = 0 0	phi2 = 1 57
Ixx = 2 5 e+5 kg-m^2	d = 10 m	
Iyy = 2 5 e+5 kg-m^2		
Izz = 2.0 e+5 kg-m^2		

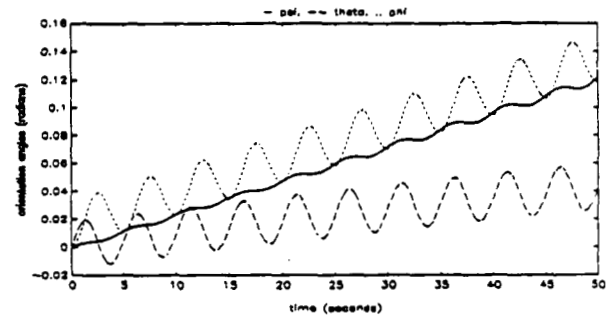**Figure 4**: Body Rates for Rigid Body with Moveable Masses

Figure 5: Attitude Drift for Rigid Body with Moveable Masses

This example illustrates the orientation drift that can occur in three dimensions.

Example 3: Non-planar 5 Link Model

Finally, consider a model for a deployment or construction sequence where large elements are manipulated by a robotic arm. The robot arm is constructed with single degree of freedom rotational joints; the overall system is represented by five links (see figure 6). The overall dynamics of this system for general link motions is very complicated. However, we consider a specific sequence of relative motions so that at any instant the motion is planar, but the plane of the motion changes periodically. Again, this system is characterized by (1) the distance between each link center of mass and the connecting link hinge points, (2) the mass and inertia of each link. The configuration space is now given by the four hinge angles and suitable orientation parameters (in SO(3)) for one of the links. The shape space is given by the four hinge angles.
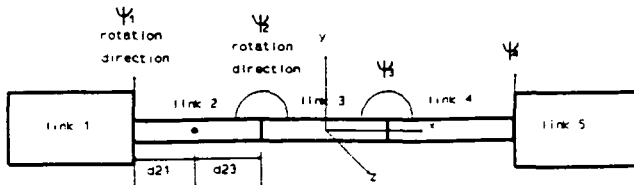
Figure 6: Five Link Nonplanar System

A sequence of three major motions is performed. Each motion segment consists of a movement of the two coplanar hinges while the other two hinges are held fixed. For simplicity, we choose motions that consist of square paths in the shape space. The entire sequence consists of a segment using the inner joints, then the outer joints, and finally the inner joints again (figure 7). Parameters for this example are shown in table 3.
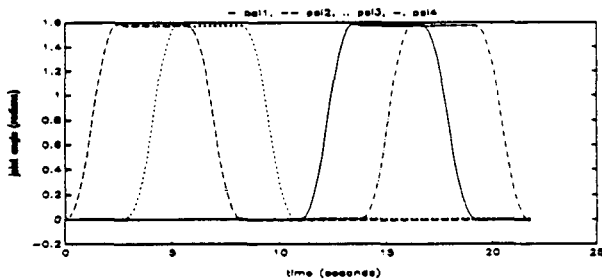

Figure 7: One Segment of Four Joint Motions

Table 3: Parameters for 5 Link System

Link 1	Link 2	Link 3	Link 4	Link 5
$d11 = 12\ m$	$d21 = 2.0\ m$	$d32 = 2.0\ m$	$d43 = 2.0\ m$	$d54 = 12.0\ m$
$d12 = 12\ m$	$d23 = 2.0\ m$	$d33 = 2.0\ m$	$d45 = 2.0\ m$	$d55 = 12.0\ m$
$m1 = 5000\ kg$	$m2 = 100\ kg$	$m3 = 100\ kg$	$m4 = 100\ kg$	$m5 = 10000\ kg$
$I1 = 4.8\ e+6$ $kg\text{-}m2$	$I2 = 134\ kg\text{-}m2$	$I3 = 134\ kg\text{-}m2$	$I4 = 134\ kg\text{-}m2$	$I5 = 9.6\ e+6$ $kg\text{-}m2$

The angular momentum equation is identical to example 1 for each of the motion segments. Equation (1) is numerically integrated for the given internal motions, using the appropriate kinematic parameters, to obtain the ω_y and ω_z body rates. In order to illustrate how these body rates effect the overall attitude of the base body, consider an Euler 3-2-3 system represented by (ψ,θ,ϕ) attached to the center link. This system is chosen since a rotation of the inner set of joints results in a change of the 3rd orientation angle directly and a rotation of the outer set of joints results in a change of the 2nd orientation angle. Note that $\omega_x=0$ for all motions in this case. The orientation angle system is defined with the z axis pointing vertically upward before the first rotation. The system is initially at $(\pi/3,\pi/3,\pi/3)$. The following transformation equations from the body rates to the Euler rates

are integrated to obtain the base body attitude, expressed in Euler coordinates:

$$\dot{\psi} = -\omega_y\frac{\sin(\phi)}{\sin(\theta)}$$

$$\dot{\theta} = \omega_y\cos(\phi) \qquad (8)$$

$$\dot{\phi} = -\omega_y\frac{\sin(\phi)}{\tan(\theta)} + \omega_z$$

Several cycles of the joints were used in order to illustrate the orientation change. The resulting motion of the orientation angles is shown in figure 8.
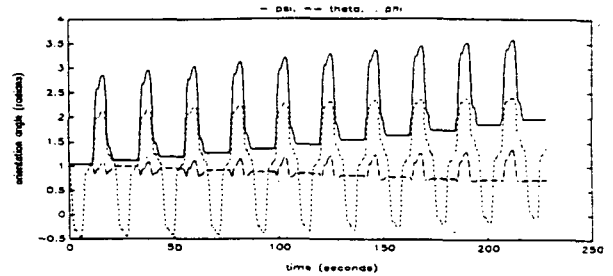

Figure 8: Attitude Drift for Five Link Nonplanar System

Note that all three orientation angles experience a drift. It can be shown for this system that any final arbitrary attitude can be achieved for the overall system through a series of planned motions as described in figure 7. In this example, the system returned to the same internal configuration at several times during the manipulation sequence; each time a new overall orientation was achieved.

Discussion

These three examples illustrate different types of internal motions for a large structure such as the space station, however, the models have important similarities. The fundamental relation in all three cases arises from the conservation of angular momentum, involving both internal velocities and external orientation. The internal motions, although possibly locally repetitive, are asynchronous or out of phase with respect to each other.

The examples have intentionally exaggerated the orientation disturbance effect for illustration purposes. The actual disturbance effect for a given system may be quite small for a single cycle of internal motions. However, for the space station, some of these disturbances are persistent, acting throughout each orbit. The net effect of these disturbances over a long period of time is additive and can result in significant attitude errors, resulting in greater than anticipated demand on the momentum management system.

There are internal motions which result in no orientation change. For instance, in examples 1 and 3, motions which are symmetric or antisymmetric about the origin in joint space result in no orientation change, independent of the magnitude of the motions.

In general, the effect of any motion on the system orientation can be analyzed using the angular momentum expression and Stoke's theorem. For planar multibody systems, this has been done previously in [5], [9], and [10]. The equation of interest is given as:

$$\Delta \theta = \iint \left(\frac{\partial f_1}{\partial \psi_2} - \frac{\partial f_2}{\partial \psi_1} \right) d\psi_1 d\psi_2 \qquad (9)$$

where

$$f_1 = \frac{N_1}{J}$$
$$f_2 = \frac{N_2}{J}$$

The integrand of this function can be plotted versus the joint angles for the parameters used in example 1 (figure 9). For example 2, a similar result can be obtained where the body axis rate components are found as a function of the two mass incremental motions. From the function shown in figure 9, it is apparent that motions which contain an area with nonzero integral will result in an orientation change.
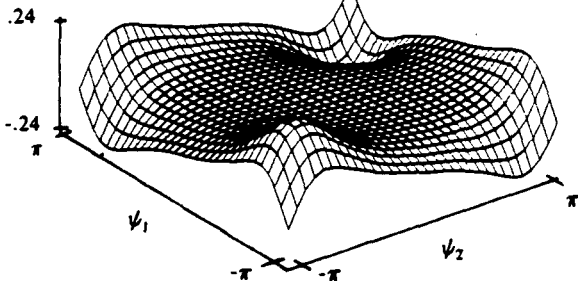

Figure 9: Integrand of Equation 9 versus Joint Angles

For general space manipulator systems, paths of minimum and maximum disturbance can be analyzed according to an enhanced disturbance map [11]. This map represents the change in attitude which is experienced from an incremental change in joint variables, i.e. the angular momentum expression in differential form. Graphical techniques are used with the enhanced disturbance map to visualize low and high disturbance paths. Motions are planned to cross zero disturbance lines in regions of low disturbance effect and are planned to move parallel to zero disturbance lines in high disturbance areas.

There are system characteristics and internal motion characteristics that accentuate or attenuate the attitude disturbance. For multibody systems, the effect is intensified through manipulating large inertias through large motions. Since the attitude disturbance effect arises as a consequence of conservation of angular momentum, similar results hold for any large space structure. Also, some internal motions can be planned to minimize the attitude disturbance or to cancel disturbances due to uncontrollable effects. These types of planning strategies could be performed using maps similar to figure 8.

Implications for the Space Station

Models of the complete space station are needed in order to perform a complete investigation of the internal motions which may disturb the space station attitude. From the examples here, multibody spacecraft and large platforms with articulating and moving elements can have significant attitude changes resulting from internal motions. The magnitude of the effect will depend on the mass distributions, the amplitudes of the motions, and the path the motions take in shape space. The analysis involves consideration of the overall angular momentum and how it is exchanged during a motion, keeping overall momentum constant.

Some planning for "controllable" motions like robot and astronaut paths can mitigate some of the disturbance effects. These might be analyzed using equation 9 or the enhanced disturbance map given in [11]. The investigation of attitude changes from internal motions is important to minimize fuel required to operate the momentum management system on the space station.

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