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INTEGRALS AND INTEGRAL EQUATIONS IN LINEARIZED WING THEORY [1]

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SUMMARY

The formulas of subsonic and supersonic wing theory for source, doublet, and vortex distributions are reviewed, and a systematic presentation is provided which relates these distributions to the pressure and to the vertical induced velocity in the plane of the wing. It is shown that care must be used in treating the singularities involved in the analysis and that the order of integration is not always reversible. Concepts suggested by the irreversibility of order of integration are shown to be useful in the inversion of singular integral equations when operational techniques are used. A number of examples are given to illustrate the methods presented, attention being directed to supersonic flight speeds.

INTRODUCTION

One of the most fundamental approaches to the analytical investigation of linearized wing theory, throughout the subsonic and supersonic ranges, stems from the use of certain elementary mathematical expressions that are identified with the physical properties of sources, doublets, and elementary horseshoe vortices. By means of these expressions boundary-value problems involving wings with thickness, camber, and angle of attack can be solved. These problems naturally fall into two categories: one, involving bodies with symmetrical thickness and no lift, is analyzed by means of source distributions; and the other, involving lifting plates without thickness, is analyzed by means of doublet and vortex distributions.

All these distributions require the treatment of singularities in the mathematical analysis. Thus, for subsonic Mach numbers, the concept of the generalized principal part plays an important role in the calculation of the induced velocities in the plane of a vortex sheet. In supersonic wing theory, the generalized principal part is again used in the analysis of vortex distributions, and it has further application in the treatment of conical-flow problems. However, the existence in supersonic flow of pressure discontinuities (due to Mach, or linearized shock, waves) brings about another type of singularity the mathematical analysis of which leads to the introduction of the finite-part concept. The integrals in both subsonic and supersonic wing theory thus require careful attention to the discontinuities in the integrand and, as an illustration, indiscriminate use of such standard devices as inversion of the order of integration can lead to incorrect results.

When direct problems are involved, that is, when prescribed functions are to be integrated (as in the problem of finding the pressure on a wing with symmetrical thickness), a guide to the proper method of calculation is often furnished by physical intuition. However, when inverse problems arise, that is, when integral equations are to be inverted (as for the flat plate of arbitrary plan form), the mathematical methods are more abstract. Nevertheless, the solutions to several types of inverse aerodynamic problems have been obtained by reasoning that required an understanding of the physical nature of the flow field. This method of solution may be sufficient for the particular problem involved but it is difficult to generalize. By using the aerodynamic data to construct mathematical boundary-value problems requiring the inversion of singular integral equations and by obtaining these inversions from a purely mathematical (operational) basis, a technique evolves whereby the existing solutions for two-dimensional subsonic, and three-dimensional supersonic wing problems (e. g., thin airfoil, conical flow, and Evvard solutions) are synthesized. Furthermore, the solution to the general supersonic wing problem is suggested.

The purpose of the present report is: First, to review the formulas of linearized wing theory in which source, doublet, and elementary horseshoe vortex distributions are introduced and to relate these distributions to the pressure and vertical induced velocity in the plane of the wing; second, to present an operational technique that can be used to invert the singular integral equations appearing in the application of the above formulas; and finally, to present certain special examples which will illustrate the basic concepts.

LIST OF IMPORTANT SYMBOLS

B_n $\dfrac{1}{k_n^2}(E_n - k_n'^2 K_n)$

c chord of a wing

C_D drag coefficient $\left(\dfrac{\text{drag}}{\frac{1}{2}\rho_o V_o^2 S}\right)$

C_p pressure coefficient $\left(-2\dfrac{u}{V_o}\right)$

E complete elliptic integral of second kind, modulus k

K complete elliptic integral of first kind, modulus k

K_n, E_n complete elliptic integrals of first and second kinds, respectively, with moduli k_n

k, k_n moduli of elliptic integrals

k', k'_x Complementary moduli ($\sqrt{1-k^2}$, $\sqrt{1-k_x^2}$)

L lift

M_o Mach number in free stream

m slope of wing leading edge

m_1 cotangent of angle between the η and x axes

m_2 cotangent of angle between the ξ and x axes

$\dfrac{\Delta p}{q}$ loading coefficient (pressure on the lower surface minus pressure on the upper surface, divided by free-stream dynamic pressure)

r, s, z characteristic coordinates

r_c $\{(x-x_1)^2+(1-M_o^2)[(y-y_1)^2+z^2]\}^{1/2}$

r_o $[(x-x_1)^2+(1-M_o^2)(y-y_1)^2]^{1/2}$

S wing area

t maximum thickness of a wing

u, v, w perturbation velocities in x, y, z directions, respectively

V_o free-stream velocity

x, y, z Cartesian coordinates

α angle of attack of wing $\left(-\dfrac{w}{V_o}\right)$

β $\sqrt{|1-M_o^2|}$

Δ jump in value of the quantity considered across the $z=0$ plane

λ streamwise slope of surface $\left(\dfrac{w}{V_o}\right)$

μ_1 $\sqrt{1+m_1^2}$

μ_2 $\sqrt{1+m_2^2}$

μ $\dfrac{1+m\beta}{1-m\beta}$

ν_c $\dfrac{x-x_1}{(y-y_1)^2+z^2}$

ξ, η, z oblique coordinates

ρ_o density of free stream

τ area of integration

φ perturbation velocity potential

ω_c $\dfrac{x-x_1}{\beta\sqrt{(y-y_1)^2+z^2}}$

SUBSCRIPTS

l value of a quantity on the lower surface of a wing ($z=0$ plane)

u value of a quantity on the upper surface of a wing ($z=0$ plane)

PART I—THE THREE FUNDAMENTAL FORMULAS

SOME BASIC MATHEMATICAL FORMULAS

FIELD EQUATION FOR SUBSONIC FLOW

The basic linearized partial differential equation governing a subsonic flow field is derived under the assumption that perturbation velocity components are small relative to the free-stream velocity V_o. Written in terms of the perturbation velocity potential $\varphi(x, y, z)$ the equation is

$$(1-M_o^2)\varphi_{xx}+\varphi_{yy}+\varphi_{zz}=0 \qquad (1)$$

where M_o is the free-stream Mach number and the x axis is parallel to the free-stream direction. Equation (1) is, in its normalized form, Laplace's equation in three dimensions. If a sufficiently thin wing at a small angle of attack is situated on or in the immediate vicinity of the xy plane, the boundary conditions in the resulting linearized theory may be assumed specified at $z=0$ and, by means of Green's theorem (see, e. g., reference 1), a solution to equation (1) can be written in the form

$$\varphi(x, y, z)=\frac{-1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\Delta w(x_1, y_1)\frac{1}{r_c}-\Delta\varphi(x_1, y_1)\frac{\beta^2 z}{r_c^3}\right]dx_1\,dy_1 \qquad (2)$$

where

$$\beta^2=1-M_o^2$$

and

$$r_c=\{(x-x_1)^2+(1-M_o^2)[(y-y_1)^2+z^2]\}^{1/2}$$

Equation (2) relates the perturbation velocity potential at a point (x, y, z) in space to the discontinuities in the potential and vertical induced velocity at the "plane of the wing." Thus, $\Delta\varphi=\varphi_u-\varphi_l$ and $\Delta w=w_u-w_l$, where the subscripts u and l denote conditions on the upper and lower side of the xy plane.

In a later section, equation (2) will be used to obtain expressions for source, vortex, and doublet distributions in subsonic flow.

FIELD EQUATION FOR SUPERSONIC FLOW

The form of the basic linearized partial differential equation governing supersonic flow fields can be written in terms of the perturbation velocity potential as

$$(M_o^2-1)\varphi_{xx}-\varphi_{yy}-\varphi_{zz}=0 \qquad (3)$$

Since M_o is now greater than one, equation (3) is, in its normalized form, the wave equation. A solution to equation (3) that relates the potential in space to its jump $\Delta\varphi$ and the jump of the vertical velocity Δw across the $z=0$ plane has also been derived by means of Green's theorem. A form of such a solution, due to Volterra (reference 2), can be written

$$\varphi(x,y,z)=-\frac{1}{2\pi}\frac{\partial}{\partial x}\int_{\tau}\int\left\{\Delta w(x_1, y_1)\ \text{arc cosh}\ \frac{x-x_1}{\beta\sqrt{(y-y_1)^2+z^2}}\right.$$
$$\left.\Delta\varphi(x_1, y_1)\frac{z(x-x_1)}{[y-y_1)^2+z^2]r_c}\right\}dx_1\,dy_1 \qquad (4)$$

where

$$\beta^2=M_o^2-1$$

and

$$r_c=\{(x-x_1)^2-(M_o^2-1)[(y-y_1)^2+z^2]\}^{1/2}$$

The area τ is that part of the $z=0$ plane contained within the Mach forecone from the point (x, y, z); that is, the area bounded by the line $x_1=-\infty$ and the hyperbola $(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2 z^2=0$.

Obviously equations (2) and (4) are not similar although the basic equations to which they apply are. A formal similarity can be obtained, however, through the introduction of an integral operator, originated by Hadamard (reference 3), and referred to as the "finite part." The use of the finite part and certain other techniques necessary to reduce equation (4) to a form similar to equation (2) requires some attention and a discussion of the mathematical

difficulties involved will be given in the following section. Then an application of these techniques in subsequent sections will make it possible to obtain expressions for source, vortex, and doublet distributions in supersonic flow.

THE FINITE PART OF AN INTEGRAL

In the study of linearized supersonic flow problems, one is continually confronted with expressions of the form

$$\frac{\partial}{\partial x}\int_a^x \frac{f(x,y)\,dy}{\sqrt{x-y}}, \quad a<x \tag{5}$$

The integrals are of this form in the sense that the integrand is infinite at one (or both) of the limits and this limit is a function of the variable by which the partial derivative is to be taken. Such an expression is annoying because the derivative cannot be "moved" through the integral sign according to the usual rule, namely,

$$\frac{\partial}{\partial x}\int_a^x F(x,y)\,dy = F(x,x) + \int_a^x \frac{\partial F(x,y)}{\partial x}\,dy \tag{6}$$

Direct application of equation (6) to equation (5) obviously yields an unacceptable indeterminate form since the term corresponding to $F(x, x)$ is infinite. One way of avoiding the difficulty is to integrate equation (5) by parts so that the radical appears in the numerator of the integral and then to apply equation (6) to the resulting expression. Such a procedure can be carried out without the introduction of any new mathematical symbol or concept. However, this involves unnecessary restrictions on the integrand and often leads to unwieldy forms since the derivative of the funtion $f(x, y)$ with respect to y can be cumbersome.

Definition.—A more direct way of taking the derivative through the integral sign in equation (5) is accomplished by using the integral operator known as the finite part. Consider the simple equality

$$\frac{\partial}{\partial x}\int_a^x \frac{dy}{\sqrt{x-y}} = \frac{1}{\sqrt{x-a}}$$

The finite-part sign can be introduced by the definition

$$\fint_a^x \frac{\partial}{\partial x}\frac{dy}{\sqrt{x-y}} = \frac{\partial}{\partial x}\int_a^x \frac{dy}{\sqrt{x-y}} \tag{7}$$

from which it follows

$$\fint_a^x \frac{dy}{(x-y)^{3/2}} = \frac{-2}{\sqrt{x-a}} \tag{8}$$

The natural extension of this idea is to consider

$$J = \fint_a^x \frac{\partial}{\partial x}\frac{A(y)\,dy}{\sqrt{x-y}} \equiv \frac{\partial}{\partial x}\int_a^x \frac{A(y)\,dy}{\sqrt{x-y}} \tag{9}$$

where $A(y)$ is continuous at $x=y$ and is integrable elsewhere in the range of integration. The evaluation of J can be reduced to a form that requires only the definition introduced by equation (7). Thus, by adding and subtracting the same term, J can be written

$$J = \frac{\partial}{\partial x}\left[\int_a^x \frac{A(y)-A(x)}{\sqrt{x-y}}\,dy + A(x)\int_a^x \frac{dy}{\sqrt{x-y}}\right]$$

and it follows that

$$\fint_a^x \frac{A(y)\,dy}{(x-y)^{3/2}} = \int_a^x \frac{A(x)-A(y)}{(x-y)^{3/2}}\,dy + A(x)\fint_a^x \frac{dy}{(x-y)^{3/2}} \tag{10}$$

The final generalization of the definitions given by equations (7) and (9) is accomplished by considering the nth derivative of the integrals and, furthermore, by allowing a functional dependence on x of the integrand A. First consider the definition

$$\fint_a^x \left(\frac{\partial}{\partial x}\right)^n \frac{A(y)\,dy}{\sqrt{x-y}} = (-1)^n \frac{1\cdot3\cdots(2n-1)}{2^n}\fint_a^x \frac{A(y)\,dy}{(x-y)^{n+1/2}}$$
$$\equiv \left(\frac{\partial}{\partial x}\right)^n \int_a^x \frac{A(y)\,dy}{\sqrt{x-y}} \tag{11}$$

The second integral can be expressed in the form

$$\fint_a^x \frac{A(y)\,dy}{(x-y)^{n+1/2}} = \int_a^x \frac{A(y)-B(x,y)}{(x-y)^{n+1/2}}\,dy + \fint_a^x \frac{B(x,y)\,dy}{(x-y)^{n+1/2}} \tag{12}$$

where

$$B(x,y) = A(x) - A'(x)(x-y) + \cdots + \frac{(-1)^{n-1}A^{(n-1)}(x)}{(n-1)!}(x-y)^{n-1}$$

and

$$\fint_a^x \frac{dy}{(x-y)^{i+1/2}} = \frac{(-1)^i 2^i}{1\cdot3\cdots(2i-1)}\left(\frac{\partial}{\partial x}\right)^i \int_a^x \frac{dy}{\sqrt{x-y}}$$
$$= \frac{-2}{(2i-1)(x-a)^{i-1/2}}$$

Finally replacing $A(y)$ by $A(x, y)$, equation (11) again defines uniquely a finite-part integral provided that

$$\lim_{y\to x}\left[\sqrt{x-y}\left(\frac{\partial}{\partial x}\right)^n A(x,y)\right] = 0$$

Methods of evaluation.—If expressions of the type presented in equation (5) appear in an analytical development, it is now possible, by using the finite-part symbol, to take the partial derivative operation through the integral sign. Such a process needs no further amplification. In applying the results of such an analysis to the solution of some specific problem, however, one is confronted with the inverse operation, that is, the problem of evaluating the finite-part integrals. This can always be done, of course, by means of the definitions already given. Often, though, such evaluations can be simplified by using one of the two following processes.

The first process is readily outlined. Rewrite equations (8) and (10) in the form

$$\fint_a^x \frac{f(x,y)\,dy}{(x-y)^{3/2}} = \lim_{\xi\to x}\int_a^\xi \frac{f(x,y)-f(x,x)}{(x-y)^{3/2}}\,dy - \frac{2f(x,x)}{\sqrt{x-a}}$$

and set the indefinite integral of

$$\int \frac{f(x,y)\,dy}{(x-y)^{3/2}}$$

equal to $F(x, y) + C$. It follows that

$$\int_a^x \frac{f(x, y)\,dy}{(x-y)^{3/2}} = -[F(x, a) + C] \tag{13}$$

where

$$C = \lim_{y \to x} \left[\frac{2f(x, y)}{\sqrt{x-y}} - F(x, y) \right] \tag{14}$$

The second technique avoids the necessity of evaluating the constant C. It depends on the use of the complex variable and is valid only when $f(x, y)$ is real in the interval $a \le y \le b$ where b is some number greater than x. Again set the indefinite integral of

$$\int \frac{f(x, y)\,dy}{(x-y)^{3/2}}$$

equal to $F(x, y) + C$. Now if r. p. stands for the real part of a function, the evaluation of the finite-part integral is provided by the equality

$$\int_a^x \frac{f(x, y)\,dy}{(x-y)^{3/2}} = \text{r. p.} \int_a^b \frac{f(x, y)\,dy}{(x-y)^{3/2}} \equiv \text{r. p.}\, [F(x, b) - F(x, a)], \quad a < x < b \tag{15}$$

As an example of the second technique, consider the integral

$$I = \int_0^x \frac{y^2\,dy}{(x^2-y^2)^{3/2}} = \text{r. p.} \int_0^b \frac{y^2\,dy}{(x^2-y^2)^{3/2}}$$

where $x < b$. From the relation

$$\int \frac{y^2\,dy}{(x^2-y^2)^{3/2}} = \frac{y}{\sqrt{x^2-y^2}} - \text{arc sin} \frac{y}{x}$$

together with equation (15), it follows that

$$I = \text{r. p.} \left(\frac{b}{\sqrt{x^2-b^2}} - \text{arc sin} \frac{b}{x} \right) = -\frac{\pi}{2}$$

A simple extension of this result yields

$$\int_{-x}^x \frac{y^2\,dy}{(x^2-y^2)^{3/2}} = \text{r. p.} \int_{-b}^b \frac{y^2\,dy}{(x^2-y^2)^{3/2}} = -\pi$$

An application.—The above methods can be applied to give the following simple but useful result. Let $Y = a + by + cy^2 = (\lambda_2 - y)(y - \lambda_1)(-c)$ and $q = 4ac - b^2$, then

$$\int_t^{\lambda_2} \frac{(c_0 + c_1 y)\,dy}{Y^{3/2}} = - \int_{\lambda_1}^t \frac{(c_0 + c_1 y)\,dy}{Y^{3/2}} = \frac{-2}{q\sqrt{a+bt+ct^2}} [c_0(2ct + b) - c_1(bt + 2a)] \tag{16}$$

Included in this result is the equality

$$\int_{\lambda_1}^{\lambda_2} \frac{c_0 + c_1 y}{Y^{3/2}}\,dy = 0 \tag{17}$$

which contains the very important identity

$$\int_{\lambda_1}^{\lambda_2} \frac{dy}{(\lambda_2-y)^{3/2}\sqrt{y-\lambda_1}} = \int_{\lambda_1}^{\lambda_2} \frac{dy}{(y-\lambda_1)^{3/2}\sqrt{\lambda_2-y}} = 0 \tag{18}$$

The case of multiple integrals.—When applied to the analysis of single integrals, the above defintions of the finite part coincide with, or are a re-expression of, those given originally by Hadamard (reference 3). Hence,

$$\overline{\int_a^x \frac{f(x, y)\,dy}{(x-y)^{3/2}}} = \int_a^x \frac{f(x, y)\,dy}{(x-y)^{3/2}}$$

where $\sqcap$ was the symbol used by Hadamard to denote evaluation by finite-part methods. When applied to double integrals, however, the signs $\sqcap$ and $\!\!\!\fint$ are no longer equivalent. Hadamard, as well as A. Robinson (reference 4), maintains the convention that the order of integration in the operation $\sqcap$ is reversible; that is,[2]

$$\overline{\int dy \int dx\, f(x, y)} = \overline{\int dx \int dy\, f(x, y)}$$

Such a convention requires that all singularities for which the order of integration is irreversible must be excluded from the area of integration. These singular regions are then treated separately. This convention has the disadvantage that, in evaluating multiple integrals, the value of a given integral is not independent of succeeding integrals. The operator $\!\!\!\fint$ avoids difficulties of this kind and each definite integral is independent of succeeding operations. For example,

$$\fint_0^\xi \frac{d\eta}{(\xi-\eta)^{3/2}\sqrt{\eta}} = \int_0^x d\xi \fint_0^\xi \frac{d\eta}{(\xi-\eta)^{3/2}\sqrt{\eta}} = 0$$

but, according to reference 3,

$$\overline{\int_0^\xi \frac{d\eta}{(\xi-\eta)^{3/2}\sqrt{\eta}}} = 0$$

whereas, according to the same reference,

$$\overline{\int_0^x d\xi \int_0^\xi \frac{d\eta}{(\xi-\eta)^{3/2}\sqrt{\eta}}} = -2\pi$$

Although the use of the symbol $\!\!\!\fint$ makes each integration independent of subsequent integrations, the order of integration for operations involving the sign $\!\!\!\fint$ cannot be reversed. Hence,

$$\int dy \fint dx\, f(x, y) \ne \int dx \fint dy\, f(x, y)$$

For example, the relation

$$\int_0^x d\xi \fint_0^\xi \frac{d\eta}{(\xi-\eta)^{3/2}\sqrt{\eta}} = 0$$

holds while the same integral taken over the same area but in reversed order is

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[2] Since the order of integration plays an important role in the following development, integration first with respect to x and then with respect to y will be denoted $\int dy \int dx\, f(x, y)$ while integration first with respect to y and then with respect to x will be denoted $\int dx \int dy\, f(x, y)$. When the notation $\int \int f(x, y)\, dy\,dx$ is used, the order of integration is immaterial.

$$\int_0^x d\eta \fint_\eta^x \frac{d\xi}{(\xi-\eta)^{3/2}\sqrt{\eta}} = -2\pi$$

The operation defined by the symbol $\fint$ will be used consistently throughout the present report. Hence attention must always be paid to the order but not to the multiplicity of integrations. It can be seen that the finite part of a conventional-type integral coincides with the value found by standard methods.

THE GENERALIZED PRINCIPAL PART OF AN INTEGRAL

Another type of important operation appearing in the development of both subsonic and supersonic wing theory appears implicity in the expression

$$I_o = \lim_{z\to 0+} \int_a^b \frac{\partial}{\partial z} \left[\frac{z f(y_1, z)}{z^2+(y-y_1)^2} \right] dy_1 \tag{19}$$

where $f(y, z)$ and its derivatives are bounded and continuous in the interval $a \le y \le b$. In the attempt to simplify I_o by letting z approach zero before performing the integration, a second special integral operator will be introduced.

To simplify I_0, first integrate by parts

$$I_o = \lim_{z\to 0+} \frac{\partial}{\partial z} \left[f(a,z) \arctan \frac{y-a}{z} - f(b,z) \arctan \frac{y-b}{z} + \int_a^b \frac{\partial f(y_1, z)}{\partial y_1} \arctan \frac{y-y_1}{z} dy_1 \right]$$

Then since

$$\lim_{z\to 0+} \int_a^b \frac{\partial f_z(y_1, z)}{\partial y_1} \arctan \frac{y-y_1}{z} dy_1 = \pi f_z(y,o) - \frac{\pi}{2} [f_z(a,o) + f_z(b,o)]$$

I_0 becomes

$$I_o = \frac{f(b,o)}{y-b} - \frac{f(a,o)}{y-a} - \int_a^b \frac{f_{y_1}(y_1,o) dy_1}{y-y_1} + \pi f_z(y,o) \tag{20}$$

Definition of the generalized principal part.—The expression of I_0 given in equation (20) contains the integral of the function $f'(y_1)/(y-y_1)$. Such an integration is not, in general, convergent; however, when the integral is so written without further qualification it is generally accepted that the singularity occurring in the integrand is to be treated using Cauchy's principal part. Evaluation by such a method is often indicated by the symbol $\oint$ and is defined by the equation

$$\oint_a^b \frac{A(y_1) dy_1}{y_1-y} \equiv \lim_{\epsilon\to 0} \left[\int_a^{y-\epsilon} \frac{A(y_1) dy_1}{y_1-y} + \int_{y+\epsilon}^b \frac{A(y_1) dy_1}{y_1-y} \right] \tag{21}$$

or, alternatively,

$$\oint_a^b \frac{A(y_1) dy_1}{y_1-y} \equiv -\frac{\partial}{\partial y} \int_a^b A(y_1) \ln|y_1-y| dy_1 \tag{22}$$

To assure the convergence of the right-hand side of equations (21) and (22) it is sufficient but not necessary to assume that $A(y_1)$ is differentiable at the point $y_1=y$ and that

elsewhere within the region of integration $A(y_1)$ is either continuous or possesses integrable singularities. The concept of the Cauchy principal part is so well known that the symbol on this integral is often omitted, as shall be done here.

The differential operator in equation (22) lends itself readily to a generalization of the principal-part concept. Thus, for the next higher order, the definition (see also, in this connection, reference 5)

$$\fint_a^b \frac{A(y_1) dy_1}{(y_1-y)^2} \equiv -\frac{\partial^2}{\partial y^2} \int_a^b A(y_1) \ln|y-y_1| dy_1 = \frac{\partial}{\partial y} \int_a^b \frac{A(y_1) dy_1}{y_1-y} \tag{23}$$

applies and, in general,

$$n! \fint_a^b \frac{A(y_1) dy_1}{(y_1-y)^{n+1}} \equiv -\frac{\partial^{n+1}}{\partial y^{n+1}} \int_a^b A(y_1) \ln|y-y_1| dy_1 = \frac{\partial^n}{\partial y^n} \int_a^b \frac{A(y_1) dy_1}{y_1-y} \tag{24}$$

The generalized principal part can also be expressed in the form

$$\fint_a^b \frac{A(y_1) dy_1}{(y_1-y)^{n+1}} \equiv \int_a^b \frac{A(y_1)-B(y, y_1)}{(y_1-y)^{n+1}} dy_1 + \fint_a^b \frac{B(y, y_1) dy_1}{(y_1-y)^{n+1}}$$

where

$$B(y, y_1) = A(y) + \frac{A'(y)}{1}(y_1-y) + \ldots + \frac{A^{(n-1)}(y)}{(n-1)!}(y_1-y)^{n-1}$$

and

$$\fint_a^b \frac{dy_1}{(y_1-y)^{i+1}} = \frac{1}{i!} \left(\frac{\partial}{\partial y} \right)^i \int_a^b \frac{dy_1}{y_1-y} = \frac{-1}{i} \left[\frac{1}{(b-y)^i} - \frac{1}{(a-y)^i} \right], \quad i \ge 1.$$

The first n derivatives of $A(y_1)$ are assumed to exist and be single valued at $y_1=y$ while elsewhere in the range of integration $A(y_1)$ may possess integrable singularities. This definition is in a form that involves no extension beyond the concept of Cauchy's principal part.

It is possible to extend the definition contained in equation (24) to include a functional dependency on y in the numerator of the integrands. Thus, replacing $A(y_1)$ by $A(y, y_1)$, equation (24) again defines uniquely a principal-part integral provided the first n derivatives of $A(y, y_1)$ with respect to y and y_1 exist at $y_1=y$.

Method of evaluation.—Operations involving the symbol $\fint$ can always be performed by means of the definitions just given. However, another method can be used, which is often simpler to apply. If the indefinite integral of $A(y_1)/(y_1-y)^{n+1}$ exists such that

$$\int \frac{A(y_1) dy_1}{(y_1-y)^{n+1}} = G(y_1, y) + C \tag{25}$$

then the value of the generalized principal part can be found by following the conventional rules for substitution of limits; thus

$$\underset{a}{\overset{b}{\mathcal{f}}} \frac{A(y_1)\,dy_1}{(y_1-y)^{n+1}} = G(b,y) - G(a,y), \quad y \neq a, b \qquad (26)$$

The proof of this result can be obtained by mathematical induction.

An application.—Returning to equation (20), since an integration by parts yields the relation

$$\underset{a}{\overset{b}{\mathcal{f}}} \frac{f(y_1,0)\,dy_1}{(y_1-y)^2} = \frac{f(b,0)}{y-b} - \frac{f(a,0)}{y-a} - \int_a^b \frac{f_{y_1}(y_1,0)\,dy_1}{y-y_1}$$

the limiting process symbolized by I_0 can be expressed as an integral that contains only the function evaluated at $z=0$ and not its derivative. Thus, finally

$$\lim_{z \to 0_+} \int_a^b \frac{\partial}{\partial z} \left[\frac{z f(y_1,z)}{z_2 + (y-y_1)^2} \right] dy_1 = \underset{a}{\overset{b}{\mathcal{f}}} \frac{f(y_1,0)dy_1}{(y_1-y)^2} + \pi f_z(y,0) \qquad (27)$$

THE OBLIQUE COORDINATE SYSTEM

Equations (1) through (4) gave the basic partial differential equations of wing theory, together with their solutions, in terms of the usual Cartesian coordinate system, the x axis extending in the direction of the undisturbed flow and the y and z axes oriented normal to this direction in such a way that boundary conditions for the wing can be specified in the $z=0$ plane. In the study of supersonic flow fields it is at times mathematically convenient to introduce in the $z=0$ plane new coordinate axes making arbitrary angles with the x and y axes.

The general case. Consider the ξ, η, z coordinate system (fig. 1) such that the z axis is normal to the plane supporting
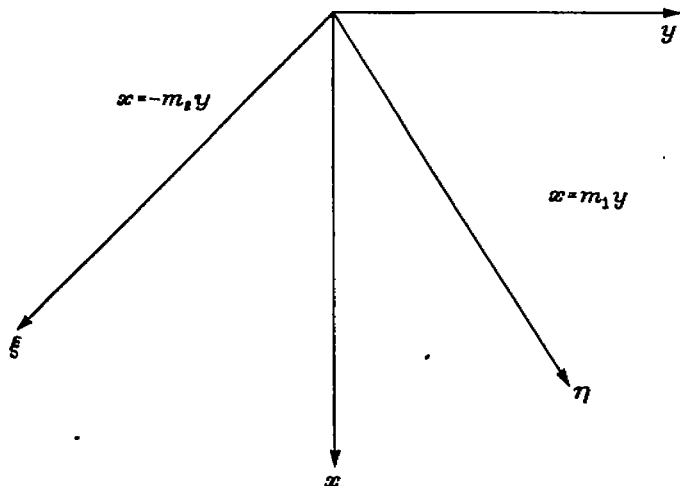

FIGURE 1.—Coordinate systems.

the boundary conditions while ξ and η are normal to z and coincident with the lines $x=-m_2 y$ and $x=m_1 y$. If

$$\mu_1 = \sqrt{1+m_1^2}, \quad \mu_2 = \sqrt{1+m_2^2} \qquad (28)$$

the equations relating the two systems of coordinates are

$$\left. \begin{array}{ll} x = \dfrac{\eta m_1}{\mu_1} + \dfrac{\xi m_2}{\mu_2}, & \xi = \dfrac{\mu_2(x-m_1 y)}{m_1+m_2} \\[2ex] y = \dfrac{\eta}{\mu_1} - \dfrac{\xi}{\mu_2}, & \eta = \dfrac{\mu_1(x+m_2 y)}{m_1+m_2} \\[2ex] z = z, & z = z \end{array} \right\} \qquad (29)$$

while the relation between the differential areas, as determined from the Jacobian of the transformations, is

$$dx_1 dy_1 = \frac{m_1+m_2}{\mu_1 \mu_2} d\xi_1 d\eta_1 \qquad (30)$$

The value of r_c, as defined under equation (4), becomes

$$r_c = \left[\frac{(\eta-\eta_1)^2(m_1^2-\beta^2)}{\mu_1^2} + \frac{2(\eta-\eta_1)(\xi-\xi_1)(m_1 m_2+\beta^2)}{\mu_1 \mu_2} + \frac{(\xi-\xi_1)^2(m_2^2-\beta^2)}{\mu_2^2} - \beta^2 z^2 \right]^{1/2} \qquad (31a)$$

where $\beta^2 = M_0^2 - 1$. If the variable ν_c is introduced such that

$$\nu_c = \frac{x-x_1}{(y-y_1)^2+z^2} \qquad (31b)$$

it follows that in the transformed coordinates this variable becomes

$$\nu_c = \frac{(\eta-\eta_1)\dfrac{m_1}{\mu_1} + (\xi-\xi_1)\dfrac{m_2}{\mu_2}}{\left[(\eta-\eta_1)\dfrac{1}{\mu_1} - (\xi-\xi_1)\dfrac{1}{\mu_2}\right]^2 + z^2} \qquad (31c)$$

Finally, the differential operator $\partial/\partial x$ transforms to

$$\frac{\partial}{\partial x} = \frac{1}{m_1+m_2}\left(\mu_2 \frac{\partial}{\partial \xi} + \mu_1 \frac{\partial}{\partial \eta} \right) \qquad (31d)$$

The area τ over which the integration of equation (4) is to be taken is still, of course, bounded by the hyperbola $r_c^2=0$ and the line $x_1=-\infty$. The asymptotes to the hyperbola become, however,

$$\left. \begin{array}{l} \eta-\eta_1 = -\dfrac{\mu_1}{\mu_2}(\xi-\xi_1)\dfrac{m_2+\beta}{m_1-\beta} \\[2ex] \eta-\eta_1 = -\dfrac{\mu_1}{\mu_2}(\xi-\xi_1)\dfrac{m_2-\beta}{m_1+\beta} \end{array} \right\} \qquad (32)$$

Figure 2 shows how the area τ in the xy plane transforms to the $\xi\eta$ plane.

(Although the ξ and η axes are oblique with respect to the original axial system, no inconsistency results if the equations in the transformed variables are plotted relative to orthogonal axes.) In case both m_1 and m_2 are less than β (the case for which the sketch was drawn), the asymptotes are straight lines with positive slopes, and the area is bounded in both the ξ and η directions by the maximum points ξ_a, η_a and ξ_b, η_b, respectively. These maximum points are shown in the figure and their analytical expressions are

$$\left. \begin{array}{ll} \xi_a = \xi - \dfrac{z\mu_2\sqrt{\beta^2-m_1^2}}{m_1+m_2}, & \eta_a = \eta - \dfrac{z\mu_1(m_1 m_2+\beta^2)}{(m_1+m_2)\sqrt{\beta^2-m_1^2}} \\[2ex] \xi_b = \xi - \dfrac{z\mu_2(m_1 m_2+\beta^2)}{(m_1+m_2)\sqrt{\beta^2-m_2^2}}, & \eta_b = \eta - \dfrac{z\mu_1\sqrt{\beta^2-m_2^2}}{m_1+m_2} \end{array} \right\} \qquad (33)$$

If the oblique coordinate system is chosen such that m_1 is less than and m_2 is greater than β, one of the asymptotes in equation (32) has a slope of opposite sign and the area τ is unbounded in the η_1 direction although ξ_a, η_a is still the point which determines the farthest extent of τ in the ξ_1 direction.

(a) x_1, y_1 plane.
(b) ξ_1, η_1 plane.
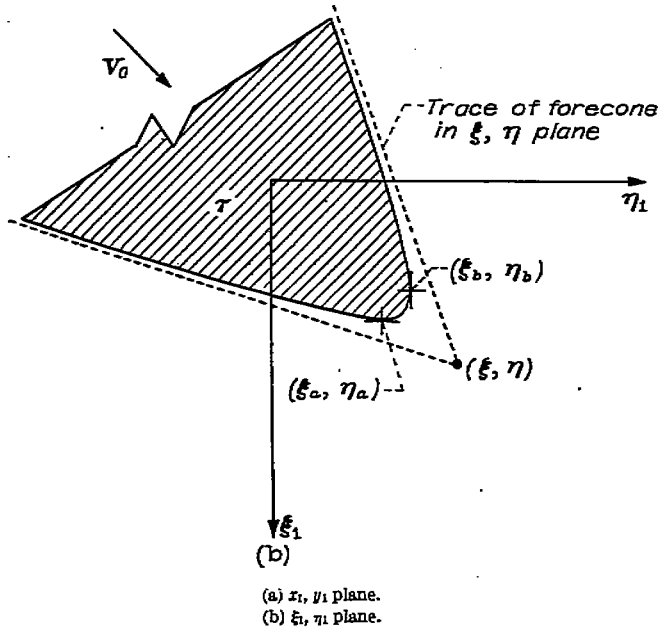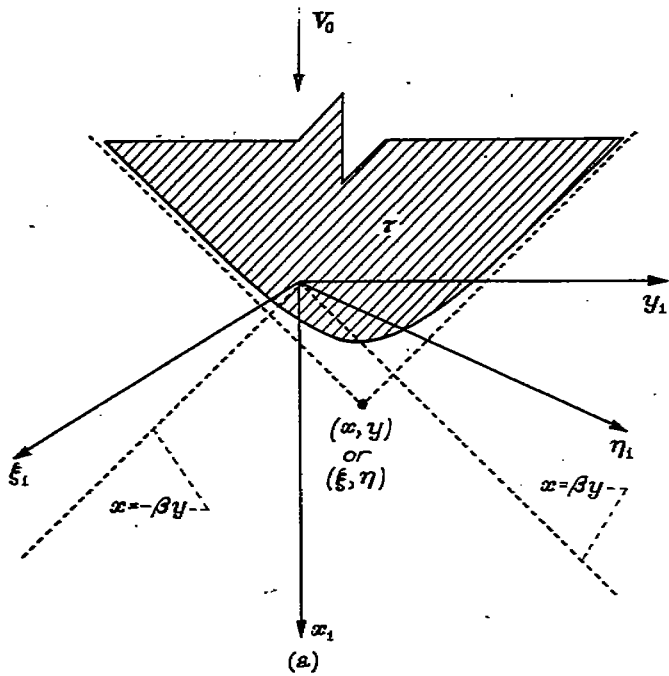
FIGURE 2.—Area of integration τ.

$$x = \frac{\beta}{M_0}(s+r), \qquad r = \frac{M_0}{2\beta}(x - \beta y)$$

$$y = \frac{1}{M_0}(s-r), \qquad s = \frac{M_0}{2\beta}(x + \beta y) \quad\Bigg\} \qquad (34)$$
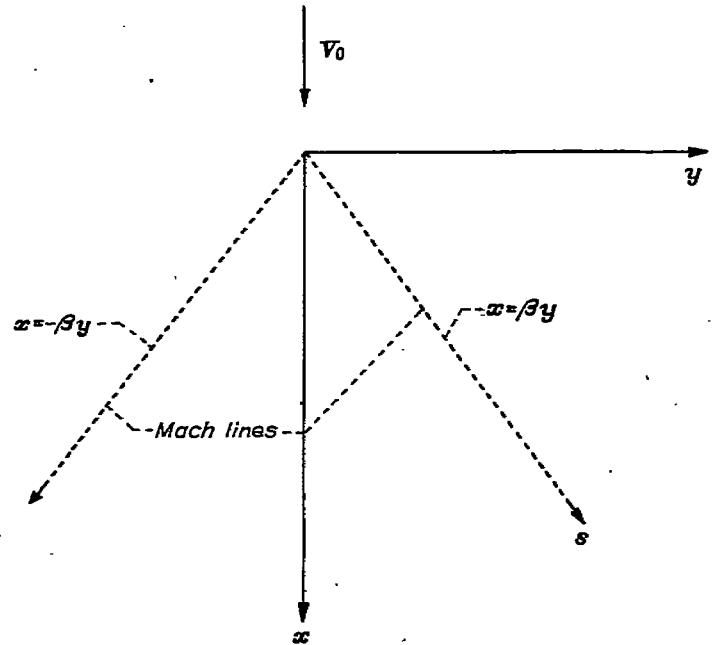
$$z = z, \qquad\qquad z = z$$

FIGURE 3.—Characteristic coordinate system.

When z is set equal to zero, the equation for r_c and the form of the area τ become especially simple. Thus

$$(r_c)_{z=0} = r_0 = \frac{2\beta}{M_0}\sqrt{(r - r_1)(s - s_1)} \qquad (35)$$

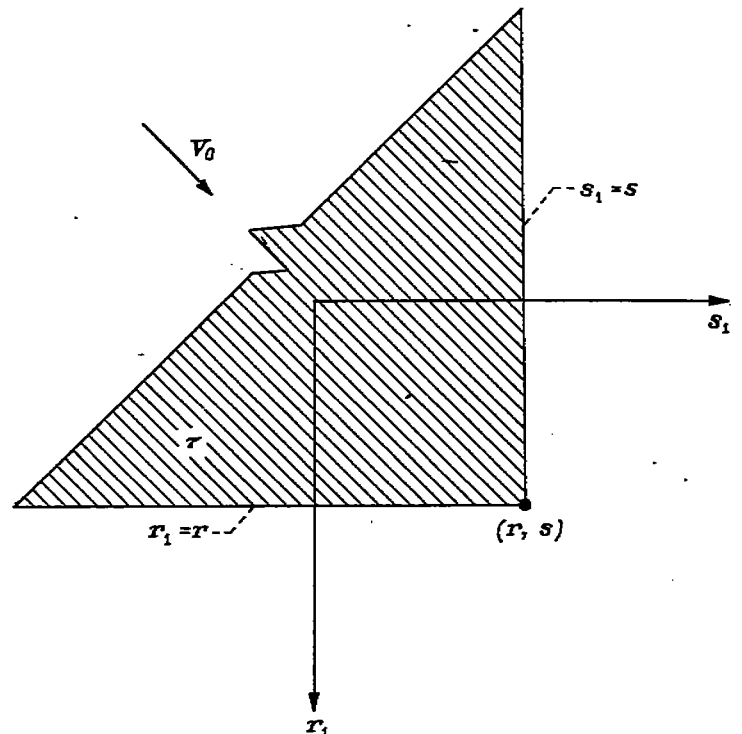and the area τ, shown in figure 4, is bounded by the straight lines $r_1 = r$, $s_1 = s$ and $r_1 = s_1 = -\infty$.

FIGURE 4.—Integration area in characteristic system for $z=0$.

On the other hand, if m_2 is less than and m_1 is greater than β, τ extends infinitely in the ξ_1 direction and ξ_b, η_b represents its upper bound in terms of η_1. Finally, if both m_1 and m_2 are greater than β, the area τ is not bounded for either negative or positive ξ_1 or η_1.

The characteristic coordinate system.—Another special case of the ξ, η, z system (the x, y, z system is, of course, a special case obtained when $m_1 = 0$ and $m_2 = \infty$) that is important enough to receive a particular notation is the one obtained when the ξ and η axes are parallel to the Mach lines (i. e., the traces of the characteristic cones) in the $z = 0$ plane. These axes are shown in figure 3, will be designated r and s, and are given by equations (29) when $m_1 = m_2 = \beta$ and $\mu_1 = \mu_2 = M_0$; thus

THE THREE FUNDAMENTAL FORMULAS IN SUBSONIC FLOW

The parallelism between the basic formulas in subsonic and supersonic wing theory is so obvious that it is advantageous to present first the somewhat more classical results applying to the purely subsonic regime. The immediate objective is therefore to present as briefly as possible the expressions for the perturbation velocity potential due to a distribution of sources, doublets, or elementary horseshoe vortices and then, by means of these expressions, to relate the pressure and vertical induced velocity in the $z=0$ plane to the weight of these distributions.

THE PERTURBATION POTENTIAL AT A POINT IN SPACE

The linearized form of the perturbation velocity potential due to a unit source, elementary horseshoe vortex, or doublet situated in a free stream moving at a uniform subsonic velocity V_o is given as follows:

Unit source_____ $\varphi=-1/4\pi r_c$
Unit elementary horseshoe vortex_ $\varphi=-zv_c/4\pi r_c$
Unit doublet_____ $\varphi=-z\beta^2/4\pi r_c^3$

where r_c is defined in equation (31b) and $\beta^2=1-M_0^2$.

It is well known that a distribution of sources in the $z=0$ plane splits the streamlines and forms a field symmetrical in u, v, and φ above and below the source plane. Hence, the strength of the sources is related to the term Δw (which, in turn, is related to the gradient of thickness of the simulated body) while the variables u, v, and φ are continuous. On the other hand, a distribution of elementary vortices or doublets causes a discontinuity in the streamwise induced velocity (or, what amounts to the same thing, the perturbation potential) across the reference plane but, at the same time, causes no division of the streamlines. The strengths of the vortices and doublets are therefore related to the terms Δu and $\Delta\varphi$, respectively, (which, in turn, are related to the wing loading), and produce no discontinuities in w. The exact analytical form of these distributions can be obtained readily from equation (2).

The source distribution.—The velocity potential induced by a distribution of sources over the $z=0$ plane follows immediately from equation (2) since, by symmetry, $\Delta\varphi$ must be zero. In practice, the area over which the sources are distributed is limited to the area S defined by the plan form. Hence,

$$\varphi(x,y,z)=\frac{-1}{4\pi}\iint_S \frac{\Delta w}{r_c}dx_1dy_1 \qquad (36)$$

The elementary-vortex distribution.—Equation (1) was written in terms of the perturbation potential φ. It could, however, after differentiation have been expressed in terms of any one of the induced velocity components and the solution in equation (2) would then also be expressed in terms of the particular velocity component chosen. Consider such a case, taking for the dependent variable the streamwise perturbation velocity instead of φ. Equation (2) then becomes

$$u(x,y,z)=\frac{-1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial\Delta u}{\partial z}\frac{1}{r_c}-\Delta u\frac{\beta^2z}{r_c^3}\right)dx_1dy_1$$

If the field is to be without sources, both Δw and $\partial\Delta w/\partial x$ vanish. But for an irrotational field the equality $\partial\Delta w/\partial x=\partial\Delta u/\partial z$ holds and the first term in the integrand of the above equation is zero. By definition

$$\int_{-\infty}^{x} u(x_1,y,z)dx_1=\varphi(x,y,z)$$

from which it follows that if the operator $\int_{-\infty}^{x} dx$ is applied to both sides of the resulting equation, the relation

$$\varphi(x,y,z)=\frac{-1}{4\pi}\iint_S \frac{z(x-x_1)\Delta u}{[(y-y_1)^2+z^2]r_c}dx_1dy_1 \qquad (37)$$

follows where the area of integration is limited to the wing plan form. This result expresses the perturbation velocity potential due to a distribution of elementary horseshoe vortices over a wing plan form in the $z=0$ plane.

The doublet distribution.—The solution for a doublet distribution, just as in the case of the sources, follows immediately from equation (2). Since the streamlines are not divided by the doublets the term containing Δw vanishes. The doublet distribution exists, however, not only over the wing area but also over the vortex wake streaming downstream behind the wing since the discontinuity in the potential persists in this region. Designating the wake area by W, the final expression for the perturbation potential associated with the doublet sheet becomes

$$\varphi(x,y,z)=\frac{\beta^2z}{4\pi}\iint_{S+W} \frac{\Delta\varphi}{r_c^3}dx_1dy_1 \qquad (38)$$

REDUCTION TO THE PLANE OF THE WING

The aerodynamicist is usually interested in the forces on the surface of the wing itself and, as a consequence, it is pertinent to consider each of the above formulas in the limiting case as z approaches zero. An explicit expression of these results is given below.

The source distribution.—The limiting value of equation (36) as z approaches zero is obtained immediately by simply setting z equal to zero. The resulting expression is

$$\varphi(x,y,0)=\frac{-1}{4\pi}\iint_S \frac{\Delta w}{r_0}dx_1dy_1 \qquad (39)$$

where

$$r_0=[(x-x_1)^2+\beta^2(y-y_1)^2]^{1/2}$$

Practical interest is usually concentrated on the relation between the pressure on the wing surface and the wing shape. Since in linearized theory pressure coefficient C_p and wing slope λ_u of the upper surface are known to be

$$C_p=\frac{-2u}{V_0}, \quad \lambda_u=\frac{w_u}{V_0}=\frac{1}{2}\frac{\Delta w}{V_0} \qquad (40)$$

it follows, after differentiation of equation (39) with respect to x, that pressure coefficient is

$$C_p=\frac{1}{\pi}\frac{\partial}{\partial x}\iint_S \frac{\lambda_u}{r_0}dx_1dy_1 \qquad (41)$$

The elementary-vortex distribution.—When the strength of the elementary horseshoe vortices is known over a wing plan form, there is no difficulty in finding the potential in the $z=0$ plane since it follows from a direct integration of the vortex strength. The pertinent question is, rather, to determine the vertical induced velocity in the plane of the wing from the given vortex strength. If load coefficient is defined in the usual way

$$\frac{\Delta p}{q}=\frac{2\Delta u}{V_0}=\frac{4u_u}{V_0} \qquad (42)$$

the answer to this question requires the evaluation of the following limiting process:

$$\frac{w}{V_0}=\lim_{z\to 0}\frac{-1}{8\pi}\iint_S\frac{\partial}{\partial z}\frac{z(x-x_1)}{[(y-y_1)^2+z^2]r_c}\frac{\Delta p}{q}\,dx_1dy_1$$

If, as in equation (27), the generalized principal part is introduced, the required expression becomes

$$\frac{w}{V_0}=\frac{-1}{8\pi}\underset{S}{\int\!\!\!\int}\frac{(x-x_1)}{(y-y_1)^2r_0}\frac{\Delta p}{q}\,dx_1dy_1 \qquad (43)$$

The doublet distribution.—In the case of doublet distributions, the relevant problem requires the expression of vertical induced velocity in the plane of the wing as a function of the doublet strength $\Delta\varphi$. If equation (38) is differentiated with respect to z and z is then set equal to zero, one finds without difficulty the final formula

$$\frac{w}{V_0}=\frac{\beta^2}{4\pi V_0}\iint_{S+w}\frac{\Delta\varphi}{r_0^3}\,dx_1dy_1 \qquad (44)$$

THE THREE FUNDAMENTAL FORMULAS IN SUPERSONIC FLOW

The purpose of this section is to repeat for supersonic wing theory the developments presented in the preceding section for subsonic theory. In order to maintain the formal analogy, it is necessary to introduce the concept of the finite part. This latter concept, in turn, introduces into the analysis integral expressions containing certain *inherent singularities*. Such singularities are, by definition, points across which the order of integration cannot be reversed.[3] The study of these singularities and their effect on the fundamental formulas suggests the introduction of an oblique coordinate system defined in the first section of this report. Hence, the following analysis will be presented in the ξ, η, z system, while transformation to the Cartesian and characteristic systems, it will be remembered, can be made by considering the special cases

$$\left.\begin{array}{lll} For & Let & \\ x,\,y,\,z & m_1=0,\ m_2=\infty & \\ r,\,s,\,z & m_1=m_2=\beta & \end{array}\right\} \qquad (45)$$

Following the development of the basic formulas, a summary of results will be given in terms of the x, y and r, s coordinates.

[3] More precisely, if an inherent singularity exists in the area over which a double integral is to be evaluated, the difference between the values of the integration made in one order and then over the same area but with reversed order is not zero.

THE PERTURBATION POTENTIAL AT A POINT IN SPACE

The velocity potential due to a unit source, a unit elementary horseshoe vortex, and a unit doublet is given as follows:

$$\left.\begin{array}{l} \text{Unit source}\ldots\ldots\ldots\ \varphi=-1/2\pi r_c \\ \text{Unit elementary horseshoe} \\ \qquad\text{vortex}\ldots\ldots\ldots\ldots\varphi=-zv_c/2\pi r_c \\ \text{Unit doublet}\ldots\ldots\ldots\ \varphi=z\beta^2/2\pi r_c^3 \end{array}\right\} \qquad (46)$$

where r_c and ν_c are defined in equations (31) and $\beta^2=M_0^2-1$. In this notation, the only difference between these expressions and the corresponding ones for subsonic flow is the factor 2. A much more important difference exists, however, in the effect of the change of sign of $1-M_0^2$ on r_c and ν_c.

The first task is the expression of the perturbation velocity potential at a point in space in terms of distributions of the elementary solutions and this can be accomplished by an appropriate analysis of equation (4). Just as in the subsonic case, the source strength is given by Δw, the vortex strength by Δu, and the doublet strength by $\Delta\varphi$.

The source distribution.—The potential induced by a source distribution can be obtained from equation (4) by setting the term containing $\Delta\varphi$ equal to zero (i. e., by removing all the circulation from the flow field). By means of the notation

$$\omega_c=\frac{x-x_1}{\beta\sqrt{(y-y_1)^2+z^2}}=\frac{\frac{m_1}{\mu_1}(\eta-\eta_1)+\frac{m_2}{\mu_2}(\xi-\xi_1)}{\beta\sqrt{\left[\frac{1}{\mu_1}(\eta-\eta_1)-\frac{1}{\mu_2}(\xi-\xi_1)\right]^2+z^2}} \qquad (47)$$

The remaining expression can be written in the ξ, η, z coordinate system simply as

$$\varphi(\xi,\eta,z)=\frac{-1}{2\pi\mu_1\mu_2}\left(\mu_2\frac{\partial}{\partial\xi}+\mu_1\frac{\partial}{\partial\eta}\right)\iint_\tau\Delta w(\xi_1,\eta_1)\text{ arc cosh }\omega_c d\xi_1d\eta_1 \qquad (48)$$

It is possible to "move" the partial derivative operations through the integral signs if the equality

$$\frac{\partial}{\partial x}\int_{b(x)}^{a(x)}f(x,y)\,dy=f(a,y)\frac{\partial a}{\partial x}-f(b,y)\frac{\partial b}{\partial x}+\int_b^a\frac{\partial f(x,y)}{\partial x}\,dy \qquad (49)$$

is used. Obviously, if the operation indicated by equation (49) is to be applied to equation (48), the order of integration and the limits on the latter equation must be specified.

Consider the case when the ξ and η axes are chosen so that both m_1 and m_2 are less than β. The area τ is then the one shown in figure 2. Further consider the case when the first integration is made with respect to η_1. Then, equation (48) becomes

$$\varphi(\xi,\eta,z)=\frac{-1}{2\pi\mu_1\mu_2}\left(\mu_2\frac{\partial}{\partial\xi}+\mu_1\frac{\partial}{\partial\eta}\right)$$
$$\int_{-\infty}^{\xi_a}d\xi_1\int_{\lambda_1}^{\lambda_2}d\eta_1\Delta w(\xi_1,\eta_1)\text{ arc cosh }\omega_c \qquad (50)$$

where λ_1 and λ_2 are roots of the quadratic expression $r_c^2(\eta_1)=0$ and ξ_a, η_a are defined in equation (33). It can be shown that when $\xi_1=\xi_a$, the equality $\lambda_1=\lambda_2$ applies, and, further, that the integrand of equation (50) does not vanish. Hence,

the two partial derivatives can be moved through the first integral sign without the appearance of the additional term involving the derivative of the upper limit. It is also not difficult to show the value of ω_c at η_1 equal to either λ_1 or λ_2 is just 1. Hence, the two partial derivatives can also be moved through the second integral sign without the appearance of additional terms. Finally, since

$$\frac{1}{m_1+m_2}\left(\mu_2\frac{\partial}{\partial\xi}+\mu_1\frac{\partial}{\partial\eta}\right)\text{ arc cosh }\omega_c=\frac{\partial}{\partial x}\text{ arc cosh }\omega_c=\frac{1}{r_c}$$

the velocity potential at a point (ξ,η,z) due to a distribution of sources in the $z=0$ plane can be written

$$\varphi(\xi,\eta,z)=-\frac{m_1+m_2}{2\pi\mu_1\mu_2}\iint_\tau\frac{\Delta w(\xi_1,\eta_1)}{r_c}d\xi_1 d\eta_1 \qquad (51)$$

The elementary-vortex distribution.—The potential induced by a distribution of elementary horseshoe vortices can be derived in a manner analogous to the derivation for the subsonic case. Thus, the solution given by equation (4) is written for the induced velocity u rather than the velocity potential.[4] Since the flow field contains no sources, $\partial\Delta u/\partial z$ is zero (by the same argument presented for the subsonic derivation) and the solution can be expressed in the form

$$u=\frac{1}{2\pi}\frac{\partial}{\partial x}\iint_\tau\frac{zv_c}{r_c}\Delta u(x_1,y_1)dx_1 dy_1$$

However, by definition

$$\varphi=\int_{-\infty}^x u\,dx$$

so the relation becomes

$$\varphi=\frac{1}{2\pi}\iint_\tau\frac{zv_c}{r_c}\Delta u(x_1,y_1)dx_1 dy_1$$

Finally, in terms of the ξ, η, z coordinate system the equation for the velocity potential due to a distribution of elementary horseshoe vortices in the $z=0$ plane can be written

$$\varphi=\frac{(m_1+m_2)z}{2\pi\mu_1\mu_2}\iint_\tau\frac{v_c}{r_c}\Delta u(\xi_1,\eta_1)d\xi_1 d\eta_1 \qquad (52)$$

The doublet distribution.—The potential induced by a sheet of doublets can be obtained from equation (4) by setting the term involving Δw equal to zero (i. e., by removing all sources from the flow field). Expressing the result in the ξ, η, z coordinates, one has

$$\varphi=\frac{z}{2\pi}\left(\frac{1}{\mu_2}\frac{\partial}{\partial\eta}+\frac{1}{\mu_1}\frac{\partial}{\partial\xi}\right)\iint_\tau\frac{v_c}{r_c}\Delta\varphi(\xi_1,\eta_1)d\xi_1 d\eta_1 \qquad (53)$$

Again it is necessary to carry the partial derivative operations through the two integral signs. And again, just as was the case for the sources, this requires that the order of integration and the limits on the integrals be specified.

[4] The fact that equation (4) can be written for u as well as φ does not follow immediately in the case of supersonic flow because of the presence of the discontinuities in u along Mach waves emanating from supersonic edges. Proof of the validity for cases of interest herein is given in reference 6.

Once more consider the case when m_1 and m_2 are both less than β, the area τ is the same as that shown in figure 2, and the first integration is made with respect to η_1. Then equation (53) becomes

$$\varphi=\frac{z}{2\pi}\left(\frac{1}{\mu_2}\frac{\partial}{\partial\eta}+\frac{1}{\mu_1}\frac{\partial}{\partial\xi}\right)\int_{-\infty}^{\xi_a}d\xi_1\int_{\lambda_1}^{\lambda_2}d\eta_1\frac{v_c}{r_c}\Delta\varphi(\xi_1,\eta_1) \qquad (54)$$

More caution is necessary in moving the derivatives through the integrals than was required in the study of the source case, since at $\xi_1=\xi_a$ the η_1 integral is *indeterminate*. It is true that the interval of integration λ_2, λ_1 is zero, but it is also true that the integrand is infinite. The value of such an indeterminate form must be obtained by some process such as the following.

The upper limit to the ξ_1 integral, ξ_a, is a function of the two variables z and ξ (see equation (33)); that is, in functional notation $\xi_a=\xi_a(\xi,z)$. Replace z in ξ_a by $\sqrt{z^2+\epsilon^2}$, then when ϵ is zero ξ_a has not changed. But if $\xi_a(\xi,z)$ is replaced by $\xi_a(\xi,\sqrt{z^2+\epsilon^2})$ in equation (54) and the *limit taken as ϵ approaches zero*, the indeterminate form mentioned above can be evaluated. Hence, consider

$$\varphi(\xi,\eta,z)=\lim_{\epsilon\to0}\frac{z}{2\pi}\left(\frac{1}{\mu_2}\frac{\partial}{\partial\eta}+\frac{1}{\mu_1}\frac{\partial}{\partial\xi}\right)$$
$$\int_{-\infty}^{\xi_a(\xi,\sqrt{z^2+\epsilon^2})}d\xi_1\int_{\lambda_1}^{\lambda_2}d\eta_1\frac{v_c}{r_c}\Delta\varphi(\xi_1,\eta_1) \qquad (55)$$

By applying the operation described by equation (49) and letting a primed function symbolize its value at $\xi_1=\xi_a(\xi,\sqrt{z^2+\epsilon^2})$, the expression

$$\varphi(\xi,\eta,z)=\lim_{\epsilon\to0}\frac{z}{2\pi\mu_1}\int_{\lambda_1'}^{\lambda_2'}d\eta_1\frac{v_c'}{r_c'}\Delta\varphi'(\xi_a,\eta_1)+$$
$$\frac{z}{2\pi}\int_{-\infty}^{\xi_a}d\xi_1\left(\frac{1}{\mu_2}\frac{\partial}{\partial\eta}+\frac{1}{\mu_1}\frac{\partial}{\partial\xi}\right)\int_{\lambda_1}^{\lambda_2}d\eta_1\frac{v_c}{r_c}\Delta\varphi(\xi_1,\eta_1) \qquad (56)$$

is obtained.

The first term in equation (56) can be greatly simplified by use of the mean value theorem. In the limit as ϵ goes to zero both λ_1' and λ_2' approach the common value of η_a. Thus, for ϵ very small the variation of $\Delta\varphi'$ and v_c' in the range $\lambda_1'<\eta_1<\lambda_2'$ is slight. The same cannot, of course, be said of $1/r_c'$, since λ_1' and λ_2' are the roots of $r_c'=0$. Using the functional notation $v_c'=v_c'(\eta_1)$ and applying the mean value theorem, one can write for the first term in equation (56)

$$\lim_{\epsilon\to0}\frac{z}{2\pi\mu_1}v_c'(\theta)\Delta\varphi'(\xi_a,\theta)\int_{\lambda_1'}^{\lambda_2'}\frac{d\eta_1}{r_c'}, \quad \lambda_1'<\theta<\lambda_2' \qquad (57)$$

Now from the definition of r_c given by equation (31a)

$$\int_{\lambda_1'}^{\lambda_2'}\frac{d\eta_1}{r_c'}=\frac{\mu_1}{\sqrt{\beta^2-m_1^2}}\int_{\lambda_1'}^{\lambda_2'}\frac{d\eta_1}{\sqrt{(\lambda_2'-\eta_1)(\eta_1-\lambda_1')}}=\frac{\pi\mu_1}{\sqrt{\beta^2-m_1^2}}$$

which is independent of ϵ. Hence, the expression given in equation (57) reduces to

$$\frac{1}{2}\Delta\varphi(\xi_a,\eta_a)$$

Moving the partial derivatives through the second integral sign in the last term of equation (56) can be accomplished by introducing the finite-part sign defined previously. Since

$$\frac{\mu_1\mu_2}{m_1+m_2}\left(\frac{1}{\mu_2}\frac{\partial}{\partial\eta}+\frac{1}{\mu_1}\frac{\partial}{\partial\xi}\right)\frac{\nu_c}{r_c}=\frac{-\beta^2}{r_c^3}$$

it follows that

$$\left(\frac{1}{\mu_2}\frac{\partial}{\partial\eta}+\frac{1}{\mu_1}\frac{\partial}{\partial\xi}\right)\int_{\lambda_1}^{\lambda_2}d\eta_1\frac{\nu_c}{r_c}\Delta\varphi(\xi_1,\eta_1)=$$
$$-\int_{\lambda_1}^{\lambda_2}\frac{\Delta\varphi(\xi_1,\eta_1)d\eta_1}{r_c^3}\left(\frac{m_1+m_2}{\mu_1\mu_2}\right)$$

By means of these equalities, equation (54) can finally be written in the form

$$\varphi(\xi,\eta,z)=\frac{1}{2}\Delta\varphi(\xi_a,\eta_a)-\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\int d\xi_1\!\!\!\!\!\!\fint_\tau d\eta_1\frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}$$

(58)

If the area of integration is not changed but if the order of integration is reversed, it can be shown by a process identical to the one just described that

$$\varphi(\xi,\eta,z)=\frac{1}{2}\Delta\varphi(\xi_b,\eta_b)-\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\int d\eta_1\!\!\!\!\!\!\fint_\tau d\xi_1\frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}$$

(59)

The area τ used in equations (58) and (59) has been defined as that shown in figure 2, the axes ξ and η both chosen so as to lie outside the Mach cone from the origin of the Cartesian system. As the ξ and η axes approach the Mach lines in the x, y plane, that is, as m_2 and m_1 approach β, the residual terms in equations (58) and (59) approach ($\frac{1}{2}$) $\Delta\varphi(\xi_a, -\infty)$ and ($\frac{1}{2}$)$\Delta\varphi(-\infty, \eta_b)$, respectively, which represent the jump in potential infinitely far ahead of and to one side of the point (ξ, η, z) at which the potential is being measured. Hence, in aerodynamic applications, the ($\frac{1}{2}$)$\Delta\varphi(\xi_a, -\infty)$ and ($\frac{1}{2}$)$\Delta\varphi(-\infty, \eta_b)$ can be taken as zero. Thus, when the ξ, η axes lie along the Mach lines, thereby becoming the r, s axes of equations (34), the expressions for φ are without the residue terms and the order of integration is immaterial. When m_1 and m_2 are greater than β the same is true (i. e., the terms ($\frac{1}{2}$)$\Delta\varphi(\xi_a, \eta_a)$ and ($\frac{1}{2}$)$\Delta\varphi(\xi_b, \eta_b)$ are missing from equations (58) and (59) ,respectively) so that the effect of a distribution of doublets on the velocity potential can be summarized as follows:

For $0\leq m_1<\beta,\ 0\leq m_2<\beta$

$$\varphi=\frac{1}{2}\Delta\varphi(\xi_a,\eta_a)-\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\fint d\xi_1\!\!\!\!\fint_\tau d\eta_1\frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}$$ (60a)

$$=\frac{1}{2}\Delta\varphi(\xi_b,\eta_b)-\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\fint d\eta_1\!\!\!\!\fint_\tau d\xi_1\frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}$$ (60b)

For $0\leq m_1<\beta,\ \beta\leq m_2\leq\infty$

$$\varphi=\frac{1}{2}\Delta\varphi(\xi_a,\eta_a)-\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\fint d\xi_1\!\!\!\!\fint_\tau d\eta_1\frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}.$$ (60c)

$$=-\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\fint d\eta_1\!\!\!\!\fint_\tau d\xi_1\frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}$$ (60d)

For $\beta\leq m_1\leq\infty,\ 0\geq m_2<\beta$.

$$\varphi=\frac{1}{2}\Delta\varphi(\xi_b,\eta_b)-\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\fint d\eta_1\!\!\!\!\fint_\tau d\xi_1\frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}$$ (60e)

$$=-\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\fint d\xi_1\!\!\!\!\fint_\tau d\eta_1\frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}$$ (60f)

For $\beta\leq m_1\leq\infty,\ \beta\leq m_2\leq\infty$

$$\varphi=-\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2}\iint_\tau\frac{\Delta\varphi(\xi_1,\eta_1)}{r_c^3}d\xi_1 d\eta_1$$ (60g)

There exists an interesting corollary obtained by subtracting equation (60a) from (60b); namely, that the difference between an integration of supersonic doublets made first in one order and then in the reverse order is equal to the difference in the magnitude of the distribution at two points in the plane.

REDUCTION TO THE PLANE OF THE WING

The next problem is to consider the above formulas in the limiting case as z approaches zero.

The source distribution.—The potential in the plane of the disturbing source sheets follows immediately from equation (51) by simply setting z equal to zero; in this way

$$\varphi(\xi,\eta,0)=\frac{-(m_1+m_2)}{2\pi\mu_1\mu_2}\iint_\tau\frac{\Delta w(\xi_1,\eta_1)}{r_0}d\xi_1 d\eta_1$$ (61)

In order to relate the pressure coefficient C_p to the slope λ_u of the upper surface of the wing (where both C_p and λ_u are defined in equation (40)) the operator

$$-\frac{2}{V_0}\frac{\partial}{\partial x}=-\frac{2\mu_1\mu_2}{V_0(m_1+m_2)}\left(\frac{1}{\mu_1}\frac{\partial}{\partial\xi}+\frac{1}{\mu_2}\frac{\partial}{\partial\eta}\right)$$

must be applied to both sides of equation (61). Hence,

$$C_p=\frac{2}{\pi}\left(\frac{1}{\mu_1}\frac{\partial}{\partial\xi}+\frac{1}{\mu_2}\frac{\partial}{\partial\eta}\right)\iint_\tau\frac{\lambda_u(\xi_1,\eta_1)}{r_0}d\xi_1 d\eta_1$$ (62)

The task of moving the partial derivatives through the two integral signs presents a problem identical to the one studied in reducing equation (53) to equations (60). Two inherent singularities in the area occur at the points ξ_a, η_a and ξ_b, η_b and form, just as in equations (60), certain residuals there. In the present discussion, interest is confined to the case when $z=0$. Equation (33) shows immediately

that the values of ξ_a, η_a and ξ_b, η_b for z equal to zero can be written

$$(\xi_a)_{z=0}=(\xi_b)_{z=0}=\xi \\ (\eta_a)_{z=0}=(\eta_b)_{z=0}=\eta \Bigg\} \qquad (63)$$

It can be shown, therefore (the details being omitted since they are precisely the same as those described in the reduction of equation (53)), that the following relations hold:

For $0 \leq m_1 < \beta, 0 \leq m_2 < \beta$

$$C_p = \frac{2}{\sqrt{\beta^2 - m_1^2}} \lambda_u(\xi, \eta) -$$

$$\frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \oint d\xi_1 \int_\tau d\eta_1 \frac{\frac{m_1}{\mu_1}(\eta-\eta_1)+\frac{m_2}{\mu_2}(\xi-\xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \qquad (64a)$$

$$= \frac{2}{\sqrt{\beta^2 - m_2^2}} \lambda_u(\xi, \eta) -$$

$$\frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \oint d\eta_1 \int_\tau d\xi_1 \frac{\frac{m_1}{\mu_1}(\eta-\eta_1)+\frac{m_2}{\mu_2}(\xi-\xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \qquad (64b)$$

For $0 \leq m_1 < \beta, \beta \leq m_2 \leq \infty$

$$C_p = \frac{2}{\sqrt{\beta^2 - m_1^2}} \lambda_u(\xi, \eta) -$$

$$\frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \oint d\xi_1 \int_\tau d\eta_1 \frac{\frac{m_1}{\mu_1}(\eta-\eta_1)+\frac{m_2}{\mu_2}(\xi-\xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \qquad (64c)$$

$$= -\frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \oint d\eta_1 \int_\tau d\xi_1 \frac{\frac{m_1}{\mu_1}(\eta-\eta_1)+\frac{m_2}{\mu_2}(\xi-\xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \qquad (64d)$$

For $\beta \leq m_1 \leq \infty, 0 \leq m_2 < \beta$

$$C_p = \frac{2}{\sqrt{\beta^2 - m_2^2}} \lambda_u(\xi, \eta) -$$

$$\frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \oint d\eta_1 \int_\tau d\xi_1 \frac{\frac{m_1}{\mu_1}(\eta-\eta_1)+\frac{m_2}{\mu_2}(\xi-\xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \qquad (64e)$$

$$= -\frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \oint d\xi_1 \int_\tau d\eta_1 \frac{\frac{m_1}{\mu_1}(\eta-\eta_1)+\frac{m_2}{\mu_2}(\xi-\xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \qquad (64f)$$

For $\beta \leq m_1 \leq \infty, \beta \leq m_2 \leq \infty$

$$C_p = -\frac{2(m_1+m_2)}{\pi\mu_1\mu_2} \iint_\tau \frac{\frac{m_1}{\mu_1}(\eta-\eta_1)+\frac{m_2}{\mu_2}(\xi-\xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) d\xi_1 d\eta_1 \qquad (64g)$$

The elementary-vortex distribution.—The reduction of equation (52), the formula expressing the potential due to a sheet of elementary horseshoe vortices, to the plane of the wing is of little interest since if Δu is known $\varphi_{z=0}$ can be obtained by the simple relation $\varphi_{z=0} = \frac{1}{2}\int_{-\infty}^x \Delta u \, dx$. However,

if the derivative with respect to z is determined on both sides of equation (52) and the limit is found as z approaches zero, the vertical induced velocity in the plane of the wing will be related to the vortex strength there. In a more physical sense, this will relate the slope of the lifting surface to the load distribution it supports. Such an equation is of basic importance.

The mathematical expression to be studied is

$$w = \lim_{z\to 0} \frac{\partial}{\partial z} \frac{(m_1+m_2)V_0}{4\pi\mu_1\mu_2} \iint_\tau \frac{z\nu_c}{r_c} \frac{\Delta p(\xi_1, \eta_1)}{q} d\xi_1 d\eta_1 \qquad (65)$$

The evaluation of w can be divided into two steps: First, the procedure necessary in order to carry the derivative through the first integral and, second, the calculation of I where

$$I = \lim_{z\to 0} \frac{(m_1+m_2)V_0}{4\pi\mu_1\mu_2} \int \frac{\partial}{\partial z} \int_\tau \frac{z\nu_c}{r_c} \frac{\Delta p(\xi_1, \eta_1)}{q} d\xi_1 d\eta_1 \qquad (66)$$

Again the order of integration is important. As in the preceding discussion of the doublet sheet (equations (53) through (60)) and the source sheet (equations (61) through (64)), assume first that the area of integration is the one given in figure 2. Thus, m_1 is less than β and the η_1 integration is performed first.

If ϵ is introduced in order to evaluate the indeterminate form, equation (65) is expressible in the form

$$w = \lim_{\substack{z\to 0 \\ \epsilon\to 0}} \frac{(m_1+m_2)V_0}{4\pi\mu_1\mu_2} \frac{\partial}{\partial z} \int_{-\infty}^{\xi_a(\xi, \sqrt{z^2+\epsilon^2})} d\xi_1 \int_{\lambda_1}^{\lambda_2} d\eta_1 \frac{z\nu_c}{r_c} \frac{\Delta p}{q} \qquad (67)$$

where the terms λ_1 and λ_2 are the roots of the quadratic $r_c^2 = 0$ and where the limiting process for ϵ must be performed before that for z. As before, the primed expression denotes values for the particular case $\xi_1 = \xi_a(\xi, \sqrt{z^2+\epsilon^2})$, and equation (67) reduces to

$$w = \lim_{\substack{z\to 0 \\ \epsilon\to 0}} \left(\frac{\partial \xi_a}{\partial z}\right) \frac{(m_1+m_2)V_0}{4\pi\mu_1\mu_2} \int_{\lambda_1'}^{\lambda_2'} \frac{z\nu_c'}{r_c'} \left(\frac{\Delta p}{q}\right)' d\eta_1 + I \qquad (68)$$

By means of the mean value theorem, the first term on the right-hand side of the equation can be simplified. The procedure involved in such an analysis was outlined in the derivation of equation (57). The process used here is identical and equation (68) becomes

$$w = \frac{V_0\sqrt{\beta^2 - m_1^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + I \qquad (69)$$

The term I, defined by equation (66), can be expressed in a simple form by introducing the notation for the generalized principal part (see equation (27)). This term becomes

$$I = \frac{(m_1+m_2)V_0}{4\pi\mu_1\mu_2} \oint d\xi_1 \oint d\eta_1 \frac{\nu_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \qquad (70)$$

If the integration of the above expressions had been taken in the opposite order or if the range of m_1 and m_2 had been different, the residual term would change. This phenomena has been presented in connection with both doublet and

source distributions and is by now familiar. Finally, therefore, expressions relating the slope of the wing surface to its loading can be written:

For $0 \leq m_1 < \beta$, $0 \leq m_2 < \beta$

$$\frac{w}{V_0} = -\frac{\sqrt{\beta^2 - m_1^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \int d\xi_1 \int_\tau d\eta_1 \frac{\nu_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \tag{71a}$$

$$= -\frac{\sqrt{\beta^2 - m_2^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \int d\eta_1 \int_\tau d\xi_1 \frac{\nu_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \tag{71b}$$

For $0 \leq m_1 < \beta$, $\beta \leq m_2 \leq \infty$

$$\frac{w}{V_0} = -\frac{\sqrt{\beta^2 - m_1^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \int d\xi_1 \int_\tau d\eta_1 \frac{\nu_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \tag{71c}$$

$$= \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \int d\eta_1 \int_\tau d\xi_1 \frac{\nu_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \tag{71d}$$

For $\beta \leq m_1 \leq \infty$, $0 \leq m_2 < \beta$

$$\frac{w}{V_0} = -\frac{\sqrt{\beta^2 - m_2^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \int d\eta_1 \int_\tau d\xi_1 \frac{\nu_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \tag{71e}$$

$$= \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \int d\xi_1 \int_\tau d\eta_1 \frac{\nu_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \tag{71f}$$

For $\beta \leq m_1 \leq \infty$, $\beta \leq m_2 \leq \infty$

$$\frac{w}{V_0} = \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \int_\tau \int \frac{\nu_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} d\xi_1 d\eta_1 \tag{71g}$$

The doublet distribution.—The pertinent problem in this case is to find the vertical induced velocity in the plane of the wing as a function of the jump in potential across the plane.

Consider equations (60) and take the partial derivative of both sides with respect to z; then find the limit of the resulting expression as z approaches zero. If equation (60a) is used, for example, there results for the first term

$$\lim_{z \to 0} \frac{\partial}{\partial z} \frac{1}{2} \Delta\varphi(\xi_a, \eta_a) = \frac{1}{2}\left(\frac{\partial\Delta\varphi}{\partial\xi_a}\right)_{z=0} \frac{\partial\xi_a}{\partial z} + \frac{1}{2}\left(\frac{\partial\Delta\varphi}{\partial\eta_a}\right)_{z=0} \frac{\partial\eta_a}{\partial z}$$

which becomes

$$\frac{-1}{2(m_1+m_2)\sqrt{\beta^2 - m_1^2}}$$
$$\left[\mu_2(\beta^2 - m_1^2) \frac{\partial}{\partial\xi}\Delta\varphi(\xi, \eta) + \mu_1(m_1 m_2 + \beta^2) \frac{\partial}{\partial\eta}\Delta\varphi(\xi, \eta)\right] \tag{72}$$

The second term can be written

$$-\lim_{z \to 0} \frac{\partial}{\partial z} \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_\tau d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3}$$

and this reduces to

$$-\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_\tau d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_0^3} -$$
$$\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \lim_{z \to 0} z \frac{\partial}{\partial z} \int d\xi_1 \int_\tau d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} \tag{73}$$

Since, by equation (17),

$$\int_{\lambda_1}^{\lambda_2} \frac{d\eta_1}{[(\lambda_2 - \eta_1)(\eta_1 - \lambda_1)]^{3/2}} = 0$$

the second term in expression (73) vanishes. Finally, therefore, the vertical induced velocity in the plane of the wing becomes:

For $0 \leq m_1 < \beta$, $0 \leq m_2 < \beta$

$$w = -\frac{1}{2(m_1+m_2)\sqrt{\beta^2 - m_1^2}}\left[\mu_2(\beta^2 - m_1^2) \frac{\partial}{\partial\xi}\Delta\varphi(\xi, \eta) + \mu_1(m_1 m_2 + \beta^2) \frac{\partial}{\partial\eta}\Delta\varphi(\xi, \eta)\right] - \frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_\tau d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_0^3} \tag{74a}$$

$$= -\frac{1}{2(m_1+m_2)\sqrt{\beta^2 - m_2^2}}\left[\mu_2(m_1 m_2 + \beta^2) \frac{\partial}{\partial\xi}\Delta\varphi(\xi, \eta) + \mu_1(\beta^2 - m_2^2) \frac{\partial}{\partial\eta}\Delta\varphi(\xi, \eta)\right] - \frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\eta_1 \int_\tau d\xi_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_0^3} \tag{74b}$$

For $0 \leq m_1 < \beta$, $\beta \leq m_2 \leq \infty$

$$w = -\frac{1}{2(m_1+m_2)\sqrt{\beta^2 - m_1^2}}\left[\mu_2(\beta^2 - m_1^2) \frac{\partial}{\partial\xi}\Delta\varphi(\xi, \eta) + \mu_1(m_1 m_2 + \beta^2) \frac{\partial}{\partial\eta}\Delta\varphi(\xi, \eta)\right] - \frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_\tau d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_0^3} \tag{74c}$$

$$= -\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\eta_1 \int_\tau d\xi_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_0^3} \tag{74d}$$

For $\beta \leq m_1 \leq \infty$, $0 \leq m_2 < \beta$

$$w = -\frac{1}{2(m_1+m_2)\sqrt{\beta^2 - m_2^2}}\left[\mu_2(m_1 m_2 + \beta^2) \frac{\partial}{\partial\xi}\Delta\varphi(\xi, \eta) + \mu_1(\beta^2 - m_2^2) \frac{\partial}{\partial\eta}\Delta\varphi(\xi, \eta)\right] - \frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\eta_1 \int_\tau d\xi_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_0^3} \tag{74e}$$

$$= -\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_\tau d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_0^3} \tag{74f}$$

For $\beta \leq m_1 \leq \infty$, $\beta \leq m_2 \leq \infty$

$$w = -\frac{\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int_\tau \int \frac{\Delta\varphi(\xi_1, \eta_1)}{r_0^3} d\xi_1 d\eta_1 \tag{74g}$$

SUMMARY OF RESULTS FOR φ, C_p, AND, w IN THE PLANE OF THE WING; CARTESIAN-COORDINATE SYSTEM

The special forms of equations (64), (71), and (74) when the ξ,η axes become the Cartesian axes are given in the following sections. In these cases, $\xi \to x$, $\eta \to y$, $m_1 \to 0$, and $m_2 \to \infty$.

Potential in terms of vertical velocity, nonlifting case.—

$$\varphi(x,y) = -\frac{1}{\pi} \int_\tau \int \frac{w_u(x_1,y_1)}{\sqrt{(x-x_1)^2 - \beta^2(y-y_1)^2}} \, dx_1\,dy_1 \qquad (75)$$

Pressure coefficient in terms of surface slope, nonlifting case.—

$$C_p = \frac{2}{\beta} \lambda_u(x,y) - \frac{2}{\pi} \int dx_1 \int_\tau dy_1 \frac{(x-x_1)\lambda_u(x_1,y_1)}{[(x-x_1)^2 - \beta^2(y-y_1)^2]^{3/2}} \qquad (76a)$$

$$C_p = -\frac{2}{\pi} \int dy_1 \int_\tau dx_1 \frac{(x-x_1)\lambda_u(x_1,y_1)}{[(x-x_1)^2 - \beta^2(y-y_1)^2]^{3/2}} \qquad (76b)$$

Vertical velocity in terms of loading coefficient, lifting case.—

$$\frac{w_u}{V_0} = -\frac{\beta}{4} \frac{\Delta p(x,y)}{q} + \frac{1}{4\pi} \int dx_1 \int_\tau dy_1 \frac{(x-x_1)\frac{\Delta p(x_1,y_1)}{q}}{(y-y_1)^2 \sqrt{(x-x_1)^2 - \beta^2(y-y_1)^2}} \qquad (77a)$$

$$\frac{w_u}{V_0} = \frac{1}{4\pi} \int dy_1 \int_\tau dx_1 \frac{(x-x_1)\frac{\Delta p(x_1,y_1)}{q}}{(y-y_1)^2 \sqrt{(x-x_1)^2 - \beta^2(y-y_1)^2}} \qquad (77b)$$

Vertical velocity in terms of surface potential, lifting case.—

$$w_u = -\frac{\beta}{2} \Delta u(x,y) - \frac{\beta^2}{\pi} \int dx_1 \int_\tau dy_1 \frac{\varphi_u(x_1,y_1)}{[(x-x_1)^2 - \beta^2(y-y_1)^2]^{3/2}} \qquad (78a)$$

$$w_u = -\frac{\beta^2}{\pi} \int dy_1 \int_\tau dx_1 \frac{\varphi_u(x_1,y_1)}{[(x-x_1)^2 - \beta^2(y-y_1)^2]^{3/2}} \qquad (78b)$$

SUMMARY OF RESULTS FOR φ, C_p, AND w IN THE PLANE OF THE WING; CHARACTERISTIC COORDINATE SYSTEM

The special forms of equations (64), (71), and (74) when the ξ, η axes become the characteristic axes are given in the following sections. In these cases, $\xi \to r$, $\eta \to s$, $m_1 \to \beta$, and $m_2 \to \beta$.

Potential in terms of vertical velocity, nonlifting case —

$$\varphi = -\frac{1}{\pi M_0} \int_\tau \int \frac{w_u(r_1,s_1)}{\sqrt{(r-r_1)(s-s_1)}} \, dr_1\,ds_1 \qquad (79)$$

Pressure coefficient in terms of surface slope, nonlifting case.—

$$C_p = -\frac{1}{2\pi\beta} \int_\tau \int \frac{(r-r_1)+(s-s_1)}{[(r-r_1)(s-s_1)]^{3/2}} \lambda_u(r_1,s_1) \, dr_1\,ds_1 \qquad (80)$$

Vertical velocity in terms of loading coefficient, lifting case.—

$$\frac{w_u}{V_0} = \frac{\beta}{4\pi} \int_\tau \int \frac{(r-r_1)+(s-s_1)}{[(s-s_1)-(r-r_1)]^2 \sqrt{(r-r_1)(s-s_1)}} \frac{\Delta p(r_1,s_1)}{q} \, dr_1\,ds_1 \qquad (81)$$

Vertical velocity in terms of surface potential, lifting case.—

$$w_u = -\frac{M_0}{4\pi} \int_\tau \int \frac{\varphi_u(r_1,s_1)}{[(r-r_1)(s-s_1)]^{3/2}} \, dr_1\,ds_1 \qquad (82)$$

PART II—THE DIRECT PROBLEM

DISCUSSION

The term "direct problem" shall be defined herein as a problem requiring for its solution the evaluation of integrals with known integrands. The three fundamental formulas presented in part I, equations (64), (71), and (74), apply, respectively, to source, vortex, and doublet distributions, and a consideration of them shows that there are essentially only two different boundary-value problems of wing theory that lead to the direct classification. In the first of these problems (see equation (64)), the pressure coefficient is given by an integral involving the shape of a wing having thickness, but no angle of attack, twist, or camber. The other direct problem is represented by equations (71) and (74), where the angle of attack, twist, or camber of a wing having no thickness is given in terms of an integration involving, respectively, the wing loading or its streamwise integral, the discontinuity in velocity potential. The circumstances of the particular problem will determine which of the two alternative formulas is to be used.

THE AERODYNAMIC PROBLEM

The statement of the two problems can be given from a physical viewpoint as follows.

The thickness case.—The thickness of a wing that is symmetrical above and below a horizontal plane is given and the pressure distribution over the wing is to be determined. Such problems are of special interest in the study of wings in a supersonic flow since their evaluation is necessary for the calculation of the wave drag.

The lifting case.—The load distribution on a lifting plane, a surface without thickness, is given and the slope of the surface that will support such a loading is to be determined.

THE MATHEMATICAL PROBLEM

The mathematical statement of the two problems can be made by referring to the equations expressing the three fundamental formulas in the plane of the wing. For the thickness case, equations (41) or (64) apply, λ_u is given over the wing plan form and C_p is to be determined. For the lifting case, equations (43), (44), (71) or (74) apply, $\Delta p/q$ or $\Delta \varphi$ is given over the wing plan form and w_u is to be determined.

The solution of problems in the direct classification depends only on the analyst's facility in evaluating integrals. Although the integrations may be quite difficult to perform, this nevertheless must be regarded as a question of technique and, in a mathematical sense, a direct problem is solved.

PART III—THE INVERSE PROBLEM

INTRODUCTION

The term "inverse problem" shall be defined herein as a problem requiring for its solution the inversion of an integral equation. In application to the study of problems in aerodynamic wing theory, two different boundary-value problems appear in the inverse classification. These are provided, as was discussed in the presentation of the direct problem, by the two basic relationships that exist in the three fundamental formulas. In those equations the two basic relationships are: First, the pressure is given in terms of an integration involving the shape of a wing having thickness but with no angle of attack, twist, or camber; second, the angle of attack, twist, and camber of a wing having no thickness is given in terms of an integration involving either the wing loading or the discontinuity in the velocity potential.

THE AERODYNAMIC PROBLEM

The physical interpretation of the two types of inverse boundary-value problems is made as follows.

The thickness case.—The pressure distribution over a wing that is symmetrical above and below a horizontal plane is given and the shape of the wing is to be determined. To this bare statement of the problem, however, must be added certain auxiliary considerations. For example, it is physically evident that solutions yielding wings with negative volumes must be excluded. Consideration must also be given to the question of wing closure. It is apparent that these two conditions will serve to restrict the arbitrariness of the pressure distributions which can be prescribed. Finally, the question of the uniqueness of the wing shape arises. For example, it is known that the thin-airfoil-theory solution in the two-dimensional case is unique, provided the prescribed pressure distribution is one leading to a real and closed wing section. In the supersonic three-dimensional case, however, these conditions are no longer sufficient to guarantee a unique shape from a given pressure distribution (although the reverse is always true, i. e., a given shape produces a unique pressure). This fact will be illustrated later (Part IV) in connection with quasi-conical flow problems.

The lifting case.—The slope of a lifting plate, a surface without thickness, is given and the resulting load distribution is to be determined. To insure uniqueness in problems of this type it is sometimes necessary to impose an additional condition. For example, it is necessary to assume that the Kutta condition applies to all trailing edges for which the normal component of the free-stream velocity is subsonic.

THE MATHEMATICAL PROBLEM

The mathematical statement of the two problems can be made at once. Thus, for the thickness case, equations (41) or (64) apply, C_p is given over the area occupied by the wing plan form and λ_u is to be determined. For the lifting case, equations (43), (44), (71), or (74) apply, w_u is given over the wing plan form and $\Delta p/q$ is to be determined.

Of course, by definition, the solutions of both of these problems require the inversion of an integral equation. Further, these particular equations are known as singular

integral equations. Complete inversions to all the cases considered have not as yet been obtained. Some progress has been made, however, and the following section outlines one method by means of which certain singular integral equations can be inverted.

ON THE INVERSION OF SINGULAR INTEGRAL EQUATIONS

DEFINITIONS

An integral equation.—Consider the equation

$$\int_{L_1} g(x) K(x, y) dx = w(y) \tag{83}$$

If $w(y)$ and $K(x, y)$ are given functions and $g(x)$ is unknown, equation (83) is known as integral equation, and more specifically as an integral equation of the first kind. The path of integration L_1 lies along the x axis (in this report only real variables are considered although the methods and results can be generalized to include complex variables) and, in general, can depend on y. The term $K(x, y)$ is known as the kernel of the integral equation.

A singular integral equation.—An integral equation is referred to as singular either when the path of integration, L_1, has infinite extent, or when the kernel, $K(x, y)$, is infinite at points of the interval L_1. In other words, equation (83) is a singular integral equation if $K(x, y)$ is unbounded somewhere on L_1.

An integral transform.—Again consider equation (83). If both sides of this equation are multiplied by the function $H(\lambda, y)$ and integrated with respect to y along the interval L_2 (which is, of course, independent of y but can be a function of λ), the equation is said to have been transformed and the operator

$$\int_{L_2} H(\lambda, y) dy \tag{84}$$

is referred to as an integral transform. The resulting expression

$$\int_{L_2} dy \int_{L_1} dx\, g(x) H(\lambda, y) K(x, y) = \int_{L_2} w(y) H(\lambda, y) dy \tag{85}$$

is obviously a function only of λ, both x and y being dummy variables of integration.

INHERENT SINGULARITIES

An inherent singularity can be defined first in terms of a function of two variables. Consider the function $f(x, y)$ and let the point a, b lie somewhere in the x, y plane. Then an inherent singularity will be said to exist at the point a, b if

$$\lim_{\epsilon \to 0} \epsilon^2 f(\epsilon \xi + a, \epsilon \eta + b) \neq 0$$

where

$$x - a = \epsilon \xi, \quad y - b = \epsilon \eta$$

In other words, a square of width 2ϵ is first placed on the x, y plane with a, b at its center, the function $f(x, y)$ is then evaluated at any point on the boundary of the square and multiplied by one-fourth the area of the square. Finally, in the limit as the width of the square vanishes, if this product is not

zero, the function $f(x, y)$ contains an inherent singularity at the point a, b.

Such a concept can obviously be generalized to include functions of three and more variables. For example, the function $f(x_1, x_2, ..., x_n)$ contains an inherent singularity at the point $a_1, a_2, ..., a_n$ if

$$\lim_{\epsilon \to 0} \epsilon^n f(\epsilon \xi_1 + a_1, \epsilon \xi_2 + a_2, ..., \epsilon \xi_n + a_n) \neq 0$$

where

$$x_i - a_i = \epsilon \xi_i$$

RESIDUALS

Consider a double integration with respect to x and y of the function $f(x,y)$ over the area S in the x, y plane. Perform the integration of the same function over the same area but with the order of integrations reversed. The difference between the results of these two operations will be defined as the residual. Thus

$$R \equiv \int dy \int_S dx\, f(x, y) - \int dx \int_S dy\, f(x, y) \qquad (86)$$

Ordinarily the residual R is zero, since the order of integration for a double integral is usually immaterial. In the manipulation of singular integrals and singular integral transforms, however, a nonvanishing residual often exists. The evaluation of the residual can be accomplished in the following manner. Let the point a_1, b_1 be an inherent singularity in the area S. Then the residual from such a point is

$$R_1 = \lim_{\epsilon \to 0} \left[\int_{b_1 - \epsilon}^{b_1 + \epsilon} dy \int_{a_1 - \epsilon}^{a_1 + \epsilon} dx\, f(x, y) - \int_{a_1 - \epsilon}^{a_1 + \epsilon} dx \int_{b_1 - \epsilon}^{b_1 + \epsilon} dy\, f(x, y) \right]$$

Setting $\epsilon \xi + a_1 = x$ and $\epsilon \eta + b_1 = y$, one can write

$$R_1 = \int_{-1}^{1} d\eta \int_{-1}^{1} d\xi \lim_{\epsilon \to 0} \epsilon^2 f(\epsilon \xi + a_1, \epsilon \eta + b_1) -$$
$$\int_{-1}^{1} d\xi \int_{-1}^{1} d\eta \lim_{\epsilon \to 0} \epsilon^2 f(\epsilon \xi + a_1, \epsilon \eta + b_1) \qquad (87)$$

Hence the necessary condition for the existence of a residual is the occurrence of an inherent singularity in the area of integration over which the double integration is performed. The total residual is the sum of the residuals from each inherent singularity in the area involved.

THE NULL TRANSFORM

Definition.—The integral operator $\int_{L_3} H(\lambda, y)\, dy$ is said to be a null transform of order n to the function $K(x, y)$ in the interval L_3 if

$$\left(\frac{\partial}{\partial x}\right)^n \int_{L_3} H(\lambda, y) K(x, y)\, dy = 0 \qquad (88)$$

where x, λ and, of course, y are on L_3.

Examples.—The operator

$$\int_{L_3} H(\lambda, y)\, dy = \int_x^\lambda \frac{dy}{(\lambda - y)^{3/2}}$$

is a null transform of order zero to the function

$$K(x, y) = 1/\sqrt{y - x}$$

in the interval $x \leq y \leq \lambda$. Thus, (see equation (18)),

$$\int_x^\lambda \frac{dy}{\sqrt{(y - x)(\lambda - y)^3}} = 0 \qquad (89)$$

The operator

$$\int_{L_3} H(\lambda, y)\, dy = \int_\lambda^1 \frac{dy}{\sqrt{(1 - y)(y - \lambda)}}$$

is a null transform of order zero to the function $K(x, y) = 1/(y - x)$ in the interval $\lambda \leq y \leq 1$. Thus

$$\int_\lambda^1 \frac{dy}{(y - x)\sqrt{(1 - y)(\lambda - y)}} = 0, \lambda < x < 1 \qquad (90)$$

Finally, it can be shown from equation (90) that

$$\int_{L_3} H(\lambda, y)\, dy = \int_a^b \frac{\sqrt{(b - y)(y - a)}}{\lambda - y}\, dy, a < b$$

where a and b are constants, is a null transform of order one to the function $K(x, y) = 1/(y - x)$ in the interval $a < y < b$. Thus

$$\frac{\partial}{\partial x} \int_a^b \frac{\sqrt{(b - y)(y - a)}}{(y - x)(\lambda - y)}\, dy = 0, \begin{Bmatrix} a < x < b \\ a < \lambda < b \end{Bmatrix} \qquad (91)$$

THE INVERSION OF SINGULAR INTEGRAL EQUATIONS BY MEANS OF NULL TRANSFORMS

Consider an integral equation of the first kind

$$w(y) = \int_{L_1} g(x) K(x, y)\, dx \qquad (92)$$

such that the kernel $K(x, y)$ tends to infinity as x approaches y and let the point $x = y$ lie in L_1. Equation (92) is, by definition, a singular integral equation of the first kind. Apply to both sides of this equation the integral transform

$\int_{L_2} H(\lambda, y)\, dy$ so that

$$\int_{L_2} w(y) H(\lambda, y)\, dy = \int_{L_2} dy \int_{L_1} dx\, g(x) H(\lambda, y) K(x, y) \qquad (93)$$

Suppose that the area $L_1 + L_2$ of the double integral is bounded by a simple closed curve having the property that any line parallel to the x or y axis crosses its boundary at most twice. For such an area, it is always possible to write the reversed form of the double integral in equation (93) as

$$\int_{L_4} dx \int_{L_3} dy\, g(x) H(\lambda, y) K(x, y) \qquad (94)$$

where L_3 can be a function of x and λ, and L_4 can be a function only of λ.

Subtracting expression (94) from the double integral in equation (93), one finds

$$\int_{L_2} dy \int_{L_1} dx\, g(x) H(\lambda, y) K(x, y) =$$

$$\int_{L_4} dx \int_{L_3} dy\, g(x) H(\lambda, y) K(x, y) + R(\lambda) \qquad (95)$$

where $R(\lambda)$ is, by definition, the residual. Hence, equation (93) can be rewritten in the form

$$\int_{L_2} w(y) H(\lambda,\, y) dy =$$

$$\int_{L_4} dx \int_{L_3} dy\, g(x) H(\lambda, y) K(x, y) + R(\lambda) \qquad (96)$$

One can now show that if $H(\lambda, y) K(x, y)$ contains an inherent singularity at the point $x=y=\lambda$, equation (96) is the inversion to equation (92) when $H(\lambda, y)$ is a null transform of order one or zero to the kernel $K(x, y)$ in the interval L_3.

First, it is necessary to relate $R(\lambda)$ to $g(\lambda)$. By the definition given as equation (87), $R(\lambda)$ can be written

$$R(\lambda) = R_0(\lambda) + \int_{-1}^{1} d\eta \int_{-1}^{1} d\xi \lim_{\epsilon \to 0} g(\epsilon\xi + \lambda) K(\epsilon\xi + \lambda, \epsilon\eta + \lambda) H(\lambda, \epsilon\eta + \lambda) -$$

$$\int_{-1}^{1} d\xi \int_{-1}^{1} d\eta \lim_{\epsilon \to 0} g(\epsilon\xi + \lambda) K(\epsilon\xi + \lambda, \epsilon\eta + \lambda) H(\lambda, \epsilon\eta + \lambda) \quad (97)$$

where $R_0(\lambda)$ is the sum of the remaining residuals (if there are any) from the other inherent singularities that might exist in the area $L_1 + L_2$. This reduces to

$$R(\lambda) = R_0(\lambda) + g(\lambda) R^*(\lambda) \qquad (98)$$

where

$$R^*(\lambda) = \int_{-1}^{1} d\eta \int_{-1}^{1} d\xi \lim_{\epsilon \to 0} K(\epsilon\xi + \lambda, \epsilon\eta + \lambda) H(\lambda, \epsilon\eta + \lambda) -$$

$$\int_{-1}^{1} d\xi \int_{-1}^{1} d\eta \lim_{\epsilon \to 0} K(\epsilon\xi + \lambda, \epsilon\eta + \lambda) H(\lambda, \epsilon\eta + \lambda) \qquad (99)$$

If inherent singularities other than the one at $x=y=\lambda$ exist in $L_1 + L_2$, they must be at a point on the line $x=y$. The residual at such a point say $x=y=a$, would be the product of $g(a)$ and $R_0^*(a)$, the difference between the two appropriate double integrals. Hence, $R_0(\lambda)$ cannot contain $g(\lambda)$.

Now, if $\int_{L_3} H(\lambda, y)\, dy$ is a null transform of zero order, equation (96) becomes

$$\int_{L_2} w(y) H(\lambda, y) dy = R_0(\lambda) + g(\lambda) R^*(\lambda)$$

or

$$g(\lambda) = \frac{1}{R^*(\lambda)} \left[-R_0(\lambda) + \int_{L_2} w(y) H(\lambda, y) dy \right] \qquad (100)$$

which is an inversion of equation (92). Further, if

$$\int_{L_3} H(\lambda, y) dy$$

is a null transform of the first order, so that

$$\int_{L_3} H(\lambda, y) K(x, y) dy = C(\lambda)$$

equation (96) becomes

$$\int_{L_2} w(y) H(\lambda, y) dy = C(\lambda) \int_{L_4} g(x) dx + R_0(\lambda) + g(\lambda) R^*(\lambda) \qquad (101)$$

If L_4 is independent of λ this already is an inversion of equation (92). However, if L_4 contains λ then, after dividing through by $C(\lambda)$ and taking the derivative with respect to λ, a first-order differential equation in $g(\lambda)$ results. This is considered to be an inversion.

THE INVERSION OF SOME PARTICULAR SINGULAR INTEGRAL EQUATIONS

ABEL'S INTEGRAL EQUATION

Consider the special form of Abel's integral equation

$$w(y) = \int_{a}^{y} \frac{g(x) dx}{\sqrt{y-x}}, \quad a < y \qquad (102)$$

It has been shown (see equation (89)) that

$$\int_{x}^{\lambda} \frac{dy}{(\lambda - y)^{3/2}}$$

is a null transform of order zero to the kernel $1/\sqrt{y-x}$ in the interval $x \leq y \leq \lambda$. Hence, applying the transform

$$\int_{a}^{\lambda} \frac{dy}{(\lambda - y)^{3/2}}$$

to both sides of equation (102), reversing the order of integration, and noting that

$$\int_{a}^{\lambda} dy \int_{a}^{y} dx \frac{g(x)}{\sqrt{y-x}\,(\lambda-y)^{3/2}} =$$

$$\int_{a}^{\lambda} g(x)\, dx \int_{x}^{\lambda} dy \frac{1}{\sqrt{y-x}\,(\lambda-y)^{3/2}} + R(\lambda) = R(\lambda) \qquad (103)$$

leads one to the result

$$\int_{a}^{\lambda} \frac{w(y) dy}{(\lambda - y)^{3/2}} = R(\lambda) \qquad (104)$$

The only inherent singularity that appears in the area of integration occurs at the point $x=y=\lambda$. The residual is obtained by integrating over the shaded area in figure 5 and finding the limit as ϵ goes to zero.
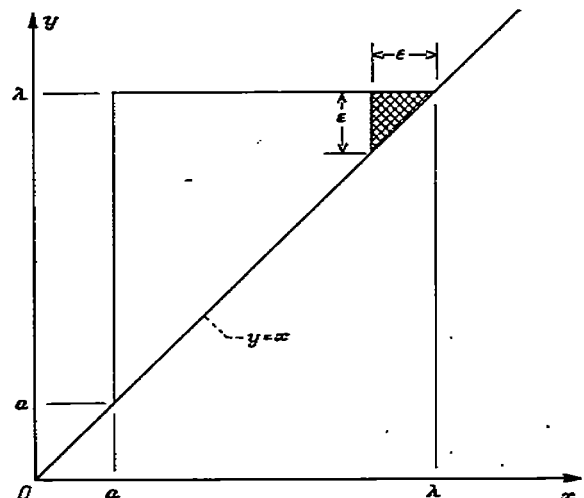

FIGURE 5.—Region of integration for equation (105a).

Thus

$$R(\lambda) = \lim_{\epsilon \to 0} \left[\oint_{\lambda-\epsilon}^{\lambda} dy \int_{\lambda-\epsilon}^{y} dx \, \frac{g(x)}{\sqrt{y-x} \, (\lambda-y)^{3/2}} - \int_{\lambda-\epsilon}^{\lambda} g(x) \, dx \oint_{x}^{\lambda} dy \, \frac{1}{\sqrt{y-x} \, (\lambda-y)^{3/2}} \right] \quad (105a)$$

and, since the second double integral is zero, the transformations $\epsilon \xi = \lambda - x$ and $\epsilon \eta = \lambda - y$ reduce this to

$$R(\lambda) = \lim_{\epsilon \to 0} \oint_{1}^{0} d\eta \int_{1}^{\eta} d\xi \, \frac{g(-\epsilon\xi+\lambda)}{\sqrt{\xi-\eta} \, \eta^{3/2}}$$

Finally, in the limit as ϵ goes to zero,

$$R(\lambda) = g(\lambda) \oint_{1}^{0} d\eta \int_{1}^{\eta} \frac{d\xi}{\eta^{3/2} \sqrt{\xi-\eta}} = -2\pi g(\lambda) \quad (105b)$$

Substitute equation (105b) into equation (104) and the inversion of the integral equation (102) can be written

$$g(\lambda) = \frac{-1}{2\pi} \oint_{a}^{\lambda} \frac{w(y) \, dy}{(\lambda-y)^{3/2}} \quad (106)$$

By applying the definition of the finite part, one can rewrite equation (106) in the alternative form

$$g(\lambda) = \frac{1}{\pi} \frac{d}{d\lambda} \int_{a}^{\lambda} \frac{w(y) \, dy}{\sqrt{\lambda-y}} \quad (107)$$

which is the form of the inversion usually presented.

THE AIRFOIL EQUATION

The study of the singular integral equation known as the airfoil equation is closely associated with the study of boundary-value problems related to Laplace's equation in two dimensions. These boundary conditions are sometimes given along a straight line as is the case, for example, in the linearized study of two-dimensional subsonic wings. If the boundary conditions are given along a suitably prescribed curve, the curve can, by application of complex variable concepts, be mapped onto a straight line. For example, the Joukowski transformation maps a circle in one plane onto a straight line in another and in both planes the governing formula is the two-dimensional form of Laplace's equation. The solution to such boundary-value problems can be reduced to the inversion of the following singular integral equation:

$$w(y) = \int_{a}^{b} \frac{g(x) \, dx}{y-x}, \quad a < y < b \quad (108)$$

where a and b are constants.

It has been shown (see equation (91)) that

$$\int_{a}^{b} \frac{\sqrt{(b-y)(y-a)}}{\lambda-y} \, dy$$

is a null transform of order one to the kernel $1/(y-x)$ in the interval $a \le y \le b$. Applying this transform to equation (108), and using the definition of the residual, one obtains

$$\int_{a}^{b} \frac{w(y) \sqrt{(b-y)(y-a)}}{\lambda-y} \, dy = \int_{a}^{b} g(x) \, dx \int_{a}^{b} \frac{\sqrt{(b-y)(y-a)}}{(\lambda-y)(y-x)} \, dy + R(\lambda)$$

which reduces to the form

$$\int_{a}^{b} \frac{w(y) \sqrt{(b-y)(y-a)}}{\lambda-y} \, dy = \pi \int_{a}^{b} g(x) \, dx + R(\lambda) \quad (109)$$

Again the only inherent singularity in the area of integration is at the point $x = y = \lambda$. Evaluating the residual according to equation (87), one finds

$$R(\lambda) = g(\lambda) \sqrt{(b-\lambda)(\lambda-a)} \left[\int_{-1}^{1} d\eta \int_{-1}^{1} \frac{d\xi}{\eta(\xi-\eta)} - \int_{-1}^{1} d\xi \int_{-1}^{1} \frac{d\eta}{\eta(\xi-\eta)} \right]$$

and since

$$\int_{-1}^{1} \frac{d\eta}{\eta(\xi-\eta)} = \frac{-1}{\xi} \ln \frac{1-\xi}{1+\xi}$$

this becomes

$$R(\lambda) = 2g(\lambda) \sqrt{(b-\lambda)(\lambda-a)} \int_{-1}^{1} \frac{d\eta}{\eta} \ln \frac{1-\eta}{1+\eta} = -\pi^2 g(\lambda) \sqrt{(b-\lambda)(\lambda-a)} \quad (110)$$

By the combination of equation (110) with (109), the inversion to the airfoil integral equation (108) thus becomes

$$g(\lambda) = \frac{1}{\pi^2 \sqrt{(b-\lambda)(\lambda-a)}} \left[\pi \int_{a}^{b} g(x) \, dx - \int_{a}^{b} \frac{w(y) \sqrt{(b-y)(y-a)} \, dy}{\lambda-y} \right] \quad (111)$$

It is apparent that the inversion to equation (108) provided by equation (111) is not unique (because of the existence of the term $\int_{a}^{b} g(x) \, dx$ which can be thought of as an arbitrary constant). Hence, in the application of equation (108) to physical boundary-value problems it is not sufficient to specify the value of w along the y axis; some additional condition must also be supplied. Examples of such additional conditions in the study of aerodynamic problems are the specification of closure in the study of two-dimensional sections and the assumption of the Kutta condition along the trailing edge of two-dimensional lifting surfaces.

THE SUPERSONIC DOUBLET EQUATION

The general concepts of the method just applied to the solution of single-integral equations with a singular kernel can also be used to invert double-integral equations with singular kernels. Success in solving these more complicated forms depends again on the discovery of an appropriate integral transform—in this case a double integral transform— and the usefulness of these operators depends, in turn, on both the structure of their integrand and, what is just as important, the space (now four-dimensional) of integration. As it turns out, however, inversions can be obtained in many cases that are of importance in the study of supersonic aerodynamic problems.

Case 1—Supersonic leading edge.—Consider the equation that gives the vertical induced velocity in the plane of the wing in terms of the perturbation velocity potential on the upper surface of a lifting surface in a supersonic free stream, equation (82),

$$w_u(r_2,s_2)=\frac{-M_o}{4\pi}\int\!\!\!\int\limits_{\tau(r_2,s_2)}\frac{\varphi_u(r_1,s_1)\,dr_1\,ds_1}{[(r_2-r_1)(s_2-s_1)]^{3/2}} \qquad (112)$$

The area $\tau(r_2,s_2)$ represents the area within the forecone from the point r_2,s_2 at which the vertical induced velocity is being measured. The details of the solution to equation (112) will now be presented for two different types of boundaries to the area $\tau(r_2,s_2)$, that is, from an aerodynamic standpoint, for two different types of wing plan forms.

First consider the case when $\tau=\tau_0$ is an area such as the one shown in figure 6. The two lines $r_1=r_2$ and $s_1=s_2$, which represent the traces of the Mach forecones from the point r_2, s_2, form two bounds of the area while the third is given by the wing leading edge, the equation for which may be written as either $s_1=f_0(r_1)$ or $r_1=f_0^*(s_1)$.

(a) x_1, y_1 plane.
(b) r_1, s_1 plane.
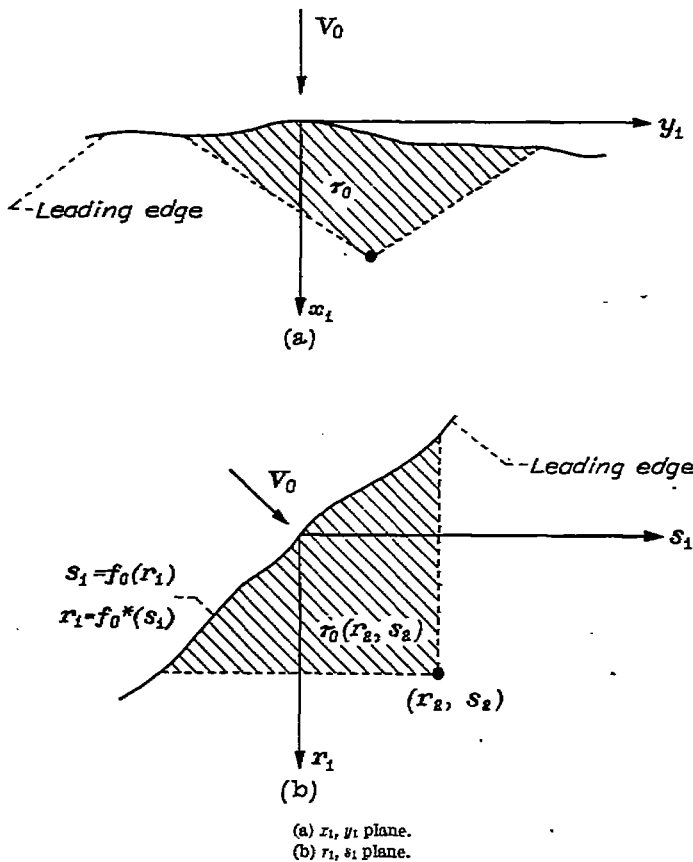
FIGURE 6.—Region of integration for wing with supersonic leading edge.

In the present case, it is assumed, furthermore, that $s_1=f_0(r_1)$ is a continuous monotonic function with a negative slope, or, physically, that the wing has a supersonic type leading edge.[5]

The kernel of equation (112) is formed by the product of two functions, one independent of r_1 and r_2 and the other independent of s_1 and s_2. It follows, therefore, that if an integral operator is used that contains the product of two linear functions of s_2 and r_2 to the $-1/2$ power, there exists the possibility of obtaining a null transform of order zero according to the equality given as equation (18). One can show, in fact, that the operator

$$\int\!\!\!\int\limits_{\tau_0(r,s)}\frac{dr_2\,ds_2}{\sqrt{(r-r_2)(s-s_2)}}=\frac{2\beta}{M_o}\int\!\!\!\int\limits_{\tau_0(r,s)}\frac{dr_2\,ds_2}{r_0(r,s;r_2,s_2)} \qquad (113)$$

is a null transform of order zero to the right-hand side of equation (112) where $\tau_0(r,s)$ is the same area that appears in the integral equation.

If the transform given by equation (113) is applied to equation (112) the resulting expression is

$$\int\!\!\!\int\limits_{\tau_0(r,s)}\frac{w_u\,dr_2\,ds_2}{r_0}=$$

$$\frac{-2\beta^3}{\pi M_0^2}\int\!\!\!\int\limits_{\tau_0(r,s)}dr_2\,ds_2\oint\!\!\!\oint\limits_{\tau_0(r_2,s_2)}\frac{\varphi_u(r_1,s_1)\,dr_1\,ds_1}{r_0(r,s;r_2,s_2)r_0^3(r_2,s_2;r_1,s_1)} \qquad (114)$$

Consider next the right-hand side of this equation but with the r_2, s_2 integrals taken first. Then by definition of the residual

$$\frac{-2\beta^3}{\pi M_0^2}\int\!\!\!\int\limits_{\tau_0(r,s)}dr_2\,ds_2\oint\!\!\!\oint\limits_{\tau_0(r_2,s_2)}\frac{\varphi_u(r_1,s_1)\,dr_1\,ds_1}{r_0(r,s;r_2,s_2)r_0^3(r_2,s_2;r_1,s_1)}=$$

$$\frac{-2\beta^3}{\pi M_0^2}\int\!\!\!\int\varphi_u(r_1,s_1)\,dr_1\,ds_1\oint\!\!\!\oint\limits_{[\tau_0(r,s)+\tau_0(r_1,s_1)]}\frac{dr_2\,ds_2}{r_0(r,s;r_2,s_2)r_0^3(r_2,s_2;r_1,s_1)}+$$

$$R(r,s) \qquad (115)$$

When the integration is made first with respect to r_2 and s_2, the area of integration for these two variables can be visualized with the aid of figure 7. In such a case the points r_1, s_1 and r, s are fixed and the area is simply the one which lies in the forecone from the point r, s and in the aftercone from the point r_1, s_1. This is represented by the shaded area in the figure. It is apparent from a study of figure 7 that, when the edge $s_2=f_0(r_2)$ is a monotonic function as previously defined, *the r_2 and s_2 integrals are always taken between the limits r_1 and r and s_1 and s, respectively.* Hence, according to equation (18) the inte-
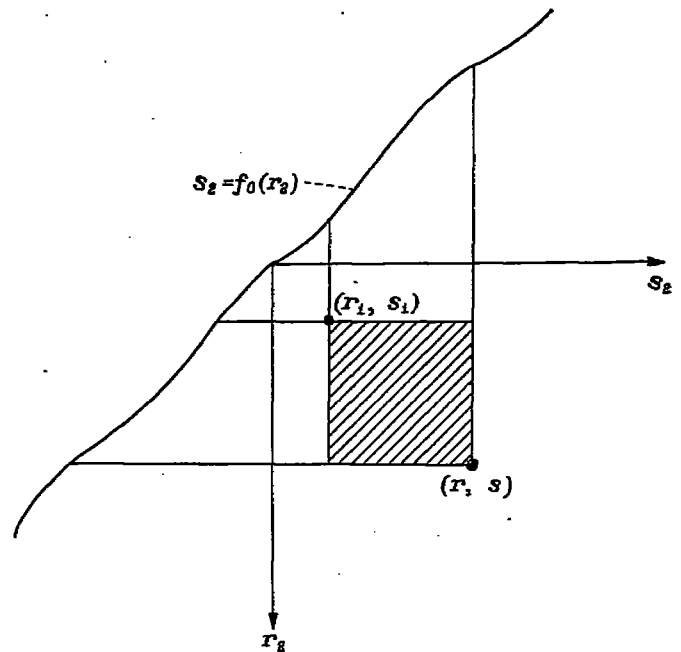

FIGURE 7.—Area of integration for fixed r_1, s_1, r, s; equation (115).

gral term on the right-hand side of equation (115) vanishes and equation (114) becomes

$$\int\int_{r_0(r,s)} \frac{w_u dr_2 ds_2}{r_0} = R(r,s) \tag{116}$$

The complete inversion of the integral equation can now be realized if an expression for $R(r,s)$ can be obtained in terms of $\varphi_u(r, s)$. The evaluation of this residual term follows along lines quite similar to those used in calculating the residual for the special form of Abel's integral equation given previously. Now, however, there is no longer a single inherent singularity; rather, the lines $r_1=r_2=r$ and $s_1=s_2=s$ are densely covered with them. First consider an integration made over the region close to the line $s_1=s_2=s$ (i. e., the sum of the areas a and b shown in fig. 8).
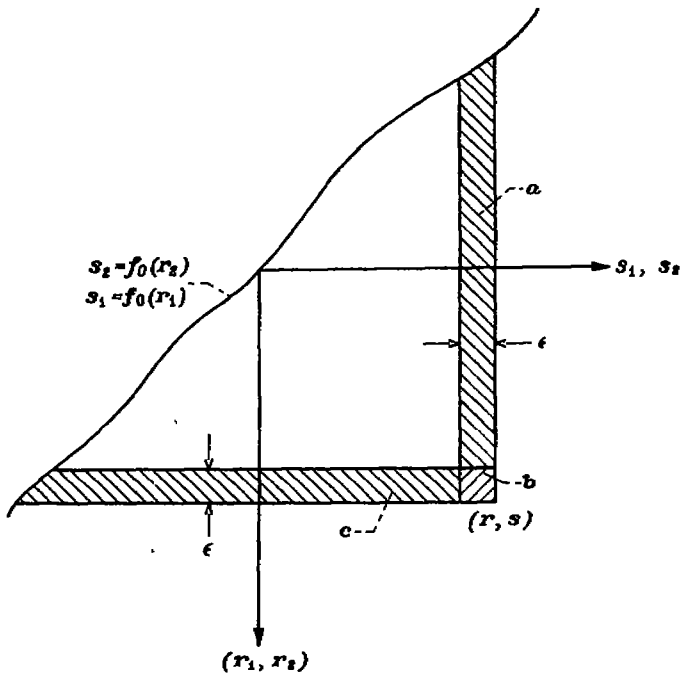
FIGURE 8.—Region of integration for evaluation of residual, equation (116).

Make the substitutions $s_1=s-\epsilon\sigma_1$ and $s_2=s-\epsilon\sigma_2$ and take the limit as ϵ goes to zero. There results for the portion of the residual due to the inherent singularities $R_{a+b}(r, s)$ along the line $s_1=s_2=s$ the equation [6]
$$R_{a+b}(r,s)=$$

$$\frac{-M_0^2}{8\pi\beta} \int_{f_0^*(s)}^r dr_2 \int_1^0 d\sigma_2 \mathrel{\rlap{\hskip2pt\text{—}}\int}_{f_0^*(s)}^{r_2} dr_1 \mathrel{\rlap{\hskip2pt\text{—}}\int}_1^{\sigma_2} d\sigma_1$$

$$\frac{\varphi_u(r_1, s)}{(\sigma_1-\sigma_2)^{3/2}(r_2-r_1)^{3/2}\sqrt{\sigma_2(r-r_2)}} \tag{117}$$

This becomes, after integrating with respect to σ_1 and σ_2 (which can be done immediately since the limits of the r_1 integral do not now contain σ_2 and, further, since the r_1, σ_2 plane contains no inherent singularity)

[6] The first term on the right-hand side of equation (115) contributes nothing to the residual since it always contains an integral equivalent to that in equation (18) and, therefore, is identically zero. This same phenomenon also appeared in the study of the residual appearing in the inversion of Abel's integral equation. (See the development of equation (105).)

$$R_{a+b}(r,s)=\frac{M_0^2}{4\beta} \int_{f_0^*(s)}^r dr_2 \mathrel{\rlap{\hskip2pt\text{—}}\int}_{f_0^*(s)}^{r_2} dr_1 \frac{\varphi_u(r_1,s)}{(r_2-r_1)^{3/2}\sqrt{r-r_2}} \tag{118}$$

It is now proposed to reverse the order of integration of the double integral in equation (118). But the r_1, r_2 plane contains an inherent singularity at the point $r_2=r_1=r$. By definition of the residual from this singularity as $R'(r,s)$, it follows that

$$R_{a+b}(r,s)=\frac{M_0^2}{4\beta}\int_{f_0^*(s)}^r dr_1\,\varphi_u(r_1,s) \mathrel{\rlap{\hskip2pt\text{—}}\int}_{r_1}^r \frac{dr_2}{(r_2-r_1)^{3/2}\sqrt{r-r_2}} +$$
$$R'(r,s)=R'(r,s)$$

since the integral term is zero by the equality that has been used repeatedly. The evaluation of $R'(r, s)$ follows the identical line of argument used in obtaining equation (105b) from (105a). Hence, setting $r_2=r-\epsilon\rho_2$ and $r_1=r-\epsilon\rho_1$, and letting ϵ go to zero, one finds

$$R_{a+b}(r,s)=R'(r,s)=$$

$$\frac{M_0^2}{4\beta}\varphi_u(r,s)\int_1^0 d\rho_2 \mathrel{\rlap{\hskip2pt\text{—}}\int}_1^{\rho_2}\frac{d\rho_1}{(\rho_1-\rho_2)^{3/2}\sqrt{\rho_2}}=\frac{-\pi M_0^2}{2\beta}\varphi_u(r,s) \tag{119}$$

The part of the residual in equation (116) contributed by the inherent singularities in the areas a and b in figure 8 is, therefore, $\dfrac{-\pi M_0^2}{2\beta}\varphi_u(r,s)$. A similar calculation shows that the singularities in the areas $c+b$ give the same result; and, finally, a calculation for the area b itself also yields $\dfrac{-\pi M_0^2}{2\beta}\varphi_u(r, s)$. The value of R in equation (116) is obtained by combining the results for the various areas. Hence,

$$R=R_{a+b}+R_{b+c}-R_b=\frac{-\pi M_0^2}{2\beta}\varphi_u(r,s) \tag{120}$$

Combining the result expressed by the last equation with equation (116), one can finally write for the inversion of the integral equation (112), when $\tau=\tau_o$, the expression

$$\varphi_u(r,s)=\frac{-1}{\pi M_0}\int\int_{\tau_i(r,s)}\frac{w_u dr_2 ds_2}{\sqrt{(r-r_2)(s-s_2)}} \tag{121}$$

Case 2—Combined subsonic and supersonic leading edges.—Consider the equation

$$w_u=\frac{-M_0}{4\pi}\int\int_{\tau_i(r_2,s_2)}\frac{\varphi_u(r_1,s_1)dr_1 ds_1}{[(r_2-r_1)(s_2-s_1)]^{3/2}} \tag{122}$$

where $\tau_i(r_2, s_2)$ is the more complicated area shown in figure 9. Again the lines $r_1=r_2$ and $s_1=s_2$, which are the traces of the forecone from the point r_2, s_2, form two bounds of the area. The remaining two boundaries are formed by the curves $s_1=f_o(r_1)$ (or in the inverse sense $r_1=f_o^*(s_1)$) and $s_1=f_1(r_1)$ (or $r_1=f_1^*(s_1)$) where f_o has the same definition it had in the study of Case 1, that is, a monotonic curve with a negative slope. The curve $s_1=f_1(r_1)$ is also a monotonic function, but with a positive slope. For convenience the origin is placed at the point of intersection of the f_o and f_1 curves.

(a)

(b)
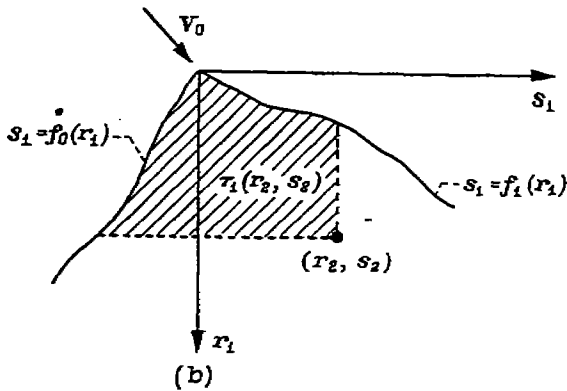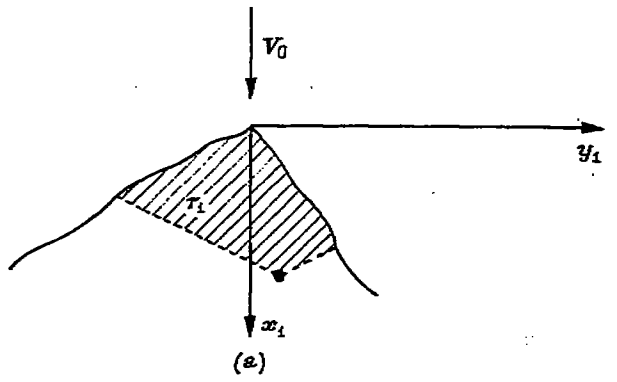
(a) x_1, y_1 plane.
(b) $(r_1, s_1$ plane.

FIGURE 9.—Region of integration for wing with combined subsonic and supersonic edges.

A physical interpretation of the area τ_1 with regard to problems in supersonic wing theory is simple. The lines $r_1=r_2$ and $s_1=s_2$, as has been mentioned, are the traces of the Mach forecone from the point r_2, s_2. The f_0 and f_1 curves are the edges of the wing plan form, the line $s_1=f_0(r_1)$ representing a supersonic leading edge and the line $s_1=f_1(r_1)$ representing a subsonic leading edge.

Proceeding exactly as in Case 1, consider the operator

$$\iint_{\tau_1(r,s)} \frac{dr_2 ds_2}{\sqrt{(r-r_2)(s-s_2)}}$$

and find whether or not it is a null transform of zero order to the kernel of equation (122), that is, whether or not the equality

$$\iint_{\tau_1(s,r)+\tau_1(r_2,s_2)]} \varphi_x(r_1, s_1)dr_1 ds_1 + \int\int \frac{dr_2 ds_2}{[(r_2-r_1)(s_2-s_1)]^{3/2}\sqrt{(r-r_2)(s-s_2)}}=0 \tag{123}$$

is satisfied. A study of figure (10) shows that it is not, since for a point r_1, s_1 located above the line $r_2=f_1^*$ (s), a portion of the s_2 integral becomes

$$\int_{s_1}^{f_1(r_2)} \frac{ds_2}{(s_2-s_1)^{3/2}\sqrt{s-s_2}}$$
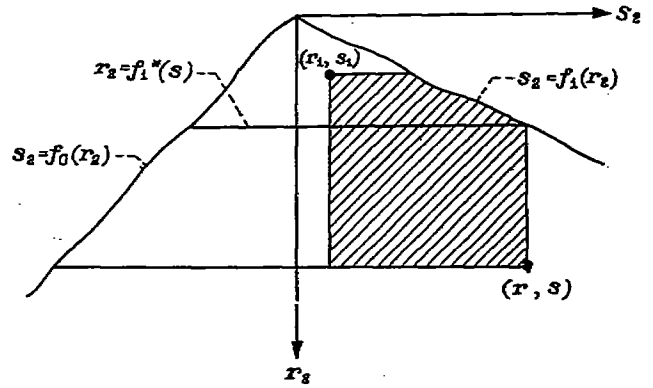
FIGURE 10.—Area of integration for fixed r_1, s_1, r, s; equation (123).

which is not zero unless $f_1(r_2)$ is identically equal to s. For the same reason, portions of the r_2 integral will not vanish for the point r_1, s_1 so located.

However, the construction of a null transform of zero order can be accomplished by studying the above failure. Thus, if the s_2 integration were carried between the limits s_1 and s, for every location of the points r_1, s_1 and r, s, the operator $\int h(r, s, r_2)dr_2 \int (s-s_2)^{-1/2}ds_2$ would be a null transform of zero order regardless of the choice of h. The area of integration producing such a transform is simply the one shown in figure 11(a), an area bounded by the lines $s_2=s$, $r_2=r$, $s_2=f_0(r_2)$ and $r_2=f_2^*(s)$ where $r_2=f_2^*(s)$ is any line such that $f_2^*(s) \geq f_1^*(s)$. This area shall be designated as $\tau_2(r, s)$. When such a transform is applied and the s_2 integration performed first, it is apparent from figure 11(b) that the limits of the s_2 integral

(a)

(b)
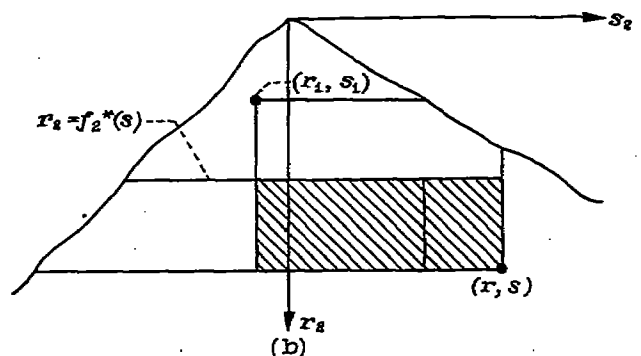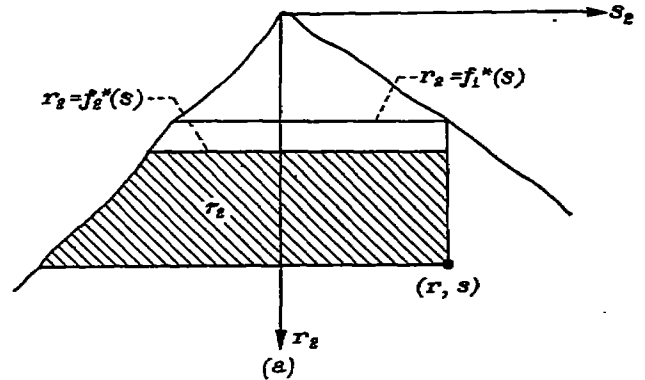
(a) Area of integration for transform.
(b) Area of integration for fixed r_1, s_1 r, s.

FIGURE 11.—Integration regions for equations (124).

are always s_1 and s, even when the point r_1, s_1 is located above the line $r_2=f_1{}^*(s)$. Hence, if the residual $R(r, s)$ is defined by the relation

$$-\frac{M_0}{4\pi}\int h(r,s,r_2)dr_2\int_{r_2(r,s)}\frac{ds_2}{\sqrt{s-s_2}}\int\int_{r_1(r_2,s_2)}\frac{\varphi_u(r_1,s_1)\,dr_1\,ds_1}{[(r_2-r_1)(s_2-s_1)]^{3/2}}+$$

$$\frac{M_0}{4\pi}\int h(r,s,r_2)\,dr_2\int\int\frac{\varphi_u(r_1,s_1)\,dr_1\,ds_1}{(r_2-r_1)^{3/2}}\int_{s_1}^{s}\frac{ds_2}{\sqrt{s-s_2}(s_2-s_1)^{3/2}}=R(r,s)$$

$$\scriptstyle [r_2(r,s)+r_1(r_2,s_2)]$$
$$\tag{124a}$$

the second term on the left-hand side is zero, and equation (122) transforms into

$$\int_{f_2{}^*(s)}^{r} h(r,s,r_2)dr_2\int_{f_0(r_2)}^{s}\frac{w_u(r_2,s_2)}{\sqrt{s-s_2}}\,ds_2=R(r,s)\tag{124b}$$

The evaluation of the residual in equation (124b) follows the same pattern used in Case 1. Thus, after isolating the line of inherent singularities, $R(r, s)$ is given by the expression

$$R(r,s)=$$

$$-\frac{M_0}{4\pi}\int_{f_2{}^*(s)}^{r} h\,dr_2\int_{s-\epsilon}^{s}\frac{ds_2}{\sqrt{s-s_2}}\int_{f_1{}^*(s_2)}^{r_2} ds_1\int_{s-\epsilon}^{s_2} dr_1\frac{\varphi_u(r_1,s_1)}{[(r_2-r_1)(s_2-s_1)]^{3/2}}$$

which becomes (after setting $s_2=s-\epsilon\sigma_2$ and $s_1=s-\epsilon\sigma_1$ and letting ϵ go to zero)

$$R=\frac{M_0}{2}\int_{f_2{}^*(s)}^{r} h\,dr_2\int_{f_1{}^*(s)}^{r_2}\frac{\varphi_u(r_1,s)}{(r_2-r_1)^{3/2}}\,dr_1\tag{125}$$

It is now apparent that after setting $f_2{}^*(s)=f_1{}^*(s)$ and $h=1/\sqrt{r-r_2}$, equation (125) can be evaluated exactly as equation (118) was evaluated, so that for these values of $f_2{}^*$ and h, $R(r,s)=-M_0\pi\varphi_u(r,s)$. Substituting into equation (124b), one finds for the inversion of equation (122) the result

$$\varphi_u(r,s)=\frac{-1}{\pi M_0}\underset{r_3(r,s)}{\int\int}\frac{w_u(r_2,s_2)\,dr_2\,ds_2}{\sqrt{(r-r_2)(s-s_2)}}\tag{126}$$

where r_3 is the shaded area in figure 12.

Case 3—Mixed boundary conditions.—Another very important kind of integral equation which can be solved directly by the proper choice of h in equation (125) is the "mixed" type problem, the boundary values of which are illustrated in figure 13. In this particular problem w_u is known over the portion of the r, s plane bounded by the curves $s=f_0(r)$ and $s=f_2(r)$ (the curve $s=f_2(r)$ is a monotonic function with a positive slope just like $s=f_1(r)$), while over another portion, bounded by the curves $s=f_2(r)$ and $s=f_1(r)$ the quantity

$$u_u(r,s)=\frac{M_0}{2\beta}\left(\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)\varphi_u(r,s)$$ is given. (This corresponds in aerodynamic applications to the specification of the vertical induced velocity over the former region and the loading over the latter.) It is also assumed that $u_u(r, s)$ is continuous across the line $s=f_2(r)$ and that $\varphi(r, s)$ vanishes along the line $s=f_1(r)$, that is, $\varphi(f_1{}^*(s), s)=0$.
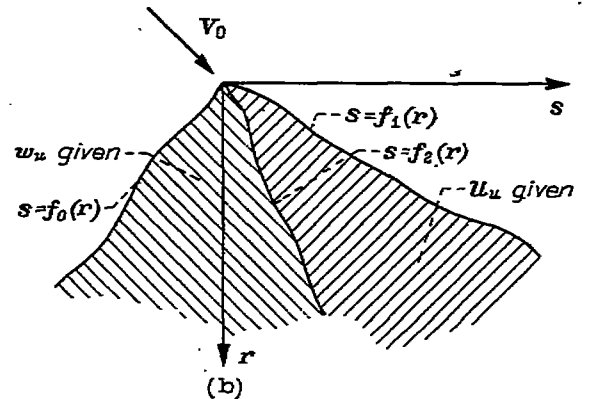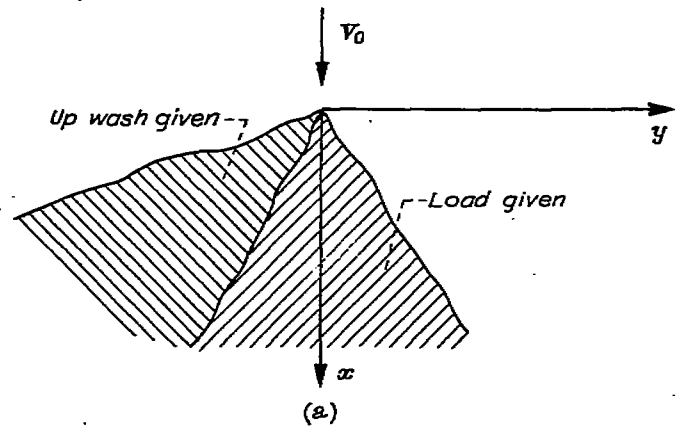
Taking for the value of h the expression

$$h=\frac{1}{\sqrt{r-r_2}}\left(\frac{\partial}{\partial r_2}+\frac{\partial}{\partial s}\right)$$
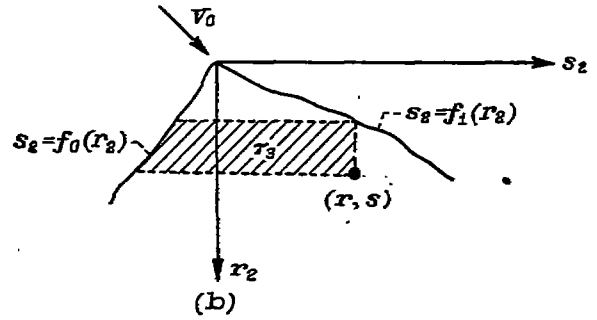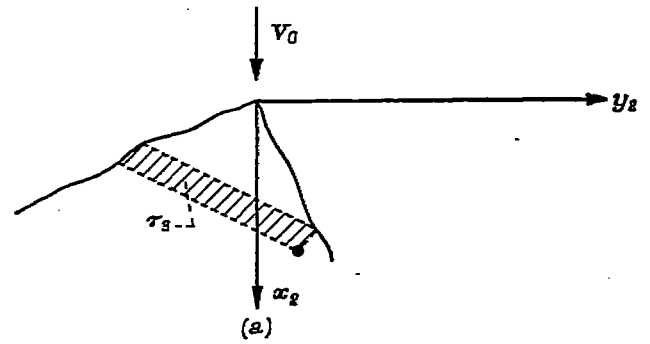
(a) x_2, y_2 plane.
(b) r_2, s_2 plane.

FIGURE 12.—Integration area r_3 (equation (126)).

(a) x, y plane.
(b) r, s plane.

FIGURE 13.—Mixed boundary conditions.

and for the area of integration the region $\tau_2(r, s)$ shown in figure 11(a), one finds from equation (125) that the residual can be written

$$R = \frac{M_0}{2} \int_{f_2^*(s)}^r \frac{dr_2}{\sqrt{r - r_2}} \left(\frac{\partial}{\partial r_2} + \frac{\partial}{\partial s} \right) \int_{f_1^*(s)}^{r_2} \frac{\varphi_u(r_1, s) dr_1}{(r_2 - r_1)^{3/2}}$$

The partial derivative can be taken through the second integral sign (first integrating by parts in the case of $\partial/\partial r_2$) since $\varphi(f_1^*(s), s) = 0$, so that

$$R = \frac{M_0}{2} \int_{f_2^*(s)}^r dr_2 \int_{f_1^*(s)}^{r_2} dr_1 \frac{\left(\frac{\partial}{\partial r_1} + \frac{\partial}{\partial s} \right) \varphi_u(r_1, s)}{\sqrt{r - r_2}(r_2 - r_1)^{3/2}}$$

Part of the boundary condition is that $u_u(r, s)$ is given over the region indicated in figure 13. Hence, the residual can be written

$$R = \beta \int_{f_2^*(s)}^r dr_2 \int_{f_1^*(s)}^{f_2^*(s)} dr_1 \frac{u_u(r_1, s)}{\sqrt{r - r_2}(r_2 - r_1)^{3/2}} +$$
$$\beta \int_{f_2^*(s)}^r dr_2 \int_{f_2^*(s)}^{r_2} dr_1 \frac{u_u(r_1, s)}{\sqrt{r - r_2}(r_2 - r_1)^{3/2}}$$

The latter of these two terms is again just like the one given in equation (118) so its evaluation is immediate. The first term may be simplified by reversing the order of integration. (There is no residual since the point $r = r_1 = r_2$ is not included in the area of integration.) Finally, R becomes

$$R = 2\beta \sqrt{r - f_2^*(s)} \int_{f_1^*(s)}^{f_2^*(s)} \frac{u_u(r_1, s) dr_1}{(r - r_1)\sqrt{f_2^*(s) - r_1}} - 2\pi \beta u_u(r, s)$$

and the inversion of this mixed type problem can be written

$$u_u(r, s) = \frac{\sqrt{r - f_2^*(s)}}{\pi} \int_{f_1^*(s)}^{f_2^*(s)} \frac{u_u(r_1, s) dr_1}{(r - r_1)\sqrt{f_2^*(s) - r_1}} -$$
$$\frac{1}{2\beta\pi} \int_{f_2^*(s)}^r \frac{dr_2}{\sqrt{r - r_2}} \left(\frac{\partial}{\partial r_2} + \frac{\partial}{\partial s} \right) \int_{f_0(r_2)}^s \frac{w_u(r_2, s_2) ds_2}{\sqrt{s - s_2}} \quad (127)$$

An equation which does not impose the condition that $u_u(r, s)$ be continuous across the line $s = f_2(r)$ can readily be developed by using the operator $h = 1/\sqrt{r - r_2}$. This leads directly to the result

$$\varphi(r, s) = \frac{\sqrt{r - f_2^*(s)}}{\pi} \int_{f_1^*(s)}^{f_2^*(s)} \frac{\varphi_u(r_1, s) dr_1}{(r - r_1)\sqrt{f_2^*(s) - r_1}} -$$
$$\frac{1}{\pi M_0} \int_{f_2^*(s)}^r dr_2 \int_{f_0(r_2)}^s ds_2 \frac{w_u(r_2, s_2)}{\sqrt{(s - s_2)(r - r_2)}} \quad (128)$$

THE SUPERSONIC SOURCE EQUATION

The null transforms to the supersonic doublet equation were constructed by applying an integral operator having an integrand which, when combined with the kernel, would produce the integral

$$\int \frac{dx}{(x - b)^{3/2}\sqrt{a - x}}$$

and having an area which, when combined with that of the integral equation and traversed in reverse order, would pro-

duce the limits a and b on the integral. If one now considers the supersonic source equation (see equation (79))

$$\varphi_u(r_2, s_2) = \frac{-1}{\pi M_0} \iint_{\tau(r_1, s_1)} \frac{w_u(r_1, s_1) dr_1 ds_1}{\sqrt{(r_2 - r_1)(s_2 - s_1)}} \quad (129)$$

it is apparent that the null transform will differ from the transform used for the supersonic doublet equation only in the exponent of the integrand. Thus, for the area $\tau_0(r_2, s_2)$ (see fig. 6) the operator

$$\oint \int_{\tau_1(r, s)} \frac{dr_2 ds_2}{[(r - r_2)(s - s_2)]^{3/2}}$$

is a null transform of order zero to equation (129), and its application yields the solution

$$w_u(r, s) = \frac{-M_0}{4\pi} \oint \int_{\tau_0(r, s)} \frac{\varphi_u(r_2, s_2) dr_2 ds_2}{[(r - r_2)(s - s_2)]^{3/2}} \quad (130)$$

For the area $\tau_1(r_2, s_2)$ (see fig. 9), the operator

$$\int \int_{\tau_1(r, s)} \frac{dr_2 ds_2}{[(r - r_2)(s - s_2)]^{3/2}}$$

is a null transform of order zero. Its application, under conditions like those specified in the development of equation (126), yields the inversion

$$w_u(r, s) = \frac{-M_0}{4\pi} \oint \int_{\tau_1(r, s)} \frac{\varphi_u(r_2, s_2) dr_2 ds_2}{[(r - r_2)(s - s_2)]^{3/2}} \quad (131)$$

Equation (130) gives vertical velocity in terms of a prescribed surface potential for a wing with a supersonic leading edge, while equation (131) does the same for a wing with a leading edge which is partly subsonic and partly supersonic. The transform with the area τ_3 (see fig. 12) can also be used to obtain inversions to equation (129) under conditions such as those imposed in the development of equations (127) and (128).

DISCUSSION OF INVERSION OF SUPERSONIC SOURCE AND DOUBLET EQUATIONS FROM A PHYSICAL BASIS

Each of the above examples used to illustrate the application of the null transform method to the inversion of double integral equations represents the solution of a class of supersonic problems. However, most of these solutions are well known and were originally obtained by reasoning that was suggested by knowledge of the physical structure of the problem. For example, the reciprocal relation between the source and doublet integral equations

$$\left. \begin{array}{l} -\varphi_u(r, s) = \frac{-1}{\pi M_0} \int_{\tau_0} \int \frac{w_u(r_1, s_1) dr_1 ds_1}{\sqrt{(r - r_1)(s - s_1)}} \\ \\ w_u(r, s) = \frac{-M_0}{4\pi} \oint_{\tau_0} \int \frac{\varphi_u(r_1, s_1) dr_1 ds_1}{[(r - r_1)(s - s_1)]^{3/2}} \end{array} \right\} \quad (132)$$

when considered with respect to the area τ_0, has a simple physical meaning. By definition τ_0 is an area bounded by the Mach forecone and a supersonic leading edge. From a

physical standpoint it is clear that the flow field at a point, affected only by a supersonic edge, on the upper surface of a wing cannot be influenced by the shape of any part of the lower surface of the wing. In other words, the upper and lower surfaces are noninteracting. Hence, the upper surface of a wing does not "know" whether it is the upper surface of a lifting plate that is supporting loading and has no thickness, or the upper surface of a wing section that is symmetrical above and below the $z=0$ plane. Thus the source equation must be the inversion to the doublet equation and vice versa, and from a physical point of view the reciprocal relation given by equation (132) is obvious.

The solution to wing problems involving one supersonic and one subsonic edge, giving an area of integration for the integral equations corresponding to the area τ_1 in the previous discussion, was originally obtained by Evvard (see reference 7). It should be noted here that the inversions to the source and doublet equations (equations (126) and (131)) considered with respect to the area τ_1, no longer form reciprocal relations.

Finally, the examples presented herein with regard to the mixed type of problem have also been derived (see references 8 and 9) using more or less physical arguments. The solutions to these mixed type problems form the basis of a lift cancellation technique that provides a very useful extension of Evvard's original discovery.

ITERATIVE METHODS OF SOLUTION

Other types of plan forms.—The question that naturally arises from a practical viewpoint is how the source and doublet equations can be inverted when the area τ_1 is not of the two special kinds discussed, or, in other words, when the wing plan forms are complicated by having more than one monotonic (in the r, s plane) subsonic edge. The answer must be that, unless null transforms with respect to these new integration areas can be discovered, the methods discussed here will not give the direct inversion to the problem. Several possibilities remain, however, so that even when the null transform cannot be found the concepts of the residual, inherent singularity, etc., can be used to simplify, if not solve, the supersonic source and doublet integral equations.

With respect to the lift cancellation techniques already mentioned, references 8 and 9 outline these methods in considerable detail and show how they can be applied to find the loading on wings in regions affected by two or more subsonic edges.

Regions influenced by multiple reflections of Mach waves.—A more direct example of how some of the concepts presented heretofore can be applied, even when the null transform is not available, is given by considering the following problem: Find the loading at the point (x, y) on the flat wing tip shown in figure 14. If the r, s coordinate system is used and the operator

$$\int_{f_2^*(s)}^{r} \frac{dr_2}{\sqrt{r-r_2}} \left(\frac{\partial}{\partial r_2} + \frac{\partial}{\partial s} \right) \int_{f_3(r)}^{s} \frac{ds_2}{\sqrt{s-s_2}}$$

(where the area of integration is region 4 in fig. 14 (b)) is applied to the doublet equation (equation (82)), two results
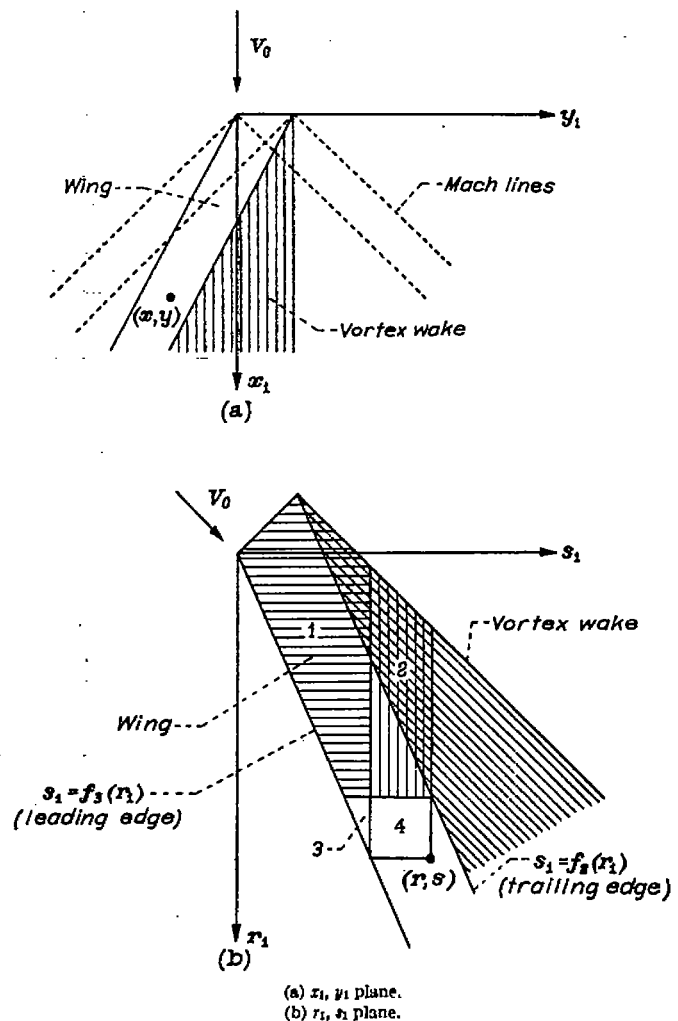

(a) x_1, y_1 plane.
(b) r_1, s_1 plane.

FIGURE 14.—Tip of swept forward wing and regions of integration.

can be anticipated: One, since the loading off the wing is zero, the residual will be $-2\pi\beta u_u(r,s)$; and the other, the transformed integral with the s_2 (in the notation of equation (124a)) integration performed first, is zero for all points r_1,s_1 lying in regions 2 and 4 in figure 14. These results follow directly from the discussion presented above in case 3 of the similar integral transform applied to the doublet equation (equation (122)) with the τ_1 area. Without proceeding further, therefore, it is apparent that the original integral equation has been reduced to: (1), an integral of the known function w_u over region 4 in figure 14; and (2), an integral of the unknown function φ_u over the regions 1 and 3 in figure 14. (It can be shown that the integration over region 3 will also vanish.) But regions 1 and 3 are ahead of the Mach forecone from (r,s). Hence, by repeating the above process for regions farther and farther up (toward the origin) the wing, the problem must eventually be reduced to one of finding the solution for an area such as τ_1; in other words, to a problem involving only two edges, one subsonic and one supersonic. Since the latter problem is solved, the one considered in this section is also (theoretically at least) solved.

Triangular plan form with subsonic leading edges.—As a final exemplification of the preceding concepts, consider the problem of finding the loading on a flat triangular wing fly-

ing at a supersonic speed but with both leading edges subsonic (see fig. 15). To the supersonic doublet equation (equation (82)) apply the integral transform
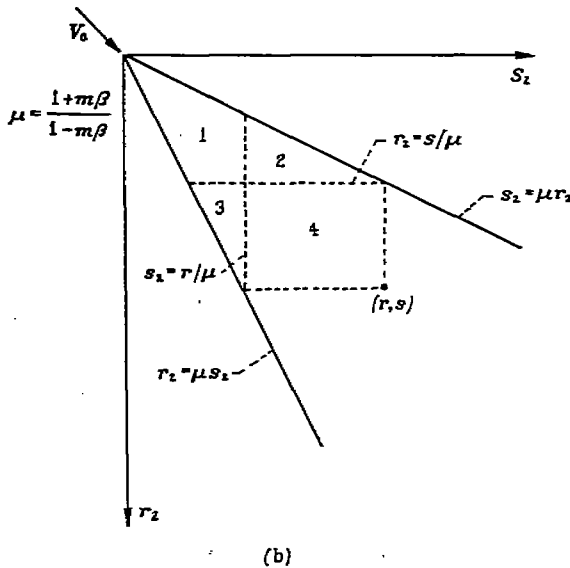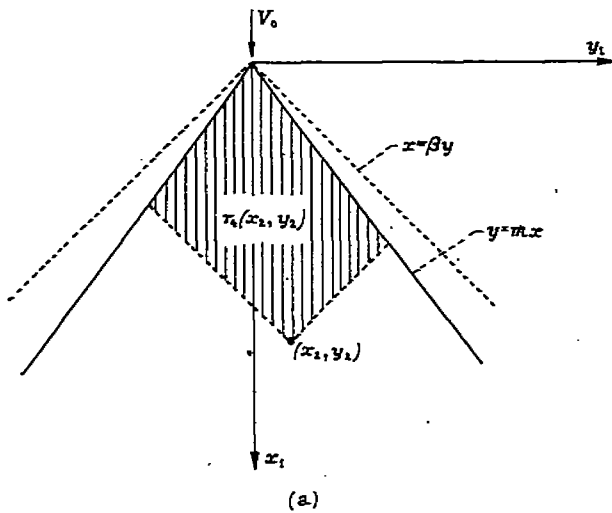
$$\frac{-1}{\pi M_0}\int_{s/\mu}^{r}dr_2\int_{r/\mu}^{s}ds_2\frac{1}{\sqrt{(r-r_2)(s-s_2)}}$$

where the area of integration is region 4 in figure 15(b). There results

$$\frac{-1}{\pi M_0}\int_{s/\mu}^{r}dr_2\int_{r/\mu}^{s}ds_2\frac{w_u(r_2,s_2)}{\sqrt{(r-r_2)(s-s_2)}}=$$
$$\frac{1}{4\pi^2}\int_{s/\mu}^{r}dr_2\int_{r/\mu}^{s}ds_2\oint\oint_{\tau_4(r_2,s_2)}\frac{\varphi_u(r_1,s_1)\,dr_1ds_1}{\sqrt{(r-r_2)(s-s_2)}\,[(r_2-r_1)(s_2-s_1)]^{3/2}}$$
(133)

where $\tau_4(r_2,s_2)$ is the original area of integration as shown in figure 15(a). Since w_u is a constant equal to $-V_0\alpha$, the left-hand side of equation (133) becomes

$$\frac{4V_0\alpha}{\pi M_0}\sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)}$$
(134)

where $\mu=(1+m\beta)/(1-m\beta)$, m being the slope of the leading edge in the x,y plane. Inverting the order of integration and evaluating the residual, one finds that equation (133) can be written

$$\varphi_u(r,s)=\frac{4V_0\alpha\sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)}}{\pi M_0}-\frac{\sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)}}{\pi^2}\int_{\tau_5}\int\frac{\varphi_u(r_1,s_1)\,dr_1ds_1}{(r-r_1)(s-s_1)\sqrt{\left(\frac{r}{\mu}-s_1\right)\left(\frac{s}{\mu}-r_1\right)}}$$
(135)

where τ_5 is the area shown as region 1 in figure 15(b).

Equation (135) is not, of course, the inversion of the supersonic doublet equation for the triangular flat plate. In fact, it simply represents the transformation of the doublet equation, which is a singular integral equation of the first kind, to a singular integral equation of the second kind. However, this latter form has the advantage that it is readily susceptible to the process of iteration. Thus, in the particular case of equation (135), it is possible to take as a first approximation to φ_u the value

$$\varphi_1(r,s)=\frac{4V_0\alpha}{\pi M_0}\sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)}$$
(136)

and, as the second, the value

$$\varphi_2(r,s)=\frac{4V_0\alpha}{\pi M_0}\sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)}\left[1-\frac{1}{\pi^2}\int_{\tau_5}\int\frac{\sqrt{\left(r_1-\frac{s_1}{\mu}\right)\left(s_1-\frac{r_1}{\mu}\right)}\,dr_1ds_1}{(r-r_1)(s-s_1)\sqrt{\left(\frac{r}{\mu}-s_1\right)\left(\frac{s}{\mu}-r_1\right)}}\right]$$
(137)

and so on. By means of the substitutions

$$\mu r_1=\frac{s+r\mu\rho_1+s\sigma_1}{1+\rho_1+\sigma_1}$$
$$\mu s_1=\frac{r+r\rho_1+s\mu\sigma_1}{1+\rho_1+\sigma_1}$$
(138)

the double integral in the equation for φ_2 can be evaluated. Thus, after some manipulation, one finds[7]

$$\int_{\tau_5}\int\frac{\sqrt{\left(r_1-\frac{s_1}{\mu}\right)\left(s_1-\frac{r_1}{\mu}\right)}dr_1ds_1}{(r-r_1)(s-s_1)\sqrt{\left(\frac{r}{\mu}-s_1\right)\left(\frac{s}{\mu}-r_1\right)}}$$
$$=2\pi[2E_1-(1-k_1^2)K_1]-\pi^2\quad(139)$$

where K_1 and E_1 are complete elliptic integrals of the first and second kinds, respectively, with moduli $k_1=1/\mu$. Using the identity

(a) x_1, y_1 plane.
(b) r_2, s_2 plane.

FIGURE 15.—Integration areas for triangular wing.

[7] Notice that the area of integration, i. e., the area τ_5, is bounded by the lines which represent the four roots to the radicals in the integrand. Notice also that the *value of this double integral is independent of r and s* and depends only on the parameter μ, the slope of the leading edges in the r,s plane.

$$2E_1-(1-k_1{}^2)K_1=\frac{2E}{1+m\beta} \qquad (140)$$

where E has the modulus $k=\sqrt{1-m^2\beta^2}$, and returning to the Cartesian coordinate system, one finds for the values of φ_1 and φ_2,

$$\varphi_1=\frac{4V_0\alpha}{\pi(1+m\beta)}\sqrt{m^2x^2-y^2} \qquad (141)$$

$$\varphi_2=\frac{4V_0\alpha}{\pi(1+m\beta)}\sqrt{m^2x^2-y^2}\left[2-\frac{4E}{\pi(1+m\beta)}\right] \qquad (142)$$

The process of iteration could be continued, and φ_u would be expressed as an infinite series of terms containing the parameter $m\beta$. However, since the terms in the series expansion are all independent of x and y, or, in the characteristic system, of r and s, it is more efficient to write φ in the form

$$\varphi_u(r,s)=A\sqrt{\left(s-\frac{r}{\mu}\right)\left(r-\frac{s}{\mu}\right)}=\frac{M_0A\sqrt{m^2x^2-y^2}}{1+m\beta} \qquad (143)$$

and determine the magnitude of A by substituting this expression into equation (135). There results the equality

$$A=\frac{4V_0\alpha}{\pi M_0}-\frac{A}{\pi^2}\left(\frac{4\pi E}{1+m\beta}-\pi^2\right)$$

from which it can be shown that

$$A=\frac{V_0\alpha}{M_0E}(1+m\beta) \qquad (144)$$

Finally, therefore, the velocity potential on the upper surface of a triangular wing with subsonic leading edges can be written

$$\varphi_u(x,y)=\frac{V_0\alpha}{E}\sqrt{m^2x^2-y^2} \qquad (145)$$

and the familiar expression for the loading coefficient (see, e. g., reference 10) follows immediately

$$\frac{\Delta p}{q}=\frac{4\alpha m^2x}{E\sqrt{m^2x^2-y^2}} \qquad (146)$$

The purpose of examining this particular problem was not, of course, to obtain the solution presented as equation (145) or (146), since that solution is by now quite well known. Rather, the purpose was to show how the supersonic doublet equation could be transformed to a singular-integral equation of the second kind and how this equation could, in turn, be solved by applying an iteration process. Such a method has far more general applications than are given here and is by no means limited to problems in which the flow is conical or quasi-conical.

PART IV—APPLICATIONS

DIRECT PROBLEMS

The following three examples will serve to illustrate how the formulas derived in Part I can be used to solve the direct problems outlined in Part II. The applications will be limited to supersonic flow problems.

RECTANGULAR WING WITH BICONVEX SECTION

Consider a rectangular, nonlifting wing with a chordwise section given everywhere by the equation

$$\lambda_u(x,y)=\frac{2t}{c^2}(c-2x) \qquad (147)$$

where λ_u is the slope of the upper surface, c is the chord, and t is the maximum thickness. The equation for the pressure on the surface of such a wing will change form in each of the four regions indicated in figure 16. In region 1 the pressure is the same as for a two-dimensional wing with the same section, and the remaining regions contain the three-dimensional or tip effects.
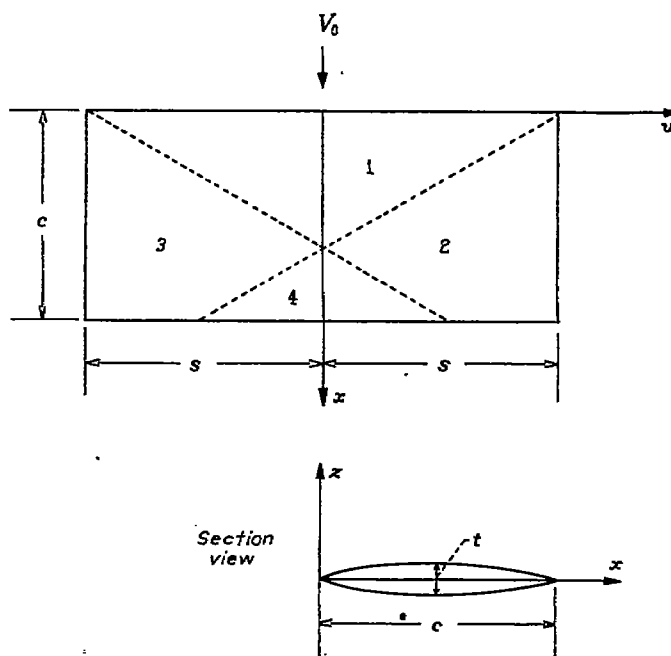

FIGURE 16.—Rectangular nonlifting wing with biconvex section.

The equation for pressure coefficient on the wing can be determined from either equation (76a) or (76b). Equation (76a), for example, becomes for region 1

$$C_p=\frac{4t}{\beta c^2}(c-2x)-\frac{4t}{\pi c^2}\int_0^x(c-2x_1)(x-x_1)dx_1\int_{y-\frac{x-x_1}{\beta}}^{y+\frac{x-x_1}{\beta}}\frac{dy_1}{r_0{}^3}$$

From the result given as equation (17), it is apparent that the integral term is zero and the pressure coefficient in region 1 of figure 16 is simply

$$C_p=\frac{4t}{\beta c^2}(c-2x) \qquad (148)$$

One can easily show that equation (76b) yields the same result.

The evaluation of pressure coefficient in regions 2, 3, and 4 can be carried out in a similar fashion. A slightly different approach can be used, however, that is useful in obtaining results in this and similar problems. Consider a wing with regions as shown in figure 17. Regions a, b, and c include all the area ahead of the line $x_1=x$. It is obvious that $r_0{}^3$ is a pure real number for all x_1, y_1 inside region b (i. e., inside
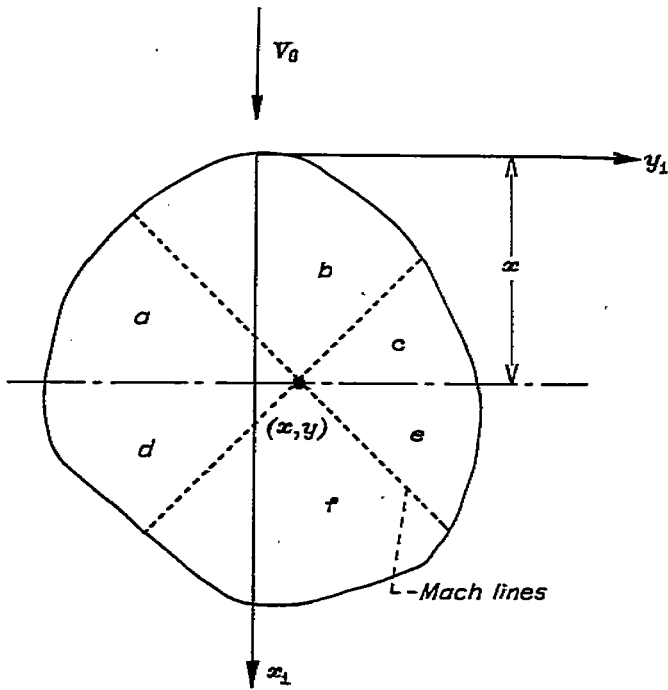
V_0

y_1

b

a

c

x

d

(x,y)

e

f

Mach lines

x_1

FIGURE 17.—Regions used in discussion of τ_t.

the forecone from x, y) and a pure imaginary number for all x_1, y_1 inside regions a and c. Further, it is clear from physical considerations that λ_u is always real on the wing plan form. Hence, if the symbol τ_t is introduced to represent the combined area in regions a, b, and c, the area τ in equations (76)

can be replaced by the area τ_t and the real part of the result will be the correct answer for the pressure coefficient.[8]

By means of this concept, pressure coefficient for all points on the wing can be written in the form

$$C_p(x,y)=\frac{4t}{\beta c^2}(c-2x)-\frac{4t}{\pi c^2}\text{ R. P.}\int_0^x dx_1 \!\!\!\!\!\!\int_{-s}^{s} dy_1 \frac{(c-2x_1)(x-x_1)}{r_0^3} \tag{149}$$

or, alternatively,

$$C_p(x,y)=\frac{-4t}{\pi c^2}\text{ R.P.}\int_{-s}^{s}dy_1\!\!\!\!\!\!\int_0^x dx_1 \frac{(c-2x_1)(x-x_1)}{r_0^3} \tag{150}$$

where the letters R. P. indicate that the real part of the integral is to be taken. Evaluating, for example, the former equation one finds

$$C_p=\frac{4t}{\beta c^2}(c-2x)+\frac{4t}{\pi c^2}\text{ R.P.}\left[-\frac{c-2x}{\beta}\text{ arc cos }\frac{\beta(s-y)}{x}+\right.$$

$$2(y-s)\text{ arc cosh }\frac{x}{\beta(s-y)}-$$

$$\left.\frac{c-2x}{\beta}\text{ arc cos }\frac{\beta(s+y)}{x}-2(y+s)\text{ arc cosh }\frac{x}{\beta(y+s)}\right] \tag{151}$$

The real and imaginary parts of the arc cosine and arc cosh terms in this equation are given in the following table for all real values of the argument.

[8] The areas d and e could also be included in the definition of τ_t since r_0^2 is imaginary within these regions also. However, region f must not be included since there the term r_0^2 is again real. The definition of τ_t adopted is usually the most convenient.

Range of x	$-1>x$	$-1<x<1$	$1<x$
arc cosh $x=\int_1^x \frac{dt}{\sqrt{t^2-1}}$	$i\pi+$arc cosh $(-x)$	i arc cos x	arc cosh x
arc cos $x=\int_x^1 \frac{dt}{\sqrt{1-t^2}}$	$\pi-i$ arc cosh $(-x)$	arc cos x	i arc cosh x
arc sin $x=\int_0^x \frac{dt}{\sqrt{1-t^2}}$	$-\frac{\pi}{2}+i$ arc cosh $(-x)$	arc sin x	$\frac{\pi}{2}-i$ arc cosh x

Equation (151) contains, at once, the entire solution to the wing shown in figure 16. Thus, in region 1 none of the terms in the braces has a real part. Hence, equation (148) follows immediately. In region 3 the solution can be written

$$C_p=\frac{4t}{\pi\beta c^2}\left\{(c-2x)\left[\pi-\text{arc cos }\frac{\beta(y+s)}{x}\right]-\right.$$

$$\left.2\beta(y+s)\text{ arc cosh }\frac{x}{\beta(y+s)}\right\}$$

The solution in region 2 follows from the one given in region 3 by symmetry; and, finally, the expression for the pressure coefficient in region 4 is given by equation (151) wherein every term is real.

DRAG REVERSIBILITY THEOREM

The well-known theorem that the drag of a symmetrical sharp-edged, nonlifting body is the same in forward and

reversed flight at the same speed (see references 11 and 12) can be derived in another way using the methods described in the above sections.

By definition, drag coefficient is

$$C_D=\frac{1}{S}\iint_S 2\lambda_u(x,y)\,C_p(x,y)\,dx\,dy \tag{152}$$

where S is the area of the wing. Using the real part concept outlined in the discussion of the preceding example, one can write

$$C_D=\frac{-4}{\pi S}\iint_S \lambda_u(x,y)\,dx\,dy\text{ R.P.}\int_{\tau_t}dy_1\!\!\!\!\!\!\int dx_1 \frac{(\dot{x}-x_1)\lambda_u(x_1,y_1)}{r_0^3} \tag{153}$$

The equation for the drag coefficient in reversed flight can be obtained by:

1. Replacing the area τ_t by τ_{tt} where $\tau_t+\tau_{tt}=S$

2. Rotating the axial system in the xy plane through $180°$
3. Reversing the signs of $\lambda_u(x, y)$ and $\lambda_u(x_1, y_1)$

There results

$$C_{D_r}=\frac{-4}{\pi S}\int\int_S \lambda_u(x, y)dxdy \text{ R.P.}\int_{\tau_{ij}} dy_1 \oint dx_1 \frac{(x_1-x)\lambda_u(x_1, y_1)}{r_0^3}$$ (154)

Subtracting equation (154) from (153) gives

$$C_D-C_{D_r}=\frac{-4}{\pi S}\text{R.P.}\int dy\int_S dx\int dy_1\oint_S dx_1 \frac{x\lambda_u(x_1, y_1)\lambda_u(x, y)}{r_0^3}+$$
$$\frac{4}{\pi S}\text{R.P.}\int dy\int_S dx\int dy_1\oint_S dx_1 \frac{x_1\lambda_u(x_1, y_1)\lambda_u(x, y)}{r_0^3}$$ (155)

Since the symbols x_1, y_1, x, y are dummy variables of integration, the last term in equation (155) can be written

$$\frac{4}{\pi S}\text{R.P.}\int dy_1\int_S dx_1\int dy\oint_S dx \frac{x\lambda_u(x, y)\lambda_u(x_1, y_1)}{r_0^3}$$

But reversing the operators $\int dy_1\int dx_1$ and $\int dy\int dx$ (always preserving the same order within the operation) and subtracting gives

$$\int dy_1\int_S dx_1\int dy\oint_S dx \frac{x\lambda_u(x, y)\lambda_u(x_1, x_1)}{r_0^3}=$$
$$\int dy\int_S dx\int dy_1\oint_S dx_1 \frac{x\lambda_u(x, y)\lambda_u(x_1, y_1)}{r_0^3}$$

since the residual is zero. Hence, the second term in equation (155) is the same expression as the first except for the sign and

$$C_D-C_{D_r}=0$$

or

$$C_D=C_{D_r}$$ (156)

as was to be shown.

LIFT ON WINGS WITH SUPERSONIC EDGES

The lift on any wing can be written

$$\frac{L}{q}=\int\int_S \frac{\Delta p}{q} dydx$$ (157)

Moreover,

$$\int_{L.E.}^{T.E.} \frac{\Delta p}{q} dx=\frac{4\varphi_{T.E}}{V_0}$$

where T. E. and L. E. denote the trailing and leading edges, respectively, and $\varphi_{T.E.}$ is the value of the velocity potential on the upper surface of the wing at the trailing edge.

Consider now a wing with all edges supersonic and a straight trailing edge not necessarily at right angles to the free-stream direction (see fig. 18). Let the wing be a plate having arbitrary twist and camber. Then, for a point on the wing, the velocity potential can be written in the x, y coordinate system on the basis of equation (121) as

$$\varphi_u=\frac{-1}{\pi}\text{R.P.}\int_{\tau_i}\int \frac{w_u(x_1,y_1)dx_1dy_1}{\sqrt{(x-x_1)^2-\beta^2(y-y_1)^2}}$$ (158)
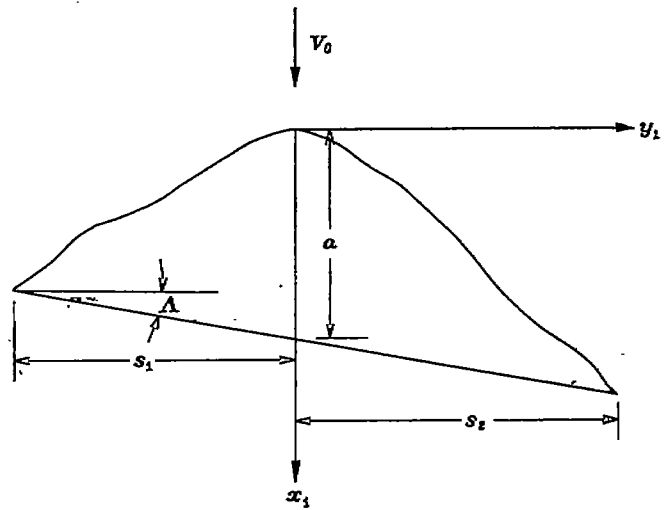

FIGURE 18.—Wing with supersonic edges.

where r_t, as in the previous examples, is the area on the wing ahead of the line $x_1=x$. If the equation of the trailing edge is

$$x_1=a+y_1 \tan \Lambda$$

where a is some constant, the value of the potential at the trailing edge can be written

$$\varphi_{T.E.}=\frac{-1}{\pi}\text{R.P.}\int\int_S \frac{w_u(x_1,y_1)dx_1dy_1}{\sqrt{(a+y \tan \Lambda-x_1)^2-\beta^2(y-y_1)^2}}$$ (159)

where the area of integration is the whole wing plan form since the trailing edge of the wing is supersonic and the aftercone from the point at which $\varphi_{T.E.}$ is being evaluated cannot intersect the wing. The total lift L on the wing can, therefore, be written in the form

$$\frac{L}{q}=\frac{-4}{\pi V_0}\text{R.P.}\int_{s_1}^{s_2} dy \int\int_S \frac{w_u(x_1,y_1)dx_1dy_1}{\sqrt{(a+y \tan \Lambda-x_1)^2-\beta^2(y-y_1)^2}}$$ (160)

The area S does not depend on y, so the y integration can be made first and, since the edges are all supersonic, the interval $s_1\leq y\leq s_2$ must always contain the roots λ_1 and λ_2 of the expression under the radical. Hence

$$\text{R.P.}\int_{s_1}^{s_2} \frac{dy}{\sqrt{(\beta^2-\tan^2\Lambda)(\lambda_1-y)(y-\lambda_2)}}=$$
$$\int_{\lambda_1}^{\lambda_2} \frac{dy}{\sqrt{(\beta^2-\tan^2\Lambda)(\lambda_1-y)(y-\lambda_2)}}$$

and since

$$\int_{\lambda_1}^{\lambda_2} \frac{dy}{\sqrt{(\lambda_1-y)(y-\lambda_2)}}=\pi$$

then

$$\frac{L}{q}=\frac{-4}{\sqrt{\beta^2-\tan^2\Lambda}}\int\int_S \frac{w_u(x_1,y_1)}{V_0}dx_1dy_1$$ (161)

Defining the average angle of attack $\bar{\alpha}$ by the expression

$$\bar{\alpha}=\frac{1}{S}\int_S\int \frac{-w_u(x_1,y_1)dx_1dy_1}{V_0}$$ (162)

one can write equation (161) in the alternative form

$$C_L = \frac{4\bar{\alpha}}{\sqrt{\beta^2 - \tan^2 \Lambda}} \qquad (163)$$

It is interesting to notice that the lift coefficient for the wing just studied is the same as that for a two-dimensional flat plate flying at an angle of attack $\bar{\alpha}$ into a free stream, the speed of which is given by the component of velocity normal to the trailing edge of the three-dimensional wing. This result has been derived previously in reference 13.

INVERSE PROBLEMS

LOW-ASPECT-RATIO RECTANGULAR WING

It will be noted in the summary of results for the fundamental formulas applicable to supersonic flow (equations (75) through (82)) that the results are presented for both orders of integration in the x, y coordinate system. While the analysis of direct problems can be carried out in all cases if, say, the x_1 integration is always performed first, it may sometimes be more convenient to perform the y_1 integration first. In the analysis of inverse problems, however, it is much more important that freedom exists in the choice of the first variable of integration. A good example of this is provided by the following approximate derivation of the loading on a slender (in the streamwise sense) rectangular flat plate.

By considering the special case when the wing chord is long compared to the span, one can obtain an approximate solution for the slender flat plate by assuming the loading coefficient has the form

$$\frac{\Delta p}{q} = 4\alpha f\left(\frac{x}{s}\right)\sqrt{1 - \left(\frac{y}{s}\right)^2} \qquad (164)$$

where s is the semispan (see fig. 19) and $f(x/s)$ is an unknown function. The function f is to be determined by the condition that w_u is constant along the center line of the wing. If the solution to such a problem is to be determined by use of the doublet or vortex equation, it is obviously important that the first integration be made with respect to y_1 since the variation of $\Delta p/q$ with y_1 is known.

Since $\Delta p/q$ is given, let the vortex equation be used and let equation (164) be placed into (77a). For $y=0$ (and for added simplicity for $\beta=1$) the area τ is shown by the shaded area in figure 19 and the resulting equation can be written for $x>s$

$$\frac{w_u}{V_0} = -\alpha f\left(\frac{x}{s}\right) + \frac{\alpha}{\pi}\int_0^{x-s}(x-x_1)f\left(\frac{x_1}{s}\right)dx_1$$

$$\fint_{-s}^{s}\frac{\sqrt{1-(y_1/s)^2}}{y_1^2\sqrt{(x-x_1)^2-y_1^2}}dy_1 + \frac{\alpha}{\pi}\int_{x-s}^{x}(x-x_1)f\left(\frac{x_1}{s}\right)dx_1$$

$$\fint_{-(x-x_1)}^{x-x_1}\frac{\sqrt{1-(y_1/s)^2}}{y_1^2\sqrt{(x-x_1)^2-y_1^2}}dy_1$$

for $0<x<s$

$$\frac{w_u}{V_0} = -\alpha f\left(\frac{x}{s}\right) + \frac{\alpha}{\pi}\int_0^{x}(x-x_1)f\left(\frac{x_1}{s}\right)dx_1$$

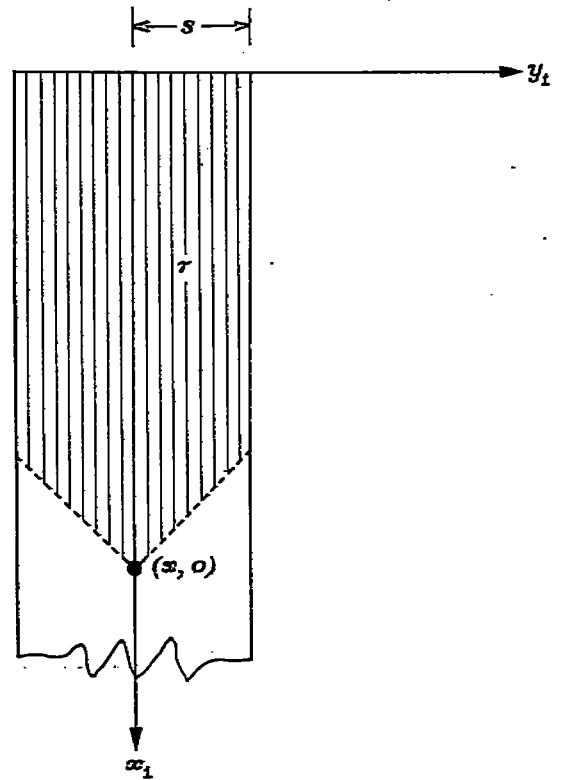$$\fint_{-(x-x_1)}^{x-x_1}\frac{\sqrt{1-(y_1/s)^2}}{y_1^2\sqrt{(x-x_1)^2-y_1^2}}dy_1$$

FIGURE 19.—Integration area for slender rectangular wing.

Introduce the notation

$$\theta_1 = \frac{x_1}{s}, \quad \theta = \frac{x}{s}, \quad k_1 = \frac{1}{\theta-\theta_1}, \quad k_2 = \theta - \theta_1$$

and these equations become,[9] since $\alpha = -w_u/V_0$, for $0<\theta<1$

$$1 = f(\theta) + \frac{2}{\pi}\int_0^{\theta}k_2 B_2 f(\theta_1)d\theta_1 \qquad (165a)$$

for $1<\theta$

$$1 = f(\theta) + \frac{2}{\pi}\int_{\theta-1}^{\theta}k_2 B_2 f(\theta_1)d\theta_1 + \frac{2}{\pi}\int_0^{\theta-1}E_1 f(\theta_1)d\theta_1 \qquad (165b)$$

Equations (165) are integral equations of the second kind (more specifically, Volterra's integral equations of the second kind) and the kernels are regular and bounded everywhere in the interval of integration. Hence, their solution can be determined readily by numerical processes. This has been done and the result in terms of the loading coefficient on the center line $\left(\frac{\Delta p}{q\alpha}\right)_0$ $\left(\text{from equation (164)} \left(\frac{\Delta p}{q\alpha}\right)_0 = 4f\left(\frac{x}{s}\right)\right)$ is shown in figure 20.

For the purpose of comparison, the exact linearized value also is shown in the interval where it is known, together with another approximate solution obtained by Stewartson (reference 14) using a different approach. Near the leading edge, where the comparison with the exact results can be made, the agreement between the exact and approximate solutions obtained herein will be poorest because in this region the spanwise variation deviates most radically from the value assumed in the construction of the integral equation.

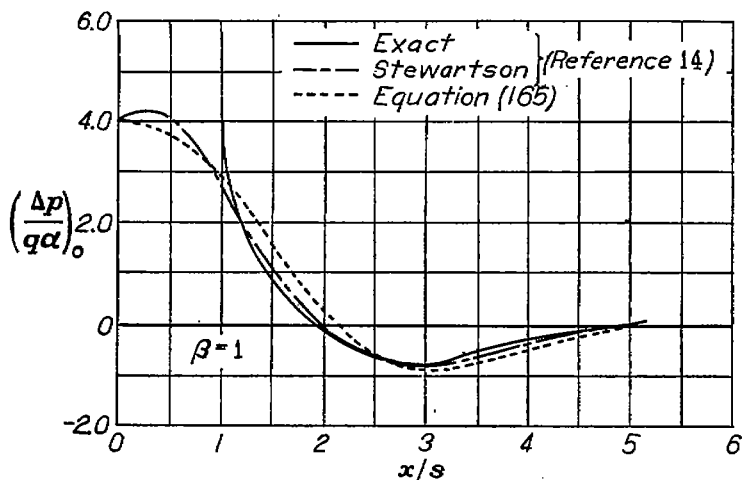[9] The symbols B and E indicate elliptic integrals. See the table of symbols.

FIGURE 20.—Load coefficient for unit angle of attack along center line of rectangular wing.

CONICAL AND QUASI-CONICAL FLOW

Equation (108) and its solution, equation (111), occur repeatedly in the study of aerodynamics. In fact, equation (108) is often referred to as the airfoil equation since it plays a dominant role in the development of linearized, two-dimensional, subsonic wing theory. It appears also in the study of slender wings (reference 15) or wings flying at near sonic speeds (reference 16) since the boundary conditions lead again to the required inversion of the same type of integral relationship. In the present section, problems arising in supersonic conical or quasi-conical flow fields will be reduced also to this basic equation.

Several methods exist whereby the solution to conical flow problems can be determined. The one to be studied here is based on the construction of conical elements extending radially from the apex of the field and inclined at an angle arc tan m to the x axis (see fig. 21(a)). In order to obtain such an element it is sufficient to subtract two plan forms of prescribed loading or thickness, each plan form having one side directed along the x axis while the other sides are inclined at angles that differ only infinitesimally.

Consider first the construction of a quasi-conical,[10] radial, lifting element that carries a load given by the expression

$$\frac{\Delta p}{q} = C y^\kappa \qquad (166)$$

where C is a constant. The upwash field of a triangular plan form such as the one shown in figure 21(b) can be found by integrating elementary horseshoe vortices over the appropriate area τ. Thus, using equation (77b), performing an integration, and making the substitutions
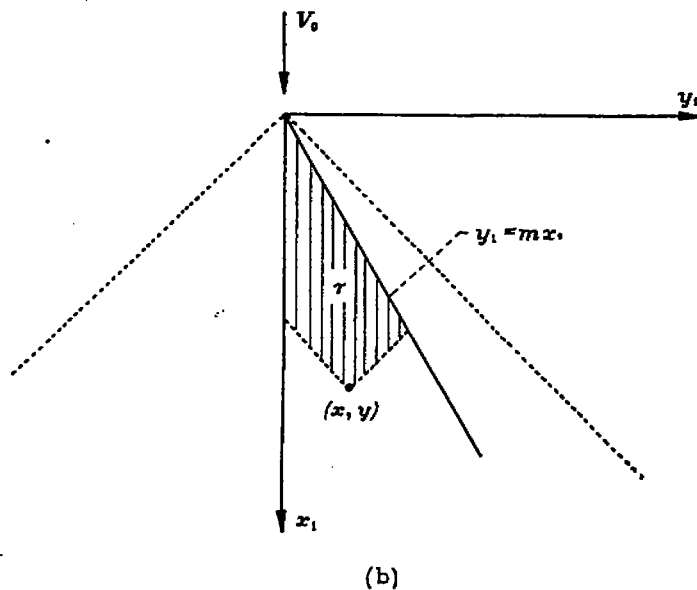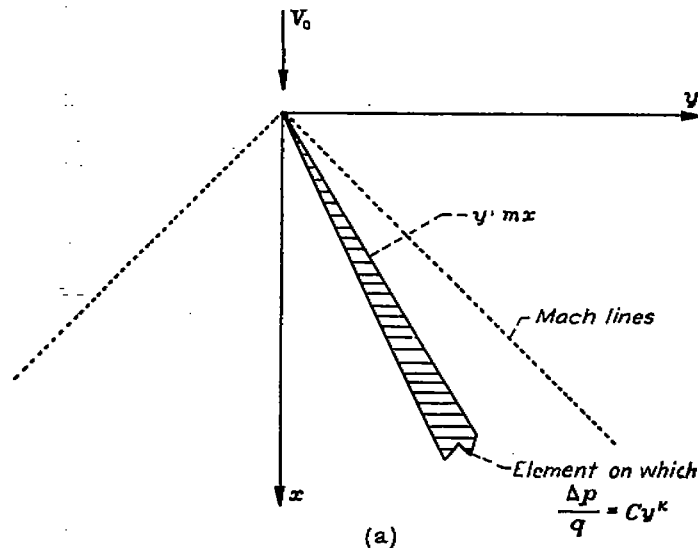
$$\theta = m\beta, \qquad \eta = \beta y/x, \qquad \eta_1 = \beta y_1/x \qquad (167)$$

one finds the result

$$\left(\frac{\beta}{x}\right)^\kappa \frac{w_u}{V_0} = \frac{\beta C}{4\pi} \int_0^{\theta \frac{1+\eta}{1+\theta}} \frac{\eta_1{}^\kappa d\eta_1}{(\eta-\eta_1)^2} \sqrt{\left(1-\frac{\eta_1}{\theta}\right)^2 - (\eta-\eta_1)^2}, \quad 0 < \theta, \ \eta < \theta \qquad (168a)$$

$$= \frac{\beta C}{4\pi} \int_0^{\theta \frac{1-\eta}{1-\theta}} \frac{\eta_1{}^\kappa d\eta_1}{(\eta-\eta_1)^2} \sqrt{\left(1-\frac{\eta_1}{\theta}\right)^2 - (\eta-\eta_1)^2}, \quad 0 < \theta, \ \theta < \eta \qquad (168b)$$

[10] A conical field is defined as one in which the induced velocities are constant along rays through a point. In the subsequent analysis this corresponds to the case when $\kappa=0$. Quasi-conical fields are those in which κ is greater than zero.

(a) Lifting element.
(b) Triangular wing.

FIGURE 21.—Construction of conical element.

Equations (168) can be written in the functional notation

$$w_u = f(\theta, \eta)$$

It follows that if the analysis were repeated for a wing with a slightly larger apex angle, there would have resulted

$$w_u = f(\theta + \Delta\theta, \eta)$$

Subtracting these two expressions for w_u gives the increment in vertical induced velocity due to a quasi-conical element

$$dw_u = \lim_{\Delta\theta \to 0} \frac{f(\theta+\Delta\theta, \eta) - f(\theta, \eta)}{\Delta\theta} \Delta\theta = \frac{\partial f}{\partial\theta} d\theta \qquad (169)$$

Carrying out the operation indicated by equation (169), making the substitution

$$\eta_1 = \theta \frac{\eta-t}{\theta-t}, \qquad t = \theta \frac{\eta-\eta_1}{\theta-\eta_1}$$

and distributing these elements between θ_1 and θ_0 with weight $C(\theta)$, one can finally show that the equation

$$\left(\frac{\beta}{x}\right)^\kappa \frac{w_u}{V_0} = \frac{-\beta(\kappa+1)}{4\pi} \int_{\theta_1}^{\theta_0} \theta^\kappa C(\theta) H(\theta,\eta) d\theta \qquad (170)$$

applies where

$$H(\theta,\eta) = \int_1^\eta \frac{(\eta-t)^\kappa \sqrt{1-t^2}}{(\theta-t)^{\kappa+2}t} dt, \quad \theta_1 \le \theta < \eta$$

$$= \int_{-1}^\eta \frac{(\eta-t)^\kappa \sqrt{1-t^2}}{(\theta-t)^{\kappa+2}t} dt, \quad \eta < \theta \le \theta_0$$

The function $H(\theta,\eta)$ has a simple pole at $\theta=\eta$ and the integral expression for w_u is therefore evaluated as a Cauchy principal part.

The boundary condition to be satisfied by equation (170) is that $\beta^\kappa w_u / x^\kappa V_0$ is a given polynomial of degree κ in the variable η. Hence, equation (170) is a singular integral equation with a pole of the same order as that in the airfoil equation. In its present form the equation appears somewhat formidable, but it can be simplified considerably by a simple operation. Since $\beta^\kappa w_u / x^\kappa V_0$ is a polynomial of degree κ, it follows that the $(\kappa+1)^{st}$ derivative of the right-hand member of equation (170) must vanish. Thus, using the concept of the generalized principal part, one has

$$0 = \frac{\sqrt{1-\eta^2}}{\eta} \int_{\theta_1}^{\theta_0} \frac{\theta^\kappa C(\theta) d\theta}{(\theta-\eta)^{\kappa+2}} \qquad (171)$$

which, by definition, can be put in the form

$$\left(\frac{\partial}{\partial\eta}\right)^{\kappa+1} \int_{\theta_1}^{\theta_0} \frac{\theta^\kappa C(\theta) d\theta}{\theta-\eta}$$

The function $\theta^\kappa C(\theta)$ is therefore to be found through the inversion of the integral equation

$$\int_{\theta_1}^{\theta_0} \frac{\theta^\kappa C(\theta) d\theta}{\theta-\eta} = \sum_{i=0}^\kappa a_i \eta^i \qquad (172)$$

But equation (172) is precisely the airfoil equation and its solution is given by equation (111). Hence,

$$\theta^\kappa C(\theta) = \frac{1}{\pi^2 \sqrt{(\theta_0-\theta)(\theta-\theta_1)}} \left[A - \int_{\theta_1}^{\theta_0} \frac{\sum_{i=0}^\kappa a_i \eta^i \sqrt{(\theta_0-\eta)(\eta-\theta_1)}}{\theta-\eta} d\eta \right] \qquad (173)$$

so that, finally,

$$\frac{\Delta p}{q} = \left(\frac{x}{\beta}\right)^\kappa \sum_{i=0}^{\kappa+1} \frac{b_i \theta^i}{\sqrt{(\theta_0-\theta)(\theta-\theta_1)}} \qquad (174)$$

where the coefficients b_i are functions of the constants θ_0 and θ_1 but not of θ. These coefficients must be determined from known conditions about the surface geometry.

Consider the unyawed (i. e., $\theta_1 = -\theta_0$) triangular wing shown in figure 22. If the loading is to be determined on a flat plate with such a plan form $\beta^\kappa w_u / x^\kappa V_0$ becomes $-\alpha$, and equation (174) reduces to
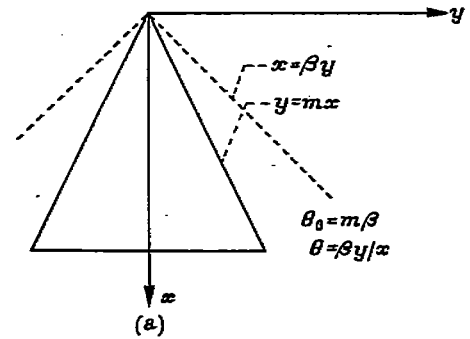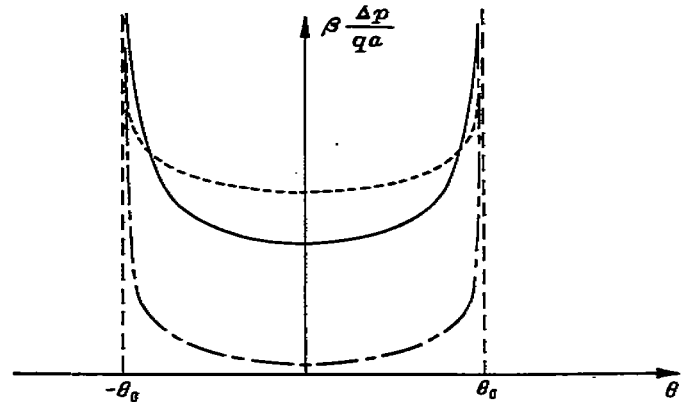
(a) Wing plan form.
(b) Load distribution at $x=1$.

FIGURE 22.—Load distribution on triangular wings with specified twist.

$$\frac{\Delta p}{q} = \frac{b_0 + b_1 \theta}{\sqrt{\theta_0^2 - \theta^2}} \qquad (175)$$

By symmetry, the coefficient b_1 in equation (175) must be zero and b_0 can be evaluated by placing equation (175) into (170) and integrating. There results the solution for the pressure coefficient shown in figure 22 where $\alpha=a$ and already presented as equation (146).

One can go on to show that if, as for a wing pitching about the apex at a rate Q, (see reference 18),

$$w_u = -Qx, \quad \frac{\beta^\kappa w_u}{x^\kappa V_0} = \frac{-\beta Q}{V_0}, \quad \kappa=1 \qquad (176)$$

then

$$\frac{\Delta p}{q} = \frac{4xQ}{V_0 \beta \sqrt{\theta_0^2 - \theta^2}} \frac{2\theta_0^2 - \theta^2}{\left[\frac{\theta_0^2 K}{1-\theta_0^2} + \frac{1-2\theta_0^2}{1-\theta_0^2} E\right]} \qquad (177)$$

where the complete elliptic integrals K and E have a modulus equal to $k = \sqrt{1-\theta_0^2}$. Further, if

$$\frac{w_u}{V_0} = hy^2, \quad \frac{\beta^\kappa w_u}{x^\kappa V_0} = h\eta^2, \quad \kappa=2 \qquad (178)$$

then

$$\frac{\Delta p}{q} = \frac{8h}{\beta}\left(\frac{x}{\beta}\right)^2 \theta_0^4$$

$$\left\{\frac{[2\theta_0^2 K - (1+\theta_0^2)E] + [-(3-\theta_0^2)K + (4-2\theta_0^2)E]\theta^2}{[-5\theta_0^4 K^2 + 8\theta_0^2(1+\theta_0^2)KE + (4\theta_0^4 - 19\theta_0^2 + 4)E^2]\sqrt{\theta_0^2 - \theta^2}}\right\} \qquad (179)$$

where $k=\sqrt{1-\theta_0{}^2}$ $(=\sqrt{1-m^2\beta^2})$ is the modulus of the complete elliptic integrals. These solutions are all shown in figure 22, where in one case $a=Q/V_0$ and in the other $a=-h/\beta^2$.

If the function $\beta^\kappa w_u/x^\kappa V_0$ is discontinuous but a polynomial in each interval of continuity, the solution given as equation (173) still applies. For example, consider the case when $\kappa=0$ and w_u/V_0 is a constant that changes sign in crossing the x axis. (See fig. 23.) For simplicity let $\theta_1=-\theta_0$, then by equation (173)

$$\frac{w_u}{V_0}=-\delta\frac{\eta}{|\eta|}, \qquad \kappa=0 \qquad (180)$$
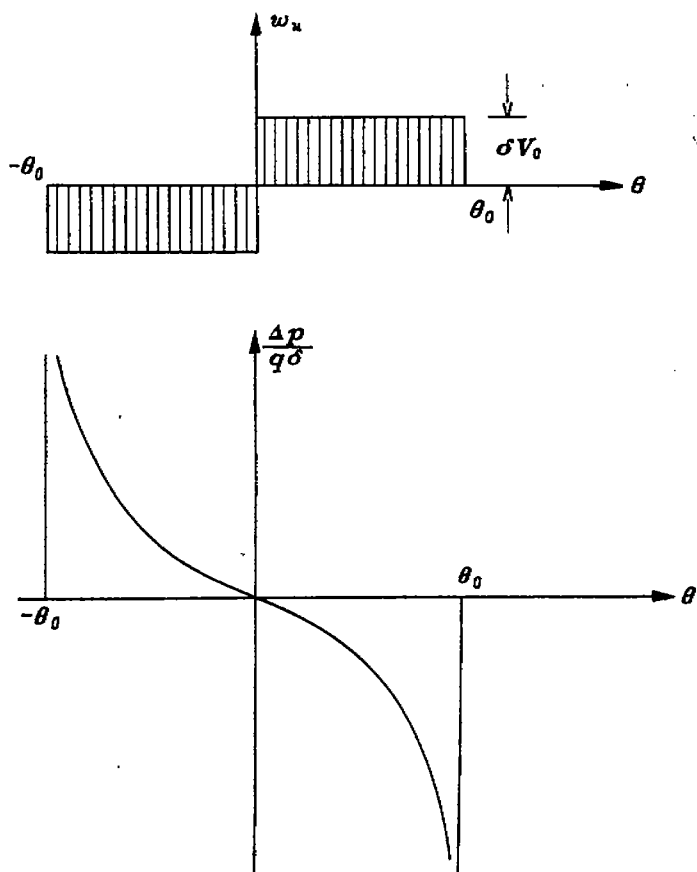
FIGURE 23.—Loading on differentially deflected triangular plate.

Again, as in the development of equation (175),

$$\frac{\Delta p}{q}=\frac{b_0+b_1\theta}{\sqrt{\theta_0{}^2-\theta^2}} \qquad (181)$$

Now, however, the solution must exhibit odd symmetry and the constant b_0 is zero. The constant b_1 can be evaluated by substituting equation (181) into equation (170). There results, finally,

$$\frac{\Delta p}{q}=\frac{8\delta\theta_0\theta}{\pi\beta\sqrt{\theta_0{}^2-\theta^2}} \qquad (182)$$

This solution is shown graphically in figure 23.

The methods described above can also be applied to problems involving wings with thickness and without lift. In these cases one constructs a radial element emanating from the origin and possessing a quasi-conical thickness distribution

$$\frac{dz_u}{dx}=\lambda_u=Cy^\kappa \qquad (183)$$

The derivation of the induced pressure field associated with the element follows closely the analysis in the lifting case. First a triangular plan form is considered with one side having the slope m and the other parallel to the free-stream direction. The thickness is assumed to have the form given by equation (183) so that the pressure coefficient can be obtained from the equation

$$C_p=\frac{-2}{\pi}\int\lambda_u\,dy\int_\tau\frac{(x-x_1)\,dx_1}{[(x-x_1)^2-\beta^2(y-y_1)^2]^{3/2}} \qquad (184)$$

Make the same notational changes as in the analysis of the lifting case; construct the element by taking the partial derivative with respect to θ; and, finally, distribute weighted elements over the wing plan form by making C a function of θ and integrating with respect to θ between the limits θ_1 and θ_0. There results

$$\left(\frac{\beta}{x}\right)^\kappa C_p=\frac{-2(\kappa+1)}{\pi\beta}\int_{\theta_1}^{\theta_0}\theta^\kappa C(\theta)H_1(\theta,\eta)\,d\theta \qquad (185)$$

where

$$H_1(\theta,\eta)=\int_1^\eta\frac{t(\eta-t)^\kappa dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}}, \quad \theta_1\leq\theta\leq\eta$$

$$=\int_{-1}^\eta\frac{t(\eta-t)^\kappa dt}{(\theta-t)^{\kappa+2}\sqrt{1-t^2}}, \quad \eta<\theta\leq\theta_0$$

The boundary conditions require $(\beta/x)^\kappa C_p$ to be a polynomial of degree κ in η. If the $(\kappa+1)^{st}$ derivative of equation (185) is set equal to zero, the relation

$$0=\int_{\theta_1}^{\theta_0}\frac{\theta^\kappa C(\theta)d\theta}{(\theta-\eta)^{\kappa+2}}=\left(\frac{\partial}{\partial\eta}\right)^{\kappa+1}\frac{1}{(\kappa+1)!}\int_{\theta_1}^{\theta_0}\frac{\theta^\kappa C(\theta)d\theta}{\theta-\eta} \qquad (186)$$

results. The function $C(\theta)$ satisfies the same integral equation that arose in the lifting case, and the solution can therefore be written immediately in the form

$$\lambda_u=y^\kappa C(\theta)=\left(\frac{x}{\beta}\right)^\kappa\sum_{i=0}^{\kappa+1}\frac{b_i\theta^i}{\sqrt{(\theta_0-\theta)(\theta-\theta_1)}} \qquad (187)$$

Again the coefficients b_i must be determined from known conditions about the surface geometry.

Consider first the case when the pressure is constant over an unyawed triangular plan form as in figure 24. Thus

$$C_p=C_{p_0}, \qquad \kappa=0 \qquad (188)$$
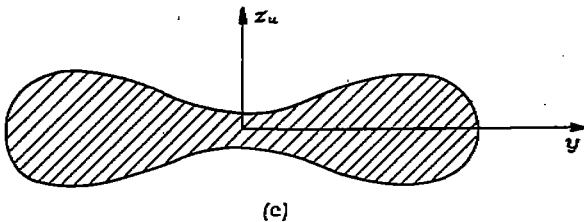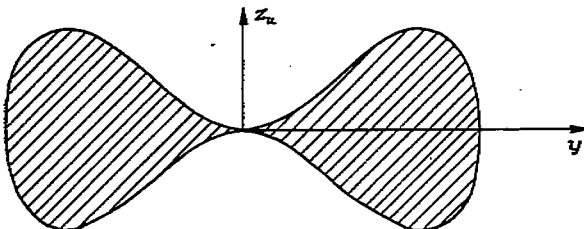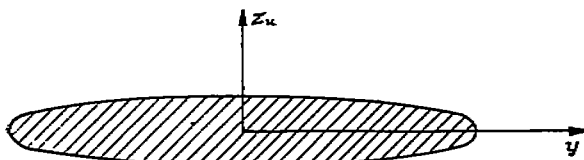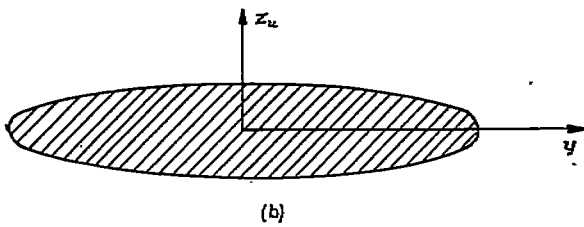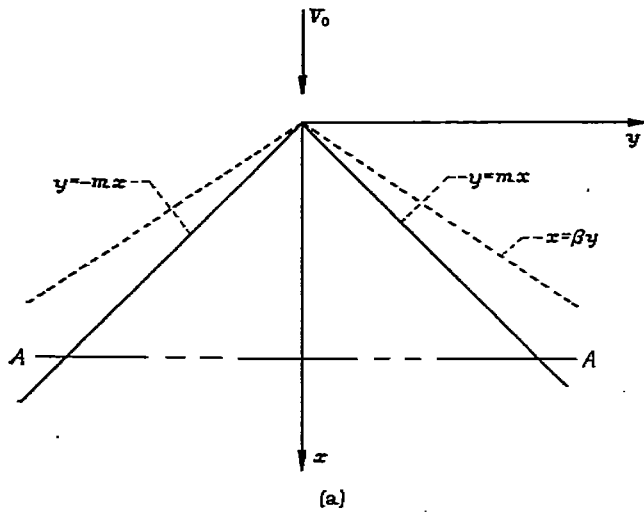
and (b_1 being zero by symmetry)

$$\lambda_u=\frac{dz_u}{dx}=\frac{b_0}{\sqrt{\theta_0{}^2-\theta^2}} \qquad (189)$$

Evaluating the constant b_0 by substituting equation (189) into (185), it can be shown that the surface ordinate is

$$z_u=\frac{(1-\theta_0{}^2)C_{p_0}}{2m^2(K-E)}\sqrt{m^2x^2-y^2} \qquad (190)$$

where the modulus of K and E is $k=\sqrt{1-\theta_0{}^2}$. This result, which is the equation of an elliptical section, is shown in figure 24.

(a) Triangular wing with subsonic edges.
(b) Section AA for C_p constant.
(c) Section AA for C_p varying linearly with x.

FIGURE 24.—Symmetrical triangular wing with specified pressure distributions.

Finally, consider an unyawed triangular wing for which the pressure varies linearly in the x direction. For such a case

$$C_p = (C_{p_0})x, \qquad \kappa = 1 \qquad (191)$$

and (b_1 again being zero by symmetry)

$$\lambda_u = \frac{dz_u}{dx} = \frac{x}{\beta} \frac{b_0 + b_2 \theta^2}{\sqrt{\theta_0^2 - \theta^2}} \qquad (192)$$

from which it immediately follows by integration

$$z_u = \frac{b_0 x}{2\beta^2 m^2} \sqrt{m^2 x^2 - y^2} + \frac{y^2}{2\beta^2 m^2} (b_0 + 2\beta^2 m^2 b_2) \text{ arc cosh } \frac{mx}{y} \qquad (193)$$

Placing equation (192) into (185) and integrating, one can eventually show

$$C_{p_0} = \frac{2x}{\beta^2 (1 - \theta_0^2)^2} \{ b_0[2K - E(3 - \theta_0^2)] + b_2[(\theta_0^4 + \theta_0^2)K - 2\theta_0^2 E] \} \qquad (194)$$

It is immediately apparent that the wing shape required to support a linear pressure gradient in the x direction is not unique, that there are, in fact, an infinite number of shapes that will induce the same pressure distribution. (The converse, however, is not true. That is, a given shape has only one possible distribution of pressure.)

Squire (reference 17) considered the thickness distribution that is obtained by neglecting the arc hyperbolic function in equation (193). His result corresponds to the case when b_0 is $-2\beta^2 m^2 b_2$ and can be written specifically

$$z_u = \frac{b_0 x}{2\beta^2 m^2} \sqrt{m^2 x^2 - y^2} \qquad (195a)$$

$$C_{p_0} = \frac{b_0 x}{\beta^2 (1 - \theta_0^2)^2} [(3 - \theta_0^2)K - E(4 - 2\theta_0^2)] \qquad (195b)$$

where b_0/β^2 can be related to the thickness chord ratio of the wing and the apex angle. Figure 24 shows how equation (194) can be used to obtain several triangular wing shapes all having the same linear pressure distribution. (It is interesting to note that no combination of b_0 and b_2 exists that will give a real wing shape with zero pressure coefficient since the resulting negative ordinates would require the surface to cross itself.)

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