A SEMI-LAGRANGIAN APPROACH TO THE SHALLOW WATER EQUATIONS

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Abstract

We present a formulation of the shallow water equations that emphasizes the conservation of potential vorticity. A locally conservative semi-Lagrangian time-stepping scheme is developed, which leads to a

system of three coupled PDE's to be solved at each time level. We describe a smoothing analysis of these equations, on which an effective multigrid solver is constructed. Some results from applying this solver to the static version of these equations are presented.

1 Formulation of the Shallow Water Equations

The shallow water equations provide a two-dimensional prototype of the equations needed for three-dimensional simulations of atmospheric motions [1] [2]. They are useful for testing the viability of new numerical schemes for atmospheric simulation because they share many of the properties with, but lack the full complexity of, a full three-dimensional system. The shallow water equations can be written as

$$\frac{du}{dt} = -\phi_x + fv, \tag{1}$$

$$\frac{dv}{dt} = -\phi_y - fu, \tag{2}$$

$$\frac{d\phi}{dt} = -\phi D, \tag{3}$$

where u and v are the velocity components of the wind, $D = u_x + v_y$ is the divergence of the velocity, f is the Coriolis parameter, and ϕ is the geopotential height, assumed to be a positive function. The derivatives are material derivatives, that is,

$$\frac{d}{dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \frac{\partial}{\partial t}. \tag{4}$$

A considerable amount of effort has gone into designing numerical methods that will solve these equations (see for example the references cited in [1]). The purpose of this paper is to study a multigrid scheme applied to a form of these equations that is of special physical interest.

There are many possible formulations of the shallow water equations. We will derive a different formulation from the one above that has certain physical and numerical advantages. To this end, we define vorticity by

$$\zeta = v_x - u_y. \tag{5}$$

Then, subtracting the y-derivative of Eq. 2 from the x-derivative of Eq. 1 gives

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)D. \tag{6}$$

Solving for D in Eq. 3 and substituting it into Eq. 6 yields

$$\frac{d}{dt}[\frac{\zeta + f}{\phi}] = 0. \tag{7}$$

Eq. 7 is important in practice because it clearly asserts that the physical quantity $(\zeta + f)/\phi$, called potential vorticity, is conserved in time along any Lagrangian trajectory.

Now adding the x-derivative of Eq. 1 and the y-derivative of Eq. 2 gives

$$\frac{dD}{dt} = -\nabla^2\phi - \nabla.(f\mathbf{k} \times \mathbf{V}) - N, \tag{8}$$

where $\mathbf{k} = (0,0,1)$, $\mathbf{V} = (u,v,0)$, and $N = (u_x)^2 + (v_y)^2 + 2v_x u_y$. It is not hard to see that Eqs. 3, 7, and 8 are equivalent to the original formulation of the shallow water equations (Eqs. 1-3), but they are not yet in the form we wish to consider.

From the point of view of a multigrid solver, we will see that it is convenient to rewrite these equations in terms of the geopotential, ϕ, the stream function, ψ, and the velocity potential, χ. The latter two variables satisfy

$$\mathbf{V} = \mathbf{k} \times \nabla\psi + \nabla\chi, \tag{9}$$

$$\zeta = \nabla^2\psi, \tag{10}$$

$$D = \nabla^2\chi. \tag{11}$$

Using these variables, we arrive at the form of the shallow water equations used in this paper:

$$\frac{d}{dt}[\frac{\nabla^2\psi + f}{\phi}] = 0. \tag{12}$$

$$\frac{d\nabla^2\chi}{dt} = -\nabla^2\phi - \nabla.(f\mathbf{k} \times \nabla\chi - f\nabla\psi) - N, \tag{13}$$

$$\frac{d\phi}{dt} = -\phi\nabla^2\chi, \tag{14}$$

where $N = (\psi_{xy} - \chi_{xx})^2 + (\psi_{xy} + \chi_{yy})^2 - 2(\psi_{yy} - \chi_{xy})(\psi_{xx} + \chi_{xy})$.

These equations have several attractive properties. As already noted, they emphasize the conservation of potential vorticity along Lagrangian trajectories. Furthermore, we shall show in Section 2 that, when a semi-Lagrangian approach is taken for the time derivatives, all of the variables appear in potential form in the resulting equations. This means that a simple vertex centered grid is sufficient to discretize the problem spatially; a staggered grid is not needed. This fact should be particularly useful when the problem is posed on a spherical domain. Finally, we shall see in Section 3 that these equations are well suited for multigrid solution.

An ideal domain for simulating atmospheric motions is a sphere. However, a spherical coordinate system introduces many difficulties that may confuse the task of developing an efficient solver for the equations at hand. Thus, as a first step in determining the feasibility of applying multigrid methods to our formulation of the shallow water equations, we have chosen to solve the system on a cylindrical domain. Specifically, we consider a domain that is periodic in the x direction with length d and includes y in the range $[0, L]$. We set $\chi = \psi = 0$ and $\phi = \phi_0$ at the y boundary, where ϕ_0 is a given constant. We assume that the Coriolis parameter may be written as

$$f = f_o + \beta y, \tag{15}$$

with f_o and β constants. This model allows us to determine the effectiveness of multigrid methods for these equations without the complications of constructing a full three-dimensional global atmospheric model.

2 A Semi-Lagrangian Time Stepping Scheme

Eqs. 12-14 are written in a Lagrangian reference frame in which the evolution of the fluid is observed along the paths of imaginary fluid particles. There are some obvious disadvantages of evolving a set of particles along Lagrangian trajectories numerically. In particular, a grid that is initially uniform will in general become very irregular, often leading to a degradation of global accuracy. As a compromise, semi-Lagrangian methods have been developed to produce numerical methods that preserve the advantages of regular grids while simultaneously taking advantage of the Lagrangian form of the equations. There is an extensive body of literature describing these

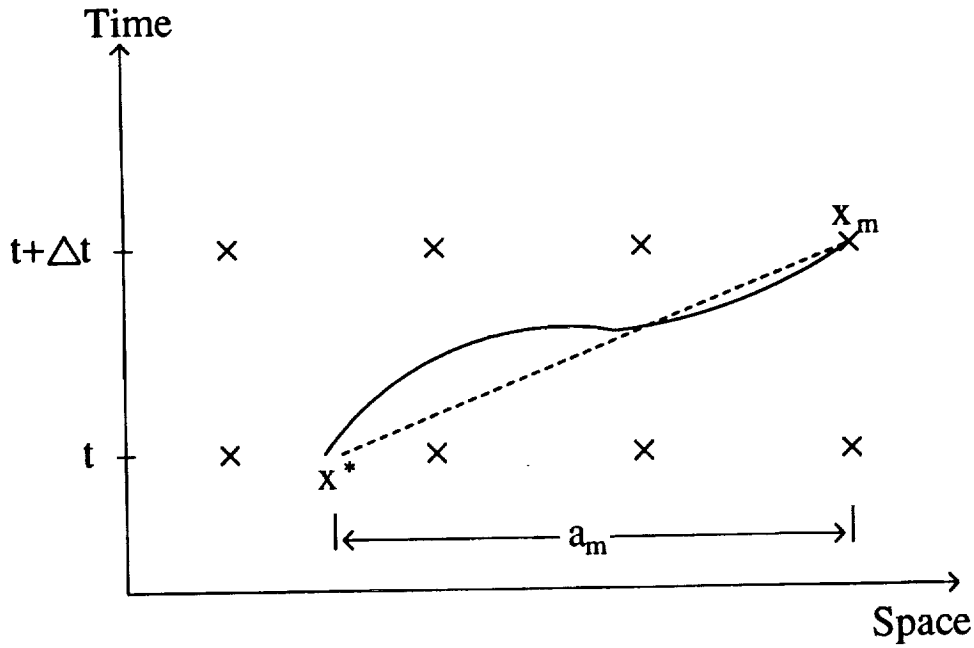

Figure 1: A schematic diagram showing the main quantities used in the calculation of the departure points for the semi-Lagrangian time-stepping scheme. The exact trajectory is represented by a solid line and the approximate trajectory with a dashed line.

methods. In particular, [1] provides an excellent review of the application of semi-Lagrangian methods to meterological problems. This reference describes in detail a semi-Lagrangian scheme for the integration of Eqs. 1-3. The scheme we describe in this section is an adaptation of this scheme to our reformulation of the shallow water equations, and the reader is urged to consult [1] for more detail.

The fundamental idea of a semi-Lagrangian scheme is to impose a regular grid at the new time level, and to backtrack the fluid trajectories to the previous time level. At the old time level, the quantities that are needed are evaluated by interpolation from their known values on a regular grid. In general, as is the case in our problem, the velocity field at the new time step is unknown, so the critical problem in this idea is the computation of the trajectory departure points.

A schematic representation of the quantities involved in computing the

departure points is shown in Fig. 1. The displacement between a grid point on the new time level, $\mathbf{x}_m(t)$, and the departure point of the trajectory leading to this point on the previous time level, $\mathbf{x}_m^*(t-\Delta t)$, is denoted by $\mathbf{a}_m$. If the velocity field is considered to be constant from $t-\Delta t$ to t, then $\mathbf{a}_m$ satisfies the equation

$$\mathbf{a}_m = \Delta t \mathbf{V}(\mathbf{x}_m - \frac{\mathbf{a}_m}{2}, t - \frac{\Delta t}{2}). \tag{16}$$

The velocity at time $t-\Delta t/2$ may be defined by extrapolation from the two previous time levels by

$$\mathbf{V}(\mathbf{x}, t - \frac{\Delta t}{2}) = \frac{3}{2}\mathbf{V}(\mathbf{x}, t-\Delta t) - \frac{1}{2}\mathbf{V}(\mathbf{x}, t-2\Delta t) + \mathcal{O}(\Delta t^2). \tag{17}$$

Eqs. 16 and 17 give an implicit equation for $\mathbf{a}_m$ in terms of the known velocity field at two previous time levels, and we may consider an iterative method for determining the correct $\mathbf{a}_m$. Assuming that a suitable approximation is made, then $\mathbf{x}_m - \mathbf{a}_m/2$ would not generally lie on a grid point, so the velocities at this point must be obtained by interplation. It has been shown [4] [5] [6] that for problems of this type it is sufficient to use linear interpolation to define the quantities in Eq. 17. It is also known [7] that succesive iteration for the solution of Eq. 16 converges provided

$$\Delta t \leq \frac{1}{\max[|u_x|, |u_y|, |v_x|, |v_y|]}. \tag{18}$$

Once the $\mathbf{a}_m$ are known, the departure point values of the variables in our equations are defined as illustrated by

$$\phi_m^*(t-\Delta t) = \phi(\mathbf{x}_m - \mathbf{a}_m, t-\Delta t). \tag{19}$$

Again, these values must be interpolated from known values at the grid points. It has been found [4] [5] [6] that it is advantageous to do this using cubic interpolation. A material time derivative may then be discretized by

$$\frac{d\phi}{dt} = \frac{1}{\Delta t}[\phi(t) - \phi^*(t-\Delta t)], \tag{20}$$

and nonderivative quantities can be represented by the simple average

$$\phi = \frac{1}{2}[\phi(t) + \phi^*(t-\Delta t)]. \tag{21}$$

Using this discretization, our formulation of the shallow water equations may be manipulated to show that the equations that determine the solution of the system at a new time level are

$$\nabla^2\psi^+ + f^+ - f_1\phi^+ = 0, \tag{22}$$

$$\nabla^2\chi^+ + \tau[\nabla^2\phi^+ + \beta\chi_x^+ - \beta\psi_y^+ - f\nabla^2\psi^+] = f_2, \tag{23}$$

$$\phi^+[1 + \tau\nabla^2\chi^+] = f_3, \tag{24}$$

where

$$f_1 = \frac{\nabla^2\psi^* + f^*}{\phi^*}, \tag{25}$$

$$f_2 = \nabla \cdot [\mathbf{V}^* - \tau(f\mathbf{k} \times \mathbf{V}^* + \nabla\phi^*)], \tag{26}$$

$$f_3 = \phi^*(1 - \tau\nabla^2\chi^*), \tag{27}$$

and $\tau = \Delta t/2$. The starred quantities are evaluated at the trajectory departure points at the previous time level, and the superscript + refers to quantities defined on a regular spatial grid at the new time level. We refer to Eqs. 22-24 as the static equations. The superscript + will be omitted in what follows.

The numerical algorithm needed to integrate our form of the shallow water equations splits naturally into two pieces. The first task is to compute the departure point quantities needed to define f_1, f_2, and f_3. This is done in the manner outlined above, using information from two previous time levels. The velocity field at any time level may be obtained from χ and ψ using $u = -\psi_y + \chi_x$ and $v = \psi_x + \chi_y$. Once the departure point quantities are known, the second task is to solve the static equations. As we shall demonstrate below, it is possible to construct an efficient multigrid solver for these equations. Note that nowhere in this method is it necessary to solve Eqs. 10 and 11 for χ and ψ in terms of u and v.

3 Coupling Analysis of the Static Equations

The coupling between the equations in any system of equations plays a pivotal role in the behavior of the system. In particular, when discretized systems of PDE's are to be solved by multigrid, the coupling of the equations

must determine the character of the relaxation schemes that are to be applied. Fortunately, a straightforward method for analyzing the coupling of a system and its relation to constructing a multigrid solver is available [8]. In this section we apply this method to the static equations derived above. Throughout the section, we use the definitions and notation of [8], to which the reader is referred for an understanding of the technique we are about to use.

The linearized static equations are given in brief as follows:

$$\begin{pmatrix} \nabla^2 & -f_1 & 0 \\ -f\tau\nabla^2 - \tau\beta\partial_y & \tau\nabla^2 & \nabla^2 + \tau\beta\partial_x \\ 0 & 1+\tau\nabla^2\chi & \tau\phi\nabla^2 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \\ \chi \end{pmatrix} = \begin{pmatrix} -f \\ f_2 \\ f_3 \end{pmatrix}. \quad (28)$$

In constructing this system, we have associated variables with equations in the natural way; that is, ψ, ϕ, and χ are associated with Eqs. 22, 23, and 24, respectively.

The order array and weight array for this system are

$$Q = \begin{bmatrix} 2 & 0 & N \\ 2 & 2 & 2 \\ N & 0 & 2 \end{bmatrix}, \quad (29)$$

and

$$W = \begin{bmatrix} N & 2 & N \\ 0 & N & 0 \\ N & 2 & N \end{bmatrix}, \quad (30)$$

respectively.

To account for finite mesh size effects, we need the scaled coefficient array

$$C = \begin{bmatrix} 1 & -f_1 & N \\ -f & 1 & \tau^{-1} \\ N & \frac{1+\tau\nabla^2\chi}{\tau\phi} & 1 \end{bmatrix}. \quad (31)$$

The computation of these arrays is straightforward. The method of [8] is almost automatic, and the arrays are included here explicitly only for completeness. From these arrays, the coupling graph may be constructed, as shown in Fig. 2.

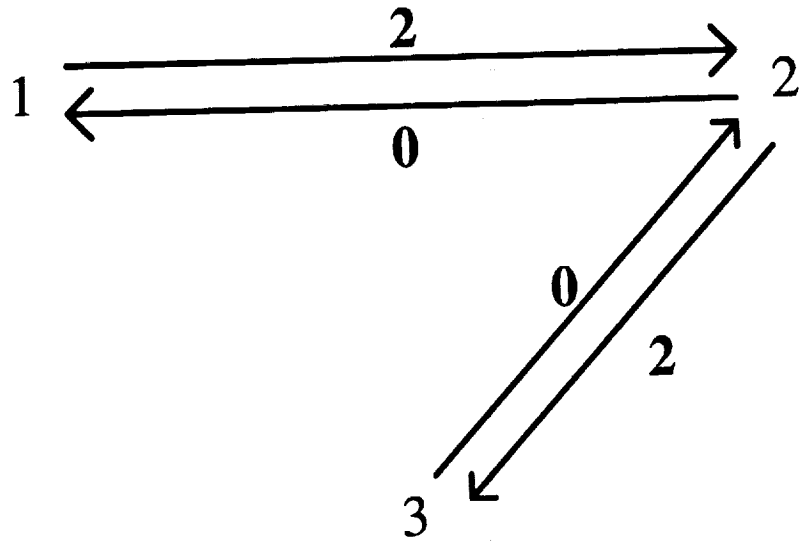

Figure 2: The coupling graph for the static equations. The finite mesh-size coupling coefficients are f_1 $(1 \rightarrow 2)$, f $(2 \rightarrow 1)$, $1/\tau$ $(2 \rightarrow 3)$, and $[1 + \tau\nabla^2\chi]/\tau\phi$ $(3 \rightarrow 2)$.

We may conclude immediately from the coupling graph that Eq. 22 is weakly coupled to Eq. 23 when

$$f_1 f h^2 << 1, \tag{32}$$

and that Eqs. 23 and 24 are weakly coupled when

$$(1 + \tau\nabla^2\chi)\frac{h^2}{\tau^2\phi} << 1. \tag{33}$$

This implies that if both of these conditions are satisfied, then each equation may be relaxed separately, as though the system were fully decoupled.

We now need to estimate the quantities in these coupling conditions using a physically realistic solution of a slightly different version of the shallow water equations. The equations we are dealing with assume that the surface of the fluid is free. To fix the surface profile of the fluid (the so-called 'rigid-lid' condition), we set $d\phi/dt = 0$ in Eq. 1. It can then be shown by direct substitution that the following is an exact form for the resulting Rossby-Haurwitz wave solution:

$$u = U - Al\cos ly \sin k(x - ct), \tag{34}$$

f	$f_0 = 1 \times 10^{-4} s^{-1}, \beta = 1.57 \times 10^{-11} m^{-1} s^{-1}$	τ	$500s$
d	$1 \times 10^7 m$	f_1	$\simeq 10^{-8}$
L	$5 \times 10^6 m$	f_2	$\simeq 10^{-7}$
ϕ_0	$1 \times 10^4 m$	f_3	$\simeq 10^4$

Table 1: Some typical physical parameters for the shallow water equations.

$$v = Ak \sin ly \cos k(x - ct), \tag{35}$$

$$\phi = \phi_0 - f_0 U y - \frac{1}{2}\beta U y^2 + A \sin k(x - ct)[f \sin ly - (c - U) l \cos ly] + \frac{1}{4} A^2 [l^2 \cos 2k(x - ct) + k^2 \cos 2ly], \tag{36}$$

where A and U are constants, $c = U - (\beta^2/(k^2 + l^2))$ is the Rossby-Haurwitz phase speed, $k = 2\pi m/L$ for integer m, and $l = n\pi/d$ for integer n. Waves of this type are the dominant feature of large scale weather motions. This solution satisfies different boundary conditions from the problem we are treating, but it is nevertheless useful for estimating the size of the parameters in our system. It can be shown for this solution that

$$\nabla^2 \chi = 0 \tag{37}$$

and

$$\nabla^2 \psi = -A(k^2 + l^2) \sin ly \sin k(x - ct). \tag{38}$$

Some typical numerical values of the parameters in the coupling conditions are shown in Table 1. A Rossby-Haurwitz wave with $n = m = 1$, $A = 3 \times 10^7 m^2 s^{-1}$, and $U = 20 m s^{-1}$, together with standard physical constants, was used to derive the data in this table. From these values it can be seen that Eq. 32 is satisfied, but Eq. 33 is certainly violated on intermediate and coarse grids. In terms of constructing a good smoother for the system, this means that Eq. 22 can be relaxed as though it were decoupled from the system, but the two remaining equations must be dealt with together, at least on coarse grids. In practise, it is easiest to use the same smoother on all grids to start with.

To deal with Eqs. 22 and 23 together, collective relaxation is used. For linear equations, this means that, when the equations are relaxed at a point,

corrections are made to all the variables associated with the equations such that the residuals of the equations become zero at that point. This may be done by replacing ϕ and χ with $\phi + \delta_\phi$ and $\chi + \delta_\chi$, respectively, in Eqs. 23 and 24 at a single point and the differential operators with their discretized counterparts and solving for the corrections δ_ϕ and δ_χ. Because Eq. 24 is nonlinear, a term proportional to $\delta_\phi \delta_\chi$ appears. We neglect this term and solve the resulting linear system directly. This method is equivalent to taking a single Newton step for these equations.

4 Preliminary Numerical Results

A preliminary code has been implemented that applies the multigrid method just described to the static equations. Eq. 22 was relaxed by red-black Gauss-Seidel iteration, and Eqs. 23 and 24 were relaxed collectively as described above in a lexicographic ordering. The equations are nonlinear, so the Full Approximation Scheme (FAS) [9] was used for the coarse-to-fine corrections. Full weighting was used for the fine to coarse grid restrictions, and linear interpolation for the coarse to fine grid transfers. Note that the grid transfers are straightforward because all the variables are defined on the same vertex centered grid. The standard five-point discretization was used for the Laplacian operator. Similarly, other derivatives were discretized using the usual finite difference formulae. At the time of writing, a semi-Lagrangian time-stepping scheme had been implemented, but the two codes had not been fully combined.

In order to test the convergence of the multigrid scheme, we set the forcing functions in the static equations to a variety of functional forms. The magnitude of these functions was indicated by the Rossby-Haurwitz wave solution introduced in the previous section. When the problem was solved on a 64×32 grid with a V(1,1) cycle, the convergence rates for the L^2 norm of the residuals were 0.22, 0.25, and 0.27 for Eqs. 22-24, respectively. When a V(2,1) cycle was used , the rates were 0.15, 0.13, and 0.14. In each of these cases, a single relaxation sweep consisted of relaxing Eq. 22 once followed by relaxing Eqs. 23 and 24 collectively once.

These results suggest that multigrid may be an efficient way of solving these equations. Clearly, there are many possible variants on the scheme described above. For instance, the coupling analysis suggests that it may be

fruitful to relax Eqs. 23 and 24 independently on fine grids and switch to collective relaxation only on coarser grids.

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