Energy and Maximum Norm Estimates for Nonlinear Conservation Laws

Pelle Olsson and Joseph Oliger

The Research Institute of Advanced Computer Science is operated by Universities Space Research Association, The American City Building, Suite 212, Columbia, MD 21044, (410) 730-2656

Work reported herein was sponsored by NASA under contract NAS 2-13721 between NASA and the Universities Space Research Association (USRA).

Abstract

We have devised a technique that makes it possible to obtain energy estimates for initial-boundary value problems for nonlinear conservation laws. The two major tools to achieve the energy estimates are a certain splitting of the flux vector derivative $f(u)_x$, and a structural hypothesis, referred to as a cone condition, on the flux vector $f(u)$. These hypotheses are fulfilled for many equations that occur in practice, such as the Euler equations of gas dynamics. It should be noted that the energy estimates are obtained without any assumptions on the gradient of the solution u. The results extend to weak solutions that are obtained as pointwise limits of vanishing viscosity solutions. As a by-product we obtain explicit expressions for the entropy function and the entropy flux of symmetrizable systems of conservation laws. Under certain circumstances the proposed technique can be applied repeatedly so as to yield estimates in the maximum norm.

1 Introduction

Most of the existing theory for nonlinear conservation laws is concerned with the initial value problem. The basic tenet of this paper is to devise a technique that makes it possible to obtain energy estimates for the initial-boundary value problem. The key to obtaining an energy estimate lies in a certain splitting of the flux vector derivative $f(u)_x$. Based on this splitting one can obtain energy estimates without assumptions on the gradient u_x, provided the flux vector f satisfies certain structural hypotheses. It should be noted that these hypotheses are fulfilled for many equations that occur in practice, such as the Euler equations of gas dynamics. In certain cases the proposed technique can be applied repeatedly so as to yield estimates in the maximum norm.

We now give a brief presentation of the contents of this paper. Section (2) deals with the scalar problem in one space dimension. First, we consider flux functions of the form $f(u) = u^j$, for which it is very easy to derive an energy estimate. Next we assume that $f(u)$ can be expanded in a convergent Taylor series and apply the technique to each individual monomial u^j. This procedure leads to a certain differential equation. Conversely, the solution to this differential equation will achieve the correct splitting of $f(u)_x$, even if f is non-analytic. To obtain an energy estimate, the "dangerous" boundary terms must be eliminated by means of the characteristic boundary conditions. This can not be done in general, however. We therefore propose a so called cone condition to single out the relevant fluxes. In section (3) we repeatedly apply the ideas developed in the previous section. The main result is a maximum norm estimate for the scalar initial-boundary value problem. The principles of section (2) can be generalized to systems in a straightforward manner. This is the major issue of section (4). To begin with, we confine ourselves to symmetric hyperbolic systems, i. e., systems where the Jacobian satisfies $f'^T = f'$. The cone condition is generalized to systems, and we are thus able to derive an energy estimate. Next, we require that the system be symmetrizable by means of a change of variables $u = u(v)$. The Euler equations belong to this latter category. Applying the splitting to the time derivative as well, we are able conclude that v satisfies a generalized energy estimate. We use the term generalized energy estimate, because the norm will, in general, depend on $u(v)$ (but *not* on the gradient of u). As a by-product we get an explicit expression of an entropy function $U(u)$ and the corresponding entropy flux $F(u)$. Section (5) is concerned with generalizations of the preceding sections to several space dimensions. Finally, in section (6) we prove energy estimates for weak solutions that can be obtained as limits of vanishing viscosity solutions.

2 Scalar Conservation Laws in One Space Dimension

Consider the conservation law

$$
\begin{aligned}
&u_t + f_x = 0, \qquad x \in (0,1) \quad t > 0 \\
&u(x,0) = \varphi(x).
\end{aligned} \tag{1}
$$

At the boundaries $x = 0, 1$ we prescribe data $\psi(t)$ for the ingoing characteristics, which are determined by the sign of $f'(u(i,t))$, $i = 0, 1$. The flux $f(u) \in \mathrm{R}$ is assumed to be a continuously differentiable function of $u \in \mathrm{R}$. It is well known that eq. (1) allows an energy estimate when $f(u) = u^2$. One way to see this is to make use of the splitting

$$
\left(u^2\right)_x = \frac{2}{3}\left[\left(u^2\right)_x + uu_x\right]
$$

which is valid for smooth functions u. If we use this splitting in eq. (1) and apply the energy method we obtain

$$
\frac{d}{dt}||u||^2 = -\frac{4}{3}\int_0^1 \left[u\left(u^2\right)_x + u^2 u_x\right] dx = -\frac{4}{3}\int_0^1 \left(u^3\right)_x dx = \frac{4}{3}\left(u(0,t)^3 - u(1,t)^3\right),
$$

where $||\cdot||$ denotes the usual norm associated with the scalar product

$$
(u,v) = \int_0^1 uv dx .
$$

If $u(0,t) > 0$, then we prescribe $u(0,t) = \psi_0(t)$. Similarly, if $u(1,t) < 0$, then we set $u(1,t) = \psi_1(t)$. We then obtain an energy estimate.

More generally, any monomial $f(u) = u^j$ satisfies the identity

$$
\left(u^j\right)_x = \frac{j}{j+1}\left[\left(u^j\right)_x + u^{j-1}u_x\right]. \tag{2}
$$

Suppose for the moment that $f(u)$ is analytic. Then there exists a convergent Taylor series such that

$$
f(u) = \sum_{j=0}^{\infty} c_j u^j .
$$

Since the convergence is uniform we get

$$
f_x = \left(\sum_{j=0}^{\infty} c_j \frac{j}{j+1} u^j\right)_x + \frac{1}{u}\left(\sum_{j=0}^{\infty} c_j \frac{j}{j+1} u^j\right) u_x \tag{3}
$$

using eq. (2). Some simple calculations show that

$$
\sum_{j=0}^{\infty} c_j \frac{j}{j+1} u^j = f - \frac{1}{u}\int_0^u f(v) dv . \tag{4}
$$

Define

$$F(u) = \frac{1}{u}\int_0^u f(v)dv = \int_0^1 f(\theta u)d\theta\,. \tag{5}$$

Eqs. (3), (4), (5) imply that

$$f_x = (f - F)_x + \frac{1}{u}(f - F)\,u_x\,. \tag{6}$$

From definition (5) it follows immediately that F satisfies the following differential equation

$$\begin{aligned} F'u &= -F + f \\ F(0) &= f(0)\,. \end{aligned} \tag{7}$$

It will henceforth be assumed that $f(0) = 0$. This is no restriction, since we can always replace $f(u)$ by $f(u) - f(0)$ without affecting eq. (1). Furthermore, most examples that occur in practice satisfy $f(0) = 0$. Combining eqs. (6) and (7) gives

$$f_x = (F'u)_x + F'u_x\,, \tag{8}$$

which we will refer to as the *canonical* splitting.

Conversely, let $F(u)$ be the solution to eq. (7), where it is no longer required that $f(u)$ be an analytic function. It is clear that $F(u)$ is given by eq. (5). Hence,

$$f_x = (f - F)_x + F_x = (F'u)_x + F'u_x\,,$$

where the second equality follows from eq. (7) and the chain rule. We have thus established the canonical splitting for all f such that eq. (7) has a C^1-solution. We shall henceforth assume that f is differentiable. We point out that if f is a linear function of u, i. e., $f(u) = au$, then $F(u) = (1/2)au$ and thus $F'(u) = (1/2)a$. Consequently,

$$f_x = \frac{1}{2}(au)_x + \frac{1}{2}au_x\,,$$

which is the usual skew-symmetric form of a linear hyperbolic PDE.

The energy method applied to eq. (1) yields

$$\frac{d}{dt}||u||^2 = -2(u, f_x) = -2(u, (F'u)_x) - 2(u, F'u_x) = -2uF'u|_0^1\,,$$

where the second equality follows from the canonical splitting (8). To obtain the actual estimate we must analyze

$$uF'u = uf - \int_0^u f(v)dv = \int_0^u f'(v)vdv\,.$$

From the last expression it is clear that, in general, knowledge about the sign of $f'(u)$, upon which the characteristic boundary conditions are based, has no bearing on the sign of $uF'u$. We must thus limit ourselves to flux functions f such that the sign of $f'(u)$ will determine the sign of

$$\int_0^u f'(v)vdv = uF'u\,, \tag{9}$$

if we are to obtain an energy estimate.

Definition 2.1 *A function $f : \Omega \to R$, $\Omega \subset R$, is said to satisfy a cone condition if*

$$\mathrm{sgn}(f'(u)) = \mathrm{sgn}\left(\int_0^u f'(v)v\,dv\right) \quad u \in \Omega\,.$$

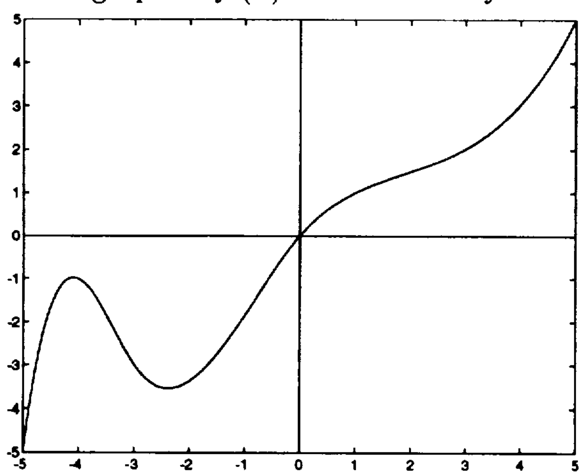
This cone condition is satisfied if

$$\mathrm{sgn}(f'(u)) = \mathrm{sgn}(u)\,, \quad u \in \Omega\,. \tag{10}$$

Suppose that $f'(u) > 0$ at the lower boundary $x = 0$. We then have an ingoing characteristic, which implies a boundary condition at $x = 0$. Now, eq. (10) implies $u > 0$, whence $0 < v < u$. But then (10) implies that $f'(v) > 0$, that is, $f'(v)v > 0$. Consequently,

$$uF'u = \int_0^u f'(v)v\,dv > 0\,,$$

which shows that the boundary energy has the needed sign. Similarly, if $f'(u) < 0$, i. e., there is an outgoing characteristic, then $u < 0$ from (10), which again leads to $f'(v)v > 0$. But the upper limit of the integral in eq. (9) is negative, and so the entire expression is negative. Below we show the graph of $f'(u)$ for a function f satisfying the condition (10).



It should be noted that the condition (10) holds for all monomials of even order; in particular, it is true for Burgers' equation. Furthermore, it is also true for the non-convex Buckley-Leverett equation for $u \in [0,1]$, which is a natural restriction since u describes the saturation of water in a two phase flow.

Another sufficient condition for the cone condition to be true is furnished by

$$\mathrm{sgn}(f'(u)) > 0 \quad (\text{or } \mathrm{sgn}(f'(u)) < 0) \quad u \in \Omega\,. \tag{11}$$

For, if $f'(u) > 0$, then the integrand of eq. (9) satisfies $\mathrm{sgn}(f'(v)v) = \mathrm{sgn}(v)$. Hence,

$$\mathrm{sgn}\left(\int_0^u f'(v)v\,dv\right) = \mathrm{sgn}\left(\int_0^u v\,dv\right) = 1\,,$$

i. e.,

$$\operatorname{sgn}(f'(u)) = \operatorname{sgn}\left(\int_0^u f'(v)v\,dv\right),$$

which shows that f satisfies a cone condition. Monomials of u of odd order, linear fluxes in particular, belong to this second category.

Before proceeding we note that the boundary data $\psi(t)$ may be restricted; for example, the cone condition (2.1) requires that f' and $uF'u$ have the same sign. This is the case for Burger's equation ($f'(u) = u, uF'u = (1/3)u^3$) in which case the condition (10) holds. If we have an ingoing characteristic at $x = 0$, then clearly $u > 0$ and we may thus only prescribe data $\psi(t) > 0$. From now on it will always be assumed that the boundary data is compatible with the cone condition.

Proposition 2.1 *Let u be a smooth solution of the initial-boundary value problem (1), where it is assumed that f is differentiable and satisfies a cone condition (2.1). Then u satisfies an energy estimate.*

Proof:
By means of the canonical splitting we get

$$\frac{d}{dt}||u||^2 = -2uF'u|_0^1,$$

where

$$uF'u = \int_0^u f'(v)v\,dv.$$

Since f satisfies a cone condition it follows that the sign of $uF'u$ is uniquely determined by that of $f'(u)$. For convenience, we assume that there is an ingoing characteristic at $x = 0$ and an outgoing one at $x = 1$, i. e., $f'(u(0,t)), f'(u(1,t)) > 0$. Hence,

$$\frac{d}{dt}||u||^2 \leq 2\int_0^{\psi(t)} f'(v)v\,dv,$$

where the data $\psi(t)$ at $x = 0$ is such that the integral in the right member is nonnegative (because of compatibility with the cone condition). Integration with respect to time yields

$$||u(\cdot,t)||^2 \leq ||\varphi||^2 + 2\int_0^t \int_0^{\psi(\tau)} f'(v)v\,dv\,d\tau,$$

which proves the proposition. □

Remark: If there are no ingoing characteristics we get an energy estimate in terms of the initial data alone. Also, if there were an ingoing characteristic at $x = 1$ ($f(u(1,t)) < 0$), then there would be a second boundary term in the energy estimate. In the case of a linear equation ($f = au$) we get $uF'u = (1/2)au^2$, and we thus recover the familiar form of the energy estimate. It should be emphasized that no gradients appear in the estimate; if $f'(v) \propto v^{p-2}$, then there is an energy estimate for all time if $\psi(t) \in L^p(0,\infty)$. □

3 Sequences of Conservation Laws

The canonical splitting (8) implies that the original conservation law (1) can be formulated as

$$u_t + (F'u)_x + F'u_x = 0\,,$$

where F satisfies eq. (7). Multiplication by u and the product rule yield

$$\left(\frac{1}{2}u^2\right)_t + (f_1)_x = 0\,,$$

where

$$f_1 = uF'u = \int_0^u f'(v)v\,dv\,.$$

This is a new conservation law for $(1/2)u^2$ associated with the flux $f_1 = uF'u$. But then we can split this new flux in canonical form

$$(f_1)_x = (F_1'u)_x + F_1'u_x\,,$$

where F_1 satisfies

$$\begin{aligned} F_1'u &= -F_1 + f_1 \\ F_1(0) &= 0\,. \end{aligned}$$

Hence, multiplication of $((1/2)u^2)_t + (f_1)_x = 0$ by u implies

$$\left(\frac{1}{3}u^3\right)_t + (f_2)_x = 0\,,$$

where

$$f_2 = uF_1'u = \int_0^u f_1'(v)v\,dv\,.$$

But

$$f_1' = (uF'u)' = (uf - uF)' = f - F - F'u + f'u = f'u\,,$$

whence

$$f_2 = \int_0^u f'(v)v^2\,dv\,.$$

The process can be repeated. A simple induction argument shows that

$$\left(\frac{1}{p+1}u^{p+1}\right)_t + (f_p)_x = 0\,, \quad p = 0, 1, \ldots\,, \tag{12}$$

where

$$f_p = \int_0^u f'(v)v^p\,dv\,, \quad p = 0, 1, \ldots\,.$$

We have thus established a sequence of conservation laws. In the following we shall restrict ourselves to the subsequence where p is odd, i. e.,

$$\left(\frac{1}{2p}u^{2p}\right)_t + (f_{2p-1})_x = 0\,, \quad f_{2p-1} = \int_0^u f'(v)v^{2p-1}\,dv\,. \tag{13}$$

The conditions (10) and (11) guarantee that

$$\operatorname{sgn}(f_{2p-1}) = \operatorname{sgn}(f')\,, \quad p = 1, 2, \ldots\,.$$

Consequently, integration of eq. (13) over $(0,1)\times(0,t)$ yields

$$||u(\cdot,t)||_{2p}^{2p} \le ||\varphi||_{2p}^{2p} + 2p\int_0^t \int_0^{\psi(\tau)} f'(v)v^{2p-1}dvd\tau\,, \quad p = 1, 2, \ldots\,, \tag{14}$$

where we have assumed $f'(u(0,t)), f'(u(1,t)) > 0$ for convenience. The elementary inequality

$$(x+y)^{\frac{1}{n}} \le x^{\frac{1}{n}} + y^{\frac{1}{n}}\,, \quad x, y \ge 0\,,$$

implies

$$||u(\cdot,t)||_{2p} \le ||\varphi||_{2p} + \left(2p\int_0^t \int_0^{\psi(\tau)} f'(v)v^{2p-1}dvd\tau\right)^{\frac{1}{2p}}\,, \tag{15}$$

where the last term of the right hand member is nonnegative since we have required that the boundary data be compatible with the cone condition. But

$$\left(2p\int_0^t \int_0^{\psi(\tau)} f'(v)v^{2p-1}dvd\tau\right)^{\frac{1}{2p}} \le \left(2p\int_0^t \int_0^{|\psi(\tau)|} |f'(v)|v^{2p-1}dvd\tau\right)^{\frac{1}{2p}}\,,$$

and

$$\left(2p\int_0^t \int_0^{|\psi(\tau)|} |f'(v)|v^{2p-1}dvd\tau\right)^{\frac{1}{2p}} \le \left(\max_{\substack{0\le v\le|\psi(\tau)| \\ 0\le\tau\le t}} |f'(v)| \int_0^t |\psi(\tau)|^{2p}d\tau\right)^{\frac{1}{2p}}\,.$$

Define the maximum norm of the boundary data $\psi \in L^\infty(0,\infty)$ as follows

$$||\psi||_\infty = \sup_{0\le\tau<\infty} |\psi(\tau)|\,. \tag{16}$$

Then

$$\left(2p\int_0^t \int_0^{\psi(\tau)} f'(v)v^{2p-1}dvd\tau\right)^{\frac{1}{2p}} \le ||\psi||_\infty \left(\max_{\substack{0\le v\le|\psi(\tau)| \\ 0\le\tau\le t}} |f'(v)|t\right)^{\frac{1}{2p}}\,.$$

Let t be fixed and let $p \to \infty$. We get

$$\limsup_{p\to\infty} \left(2p\int_0^t \int_0^{\psi(\tau)} f'(v)v^{2p-1}dvd\tau\right)^{\frac{1}{2p}} \le ||\psi||_\infty\,.$$

Similarly, it can be shown that

$$\limsup_{p\to\infty} ||\varphi||_{2p} = ||\varphi||_\infty$$

for initial data $\varphi \in L^\infty(0,1)$. From eq. (15) it then follows that

$$||u(\cdot,t)||_\infty \le ||\varphi||_\infty + ||\psi||_\infty .$$

This estimate is valid for all $t > 0$, and we have thus arrived at a maximum principle for conservation laws. The result is summarized in the following

Proposition 3.1 *Suppose that there is a smooth solution $u(x,t)$ to the initial-boundary value problem (1) for $t > 0$ and that f satisfies either of the conditions (10), (11). Then $u(x,t)$ satisfies the maximum norm estimate*

$$||u(\cdot,t)||_\infty \le ||\varphi||_\infty + ||\psi||_\infty .$$

Remark: Lax has shown in [5] that any weak solution with compact support of the *initial value* problem (1) satisfies $||u(\cdot,t_1)||_\infty \le ||u(\cdot,t_0)||_\infty$ for $t_1 > t_0$, i. e., $||u(\cdot,t)||_\infty$ is a nonincreasing function of time. □

4 Systems of Conservation Laws

We now proceed to the case where $u \in \mathrm{R}^d$ is a vector of unknowns and where $f = f(u) \in \mathrm{R}^d$ is a vector valued function of u; each component u_i is a function of $x, t \in \mathrm{R}$. We consider

$$\begin{array}{ll} u_t + f_x = 0, & x \in (0,1) \quad t > 0 \\ u(x,0) = \varphi(x) & \\ \omega_I(i,t) = S_i\omega_O(i,t) + \psi_i(t) & i = 0,1\,, \end{array} \tag{17}$$

where ω_I and ω_O are the ingoing and outgoing characteristic variables. To begin with, it will be assumed that eq. (1) is symmetric hyperbolic, i. e., $f'^T = f'$, where

$$f'(u) = \begin{pmatrix} f_{1u_1} & \dots & f_{1u_d} \\ \vdots & & \vdots \\ f_{du_1} & \dots & f_{du_d} \end{pmatrix}$$

is the Jacobian matrix of f. Thus, there exists an orthogonal matrix $Q(u)$ such that

$$Q^T(u)f'(u)Q(u) = \Lambda(u)$$

is diagonal. The characteristic variables are defined as

$$\omega = Q^T(u)u\,. \tag{18}$$

Let

$$Q(u) = (Q_I(u)\ Q_O(u)),$$

where Q_I contains the eigenvectors corresponding to ingoing characteristics; Q_O contains the remaining eigenvectors, i. e., those corresponding to outgoing characteristics and those whose eigenvalue is zero. We now define $\omega_I = Q_I^T u$ and $\omega_O = Q_O^T u$. In general, the rank d_1 of $Q_I \in \mathrm{R}^{d \times d_1}$ and the rank d_2 of $Q_O \in \mathrm{R}^{d \times d_2}$, $d_1 + d_2 = d$, will depend upon the solution u.

The key to obtaining an energy estimate for the scalar conservation law lies in the canonical splitting (8), which in turn relies upon eq. (7). For systems of conservation laws we thus consider

$$\begin{aligned} F'u &= -F + f \\ F(0) &= f(0) \end{aligned} \qquad F'^T = F', \tag{19}$$

where $u, f(u), F(u)$ are vectors; F' is the Jacobian matrix of F. As in the scalar case, we require that $f(0) = 0$. Equation (19) is the inhomogeneous Euler's differential equation (not to be confused with the Euler equations of gas dynamics). Suppose that there is a solution to (19). The symmetry condition $F'^T = F'$ implies that Euler's differential equation can be written as $f = (u^T F)'$, i. e.,

$$f' = (u^T F)'',$$

which in turn shows that $f'^T = f'$. Hence, it is necessary that the Jacobian matrix f' be symmetric. Conversely, suppose that $f'^T = f'$, then it is easy to verify that

$$F(u) = \int_0^1 f(\theta u) d\theta \tag{20}$$

solves Euler's differential equation (19). Furthermore, $f'^T = f'$ clearly implies $F'^T = F'$. Thus, eq. (19), subject to the constraint $F'^T = F'$, has a solution (20) iff $f'^T = f'$. For details on how to solve (19) we refer to [4]. Note the complete similarity between eqs. (5) and (20). With F given by (20) we achieve the canonical splitting

$$f_x = (f - F)_x + F_x = (F'u)_x + F'u_x .$$

Consequently, the solution u of eq. (17) satisfies

$$\frac{d}{dt}||u||^2 = -2(u, f_x) = -2(u, (F'u)_x) - 2(u, F'u_x).$$

Partial integration and $F'^T = F'$ imply that

$$\frac{d}{dt}||u||^2 = -2u^T F' u|_0^1, \tag{21}$$

which, formally, is an energy estimate. To get a true energy estimate we must make sure that the boundary terms

$$u^T F' u = \int_0^1 u^T f'(\theta u) \theta u d\theta$$

have the correct sign by applying the characteristic boundary conditions. As was seen in the scalar case, this will not be true for arbitrary flux functions. This remark also pertains to the vector valued case. Some kind of cone condition is needed. We generalize definition (2.1) to

Definition 4.1 *A vector valued function $f : \Omega \to R^d, \Omega \subset R^d$ is said to satisfy a cone condition if*

$$\int_0^1 u^T f'(\theta u)\theta u d\theta \geq (\leq) 0 \quad u \in M ,$$

where $M \subset \Omega$ is a submanifold such that

$$u^T f' u \geq (\leq) 0 .$$

We note that the cone condition is fulfilled if there are constants $c_0 > 0$ and $c_1 > 0$ such that

$$c_1 f' \leq F' \leq c_0 f' . \tag{22}$$

Assuming that this condition is true, then, by eq. (21) and the definition (18), we obtain

$$\frac{d}{dt}||u||^2 \leq 2c_0\omega^T \Lambda\omega(0,t) - 2c_1\omega^T\Lambda\omega(1,t) .$$

But

$$\omega^T\Lambda\omega(0,t) = \omega_+^T\Lambda_+\omega_+(0,t) + \omega_0^T\Lambda_0\omega_0(0,t) + \omega_-^T\Lambda_-\omega_-(0,t) ,$$

where Λ_+, Λ_0, Λ_- contain the positive, zero, and negative eigenvalues of f' for a given value of u. At $x = 0$ the *positive* eigenvalues correspond to ingoing characteristics; the remaining ones are treated as outgoing characteristics. With $\Lambda_I \equiv \Lambda_+$, $\Lambda_O \equiv \text{diag}(\Lambda_0 \quad -\Lambda_-)$, $\omega_I \equiv \omega_+$, and $\omega_O \equiv (\omega_0 \; \omega_-)$ it follows that

$$\omega^T\Lambda\omega(0,t) = \omega_I^T\Lambda_I\omega_I(0,t) - \omega_O^T\Lambda_O\omega_O(0,t) .$$

Similarly, at $x = 1$ we have

$$-\omega^T\Lambda\omega(1,t) = \omega_I^T\Lambda_I\omega_I(1,t) - \omega_O^T\Lambda_O\omega_O(1,t) ,$$

where $\Lambda_I \equiv -\Lambda_-$, $\Lambda_O \equiv \text{diag}(\Lambda_0 \; \Lambda_+)$, $\omega_I \equiv \omega_-$, and $\omega_O \equiv (\omega_0 \; \omega_+)$. It should be noted that $\Lambda_I > 0$ whereas $\Lambda_O \geq 0$. The characteristic boundary conditions (17) yield

$$\omega_I^T\Lambda_I\omega_I - \omega_O^T\Lambda_O\omega_O \leq 2\omega_O^T\left(S^T\Lambda_I S - \Lambda_O\right)\omega_O + 2\psi^T\Lambda_I\psi .$$

The first term of the right hand member is non-positive if $|S(u)|$ is small enough. In order to control the second term a bound on $\Lambda_I(u)$ is needed. The characteristic boundary conditions can be expressed in terms of u as

$$u = Pu + Q_I\psi , \tag{23}$$

where the projection $P(u)$ is given by

$$P(u) = Q_I(u)S(u)Q_O(u)^T + Q_O(u)Q_O(u)^T . \tag{24}$$

Hence,

$$\Lambda_I(u) = \Lambda_I(Pu + Q_I\psi) .$$

We, therefore, make the following

Assumption 4.1 *There exists a function* $\tilde{\Lambda}_I : \Omega \to R^{d_1 \times d_1}$, $\Omega \subset R^{d_1}$, *such that*

$$\Lambda_I(Pu + Q_I\psi(t)) \leq \tilde{\Lambda}_I(g(t))$$

holds at $x = 0$ *and* $x = 1$ *for some vector* $g(t) \in R^{d_1}$, *which is independent of* u.

The results can now be summarized in

Proposition 4.1 *Let* u *be a smooth solution of the initial-boundary value problem (17), where it is assumed that* f *is differentiable and that condition (22) holds. If assumption (4.1) is true, then* u *satisfies an energy estimate.*

Remark:

(i) Assumption (4.1) states that the signal speed of the ingoing characteristics can be estimated at the boundary independently of u.

(ii) Assumption (4.1) is superfluous in the case of homogeneous boundary data ($\psi \equiv 0$), since we obviously have $\psi^T \Lambda_I \psi \equiv 0$.

(iii) If all of the characteristics are ingoing, then $\Lambda_I = \Lambda$, $Q_I = Q$, $Q_O = \emptyset$, which implies $P = \emptyset$. Hence,

$$\Lambda_I(Pu + Q_I\psi(t)) = \Lambda(Q\psi(t)) .$$

Furthermore, since Q is orthogonal, there exists a vector $g(t)$ such that $\psi(t) = Q^T g(t)$, i. e., $Q\psi(t) = g(t)$, which shows that

$$\Lambda_I(Pu + Q_I\psi(t)) = \Lambda(g(t)) .$$

Thus, assumption (4.1) is true. □

We point out that, in general, the boundary data must be given in characteristic form, even if assumption (4.1) holds, if we are to obtain an energy estimate. This is true since

$$\psi^T \Lambda_I(u)\psi \leq \psi^T \tilde{\Lambda}_I(g)\psi \leq \rho(\tilde{\Lambda}_I(g))\psi^T \psi ,$$

where $\psi = Q_I^T g$ for some data vector $g \in \mathrm{R}^d$ since Q_I has full rank. Consequently,

$$\psi^T \Lambda_I(u)\psi \leq \rho(\tilde{\Lambda}_I(g)) g^T Q_I Q_I^T g\,.$$

Clearly, the right hand member will depend on u through $Q_I Q_I^T$. An exception to this is when $Q_I = Q$, in which case we have $Q_I Q_I^T = QQ^T = I$ because of orthogonality. If, however, Q is independent of u, then of course we get an energy estimate in terms of the non-characteristic data g. In this context it should be noted that the coefficient matrices of the primitive Euler equations can be symmetrized such that the resulting Q is independent of u [1, 8].

Condition (22) can be strengthened in certain cases. Suppose that the flux vector f satisfies Euler's differential equation

$$f'u = pf\,, \tag{25}$$

the solution of which can be written as [4]

$$f(u) = u_1^p g\left(\frac{u_2}{u_1}, \ldots, \frac{u_d}{u_1}\right)\,.$$

We then have

$$F = \int_0^1 f(\theta u)d\theta = \frac{1}{p}\int_0^1 f'(\theta u)\theta u d\theta = \frac{1}{p}\int_0^1 \frac{df}{d\theta}\theta d\theta\,.$$

Integration by parts yields

$$F = \frac{1}{p}f - \frac{1}{p}\int_0^1 f(\theta u)d\theta = \frac{1}{p}f - \frac{1}{p}F\,,$$

i. e.,

$$F = \frac{1}{p+1}f\,,$$

which strengthens condition (22) to

$$F' = \frac{1}{p+1}f'\,. \tag{26}$$

The flux vector of the conservative Euler equations of gas dynamics satisfies eq. (25) with $p = 1$. Unfortunately, the Euler equations are not symmetric, i. e., $f'^T \neq f'$.

Example: We consider eq. (17) where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad f(u) = \begin{pmatrix} u_1^2 + u_2^2 \\ 2u_1u_2 \end{pmatrix}\,.$$

Hence,

$$f'(u) = 2\begin{pmatrix} u_1 & u_2 \\ u_2 & u_1 \end{pmatrix}\,.$$

Obviously, $f'^T = f'$ and $f'u = 2f$. Thus, the cone condition is fulfilled. The energy method yields

$$\frac{d}{dt}||u||^2 = -\frac{2}{3}u^T f'u|_0^1 , \tag{27}$$

where we have used the canonical splitting. The eigenvalues and eigenvectors of $f'(u)$ are given by

$$\begin{array}{l} \lambda_1(u) = u_1 - u_2 \\ \lambda_2(u) = u_1 + u_2 \end{array} , \quad Q = \begin{pmatrix} q_1 & q_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

Here we have an example where Q is independent of u. Consequently, the characteristic variables are

$$\omega = Q^T u = \frac{1}{\sqrt{2}} \begin{pmatrix} u_1 - u_2 \\ u_1 + u_2 \end{pmatrix} .$$

Next, we verify that assumption (4.1) holds. For brevity we confine ourselves to the lower boundary $x = 0$; the upper boundary $x = 1$ is treated in a similar fashion. Assume that $\lambda_1(u) < 0$ and $\lambda_2(u) > 0$. Hence,

$$\Lambda_I(u) = \lambda_2(u) = \sqrt{2}\omega_2 = \sqrt{2}S\omega_1 + \sqrt{2}\psi = S\lambda_1 + \sqrt{2}\psi \leq \sqrt{2}\psi$$

if $S \geq 0$. This implies an energy estimate provided $S > 0$ is small enough. □

Remark: It is not necessary that $S \geq 0$ in the previous example since

$$\begin{aligned} u^T f'u &= \frac{1}{2}\left[(u_1 - u_2)^3 + (u_1 + u_2)^3\right] \leq \\ & \frac{1}{2}\left[(u_1 - u_2)^3 + \left(S_0(u_1 - u_2) + \sqrt{2}\psi\right)^3\right] \leq \sqrt{2}\psi^3 \end{aligned}$$

for $|S|$ small enough. □

We now consider eq. (17) when we no longer require that $f'^T = f'$. Before proceeding we recall the definition of an entropy function.

Definition 4.2 *A convex scalar function $U(u)$ is said to be an entropy function for eq. (17) if $U(u)$ satisfies*

$$U'^T f' = F'^T \tag{28}$$

for some scalar function $F(u)$ called the entropy flux. The prime denotes differentiation with respect to u.

Introducing a new dependent variable v and setting $u = u(v)$, $\tilde{f}(v) = f(u(v))$, we get

$$u_t + f_x = u_t + \tilde{f}_x = u'v_t + \tilde{f}'v_x = 0 . \tag{29}$$

Equation (17) is called symmetrizable if $u' \equiv u_v$ and $\tilde{f}' \equiv \tilde{f}_v$ of eq. (29) are symmetric matrices, and if u' is positive definite. In [3, 2] it is shown that eq. (17) is symmetrizable iff there exists an entropy function $U(u)$. Using the new variable v we can thus apply the technique used for the symmetric hyperbolic case. The only difference is that v_t is preceded by a symmetric positive definite coefficient matrix u', which implies that the change of variables $u = u(v)$ is well defined. It is therefore natural to apply the canonical splitting to the time derivative as well. Thus, let $\tilde{U}(v)$, $\tilde{F}(v) \in \mathrm{R}^d$ be the solutions to

$$\begin{array}{ll} \tilde{U}'v = -\tilde{U} + u & \tilde{U}'^T = \tilde{U}' \\ \tilde{F}'v = -\tilde{F} + \tilde{f} & \tilde{F}'^T = \tilde{F}' \end{array} . \tag{30}$$

where $\tilde{U}' \equiv \tilde{U}_v$, $\tilde{F}' \equiv \tilde{F}_v$. Hence,

$$\tilde{U}(v) = \int_0^1 u(\theta v) d\theta \quad \tilde{F}(v) = \int_0^1 \tilde{f}(\theta v) d\theta . \tag{31}$$

Applying the canonical splitting to $u_t + \tilde{f}_x$ in time and space yields

$$\left(\tilde{U}'v\right)_t + \tilde{U}'v_t + \left(\tilde{F}'v\right)_x + \tilde{F}'v_x = 0 , \tag{32}$$

which after scalar multiplication by v implies

$$\left(v^T\tilde{U}'v\right)_t + \left(v^T\tilde{F}'v\right)_x = 0 . \tag{33}$$

Integration of eq. (33) with respect to x yields

$$\frac{d}{dt}||v||^2_{\tilde{U}} = -v^T\tilde{F}v|_0^1 ,$$

where

$$||v||^2_{\tilde{U}} = \int_0^1 v^T\tilde{U}'v dx$$

defines a norm since it follows from eq. (31) that $\tilde{U}'$ is positive definite. In general, $\tilde{U}'$ will depend on v. Assume that $\tilde{f}(v)$ satisfies $\tilde{f}'v = p\tilde{f}$ for some $p > 0$. Then $\tilde{F}' = (1/(p+1))\tilde{f}'$, whence

$$\frac{d}{dt}||v||^2_{\tilde{U}} = -\frac{1}{p+1} v^T\tilde{f}'v|_0^1 .$$

Let Λ denote the eigenvalues of $\tilde{f}'$. Arguing exactly as in the symmetric hyperbolic case one obtains

$$\frac{d}{dt}||v||^2_{\tilde{U}} \leq \frac{2}{p+1}\left(\psi_0^T\Lambda_I\psi_0(0,t) + \psi_1^T\Lambda_I\psi_1(1,t)\right) . \tag{34}$$

This is not an energy estimate in the usual sense because $\tilde{U}'$ and Λ_I depend on v. However, since $\tilde{U}', \Lambda_I$ are both positive definite we shall call inequality (34) a *generalized* energy estimate. We have thus shown

Proposition 4.2 *Suppose that eq. (17) can be symmetrized by a change of variables $u = u(v)$ such that $\tilde{f}(v) = f(u(v))$ satisfies $\tilde{f}'v = p\tilde{f}$. Then any smooth solution v satisfies a generalized energy estimate (34).*

Remark: Harten [3] has shown that the Euler equations of gas dynamics can be symmetrized by a change of variables $u = u(v)$ such that $\tilde{f}'v = p\tilde{f}$. Thus, according to proposition (4.2) a generalized energy estimate is obtained. □

Equation (33) shows that eq. (17) provides an additional conservation law. It is therefore tempting to regard $v^T\tilde{U}'v$ and $v^T\tilde{F}'v$ as entropy and entropy flux. Indeed, we will show that $v^T\tilde{U}'v$ and $v^T\tilde{F}'v$ fulfill the conditions of definition (4.2). To this end we define the scalar functions

$$\begin{aligned} U(u) &\equiv v^T\tilde{U}'v = v^Tu - v^T\tilde{U} \\ F(u) &\equiv v^T\tilde{F}'v = v^Tf - v^T\tilde{F} \end{aligned} , \tag{35}$$

where the last equality is a consequence of eq. (30). The Jacobians $\tilde{U}'(v)$ and $\tilde{F}'(v)$ are symmetric by construction. Furthermore, $v' \equiv v_u$ is symmetric positive definite since it is the inverse of the symmetric positive definite matrix $u' \equiv u_v$. These symmetry relations and eq. (30) imply that

$$U'(u) = v\,, \qquad F'^T(u) = v^Tf'(u)\,.$$

Also, $U''(u) = v'(u) \geq 0$, i. e., $U(u)$ is a convex function satisfying

$$U'^T f' = F'^T\,,$$

whence $U(u)$ defined by eq. (35) is an entropy function associated with the entropy flux $F(u)$. We have thus arrived at an explicit expression for the entropy/entropy flux. We summarize the results in

Proposition 4.3 *Let eq. (17) be a symmetrizable system of conservation laws. Then an entropy function $U(u)$ and the corresponding entropy flux $F(u)$ are given by*

$$U(u) = \int_0^1 v^Tu'(\theta v)\theta v d\theta \qquad F(u) = \int_0^1 v^T\tilde{f}'(\theta v)\theta v d\theta\,, \tag{36}$$

where $v = v(u)$ is the change of variables that symmetrizes eq. (17).

Remark: In the symmetric hyperbolic case we have $u = v$, whence $\tilde{U} = (1/2)v$. This in turn implies $U(u) = (1/2)|u|^2$ in agreement with our previous results. □

In the scalar case we obtained a sequence of conservation laws (13). Define the scalar functions

$$U_p(u) = \frac{1}{2p}u^{2p}\,, \qquad p = 1, \ldots\,.$$

Hence, $U_p' = u^{2p-1}$. From eq. (13) it follows that $f_{2p-1}' = f'u^{2p-1}$ and thus

$$U_p'f' = f_{2p-1}'\,, \qquad p = 1, \ldots\,.$$

Consequently, $\{U_p, f_{2p-1}\}$ is a an entropy/entropy flux sequence. This sequence allowed us to derive an estimate in the maximum norm. *Assume* that there exists a corresponding sequence in the vector valued case (17), i. e.,

$$(U_p)_t + (F_p)_x = 0\,, \quad U_p'^T f' = F_p'^T\,, \qquad p = 1, \dots\,,$$

where the entropies satisfy $c_p|u|^{2p} \le U_p(u) \le C_p|u|^{2p}$ for some constants c_p, C_p such that

$$\limsup_{p\to\infty} \left(\frac{C_p}{c_p}\right)^{\frac{1}{2p}} = 1\,.$$

We note that these hypotheses hold in the scalar case. Also, in the symmetric hyperbolic case we have $U_1(u) = U(u) = (1/2)|u|^2$. Integration over $(-\infty, \infty) \times (0, t)$ yields

$$\int_{-\infty}^{\infty} U_p(u(x,t))dx = \int_{-\infty}^{\infty} U_p(\varphi(x))dx\,,$$

where we have assumed that u has compact support. The hypotheses on U_p imply

$$||u(\cdot,t)||_{2p} \le \left(\frac{C_p}{c_p}\right)^{\frac{1}{2p}} ||\varphi||_{2p}\,.$$

Finally, taking the limsup of both sides of this inequality leads to the maximum norm estimate

$$||u(\cdot,t)||_\infty \le ||\varphi||_\infty\,.$$

Thus, the existence of an entropy sequence implies a maximum norm estimate for the initial value problem (17), provided the entropies satisfy $c_p|u|^{2p} \le U_p(u) \le C_p|u|^{2p}$.

Remark: Lax [5, 6] has constructed a sequence of entropy functions satisfying certain growth conditions for 2×2-systems. This sequence is then used to prove that the maximum norm of the Riemann invariants is a nonincreaing function of time. □

5 Conservation Laws in Several Space Dimensions

We begin by considering scalar conservation laws

$$\begin{aligned} &u_t + \sum_i^n (f_i)_{x_i} = 0 \quad x \in \Omega \subset \mathrm{R}^n \quad t > 0 \\ &u(x,0) = \varphi(x)\,. \end{aligned} \tag{37}$$

At the boundary $\Gamma = \partial\Omega$ we prescribe the ingoing characteristics using data $\psi(x,t)$. It will be assumed that the boundary Γ is almost regular as defined in [7]. Thus, the divergence theorem can be applied to Ω. Let $\nu(x)$ denote the outward unit normal at $x \in \Gamma$. The characteristic is said to be ingoing at $x \in \Gamma$ if

$$\nu^T(x) f'(u(x,t)) \equiv \sum_{i=1}^n \nu_i(x) f_i'(u(x,t)) < 0\,. \tag{38}$$

The cone condition (2.1) is replaced by

Definition 5.1 *A function $f : \Omega_u \to R$, $\Omega_u \subset R$, satisfies an n-dimensional cone condition if*

$$\mathrm{sgn}(\nu^T f'(u)) = \mathrm{sgn}\left(\int_0^u \nu^T f'(v) v dv\right) \quad u \in \Omega_u$$

for all unit vectors $\nu \in R^n$.

This cone condition is fulfilled if we require

$$\mathrm{sgn}(\nu^T f'(u)) = \mathrm{sgn}(\nu^T f'(v)), \quad 0 \le v \le u \text{ or } u \le v \le 0. \tag{39}$$

Remark: It should be noted that the generalization of condition (10) to

$$\mathrm{sgn}(f_i'(u)) = \mathrm{sgn}(u), \quad i = 1, \ldots, n, \; u \in \Omega_u \tag{40}$$

would, in general, be insufficient to ensure the cone condition (5.1), except for the special case of $n = 1$ when condition (40) implies (39). Similarly, the generalization of (11) to

$$\mathrm{sgn}(f_i'(u)) > (<)\, 0, \quad i = 1, \ldots, n, \; u \in \Omega_u \tag{41}$$

will, in general, fail to imply cone condition (5.1). In the special case where

$$f_i(u) = c_i g(u), \quad i = 1, \ldots, n$$

condition (39) follows if $g(u)$ satisfies either condition (10) or (11). □

Partition the boundary as $\Gamma = \Gamma_I \cup \Gamma_O$, where

$$\begin{aligned} \Gamma_I &= \{x \in \Gamma : \nu^T(x) f'(u(x,t)) < 0\} \\ \Gamma_O &= \Gamma \setminus \Gamma_I, \end{aligned} \tag{42}$$

i. e., Γ_I corresponds to ingoing characteristics and Γ_O to outgoing ones.

Proposition 5.1 *Suppose that there is a smooth solution $u(x,t)$ of the initial-boundary value problem (37) for $t > 0$ and that the flux functions f_i satisfy condition (39). Then $u(x,t)$ satisfies the maximum norm estimate*

$$||u(\cdot,t)||_\infty \le ||\varphi||_\infty + ||\psi||_{\Gamma_I,\infty},$$

where $||\psi||_{\Gamma_I,\infty} = \sup_{t,x\in\Gamma_I} |\psi(x,t)|$.

Proof:
As in the one-dimensional case, the canonical splitting can be applied repeatedly to each separate flux vector $(f_i)_{x_i}$ to yield

$$\left(\frac{1}{2p} u^{2p}\right)_t + \sum_{i=1}^n \left(f_i^{(2p-1)}\right)_{x_i} = 0,$$

where

$$f_i^{(2p-1)} = \int_0^u f_i'(v)v^{2p-1}dv\,.$$

Due to condition (39) it follows that

$$\mathrm{sgn}(\nu^T f'(u)) = \mathrm{sgn}\left(\int_0^u \nu^T f'(v)v^{2p-1}dv\right)$$

for all unit vectors $\nu \in \mathrm{R}^n$ and $p = 1, \ldots$. Consequently, the cone condition holds for each p. Integration over Ω implies

$$\frac{d}{dt}||u||_{2p}^{2p} = -2p\int_\Gamma \int_0^u \nu^T f'(v)v^{2p-1}dvds$$

by the divergence theorem. Hence,

$$\frac{d}{dt}||u||_{2p}^{2p} \leq -2p\int_{\Gamma_I} \int_0^u \nu^T f'(v)v^{2p-1}dvds$$

since the cone condition holds for each p. Along Γ_I boundary data $\psi(x,t)$ is given, i. e.,

$$\frac{d}{dt}||u||_{2p}^{2p} \leq 2p\int_{\Gamma_I} \int_0^{|\psi|} |\nu^T f'(v)|v^{2p-1}dvds\,.$$

Let t be fixed. Define

$$M = \max_{\substack{0\leq v\leq|\psi(x,\tau)|\\ 0\leq\tau \\ x\in\Gamma_I}} |\nu^T f'(v)|\,.$$

Integration with respect to time results in

$$||u(\cdot,t)||_{2p}^{2p} \leq ||\varphi||_{2p}^{2p} + M\int_0^t \int_{\Gamma_I} |\psi|^{2p}dsd\tau\,.$$

Thus

$$||u(\cdot,t)||_{2p}^{2p} \leq ||\varphi||_{2p}^{2p} + |\Gamma_I|Mt||\psi||_{\Gamma_I,\infty}^{2p}\,,$$

where $|\Gamma_I|$ denotes the surface measure of Γ_I. Taking the $2p$th root of this inequality yields

$$||u(\cdot,t)||_{2p} \leq ||\varphi||_{2p} + (|\Gamma_I|Mt)^{\frac{1}{2p}}\,||\psi||_{\Gamma_I,\infty}\,.$$

Letting $p \to \infty$ proves the proposition. □

Next, we consider (37) where $u, f_i(u) \in \mathrm{R}^d$. It will be assumed that the flux vectors can be symmetrized by a change of variables $u = u(v)$, i. e., $u'(v)$ is symmetric positive definite, and $\tilde{f}_i'(v)$, $i = 1, \ldots, n$, are symmetric ($\tilde{f}_i(v) = f_i(u(v))$). The boundary conditions are defined in terms of the new variable v by eqs. (23), (24) (with u replaced by $v = v(x,t)$)

at every boundary boundary point $x \in \Gamma$. The inflow and outflow boundaries Γ_I and Γ_O are defined as

$$\begin{aligned} \Gamma_I &= \{x \in \Gamma : \lambda_j(u(v(x,t))) < 0 \text{ for some } j,\ 1 \le j \le d\} \\ \Gamma_O &= \Gamma \setminus \Gamma_I \end{aligned} \tag{43}$$

where the λ_j are the eigenvalues of

$$\tilde{f}(v, \nu) \equiv \sum_{i=1}^{n} \nu_i \tilde{f}'_i(v)\,, \tag{44}$$

i. e., $\Lambda = Q^T \tilde{f}' Q$; Q is composed of the (orthogonal) eigenvectors of $\tilde{f}'$.

Proposition 5.2 *Suppose the initial-boundary value problem (37) can be symmetrized by a change of variables $u = u(v)$ such that the flux vectors $f(u(v))_i = \tilde{f}_i(v)$ satisfy $\tilde{f}'_i v = p\tilde{f}_i$ for some $p > 0$. Then any smooth solution $v(x,t)$ satisfies a generalized energy estimate*

$$||v(\cdot,t)||^2_{\tilde{U}} \le ||\varphi||^2_{\tilde{U}} + \frac{2}{p+1}\int_0^t ||\psi(\cdot,\tau)||^2_{\Gamma_I,\Lambda_I} d\tau\,,$$

where

$$||\psi||^2_{\Gamma_I,\Lambda_I} \equiv \int_{\Gamma_I} \psi_I^T \Lambda_I \psi_I\,.$$

Proof:
Let $\tilde{U}(v), \tilde{F}_i(v) \in \mathrm{R}^d$ be solutions of

$$\begin{aligned} \tilde{U}'v &= -\tilde{U} + u \qquad & \tilde{F}'_i v &= -\tilde{F}_i + \tilde{f}_i \\ u(0) &= 0 & \tilde{F}_i(0) &= \tilde{f}_i(0)\,. \end{aligned}$$

By means of the canonical splitting we get

$$(v^T \tilde{U}' v)_t + \sum_{i=1}^{n} (v^T \tilde{F}'_i v)_{x_i} = 0\,.$$

Since $\tilde{f}_i$ satisfies $\tilde{f}'_i v = p\tilde{f}_i$ it follows that

$$\tilde{F}'_i = \frac{1}{p+1}\tilde{f}'_i$$

and thus

$$(v^T \tilde{U}' v)_t + \frac{1}{p+1}\sum_{i=1}^{n} (v^T \tilde{f}'_i v)_{x_i} = 0\,.$$

The divergence theorem yields

$$\frac{d}{dt}||v||^2_{\tilde{U}} = -\frac{1}{p+1}\int_\Gamma v^T \tilde{f}'(v,\nu) v ds = \frac{1}{p+1}\int_\Gamma \left(\omega_I^T \Lambda_I \omega_I - \omega_O^T \Lambda_O \omega_O\right) ds\,.$$

According to the definition of Γ_I and Γ_O it follows that $\Lambda_I = \emptyset$, $\Lambda_O = \Lambda \geq 0$ on Γ_O. Thus

$$\frac{d}{dt}||v||_U^2 \leq \frac{1}{p+1}\int_{\Gamma_I}\left(\omega_I^T\Lambda_I\omega_I - \omega_O^T\Lambda_O\omega_O\right) ds\,.$$

The boundary conditions on Γ_I imply

$$\frac{d}{dt}||v||_U^2 \leq \frac{2}{p+1}\int_{\Gamma_I}\psi_I^T\Lambda_I\psi_I\,.$$

Integration with respect to time yields the desired estimate. □

6 Weak Solutions

In all of the previous sections we have assumed the solutions to be smooth. This assumption allowed us to introduce the canonical splitting, which is a crucial step in the derivation of the energy estimate. Suppose now that u is a *weak* solution of

$$\begin{array}{lll} u_t + \sum_{i=1}^{n}(f_i)_{x_i} = 0 & x \in \Omega \subset \mathrm{R}^n & t > 0 \\ v = Pv + Q_I\psi & x \in \Gamma_I & \\ u(x,0) = \varphi(x)\,, & & \end{array} \tag{45}$$

where $v = v(u) \in \mathrm{R}^d$ is a change of variables that symmetrizes eq. (45); Γ_I is defined by eq. (43), and the projection P is given by eq. (24). In general, one cannot hope for an energy estimate to hold since the canonical splitting no longer exists. We therefore assume that eq. (45) can be regularized in the following sense. Let $u^\mu = u(v_\mu)$ be a smooth solution of

$$\begin{array}{lll} u_t + \sum_{i=1}^{n}(f_i)_{x_i} = \mu\sum_{i=1}^{n}u_{x_ix_i} & x \in \Omega \subset \mathrm{R}^n & t > 0 \\ \mu u_\nu + (I-P)v = Q_I\psi & x \in \Gamma & \\ u(x,0) = \varphi(x)\,, & & \end{array} \tag{46}$$

where $\mu > 0$; u_ν denotes the normal derivative of u at Γ.

Definition 6.1 *A weak solution of the symmetrizable system (45) is called a viscous limit if*

$$\lim_{\mu\to 0} u^\mu(x,t) = u(x,t) \quad (dx)\text{-a. e., } x \in \Omega, t > 0$$

$$\lim_{\mu\to 0} u^\mu(x,t) = u(x,t) \quad (ds \times dt)\text{-a. e., } (x,t) \in \Gamma \times (0,\infty),$$

and if these limits are bounded on Ω and $\Gamma \times (0,t)$ for each fixed t; dx is the n-dimensional Lebesgue measure, and ds is the $(n-1)$-dimensional surface measure on Γ induced by dx; dt is the 1-dimensional Lebesgue measure.

Proposition 6.1 *Let u be a weak solution of the symmetrizable system of conservation laws (45). If u is the viscous limit of solutions of (46), then u satisfies the entropy condition*

$$U(u)_t + \sum_{i=1}^{n} F_i(u)_{x_i} \le 0$$

in the integral sense; $U(u)$ and $F_i(u)$ are given by

$$U(u) = \int_0^1 v^T u'(\theta v)\theta v d\theta \qquad F_i(u) = \int_0^1 v^T \tilde{f}_i'(\theta v)\theta v d\theta \,.$$

Proof:
We mimic the proof of Theorem 5.6 in [6]. The regularized solution u^μ satisfies

$$u_t^\mu + \sum_{i=1}^{n} (f_i^\mu)_{x_i} = \mu \sum_{i=1}^{n} u_{x_i x_i}^\mu$$

where $u^\mu = u(v_\mu)$, $v_\mu = v(u^\mu)$, $f_i^\mu = f_i(u^\mu) = \tilde{f}_i(v_\mu)$. Since u^μ is a smooth solution we can split the hyperbolic part canonically, which after scalar multiplication by v_μ leads to

$$(v_\mu^T \tilde{U}'(v_\mu)v_\mu)_t + \sum_{i=1}^{n} \left(v_\mu^T \tilde{F}_i'(v_\mu)v_\mu \right)_{x_i} = \mu \sum_{i=1}^{n} v_\mu^T u_{x_i x_i}^\mu,$$

where

$$\tilde{U}(v_\mu) = \int_0^1 u(\theta v_\mu)d\theta \,, \qquad \tilde{F}_i(v_\mu) = \int_0^1 \tilde{f}_i(\theta v_\mu)d\theta \,.$$

Define the scalar functions

$$U(u^\mu) \equiv v_\mu^T \tilde{U}'(v_\mu)v_\mu \,, \qquad F_i(u^\mu) \equiv v_\mu^T \tilde{F}_i'(v_\mu)v_\mu \,.$$

Hence, $U'(u^\mu) = v_\mu$, $F_i'^T(u^\mu) = v_\mu^T f_i'(u^\mu)$, i. e., $U'^T(u^\mu)f_i'(u^\mu) = F_i'^T(u^\mu)$, where $U(u^\mu)$ is a convex function. We thus get

$$U(u^\mu)_t + \sum_{i=1}^{n} F_i(u^\mu)_{x_i} = \mu \sum_{i=1}^{n} U(u^\mu)_{x_i x_i} - \mu \sum_{i=1}^{n} u_{x_i}^{\mu T} U''(u^\mu) u_{x_i}^\mu \,.$$

Since U is convex it follows that $u_{x_i}^{\mu T} U''(u^\mu) u_{x_i}^\mu \ge 0$, that is,

$$U(u^\mu)_t + \sum_{i=1}^{n} F_i(u^\mu)_{x_i} \le \mu \sum_{i=1}^{n} U(u^\mu)_{x_i x_i} \,.$$

Integration over $\Omega \times (0,t)$ and the divergence theorem imply

$$\int_\Omega U(u^\mu)dx - \int_\Omega U(\varphi)dx + \int_0^t \int_\Gamma \nu^T F(u^\mu) ds d\tau \le \mu \int_0^t \int_\Gamma U_\nu(u^\mu) ds d\tau \,, \qquad (47)$$

where U_ν denotes the outward normal derivative, and

$$\nu^T F(u^\mu) = \sum_{i=1}^{n} \nu_i F_i(u^\mu) .$$

But

$$\mu \int_0^t \int_\Gamma U_\nu(u^\mu) ds d\tau = \mu \int_0^t \int_\Gamma U'^T(u^\mu) u_\nu^\mu ds d\tau .$$

The normal derivative u_ν^μ can be eliminated by means of the boundary conditions. Thus

$$\mu \int_0^t \int_\Gamma U_\nu(u^\mu) ds d\tau = \int_0^t \int_\Gamma v_\mu^T (Q_I^\mu \psi - (I - P^\mu) v_\mu) ds d\tau .$$

Using definition (6.1) we can apply the Lebesgue dominated convergence theorem as $\mu \to 0$

$$\lim_{\mu \to 0} \mu \int_0^t \int_\Gamma U_\nu(u^\mu) ds d\tau = \int_0^t \int_\Gamma v^T (Q_I \psi - (I - P) v) ds d\tau .$$

At the outflow boundary Γ_O we have $Q_I = \emptyset$, $P = I$, i. e.,

$$\lim_{\mu \to 0} \mu \int_0^t \int_\Gamma U_\nu(u^\mu) ds d\tau = \int_0^t \int_{\Gamma_I} v^T (Q_I \psi - (I - P) v) ds d\tau .$$

On Γ_I, however, the boundary conditions $v = Pv + Q_I \Psi$ are satisfied $(dt \times ds)$-a. e. Consequently,

$$\lim_{\mu \to 0} \mu \int_0^t \int_\Gamma U_\nu(u^\mu) ds d\tau = 0 .$$

The left hand side of eq. (47) can be handled analogously. We have thus arrived at

$$\int_\Omega U(u) dx - \int_\Omega U(\varphi) dx + \int_0^t \int_\Gamma \nu^T F(u) ds d\tau \leq 0 ,$$

which concludes the proof. □

As a consequence of proposition (6.1) we have

Proposition 6.2 *Let u be a weak solution of the symmetrizable system of conservation laws (45) where the flux vectors satisfy $\tilde{f}_i' v = p \tilde{f}_i$ for some $p > 0$, $i = 1, \ldots, n$. If u is the viscous limit of solutions of (46), then $v = v(u)$ satisfies a generalized energy estimate*

$$||v(\cdot, t)||_{\tilde{U}}^2 \leq ||\varphi||_{\tilde{U}}^2 + \frac{2}{p+1} \int_0^t ||\psi(\cdot, \tau)||_{\Gamma_I, \Lambda_I}^2 d\tau .$$

Proof:

From proposition (6.1) it follows that

$$\int_\Omega U(u) dx - \int_\Omega U(\varphi) dx \leq - \int_0^t \int_\Gamma \nu^T F(u) ds d\tau ,$$

where

$$U(u) = \int_0^1 v^T u'(\theta v)\theta v d\theta = v^T \tilde{U}' v\,, \qquad \tilde{U}(v) = \int_0^1 u(\theta v) d\theta$$

and

$$\nu^T F(u) = \sum_{i=1}^{n} \nu_i F_i(u) = \sum_{i=1}^{n} \nu_i \int_0^1 v^T \tilde{f}_i'(\theta v)\theta v d\theta\,.$$

Using $\tilde{f}_i' v = p\tilde{f}_i$, the identity

$$\int_0^1 v^T \tilde{f}_i'(\theta v)\theta v d\theta = \frac{1}{p+1} v^T \tilde{f}_i'(v) v$$

follows easily by integration by parts. Thus

$$\nu^T F(u) = \frac{1}{p+1}\sum_{i=1}^{n} v^T \nu_i \tilde{f}_i' v \equiv \frac{1}{p+1} v^T \tilde{f}'(v,\nu) v\,.$$

The entropy condition can then be expressed as

$$||v(\cdot,t)||_{\tilde{U}}^2 \le ||\varphi||_{\tilde{U}}^2 - \frac{1}{p+1}\int_0^t \int_\Gamma v^T \tilde{f}'(v,\nu) v ds d\tau\,.$$

The remainder of the proof is identical to that of proposition (5.2). □

The essential step in the proof of proposition (6.1) was the establishment of scalar functions $U(u)$, $F_i(u)$ ($U(u)$ convex) satisfying $U'^T f_i' = F_i'^T$. In the scalar case we know that there is a sequence $\{U_p, F_i^{(p)}\}$, $p = 1, \ldots$ such that $U_p' f_i' = F_i'^{(p)}$; U_p and $F_i^{(p)}$ are given by

$$U_p(u) = \frac{1}{2p} u^{2p}\,, \qquad F_i^{(p)}(u) = \int_0^u f_i'(v) v^{2p-1} dv\,. \tag{48}$$

For each pair $\{U_p, F_i^{(p)}\}$ proposition (6.1) is true, i. e.,

$$U_p(u)_t + \sum_{i=1}^{n} F_i^{(p)}(u)_{x_i} \le 0 \tag{49}$$

holds for all weak solutions u of eq. (37) that are viscous limits of the regularized problem. If condition (39) is met, it follows that the cone condition (5.1) is valid for all p. Equations (48), (49) thus imply an energy estimate for each p. Taking the limit as $p \to \infty$ yields

Proposition 6.3 *Let $u \in R$ be a weak solution of the scalar conservation law (37), where the flux functions f_i satisfy condition (39). If u is the viscous limit of the regularized problem, then $u(x,t)$ satisfies the maximum norm estimate*

$$||u(\cdot,t)||_\infty \le ||\varphi||_\infty + ||\psi||_{\Gamma_I,\infty}\,.$$

Thus far, it has been tacitly assumed that the sign of the eigenvalues $\lambda_j(u(x,t))$ is independent of t for each $x \in \Gamma$. This implies that for a given boundary point the rank of Q_I (Q_O) is constant in time, which means that the rank of P also is constant in time (recall that the coupling operator S has to be "small"). As a consequence, the inflow boundary Γ_I is time independent.

The estimates obtained in the previous sections have been derived from an extra conservation law (or possibly a sequence of conservation laws) subject to the same boundary conditions as the original one. To study the impact of piecewise rank constant boundary conditions it suffices to consider eq. (45). Assume that the rank of P is constant on $[0, t_1)$ and $[t_1, \infty)$. According to proposition (6.2) we have

$$||v(\cdot,t)||_{\bar{U}}^2 \leq ||\varphi||_{\bar{U}}^2 + \frac{2}{p+1} \int_0^t ||\psi^{(0)}(\cdot,\tau)||^2_{\Gamma_I^{(0)},\Lambda_I^{(0)}} d\tau \,, \quad 0 \leq t < t_1 \,,$$

and

$$||v(\cdot,t)||_{\bar{U}}^2 \leq ||v(\cdot,t_1)||_{\bar{U}}^2 + \frac{2}{p+1} \int_{t_1}^t ||\psi^{(1)}(\cdot,\tau)||^2_{\Gamma_I^{(1)},\Lambda_I^{(1)}} d\tau \,, \quad t_1 \leq t < \infty \,.$$

Now, since $v = v(u)$ is the viscous limit of a regularized problem it follows immediately that

$$\lim_{t \to t_1} ||v(\cdot,t)||_{\bar{U}} = ||v(\cdot,t_1)||_{\bar{U}} \,,$$

which yields

$$||v(\cdot,t_1)||_{\bar{U}}^2 \leq ||\varphi||_{\bar{U}}^2 + \frac{2}{p+1} \int_0^{t_1} ||\psi^{(0)}(\cdot,\tau)||^2_{\Gamma_I^{(0)},\Lambda_I^{(0)}} d\tau \,.$$

Define

$$\Gamma_I = \begin{cases} \Gamma_I^{(0)} & 0 \leq t < t_1 \\ \Gamma_I^{(1)} & t_1 \leq t \end{cases}$$

with similar definitions of ψ and Λ_I. Thus

$$||v(\cdot,t)||_{\bar{U}}^2 \leq ||\varphi||_{\bar{U}}^2 + \frac{2}{p+1} \int_0^t ||\psi(\cdot,\tau)||^2_{\Gamma_I,\Lambda_I} d\tau \,, \quad t \geq 0 \,,$$

which is identical to the estimate of proposition (6.2). The maximum norm estimates follow from a sequence of inequalities of the above type. Consequently, all energy estimates and maximum norm estimates hold when the boundary conditions have piecewise constant rank in time.

7 Discussion

Our aim has been to derive energy estimates for the initial-boundary value problem for nonlinear conservation laws by means of a few basic principles: the canonical splitting and the cone condition. These principles apply to scalar conservation laws and systems, to one or more space dimensions (even the time dimension), to smooth solutions and weak solutions that are viscous limits of vanishing viscosity solutions. For symmetrizable systems the canonical splitting gave rise to a new scalar conservation law

$$U(u)_t + \sum_{i=1}^{n} F_i(u)_{x_i} = (\leq)\ 0 \tag{50}$$

where the convex function $U(u)$ and $F_i(u)$ are related via $U'^T f_i' = F'^T$; $f_i(u)$ is the flux of the original conservation law. Thus, the canonical splitting implies an entropy condition. Furthermore, we have obtained an explicit expression for the entropy function $U(u)$ and the entropy flux $F_i(u)$.

For the initial value problem a generalized energy estimate follows immediately by integration of eq. (50), since

$$U(u) = v^T \tilde{U}'(v) v\,,$$

where $\tilde{U}'(v)$ is symmetric positive definite; $v = v(u)$ is the change of variables that symmetrizes the original conservation law. If we want an energy estimate for the initial-boundary problem, then we need to impose a cone condition on the flux vectors $f_i(u)$. The cone condition ensures that potentially dangerous boundary terms will be eliminated by enforcing the boundary conditions. The cone conditions will, in general, impose rather stringent conditions on $f_i(u)$, even in the scalar case. Convexity, however, is not needed. This contrasts with the initial value problem, in which case one can prove existence and maximum norm estimates for single conservation laws under very general hypotheses ($f(u)$ differentiable, initial data $\varphi(x) \in L^1(\mathrm{R}^n) \cap L^\infty(\mathrm{R}^n) \cap BV(\mathrm{R}^n)$)[2]. Albeit a restrictive condition, the cone condition is fulfilled for many scalar equations that occur in practice, such as the Burger's equation and the Buckley-Leverett equation. For systems a particularly important class of flux vectors is given by those that are positively homogeneous functions of v, i. e., $f'v = pf$ for some $p > 0$. The Euler equations belong to this category. Summing up, if a cone condition holds, then a generalized energy estimate follows from eq. (50) and the characteristic boundary conditions. Thus, the energy estimate appears as a special case of the entropy condition (50). In particular, there exists a generalized energy estimate iff the system is symmetrizable and a cone condition holds.

We saw for a single conservation law that it was possible to generate a sequence of entropy conditions from which we could deduce a maximum norm estimate. It is an intriguing question whether systems of conservation laws that originate from physics possess a sequence of entropy conditions such that a maximum norm estimate would follow. As mentioned earlier, Lax [5] has constructed such a sequence for 2×2-systems. Harten [3] has introduced a family of entropies for the Euler equations such that symmetry and homogeneity are preserved. It remains to be shown if a family can be found that satisfies the necessary growth conditions. Finally, we conclude this discussion by noting that many of the energy estimates carry over to the semi-discrete case by extending the technique in [9] to the nonlinear case. This is the topic of a forthcoming paper.

Acknowledgment

Many thanks are due to Prof. Bertil Gustafsson for constructive criticism and valuable suggestions.

References

[1] S. Abarbanel and D. Gottlieb. Optimal time splitting for two- and three-dimensional Navier-Stokes equations with mixed derivatives. *J. Comput. Phys.*, 41(1):1–33, 1981.

[2] E. Godlewski and P.-A. Raviart. *Hyperbolic systems of conservation laws.* Societé de Mathématiques Appliquées et Industrielles, 1991.

[3] A. Harten. On the symmetric form of systems of conservation laws with entropy. *J. Comput. Phys.*, 49:151–164, 1983.

[4] F. John. *Partial Differential Equations*, volume 1 of *Applied Mathematical Sciences.* Springer Verlag, fourth edition, 1982.

[5] P. Lax. Shock waves and entropy. In E. H. Zarantonello, editor, *Contributions to Nonlinear Functional Analysis*, volume 31, pages 603–634. University of Wisconsin, Academic Press, 1971.

[6] P. Lax. *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, volume 11 of *CBMS-NSF Regional Conference Series in Mathematics.* Society for Industrial and Applied Mathematics, 1973.

[7] L. H. Loomis and S. Sternberg. *Advanced Calculus.* Addison-Wesley Publishing Company, 1968.

[8] P. Olsson. *High-Order Difference Methods and Dataparallel Implementation.* PhD thesis, Dept. of Scientific Computing, Uppsala University, Apr. 1992.

[9] P. Olsson. Summation by parts, projections, and stability. Technical Report 93.04, RIACS, June 1993.