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A Simple Suboptimal Least-Squares Algorithm for Attitude Determination with Multiple Sensors

Thomas F. Brozenec and Douglas J. Bender

ABSTRACT

Three-axis attitude determination is equivalent to finding a coordinate transformation matrix which transforms a set of reference vectors fixed in inertial space to a set of measurement vectors fixed in the spacecraft. The attitude determination problem can be expressed as a constrained optimization problem. The constraint is that a coordinate transformation matrix must be proper, real, and orthogonal. A transformation matrix can be thought of as optimal in the least-squares sense if it maps the measurement vectors to the reference vectors with minimal 2-norm errors and meets the above constraint. This constrained optimization problem is known as Wahba's problem. Several algorithms which solve Wahba's problem exactly have been developed and used. These algorithms, while steadily improving, are all rather complicated. Furthermore, they involve such numerically unstable or sensitive operations as matrix determinant, matrix adjoint, and Newton-Raphson iterations.

This paper describes an algorithm which minimizes Wahba's loss function, but without the constraint. When the constraint is ignored, the problem can be solved by a straightforward, numerically stable least-squares algorithm such as QR decomposition. Even though the algorithm does not explicitly take the constraint into account, it still yields a nearly orthogonal matrix for most practical cases; orthogonality only becomes corrupted when the sensor measurements are very noisy, on the same order of magnitude as the attitude rotations. The algorithm can be simplified if the attitude rotations are small enough so that the approximation $\sin\theta \approx \theta$ holds.

We then compare the computational requirements for several well-known algorithms. For the general large-angle case, the QR least-squares algorithm is competitive with all other known algorithms and faster than most. If attitude rotations are small, the least-squares algorithm can be modified to run faster, and this modified algorithm is faster than all but a similarly specialized version of the QUEST algorithm.

We also introduce a novel measurement averaging technique which reduces the n-measurement case to the two measurement case for our particular application, a star tracker and earth sensor mounted on an earth-pointed geosynchronous communications satellite. Using this technique, many n-measurement problems to reduce to ≤ 3 measurements; this reduces the amount of required calculation without significant degradation in accuracy.

Finally, we present the results of some tests which compare the least-squares algorithm with the QUEST and FOAM algorithms in the two-measurement case. For our example case, all three algorithms performed with similar accuracy.

The authors are with Hughes Aircraft Company, Bldg E1, MS D110, POB 92919, Los Angeles, CA 90009

1. INTRODUCTION

This paper discusses a new sub-optimal algorithm for attitude determination. It also introduces a novel measurement averaging technique which effectively reduces the number of vector measurements that must be processed by any attitude determination algorithm. It is organized as follows: first a brief statement of the attitude determination problem is given, followed by a formal statement of Wahba's attitude determination problem. In Section 2, we introduce a new suboptimal least-squares algorithm which minimizes Wahba's loss function, but without the orthogonality constraint on the attitude matrix solution. In Section 3 the computational requirements for the new algorithm are presented and compared with several other attitude solutions (optimal & suboptimal). Section 4 discusses a technique for reducing the number of vector measurements. Finally, Section 5 presents simulation results for a specific example of a geosynchronous communications satellite with a star tracker and an earth sensor. We end with a brief conclusion.

Attitude Determination Problem Statement

Consider a set of reference vectors r_i , $i = 1 , \cdots , n$ expressed in an inertially defined coordinate system I. Consider the same set of vectors, but denoted s_i , $i = 1 , \cdots , n$ when expressed in a spacecraft body defined coordinate system B. These vectors are related by the attitude (or direction cosine) matrix A, i.e.

$$s_i = Ar_i \quad i = 1 , \cdots , n \tag{1}$$

In one application, for example, the measurement vectors s_i would be provided by earth sensor and star tracker measurements (corrupted by error sources such as noise and biases) while the reference vectors r_i would correspond to known directions of the stars and earth nadir expressed in I.

The vectors s_i and r_i (for $i = 1 , \cdots , n$) can be concatenated to form the columns of matrices

$$S = \left[s_1 \mid s_2 \mid \cdots \mid s_n\right] \quad R = \left[r_1 \mid r_2 \mid \cdots \mid r_n\right]. \tag{2}$$

Then (1) can be written as one matrix equation

$$S = AR \tag{3}$$

where S and R are $3 \times n$ matrices and A is a 3×3 proper real orthogonal matrix [1], i.e.

$$A^T = A^{-1} \quad \& \quad \det(A) = 1 \tag{4}$$

Since A satisfies the properties of (4), it represents a rotation transformation which preserves the lengths of vectors and the angles between them.

The problem is to find an estimate $\hat{A}$ of the attitude matrix A using the measurement matrix S and the reference matrix R (both possibly corrupted by noise). It is a fundamental fact that at least two non-collinear vectors are needed to determine attitude; thus, it is necessary that $n > 1$ to obtain a solution.

The problem thus described has a long history ([3]-[14]). A number of solutions have been proposed, both approximate and optimal. Approximate algorithms such as TRIAD cannot accommodate more than two observations and even throw away part of this information; therefore, they don't provide an optimal estimate of the attitude.

Optimal algorithms, on the other hand, compute a best estimate of the spacecraft attitude based on a loss function which takes into account all n measurements. One particular loss function which has found a prominent place in the literature is the so-called Wahba loss function, which was first proposed by Wahba in 1965 [2]:

$$L(\hat{A}) = \frac{1}{2}\sum_{i=1}^{n} a_i \|s_i - \hat{A}r_i\|_2^2 = \frac{1}{2}\left\| (S - \hat{A}R)\, \Lambda^{1/2} \right\|_2^2 = \frac{1}{2} tr\left[\Lambda^{1/2}(S-\hat{A}R)^T(S-\hat{A}R)\Lambda^{1/2}\right], \tag{5}$$

where $\| \cdot \|_2$ denotes the vector 2-norm, $\Lambda = diag(a_1 , a_2 , \cdots , a_n)$, and the a_i , $i = 1 , \cdots , n$ are non-negative weights, whose sum can be set to unity without loss of generality (and will be in the sequel).

Wahba's Problem

Wahba's problem is formally stated as follows: *Find a proper real orthogonal matrix satisfying (4) which minimizes the cost function (5).*

Solutions to Wahba's problem

The early solutions, as reported in [2], involve a polar decomposition of SR^T; a complete set of eigenvalues and eigenvectors of the symmetric part is then required. Therefore, these solutions required a large amount of calculation.

Davenport [4] has shown that the quadratic loss function in the attitude matrix can be transformed into a quadratic loss function in the corresponding quaternion. This is a great simplification of the problem proposed by Wahba since the quaternion is subject to fewer constraints than the nine elements of the attitude matrix. Davenport's substitution leads directly to an eigenvalue equation for the quaternion. This substitution and the resulting eigenvalue equation form the basis for much of the work presented in the literature; for example, it is the starting point for the derivation of the well-known QUEST algorithm. The QUEST algorithm provides an efficient closed-form solution to the eigenvalue problem.

The TRIAD algorithm is a deterministic, suboptimal algorithm which can accommodate only two measurements. It involves very simple and straightforward calculations, has been in existence for at least two decades, and has been implemented in a number of missions; these include, among others, Small Astronomy Satellite (SAS), Seasat, Atmospheric Explorer Mission (AEM), and Dynamics Explorer.

Markley's SVD algorithm [6] provides a very robust method for solving Wahba's problem. It is not very efficient since it requires the singular value decomposition of the attitude profile matrix (a 3×3 matrix constructed from the measurement and reference vectors). Markley's FOAM algorithm [9] provides a related solution which does not require the singular value decomposition and is, therefore, very efficient. Markley reports execution times faster even than for QUEST, previously the fastest known algorithm. In a very recent paper [10], Markley introduces a variant of FOAM. The iteration normally required for solving the usual quartic is avoided at the cost of losing orthogonality of the solution. However, orthogonality is recovered using an original technique; simulations show no loss of accuracy in most cases.

Another solution recently described is the polar decomposition (PD) algorithm of Bar-Itzhack [12]. As the name suggests, it obtains a solution by performing a polar decomposition of the attitude profile matrix; i.e. a decomposition into orthogonal and symmetric parts. The orthogonal part is precisely the solution to Wahba's problem.

In the next sections we explore the use of standard linear least-squares techniques to minimize (5) while relaxing the constraint (4).

2. THE NEW SUBOPTIMAL ALGORITHM

Least squares estimation theory can be used to obtain an approximate solution to the problem posed in Section 1. To be perfectly clear, the algorithm presented here is not a solution of Wahba's problem; it minimizes Wahba's loss function, but *without* constraint. Therefore, this solution is suboptimal. We first present the general form of the algorithm after which it is specialized for the case of small rotations. The general algorithm requires three non-collinear vector measurements in the general case, while the small-angle least-squares algorithm works with as few as two measurements.

2.1 General Case

Consider the attitude matrix A of (1). We wish to estimate A using r_i and s_i (for $i = 1, \cdots, n, n \geq 3$). Given Equation (3), the problem we address is

$$\text{minimize} \, \|S - \hat{A}R\|_2^2 \tag{6}$$

over all possible $\hat{A}$. The objective function of (6) is equivalent to Wahba's loss function with all weights a_i set to unity for simplicity; the results in this paper can easily be extended to the weighted least-squares case. The least squares estimator minimizing this objective function is given by [16]:

$$\hat{A} = SR^+ \tag{7}$$

where R^+ denotes the pseudo-inverse of R. If the columns of R are linearly independent (as they would be with ≥3 nonredundant measurements), R^+ can be written

$$R^+ = R^T \left[RR^T\right]^{-1} \tag{8}$$

In practice, rather than directly computing (8), it is much more numerically robust and computationally efficient to perform an orthogonal triangularization of R; from this the attitude estimate, $\hat{A}$, can be computed. An orthonormal matrix Q exists which transforms R to an lower triangular matrix Γ:

$$RQ = \left[\Gamma \quad 0\right] \tag{9}$$

Ref. [15] describes some standard, numerically stable algorithms exist for computing the orthogonal transformation of R; two of the more popular algorithms are Householder rotations and modified Gram-Schmidt. Equation 9 is slightly different from the "usual" orthogonal triangularization in two ways: first, in the more typical setup, Q would operate from the left instead of the right and Γ would be upper instead of lower triangular, and second, the notation is perturbed because we already used the symbol R and here we use Γ to represent the matrix that would usually be called R.

Now, partitioning Q as $[\,Q_1 \quad Q_2\,]$, yields

$$R = \left[\Gamma \quad 0\right] \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} = \Gamma Q_1^T \tag{10}$$

We now show how this factorization can be used to minimize the 2-norm of $E \triangleq S - \hat{A}R$. Denote

$$SQ = \left[C \quad D\right] \tag{11}$$

Then, calculate

$$EQ = \left[C - \hat{A}\Gamma \quad D\right] \tag{12}$$

So, the least squares solution is obtained by choosing $\hat{A}$ to satisfy

$$\hat{A}\Gamma = C \quad \text{or} \quad \hat{A} = C\Gamma^{-1} \tag{13}$$

which is a matrix equation whose matrix-valued solution can be obtained using standard techniques from linear algebra. Equations (9), (11) & (13) give an algorithm for solving the linear least squares problem (6). Equation (7) is a batch solution of a static estimation problem. It can be used to update each time step of a recursive algorithm such as a Kalman Filter, such as in [11].

Advantages of the General Least-Squares Algorithm

(1) The algorithm can be carried out using well-known, numerically stable algorithms.

Disadvantages of the General Algorithm

(1) Implementing the QR decomposition requires a relatively large number of calculations in the general case, so the algorithm trades speed for numerical robustness.

(2) The solution technique ignores the constraints of (4); thus the estimated attitude matrix will not necessarily be a true attitude matrix (i.e. orthogonal). This can be alleviated by using the orthogonalization procedure of Bar-Itzhack [11]. More importantly, as sensor errors approach zero the estimated attitude matrix will approach orthogonality. This is because the least-squares algorithm minimizes the objective function of (6); since the true attitude matrix zeros this objective function, in the error-free (no noise) case, the least squares solution will equal the exact solution to Wahba's constrained problem. We have observed this in a number of simulations.

(3) The algorithm is only usable when three or more (non-collinear) measurements are available; otherwise the inverse of (8) doesn't exist.

Summary: The general least-squares algorithm does not appear to be very interesting in its own right; its main interest is that in the special case of small attitude rotations it can be simplified, yielding a very attractive algorithm. We discuss this below.

2.2 Small Rotations Case

If small attitude rotations are assumed, the attitude matrix A of (1) can be written as

$$A \approx \begin{bmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{bmatrix} \tag{14}$$

where ϕ, θ, and ψ are the usual Euler angles and are the quantities we wish to estimate (all Euler angle sets are essentially equivalent for small angles; these are the so-called body roll, pitch, and yaw pointing errors).

Combining Equations (1) & (14) as in Bar-Itzhack [11], the least-squares problem (6) reduces to:

$$\text{minimize } \|\tilde{S}_{small} - \tilde{R}_{small}\Theta\|, \tag{15}$$

where

$$\tilde{S}_{small} \triangleq \begin{bmatrix} s_1 - r_1 \\ s_2 - r_2 \\ \vdots \\ s_n - r_n \end{bmatrix}, \quad \tilde{R}_{small} \triangleq \begin{bmatrix} 0 & -r_{13} & r_{12} \\ r_{13} & 0 & r_{11} \\ -r_{12} & r_{11} & 0 \\ 0 & -r_{23} & r_{22} \\ r_{23} & 0 & r_{21} \\ -r_{22} & r_{21} & 0 \\ \vdots & \vdots & \vdots \\ 0 & -r_{n3} & r_{n2} \\ r_{n3} & 0 & r_{n1} \\ -r_{n2} & r_{n1} & 0 \end{bmatrix}, \text{ and } \Theta \triangleq \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}.$$

Equation (15) represents a linear measurement equation where the parameters to be estimated are the desired body roll, pitch, and yaw angles. The least squares attitude estimate in this case is given by

$$\hat{\Theta} = \tilde{R}_{small}^{+}\tilde{S}_{small} = \left(\tilde{R}_{small}^{T}\tilde{R}_{small}\right)^{-1}\tilde{R}_{small}^{T}\tilde{S}_{small}. \tag{16}$$

Equation (16) gives the least-squares solution to the linear system of equations in (15); a solution exists for $n \geq 2$ measurements. Again, instead of computing (16) directly, a QR decomposition of $\tilde{R}_{small}$ is used to calculated the attitude solution. More explicitly, there exists an orthonormal matrix Q and an upper triangular matrix Γ such that

$$Q\tilde{R}_{small} = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}. \tag{17}$$

(Note that in this case, the "QR" decomposition is in its usual form except that Γ takes the place of R). Now, partition $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$; then,

$$\tilde{R}_{small} = [\, Q_1^T \;\; Q_2^T \,]\begin{bmatrix} \Gamma \\ 0 \end{bmatrix} = Q_1^T\Gamma. \tag{18}$$

As above, this factorization can be used to minimize the 2-norm of $e = \tilde{S}_{small} - \tilde{R}_{small}\Theta$. Denote

$$Q\tilde{S}_{small} = \begin{bmatrix} c \\ d \end{bmatrix}. \tag{19}$$

Then, calculate

$$Qe = \begin{bmatrix} c - \Gamma\Theta \\ d \end{bmatrix}. \tag{20}$$

The least squares solution is then obtained by choosing Θ such that

$$\Gamma\Theta = c, \tag{21}$$

which is a 3×3 system of linear equations easily solved using standard techniques such as Gaussian elimination [15]. Therefore, having computed the QR decomposition of $\tilde{R}_{small}$, Equations (19) & (21) give an algorithm for solving the linear least squares problem considered.

Advantages of the Small-Angle Least-Squares Algorithm

(1) For small angles, this algorithm provides a nearly optimal attitude estimate using an efficient and numerically stable algorithm.

(2) The small angle algorithm explicitly estimates the roll, pitch, and yaw angles (not the full attitude matrix). Thus additional calculations are not required to obtain the roll, pitch, and yaw angles from the attitude matrix.

(3) The matrix $\tilde{R}_{small}$ contains many zeros and one can take advantage of this known structure to reduce the required computation. This algorithm's computational requirement compares favorably with other algorithms as shown below.

(4) This algorithm works with $n \geq 2$ measurements, unlike the general algorithm which needs at least three measurements.

Disadvantages of the Small-Angle Algorithm

(1) The primary disadvantage is introduction of errors due to the small-angle approximation, and thus applicability limited to only those cases where small angle approximations are valid. For attitude rotations in all three coordinate axes of less than 5°, the error due to the small angle approximations is bounded by 0.006°, which is acceptable in many cases.

(2) As for the general least-squares algorithm, this algorithm is not an exact solution of the Wahba problem and does not guarantee an orthogonal attitude matrix. If no measurement noise is present (and there are no roundoff errors), the only errors are due to the small angle assumptions. If the amount of measurement noise is small compared to the size of the attitude rotations, the linearization is the main source of error and nonoptimality is not an issue.

3. COMPUTATIONAL REQUIREMENTS

In this section, we present the computational requirements for the least squares algorithm and compare these with the computational requirements of several well-known algorithms. Table 1 below summarizes the computational requirements.

For each algorithm, we address both the general case and the small rotation case. We specifically address the cases of two and three vector measurements (i.e. $n = 2, 3$). The two and three measurement cases are the most important because most typical spacecraft have either two or three sensors active at a time. Two vector measurements are sufficient to determine attitude. The three measurement case is important because many systems seeking maximal accuracy will include an orthogonal (or near orthogonal) triad of sensing devices. In the next section we present a technique for reducing more than three measurements to no more than three for cases where the additional measurements can be conveniently clustered (for example, one star tracker tracking and measuring multiple stars).

The computational requirements in this table do not include the calculations needed to compute the so-called attitude profile matrix, thus making general QUEST and SVD operations counts independent of n, the number of measurements. The least-squares operations counts, on the contrary, do grow with larger n. For the general (large-angle) least squares algorithm, only the three-measurement case is included because three vector measurements are required to obtain a solution. The SOMA (slower optimal matrix) algorithm [9], a variant of FOAM, is included for completeness.

Observations From Table 1

(1) TRIAD is the fastest two-measurement algorithm, but is error prone because it actually throws away some of the measurement information.

(2) Of small-angle algorithms (excluding TRIAD), QUEST is the fastest, both for two and three measurements. It is followed by least squares and then FOAM.

(3) Of the general large-angle algorithms, FOAM is the fastest, followed by QUEST and then least squares. SVD is the slowest.

Table 2 summarizes the primary characteristics of each algorithm. The second column indicates the nature of the solution by construction. The third column indicates the types of calculations which make up the bulk of the computation for each algorithm. We make one more observation:

Table 1 - Algorithm Computational Requirements For The Two & Three Measurement Cases							
Algorithm	Operations						
	cosine	sine	add	multiply	arccos	arcsin	sqrt
Least Squares							
General, n=3 msmts	0	0	267	264	0	0	6
Small Angles, n=2 msmts	0	0	81	78	0	0	3
Small Angles, n=3 msmts	0	0	117	114	0	0	3
TRIAD, n=2 msmts	0	0	34	59	0	0	2
QUEST							
General Case, n=2 & 3 msmts	0	0	217	209	0	0	1
Small Angles, n=2 msmts	0	0	42	71	0	0	1
Small Angles, n=3 msmts	0	0	64	95	0	0	1
SVD, n=2 & 3 msmts	0	0	299	336	0	0	0
FOAM, n=2 msmts	0	0	81	112	0	0	1
FOAM*, n=3 msmts	0	0	117	148	0	0	1
SOMA**, 3 msmts	1	0	94	155	1	0	4

* *assumes one Newton-Raphson iteration*

** *SOMA identical to FOAM in two measurement case*

(4) As a preview of Section 5, FOAM appears to have more numerical error. Both QUEST and FOAM include some computations which might be sensitive to numerical round-off errors such as trace, adjoint, determinant, and Newton-Raphson iteration.

Table 2 - Algorithm Characteristics		
Algorithm	Solution Characteristic	Primary Computations
Least Squares		
Small Angles	solves for ϕ, θ, & ψ directly; near-optimal	QR decomposition
General Case	minimizes Wahba's loss function without constraint	QR decomposition
TRIAD	non-optimal; deterministic	cross products, matrix multiply
QUEST		
Small Angles	solves Wahba's problem	solve $Ax = b$, $A \in \mathbf{R}^{3x3}$
General Case	solves Wahba's problem	trace, adjoint, determinant, Newton-Raphson iteration
SVD	solves Wahba's problem	singular value decomposition (iterative)
FOAM	solves Wahba's problem	adjoint, Newton-Raphson iteration, frobenius norm
Polar Decomposition	solves Wahba's problem	polar decomposition, linear equation solution, matrix multiply

4. AVERAGING OF MEASUREMENTS FOR NEARLY COLLINEAR SENSORS

In this section we describe a technique for combining the measurements of nearly collinear sensors to reduce the number of independent measurements which must be processed by the attitude determination algorithm. This is motivated by the following observation: given two or three (non-collinear) vector measurements, additional, independent measurements which are nearly collinear to any of the original two or three provide very little additional information. For example, consider a spacecraft with two star trackers whose boresights are well-separated from each other. Further assume that each tracker tracks several stars within its field of view and produces independent vector measurements for every star it tracks. The stars within each star tracker's boresight are separated by only a few degrees. Therefore, each star tracker is essentially a two-axis sensing device, with a weak measurement of the axis of rotation about its boresight. A much better three-axis attitude measurement is available by combining the measurements of the two star trackers, and two axes of measurement from each is sufficient.

The central concept of this section is to combine collections of nearly collinear measurements by *averaging* them, and then using that averaged measurement (along with at least one other measurement) in the attitude determination algorithm of choice.

To carry out this concept, the nearly collinear measurements (for example, the measurements of all stars in a star tracker) are averaged and normalized in Cartesian coordinates. This results in one measurement vector (and its corresponding reference vector) as follows:

$$\text{measurement vector:} \quad s_{avg}^{m} = \frac{s_1 + s_2 + \cdots + s_m}{\| s_1 + s_2 + \cdots + s_m \|_2} \tag{22}$$

$$\text{reference vector:} \quad r_{avg} = \frac{r_1 + r_2 + \cdots + r_m}{\| r_1 + r_2 + \cdots + r_m \|_2}. \tag{23}$$

It is easy to verify that if the original measurements satisfy (3), then the resulting averages from (22) & (23) also satisfy (3). Therefore, if there are at least two widely separated sets of coaligned or nearly coaligned attitude measurements, these sets of measurements can be grouped and averaged as above. This results in just a few (typically two or three) measurements, each of which might be aggregates of several sets of nearly coaligned measurements. Reducing the number of measurements this way has two main benefits:

(1) It reduces the operations necessary in the attitude determination algorithm (at least if the least squares algorithm is used; some algorithms' operations counts do not grow with additional measurements).

(2) It reduces the data flow required across the interface between the star sensor and the central processor.

5. SIMULATION RESULTS : COMPARISON OF LEAST-SQUARES, QUEST, and FOAM

In this section we present the results of a simulation which demonstrates several of the concepts in this paper. We compare the performance of the least-squares algorithm with the well-known QUEST and FOAM algorithms for the case of small attitude rotations with two vector measurements. We also demonstrate the use of the measurement averaging technique of Section 4.

The particular example studied here is a geosynchronous satellite with a star tracker and an earth sensor. The star tracker can track up to m stars, but the m star mesurements are averaged by the technique of Section 4 to produce one vector measurement. The star tracker boresight is assumed to be 53^o from the earth sensor boresight, and these two sensors provide two well-separated vector measurements for attitude determination. We simulate only the small-angle version of the least-squares algorithm since the general algorithm needs at least three measurement vectors. It is reasonable to use the small-angle algorithm, because during normal operations geosynchronous satellites typically maintain their attitudes to within small perturbations from the nominal attitude. We also simulate both the general and small-angle QUEST algorithms, and the FOAM algorithm.

To study the effects of finite precision arithmetic, the simulation includes a model of the round-off error which is exhibited in a typical processor. The algorithms are assumed to be implemented in single precision floating-point arithmetic, with numbers stored as 23-bit mantissa, sign, and 8-bit exponent. This corresponds to the popular 1750a architecture.

First, coordinate frames will be defined after which the detailed implementation of the algorithm will be described. Figure 1 depicts the geometry.

5.1 Coordinate Frame Definitions

The right-handed coordinate frames considered are defined below.

orbiting reference frame, R=(1,2,3)

body frame, B=(x,y,z)

Star sensor frame, S=(a,b,c)

inertial frame, I=(γ,II,N)

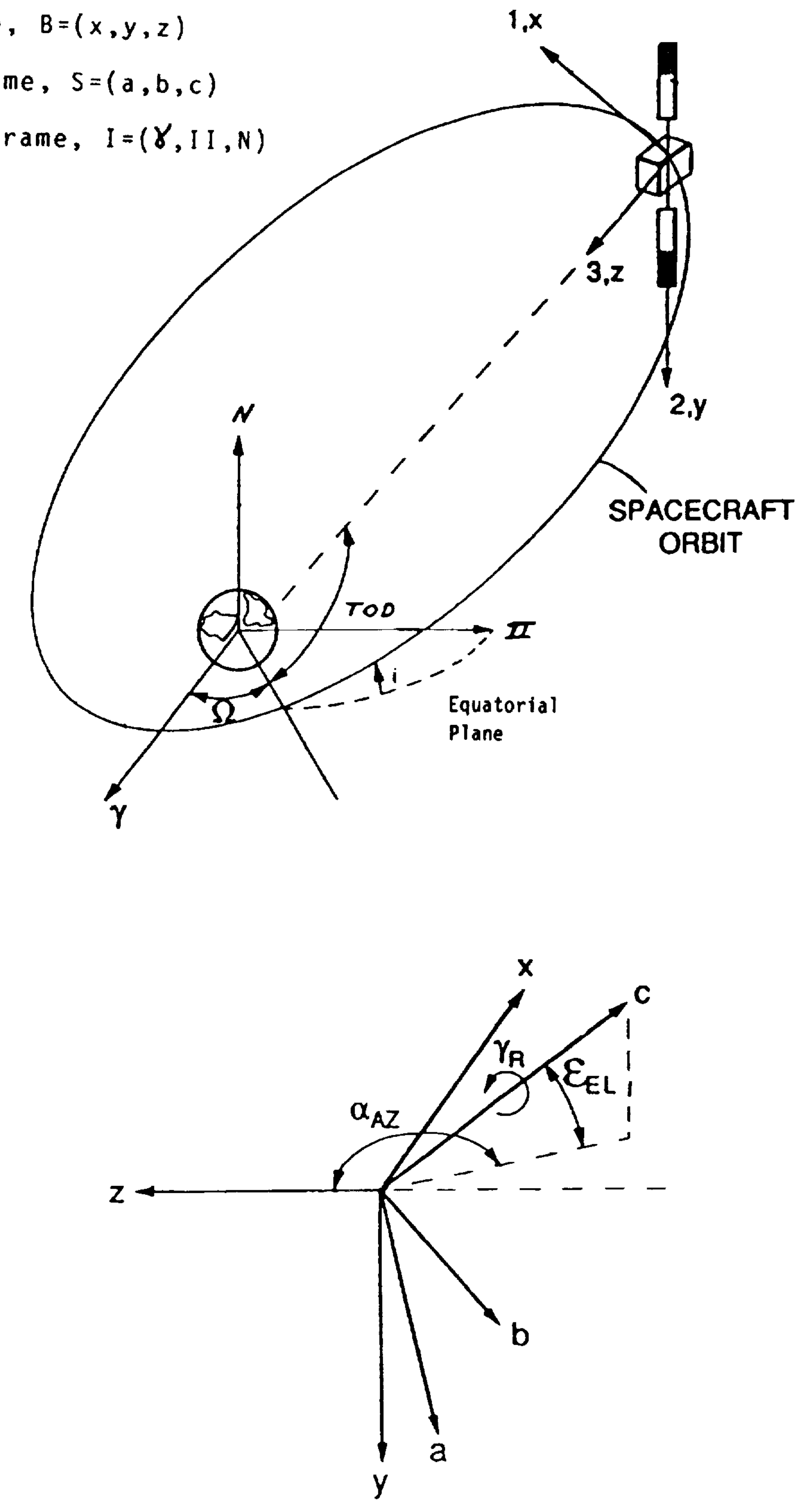

Figure 1 - Geometry & Coordinate Frames

1. **S** = Star Sensor Frame

   This frame has its 3-axis in the direction of the star sensor boresight, its 1-axis in the positive azimuth direction of the star sensor, and its 2-axis completes the right-handed triad.

2. **B** = Spacecraft Body Frame

   This frame has the spacecraft yaw axis as its 3-axis, the spacecraft pitch axis as its 2-axis, and the spacecraft roll axis as its 1-axis.

3. **R** = Orbiting Reference Frame

   This frame has its 3-axis nadir pointed, its 2-axis orbit normal south, while its 1-axis completes the right-handed triad.

4. **I** = Celestial Frame

   This frame, commonly referred to as the Earth Centered Inertial (ECI) frame, has its 3-axis parallel to the rotation axis of the earth and pointed north, its 1-axis towards the vernal equinox, and its 2-axis in the plane of the equator completing the right-handed triad.

5.2 Coordinate Frame Transformations

The coordinate frame transformations considered are defined below. A 3×3 direction cosine matrix is a transformation from one coordinate frame to another. In the following, a direction cosine matrix will be written $A = A(\alpha_1, \cdots, \alpha_n)$ where arguments α_i, $i = 1, \cdots, n$ indicate a functional dependence (not explicitly shown) of the direction cosine matrix on the arguments.

1. Transformation From Spacecraft Body To Star Sensor Coordinates

$$M : \mathbf{B} \to \mathbf{S} \quad M = M(\alpha_{AZ}, \varepsilon_{EL}, \gamma_R) \qquad (24)$$

where

$$\alpha_{AZ}, \varepsilon_{EL}, \gamma_R = \text{Azimuth, Elevation, \& Rotation Mounting Angles Of The Star Sensor}$$

2. Transformation From Orbiting Reference Frame To Body Frame

$$A : \mathbf{R} \to \mathbf{B} \quad A = A(\phi, \theta, \psi) \qquad (25)$$

3. Transformation From Celestial Frame To Orbiting Reference Frame

$$C : \mathbf{I} \to \mathbf{R} \quad C = C(i, \Omega, TOD) \qquad (26)$$

where TOD denotes the time-of-day angle, i = orbit inclination, and Ω = right ascension of the ascending node.

5.3 Measurements

The following measurements are available from the star sensor and the earth sensor to estimate the spacecraft attitude.

1. Star vector measurements expressed in the star sensor coordinate frame **S**, $s_i^m \in \mathbf{S} \quad i = 1, 2, \cdots, m$.
2. The corresponding star vectors expressed in the celestial coordinate frame **I** from the star catalog (obtained as a result of performing star identification), $r_i^c \in \mathbf{I} \quad i = 1, 2, \cdots, m$.
3. Earth vector measurements expressed in spacecraft body coordinates **B** from the earth sensor, $s_E^m \in \mathbf{B}$.

$$s_E^m = \begin{bmatrix} -\sin\theta_{AZ}\cos\phi_{EL} \\ \sin\phi_{EL} \\ \cos\theta_{AZ}\cos\phi_{EL} \end{bmatrix} \qquad (27)$$

where θ_{AZ} and ϕ_{EL} are the earth sensor azimuth and elevation measurements, respectively, with respect to earth nadir. Our earth "ephemeris" data which is analagous to star reference positions given from the catalog is $r_E^c = (0, 0, 1)^T \in \mathbf{R}$.

5.4 Collection and Averaging Of Measurements

Our ultimate goal is to find an attitude matrix $A(\phi, \theta, \psi)$ which transforms from the orbiting reference frame **R** to the body frame **B** and equates the measurement vectors in **R** to the reference vectors in **R** (see equations (6) and (24), or (30) and (32) below). The measurement and reference vectors for the earth sensor are:

$$s_E^m \in \mathbf{B} \text{ and } r_E^c \in \mathbf{R}, \tag{28}$$

and the measurement and reference vectors for the star tracker are:

$$M^T s_i^m \in \mathbf{B} \text{ and } Cr_i^c \in \mathbf{R}, \quad i = 1, 2, \cdots, m. \tag{29}$$

The star sensor measurements and reference vectors are then averaged using equations (22) and (23) to make one aggregate star sensor measurement.

5.5 Implementing the Suboptimal Algorithm

In this case, the least-squares problem (15) is to minimize the difference between the the measurements (expressed in the body frame) and the references (expressed in the orbiting reference frame) through attitude rotation A (the desired quantity). For the star sensor measurements,

$$\text{minimize } \|S_{avg} - A(\phi, \theta, \psi)\, R_{avg}\|, \tag{30}$$

where

$$S_{avg} = M^T s_{avg}^m \text{ and } R_{avg} = Cr_{avg}. \tag{31}$$

For the earth sensor measurement,

$$\text{minimize } \|s_E^m - A(\phi, \theta, \psi)\, r_E^c\|, \text{ where } r_E^c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{32}$$

Combining Equations (27), (28), and (29), rearranging terms, and removing one extraneous equation, the least-squares problem (15) reduces to

$$\text{minimize } \|\bar{S} - \bar{R}\,\Theta\|, \text{ or minimize } \left\| \begin{bmatrix} S_{avg,1} - R_{avg,1} \\ S_{avg,2} - R_{avg,2} \\ S_{avg,3} - R_{avg,3} \\ s_{E,1}^m \\ s_{E,2}^m \end{bmatrix} - \begin{bmatrix} 0 & -R_{avg,3} & R_{avg,2} \\ R_{avg,3} & 0 & -R_{avg,1} \\ -R_{avg,2} & R_{avg,1} & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right\| \tag{33}$$

which is a linear measurement equation in ϕ, θ, and ψ. $X_{avg,i}$ denotes the i'th entry of the vector X_{avg}. The least squares solution is then obtained by factoring $\bar{R}$ and implementing Equations (19) and (21) as discussed in Section 2 above.

Small angle approximations are used to provide the simple form of the attitude matrix in Equation (33). Because of the orbit normal steering law of many geosynchronous satellites, the small angle assumption is valid; the satellite body axes will not deviate from the orbiting reference frame more than a few tenths of a degree during normal operations (including stationkeeping). The star sensor measurements and references are averaged in cartesian coordinates using equations (22) and (23) providing one single "measurement" from the star sensor which reduces subsequent computation and limits data flow at the interface.

5.6 Simulation Results

For our simulations, we use the following reference vectors:

$$R_{avg\,1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad R_{avg\,2} = \begin{bmatrix} 0 \\ -\sin 53° \\ -\cos 53° \end{bmatrix} \tag{34}$$

The reference vector $R_{avg\,1}$ is the direction of earth nadir in the orbiting reference frame while $R_{avg\,2}$ is the direction of a star expressed in the orbiting reference frame. The measurement vectors are computed as

$$S_{avg\,i} = A_{att} R_{avg\,i} + n_i, \quad i = 1, 2 \tag{35}$$

where

$$A_{all} = \left[\begin{array}{ccc} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi+\sin\psi\cos\phi & -\cos\psi\sin\theta\cos\phi+\sin\psi\sin\phi \\ -\sin\psi\cos\theta & -\sin\psi\sin\theta\sin\phi+\cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi+\cos\psi\sin\phi \\ \sin\theta & -\cos\theta\sin\phi & \cos\theta\cos\phi \end{array}\right]_{\phi=\theta=\psi=\theta_{rot}} \tag{36}$$

and n_i is a vector of zero-mean Gaussian white noise measurement errors on the components of n_i. For simplicity, the measurement noise is assumed to have equal magnitude in every direction. We denote it's equivalent angular size by θ_n. In all cases, the measurement vectors thus produced are normalized to have unity 2-norm. The parameter θ_{rot} represents the size of the true attitude rotation.

The simulation results are summarized in Table 3. All algorithms were simulated with $n_i = 0$, to assess the effect of round-off errors, and with $n_i \neq 0$ to assess estimation accuracy in the presence of sensor noise (and round-off error). Algorithm accuracy is computed by

$$\varepsilon_{all} = \left[(\hat{\phi} - \theta_{rot})^2 + (\hat{\theta} - \theta_{rot})^2 + (\hat{\psi} - \theta_{rot})^2 \right]^{1/2} \tag{37}$$

Of course, when no measurement noise is included, the algorithm need be executed only once; however, when measurement noise is included, we execute the algorithm 250 times to capture the statistical effects and then take RMS values. When $n_i = 0$ (i.e. $\theta_n = 0$), this gives the effect of round-off errors; when $n_i \neq 0$, the metric of (37) is a measure of total estimation error. We stress that the numerical results are merely experimental, not validated by any rigorous numerical analysis.

Table 3 - Simulation Results : Algorithm Accuracies

CASE	ROTATION SIZE θ_{rot}, degrees	MEASUREMENT NOISE LEVEL θ_n, degrees	ESTIMATION ERROR, ε_{all}, degrees			
			SMALL ANGLE LEAST SQUARES	GENERAL QUEST	SMALL ANGLE QUEST	FOAM
A	0	0	2.0×10^{-6}	0.0	0.0	0.0
B	0.001	0	3.1×10^{-8}	4.0×10^{-6}	4.0×10^{-6}	2.5×10^{-5}
C	0.01	0	3.6×10^{-6}	1.8×10^{-6}	1.6×10^{-6}	2.8×10^{-5}
D	0.1	0	2.7×10^{-4}	1.9×10^{-6}	1.5×10^{-4}	5.8×10^{-6}
E	0.1	0.0001	2.8×10^{-4}	1.4×10^{-4}	1.9×10^{-4}	1.4×10^{-4}
F	0.1	0.01	0.014	0.014	0.014	0.014
G	0	0.001	0.0014	0.0014	0.0014	0.0014
H	0	0.01	0.014	0.014	0.014	0.014

The results summarized in Table 3 are representative of a fairly wide range of attitude errors and noises, although all would be considered "small-angle" cases. We see that, for these cases, the least-squares algorithm is less affected by finite precision arithmetic than the other algorithms (except, mysteriously, in Case A). Its accuracy is comparable with the other algorithms in the presence of sensor noise. So, from Table 3, we conclude that for this special case, the least squares algorithm provides accuracy essentially identical to QUEST and FOAM.

The FOAM algorithm appears to be most susceptible to round-off error and, thus, its estimation accuracy degrades as measurement noise decreases.

6. CONCLUSION

A number of algorithms exist for solving the attitude determination problem. In this paper we introduce a new suboptimal least-squares algorithm for the solution of Wahba's problem. It is especially attractive for cases where attitude rotations are known to be small. In the small-angle case, the least-squares algorithm is quite efficient and can be performed with numerically stable operations. The computational requirements of the least squares algorithm are compared with other well-known algorithms for the two and three measurement cases. If the angle of rotation is known *a priori* to be small, small angle QUEST is the most efficient algorithm, followed closely by the least squares algorithm. In the general n-measurement, large-angle case, QUEST and FOAM are still preferable as they are fast optimal algorithms which have been shown to work well in practice. Finally, we introduce a measurement averaging technique which reduces the number of vector measurements that any algorithm must process. Performance degradations using the averaging technique are small as long as at least two sensors with large angular separations and similar accuracies are available.

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APPENDIX - ALGORITHM OPERATIONS COUNTS

In this appendix, we present the details of some of the operations counts for the case of two measurements. These results are presented in Table 1 of the main body of this paper. This appendix is included to justify the claims of Table 1, document the assumptions used to do the operations counts for the various algorithms, and illustrate the general method used to obtain the results of Table 3 of the main body. It is not exhaustive due to lack of space.

Small Angle Least Squares Count

The operations count for the least squares solution using a QR decomposition is taken from Stewart [15] who provides a count of the number of multiplies required by each algorithm; we assume the same number of additions are required. The results are shown in Table A1.

Table A1 - Small Angle Least Squares Operations Count

OPERATION TYPE	additions	multiplies	sqrt
QR Decomposition (Stewart [15], Alg. 3.8, p. 236)	36	36	3
Use QR for Least Squares (Stewart [15], Alg. 3.7, p. 234, three times)	30	30	0
Solve upper triangular system	3	6	0
form "measurement"	6	0	0
TOTAL	81	78	3

The calculations required to solve an upper triangle 3×3 system were counted by hand. The form "measurement" operation of the last row is the calculation required to form $\bar{S}$ of Equation (15) in the main body of this paper.

Small Angle QUEST Operations Count

The bulk of the computations required for small angle QUEST are those involved with solving a 3×3 system of linear equations. We assume that Gaussian elimination is used for which Stewart [15] provides a count of the multiplies (we assume the number of additions is the same). The calculations are summarized in Table A2. The notation is from [5].

Table A2 - Small Angle QUEST Operations Count

OPERATION TYPE	additions	multiplies	sqrt
form S_0	9	24	0
calculate $A \equiv 2I - S_0$	3	0	0
form Z	9	18	0
solve $Ax = z$ using gaussian elimination (Stewart [15], Alg. 1.3, p. 131)	18	18	0
form $\bar{q}$	3	8	1
multiply elements by 2	0	3	0
TOTAL	42	71	1

General QUEST Operations Count

The general QUEST operations count is summarized in Table A3 below. The notation in the table is consistent with [5].

Table A3 - General QUEST Operations Count

OPERATION TYPE	additions	multiplies	sqrt
form S 4 times	36	0	0
form σ 4 times	8	0	0
form $\det(S)$ 4 times	20	36	0
form κ 4 times	48	72	0
form ω 4 times	16	8	0
form Z	9	18	0
form $\sigma^2 - \kappa$	1	0	0
form $\sigma^2 + Z^T Z$	3	3	0
form $\delta + Z^T SZ$	9	12	0
form $Z^T S^2 Z$	14	15	0
one newton-raphson iteration	10	4	0
form α, β, ω	5	2	0
form $(\alpha I + \beta S + S^2)Z$	36	33	0
form $\bar{q}$	2	4	1
post rotate	0	2	0
TOTAL	217	209	1

A number of the operations are done four times to find and implement pre-rotations which avoid the 180° rotation case.

FOAM Operations Count

Table A4 below summarizes the opeations count for the FOAM algorithm. Notation in the table is consistent with [9].

Table A4 - FOAM Operations Count

OPERATION TYPE	additions	multiplies	sqrt
forma λ	1	1	1
form κ	1	2	0
form ζ	0	1	0
form $\|B\|_F^2$	8	9	0
form $\|adj(B)\|_F^2$	17	27	0
form $(\kappa + \|B\|_F^2)\zeta$	1	10	0
form $\lambda adj(B^T)\zeta$	0	10	0
form $BB^T B$	36	54	0
divide above by ζ	0	9	0
add and subtract three matrices	18	0	0
TOTAL	81	112	1

For the two measurement case, no Newton-Raphson iteration is required since λ admits a closed form solution in this case.