CONVERGENCE RATE ENHANCEMENT OF NAVIER-STOKES CODES ON CLUSTERED GRIDS

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SUMMARY

Our Sensitivity-Based Minimal Residual (SBMR) method which is based on our earlier Distributed Minimal Residual (DMR) method allows each component of the solution vector in a system of equations to have its own convergence speed. Our global SBMR method was found to consistently outperform the DMR method while requiring considerably less computer memory. Recently, we have developed and tested a new Line SBMR or LSBMR method and a Time-Step-Scaling (TSS) method that are even more robust and computationally efficient than our global SBMR method especially on highly clustered computational grids in laminar and turbulent flow computations.

Distributed Minimal Residual (DMR) Method

This method predicts an optimum amount of correction to each component of the solution vector by combining the information from several previous iteration levels. Each of the corrections obtained from the past iterations is multiplied by a different weighting factor and these weighting factors are determined so that they minimize the overall future residual. Although it is based on general Krylov subspace methods, the DMR method [1] differs from them by the fact that weighting factors are different from one variable to another in the system.

Global Sensitivity-Based Minimal Residual (SBMR) Method

The residual at a grid point depends on the solution vector **Q** at the neighboring grid points including the point itself. The sensitivity of residual R_m (m=1,2,3: number of equations) with respect to Q_k (k=typical neighboring points) is $\frac{\partial R_m}{\partial Q_k}$. That sensitivity can be determined from the finite difference equation used in the scheme. For a two-dimensional incompressible flow solved using Chorin's artificial compressibility method the solution vector is $\mathbf{Q} = \{p\ u\ v\}^T$. Suppose we have calculated the solution vector **Q** at iteration levels up to t+n where n is the number of regular iteration steps between two iteration levels. Then the change in the solutions between the iteration levels can be written as follows.

$$\Delta p_k = (p_k)^{t+n} - (p_k)^t \qquad \Delta u_k = (u_k)^{t+n} - (u_k)^t, \qquad \Delta v_k = (v_k)^{t+n} - (v_k)^t, \qquad (1)$$

Using the first two terms in a Taylor series expansion in artificial time direction, each residual for a two-dimensional system after n iterations will be

$$R_m^{t+n} = R_m^t + \left[\sum_k \frac{\partial R_m^t}{\partial p_k}\Delta p_k\right] + \left[\sum_k \frac{\partial R_m^t}{\partial u_k}\Delta u_k\right] + \left[\sum_k \frac{\partial R_m^t}{\partial v_k}\Delta v_k\right] \quad ; m=1,2,3 \qquad (2)$$

Similarly, future residual at t = (t+n)+1 can be approximated by

$$R_m^{(t+n)+1} = R_m^t + \left[\sum_k \frac{\partial R_m^t}{\partial p_k}\Delta p_k\right]\alpha_p + \left[\sum_k \frac{\partial R_m^t}{\partial u_k}\Delta u_k\right]\alpha_u + \left[\sum_k \frac{\partial R_m^t}{\partial v_k}\Delta v_k\right]\alpha_v \qquad (3)$$

Here, α's are the factors that multiply Δ's to estimate the future solution vector so that it minimizes the future residual. Then, $(p_k)^{(t+n)+1} = (p_k)^t + \alpha_p \Delta p_k$ with similar expressions for u and v [2,3,4,5].

Line SBMR (LSBMR) Method

The SBMR method calculates the same α's for the entire computational domain, which cannot represent optimum α's for both coarse grid regions and fine grid regions. Therefore, a natural conclusion is to allow α's to have different values in the clustered regions. A modified formulation (LSBMR) was developed [3,5] to allow these α's to have different values from one grid line to another. The LSBMR formulation will be explained using the two-dimensional, incompressible flow Navier-Stokes equations as an example. The system has three equations (rmax=3) and three unknowns (M=3), that are p, u and v. The acceleration coefficients for those unknowns are α_p, α_u and α_v, respectively. If the grid lines are clustered in the j-direction, then each j = constant grid line will be assigned its own set of three constant α's. The residual at a grid point (i,j) incorporates α's at the neighboring grid points plus the point itself. For the given Navier-Stokes system, this yields

$$R_r^{(t+n)+1} = R_r^t + \sum_{s=i-1}^{i+1}\left[\frac{\partial R_r^t}{\partial p_{s,j-1}}\Delta p_{s,j-1}\alpha_p^{j-1} + \frac{\partial R_r^t}{\partial p_{s,j}}\Delta p_{s,j}\alpha_p^{j} + \frac{\partial R_r^t}{\partial p_{s,j+1}}\Delta p_{s,j+1}\alpha_p^{j+1}\right]$$

$$+ \sum_{s=i-1}^{i+1}\left[\frac{\partial R_r^t}{\partial u_{s,j-1}}\Delta u_{s,j-1}\alpha_u^{j-1} + \frac{\partial R_r^t}{\partial u_{s,j}}\Delta u_{s,j}\alpha_u^{j} + \frac{\partial R_r^t}{\partial u_{s,j+1}}\Delta u_{s,j+1}\alpha_u^{j+1}\right]$$

$$+ \sum_{s=i-1}^{i+1}\left[\frac{\partial R_r^t}{\partial v_{s,j-1}}\Delta v_{s,j-1}\alpha_v^{j-1} + \frac{\partial R_r^t}{\partial v_{s,j}}\Delta v_{s,j}\alpha_v^{j} + \frac{\partial R_r^t}{\partial v_{s,j+1}}\Delta v_{s,j+1}\alpha_v^{j+1}\right] \qquad (4)$$

On each j = constant line, three values (M=3) of constant a's are determined so as to minimize the L-2 norm of the future global residual:

$$2\sum_{D}\sum_{r=1}^{3}\left(R_r^{(t+n)+1}\frac{\partial R_r^{(t+n)+1}}{\partial \alpha_p^j}\right)=0 \quad 2\sum_{D}\sum_{r=1}^{3}\left(R_r^{(t+n)+1}\frac{\partial R_r^{(t+n)+1}}{\partial \alpha_u^j}\right)=0 \quad 2\sum_{D}\sum_{r=1}^{3}\left(R_r^{(t+n)+1}\frac{\partial R_r^{(t+n)+1}}{\partial \alpha_v^j}\right)=0 \tag{5}$$

for j = 1, 2,· · ·, jmax. If jmax is the total number of j = constant grid lines, then substituting the equation (4) into (5) results in (jmax) x M algebraic equations for the same number of unknown optimum α's. For a given j = constant grid line when using central differencing in j-direction, α^j appears only in $R_m^{(t+n)+1}(i,j-1)$, $R_m^{(t+n)+1}(i,j)$ and $R_m^{(t+n)+1}(i,j+1)$. Therefore, we can see that the summation over the entire domain leaves the terms only with α^{j-2}, α^{j-1}, α^j, α^{j+1} and α^{j+2} summed along the j = constant line. Now we have

$$2\sum_{i=1}^{imax}\sum_{r=1}^{3}\left[R_r^{(t+n)+1}(i,j-1)\frac{\partial R_r^{(t+n)+1}(i,j-1)}{\partial \alpha_m^j}\right]+2\sum_{i=1}^{imax}\sum_{r=1}^{3}\left[R_r^{(t+n)+1}(i,j)\frac{\partial R_r^{(t+n)+1}(i,j)}{\partial \alpha_m^j}\right]$$

$$+2\sum_{i=1}^{imax}\sum_{r=1}^{3}\left[R_r^{(t+n)+1}(i,j+1)\frac{\partial R_r^{(t+n)+1}(i,j+1)}{\partial \alpha_m^j}\right]=0 \tag{6}$$

for j = 1,2,· · ·,jmax and m = p, u, v. In this example the simultaneous system of equations (6) yields a block penta-diagonal matrix equation for (jmax) x 3 optimum α's where each block is a 3 x 3 matrix. In the general case of a two-dimensional system having M partial differential equations, the block penta-diagonal system (6) will have blocks of size M x M. Superior performance of LSBMR is demonstrated for laminar (Fig. 1) and turbulent (Fig. 2) flows.

Time Step Scaling (TSS) Technique

A simple and efficient additional convergence acceleration technique called the Time Step Scaling (TSS) was developed [3,5] for the steady-state solution of the incompressible flow on highly-clustered grids. This technique can be considered as an extension of the local time-stepping so that time steps can be different for each equation and at each grid point. In the highly-clustered grid regions, the grid spacing in the direction of grid clustering is very small and it limits the magnitude of $\Delta\tau$ to be extremely small for stability reasons (time step is proportional to the square of the grid spacing). On the other hand, the Jacobian, J, becomes large in the clustered regions. However, the

product $J\Delta\tau$ is orders of magnitude smaller in the fine grid region than in the coarse grid region. Since the transient solutions are not of interest, one may desire to use larger time steps for some equations in the highly-clustered region for faster convergence. A close investigation of the convergence characteristics of each equation reveals that it is the slow evolution of pressure field in the fine-grid region which slows the overall convergence. Therefore, the time step for the continuity equation is scaled such that $J\Delta\tau$ in the clustered regions has the same order of magnitude as that in the coarse grid regions. To do this, the maximum value of $J\Delta\tau$ along the grid line in the direction of grid clustering is located and the time step at each grid point along this grid line is scaled as follows

$$\Delta\tau(i,j) = c\,\frac{1}{J(i,j)}\left(J\Delta\tau\right)_{max} \tag{7}$$

where $\Delta\tau(i,j)$ is the calculated time step without the TSS, $J(i,j)$ is the Jacobian, c is a user specified coefficient (typically 0.8) and $\left(J\Delta\tau\right)_{max}$ is the maximum value of $J\Delta\tau$ along the grid line considered. For highly-clustered grids, magnitude of $J\Delta\tau$ in the coarse grid region is orders of magnitude larger than that in the fine-grid region. The TSS technique was found to significantly accelerate the convergence when highly clustered grids are used (Figures 1 and 2).

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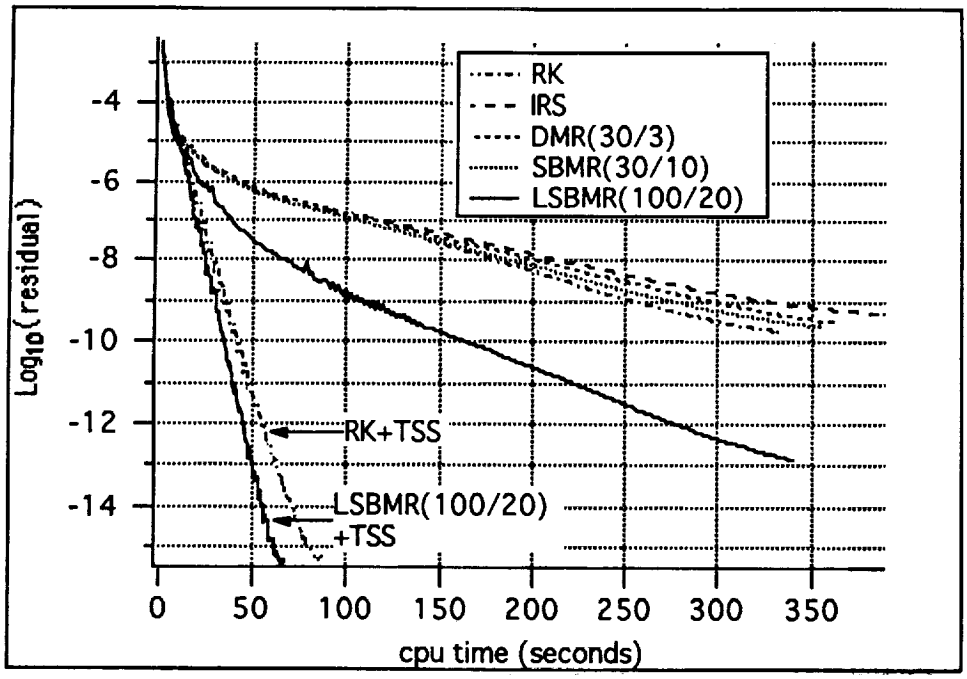

Figure 1 Convergence histories for a 2-D straight channel flow including the TSS technique. (L/H = 5, Re = 1600, 60x60 grid cells, $AR_{max} = 10^4$, $\beta = 5$, CFL = 2.8, von Neumann = 0.4)

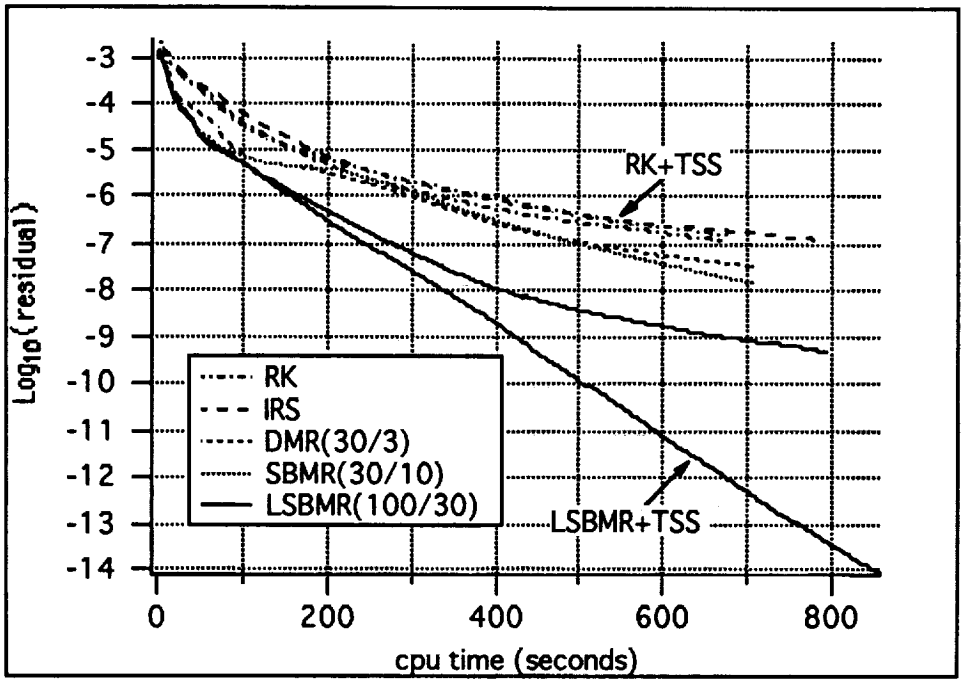

Figure 2. Convergence histories for a 2-D straight channel flow including the TSS technique. (L/H = 10, Re = 1.6×10^6, 60x120 grid cells, $AR_{max} = 10^4$, $\beta = 5$, CFL = 2.0, von Neumann = 0.4)