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Direct numerical simulation of incompressible axisymmetric flows

By Patrick Loulou[1]

1. Motivation and objectives

In the present work, we propose to conduct direct numerical simulations (DNS) of incompressible turbulent axisymmetric jets and wakes. The objectives of the study are to understand the fundamental behavior of axisymmetric jets and wakes, which are perhaps the most technologically relevant free shear flows (e.g. combuster injectors, propulsion jet). Among the data to be generated are various statistical quantities of importance in turbulence modeling, like the mean velocity, turbulent stresses, and all the terms in the Reynolds-stress balance equations. In addition, we will be interested in the evolution of large-scale structures that are common in free shear flows.

The axisymmetric jet or wake is also a good problem in which to try the newly developed b-spline numerical method. Using b-splines as interpolating functions in the non-periodic direction offers many advantages. B-splines have local support, which leads to sparse matrices that can be efficiently stored and solved. Also, they offer spectral-like accuracy and are C^{O-1} continuous, where O is the order of the spline used; this means that derivatives of the velocity such as the vorticity are smoothly and accurately represented. For purposes of validation against existing results, the present code will also be able to simulate internal flows (ones that require a no-slip boundary condition). Implementation of no-slip boundary condition is trivial in the context of the b-splines.

2. Accomplishments

To simulate these flows, we follow the procedure described in Moser *et al.* (1983) and Leonard *et al.* (1982), with b-splines replacing the Jacobi or Tchebyshev polynomials used in the radial direction.

2.1 Navier-Stokes equation

The starting point for the simulations are the incompressible Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \omega = -\nabla P - \frac{1}{Re}\Delta \mathbf{u} \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{1b}$$

Let $\mathbf{v}$ be our numerical approximation to $\mathbf{u}$, which will consist of a truncated expansion in terms of divergence-free vector functions (i.e. $\nabla \cdot \mathbf{v} = 0$). Furthermore,

1 Department of Aeronautics and Astronautics, Stanford University

let ξ be any vector function representable by another finite set of divergence-free vector functions ($\nabla \cdot \xi = 0$), satisfying $\xi = 0$ on ∂G, the boundary of the domain. By substituting $\mathbf{v}$ for $\mathbf{u}$ in (1a) and using the standard weighted residual technique with ξ as the weight functions, we obtain the discrete weak form of the Navier-Stokes equations.

$$\int_G \xi \cdot \frac{\partial \mathbf{v}}{\partial t} dV - \int_G \xi \cdot \mathbf{v} \times \omega dV = -\frac{1}{Re} \int_G \xi \cdot \Delta \mathbf{v} dV \tag{2}$$

By enforcing (2) for each ξ making up a basis for the weight functions, a coupled system of ordinary differential equations for the coefficients in the expansion for $\mathbf{v}$ are obtained, which can be solved using standard time-advance techniques (see sect. 3). This formulation has the advantage of automatically satisfying the continuity equation (1b) and eliminating the pressure.

2.2 Velocity representation and vector shape functions

Given the formulation in (2), all that remains is to select the basis vectors to represent $\mathbf{v}$ and ξ. This is facilitated by approximating the spatially developing jet or wake of interest as a time developing flow. In this case, the streamwise (axial or z) direction is homogeneous and we can approximate the flow as periodic in z with period L_z. This and the natural periodicity in θ allow the following representation of $\mathbf{v}$ and ξ:

$$\mathbf{v}(r,\theta,z,t) = \sum_j \sum_{k_z} \sum_l \alpha_{jml}(t) \mathbf{u}_l(r;j,k_z) e^{ij\theta} e^{ik_z z} \tag{3}$$

$$\xi_{j'm'l'}(r,\theta,z) = \mathbf{w}_{l'}(r;j',k_z') e^{-ij'\theta} e^{-ik_z' z} \tag{4}$$

where

$$k_z = \frac{2\pi m}{L_z}, \quad -J \le j \le J, \quad -M \le m \le M$$

Due to the continuity constraint, there are only two degrees of freedom associated with each Fourier/b-spline mode. It is therefore convenient to divide the expansion and weight vectors, $\mathbf{u}_l$ and $\mathbf{w}_{l'}$, into two distinct classes of vectors ($\mathbf{u}_l{}^+$ and $\mathbf{u}_l{}^-$) and ($\mathbf{w}_{l'}{}^+$ and $\mathbf{w}_{l'}{}^-$), with coefficients $\alpha^+_{jml}(t)$ and $\alpha^-_{jml}(t)$.

The expansion and weight vectors must be constructed such that they are divergence free and have the proper *regularity* properties at $r = 0$ (Shariff 1993). The following vectors meet both these requirements provided that $g_l(r)$ satisfy the appropriate condition at $r = 0$ (see below):

$$\mathbf{u}_l{}^+(r;j,k_z) = \begin{pmatrix} u_r{}^+{}_l \\ u_\theta{}^+{}_l \\ u_z{}^+{}_l \end{pmatrix} = \hat{\nabla} \times \begin{pmatrix} ig_l(r) \\ g_l(r) \\ 0 \end{pmatrix}, \quad \mathbf{u}_l{}^- = \hat{\nabla} \times \begin{pmatrix} -ig_l(r) \\ g_l(r) \\ 0 \end{pmatrix} \tag{5}$$

$$\mathbf{w}_{l'}{}^+(r;j',k_z') = \hat{\nabla}^* \times \hat{\nabla}^* \times \begin{pmatrix} -ig_{l'}(r) \\ g_{l'}(r) \\ 0 \end{pmatrix}, \quad \mathbf{w}_{l'}{}^- = \hat{\nabla}^* \times \hat{\nabla}^* \times \begin{pmatrix} ig_{l'}(r) \\ g_{l'}(r) \\ 0 \end{pmatrix} \tag{6}$$

where $\widehat{\nabla}\times$ is the Fourier tranformed curl operator and $\widehat{\nabla}^*\times$ is its complex conjugate. The $g_l(r)$ are b-splines expansion functions as described by de Boor (1978).

The above representation is incomplete when $k_z = 0$; so in this case, the following special representation is used:

$$\mathbf{u}_l{}^+ = \begin{pmatrix} \frac{-ijg_l(r)}{r} \\ g_l'(r) \\ 0 \end{pmatrix}, \qquad \mathbf{u}_l{}^- = \begin{pmatrix} 0 \\ 0 \\ g_l(r) \end{pmatrix} \tag{7}$$

$$\mathbf{w}_{l'}{}^+ = \begin{pmatrix} \frac{ij'g_{l'}(r)}{r} \\ g_{l'}'(r) \\ 0 \end{pmatrix}, \qquad \mathbf{w}_{l'}{}^- = \begin{pmatrix} 0 \\ 0 \\ g_{l'}(r) \end{pmatrix} \tag{8}$$

And when $j = 0$, these vectors are incomplete; so for that case ($k_z = j = 0$), we use:

$$\mathbf{u}_l{}^+ = \begin{pmatrix} 0 \\ g_l(r) \\ 0 \end{pmatrix}, \qquad \mathbf{u}_l{}^- = \begin{pmatrix} 0 \\ 0 \\ g_l(r) \end{pmatrix} \tag{9}$$

$$\mathbf{w}_{l'}{}^+ = \begin{pmatrix} 0 \\ g_{l'}(r) \\ 0 \end{pmatrix}, \qquad \mathbf{w}_{l'}{}^- = \begin{pmatrix} 0 \\ 0 \\ g_{l'}(r) \end{pmatrix} \tag{10}$$

All these additional vectors are also divergence-free and satisfy the regularity requirement.

In order to have the correct regularity property, depending on the azimuthal wave number j, $g_l(r)$ and some of its derivatives (Shariff 1993) must vanish when $r = 0$. Since this is not automatically satisfied by all the b-splines, some of the coefficients (α^+ and α^-) must be zero. In particular, for $j \geq 0$

$$\alpha_l^+ = 0, \qquad 1 \leq l \leq min(O, j) + 1 \quad \text{or} \quad lj \quad \text{is odd, while} \quad j + 3 \leq l \leq O + 1$$

and

$$\alpha_l^- = 0, \qquad 1 \leq l \leq min(O + 1, |j - 1|) \quad \text{or} \quad lj \quad \text{odd} \quad |j - 1| + 2 \leq l \leq O + 1$$

but with α_1^- unconstrained.

2.3 Boundary conditions

The present method is designed to treat both no-slip boundaries and potential boundaries (free shear flows).

2.3.1 No-slip boundary condition

Enforcing the boundary condition that $\mathbf{u}_l(R_2) = 0$, where R_2 is the outer edge of the domain, requires that

$$g_l(r = R_2) = 0, \qquad g_l'(r = R_2) = 0 \tag{11}$$

But if the g_l are the b-splines as defined by de Boor, (11) is not satisfied for $l = L$ or $l = L - 1$ (the two functions closest to the boundaries). Therefore, we impose

$$\alpha_L^+ = \alpha_L^- = \alpha_{L-1}^+ = \alpha_{L-1}^- = 0 \tag{12}$$

However, for the shape functions in (5), (11) also implies

$$\frac{\partial u_\theta}{\partial r} = 0 \tag{13}$$

at the boundary, which is too restrictive. To alleviate this, we augment (5) and (6) with the additional vectors (Moser *et al.*, 1983):

$$\mathbf{u}_{L-1}^0 = \begin{pmatrix} 0 \\ -2k_z g_{L-1}(r) \\ \frac{2jg_{L-1}(r)}{r} \end{pmatrix}, \qquad \mathbf{w}_{L-1}^0 = \begin{pmatrix} ij' g_{L-1}(r) \\ rg'_{L-1}(r) + g_{L-1}(r) \\ 0 \end{pmatrix} \tag{14}$$

With this, $\partial u_\theta / \partial r$ is unconstrained at the boundary.

2.3.2 Potential boundary condition

When simulating free shear flows, we follow the approach of Corral *et al.* (1993) and Sondergaard *et al.* (1994), where it is assumed that the vorticity in the flow is confined to a small region ($r < R_\omega$) which is to be computed. In the outer region of the flow ($r > R\omega$), the vorticity is zero, so the velocity is a potential ($\mathbf{u} = \nabla\phi$ and $\Delta\phi = 0$).

For each Fourier mode, the potential $\hat{\phi}$ is given by:

$$\hat{\phi}(r; j, k_z) \sim K_j(k_z r)$$

where $K_j(x)$ is the modified Bessel function of the second kind. At the boundary of the computational domain, $r = R_2 > R_\omega$, the following relations are satisfied since $\mathbf{u}$ is a potential:

$$u_\theta = \frac{ij}{qR_2} u_r, \qquad u_z = \frac{ik_z}{q} u_r, \qquad \text{where} \qquad q = k_z \frac{K'_j(k_z R_2)}{K_j(k_z R_2)} \tag{15}$$

In addition, for the vorticity to be zero at $r = R_2$, u_r must satisfy

$$\gamma \frac{\partial u_r}{\partial r} + u_r = 0, \qquad \text{where} \qquad \gamma = \frac{-K'_j(k_z R_2)}{k_z K''_j(k_z R_2)} \tag{16}$$

Given the representation in (5), to satisfy (15) and (16), the coefficients must satisfy:

$$\alpha_L^+ \left[\frac{j}{qR_2} + 1\right] + \alpha_L^- \left[\frac{j}{qR_2} - 1\right] = 0 \tag{17}$$

$$(\alpha^+_{L-1} + \alpha^-_{L-1}) - (\alpha^+_L + \alpha^-_L)\left[1 + \frac{1}{\gamma g'_L(R_2)}\right] = 0 \tag{18}$$

where we made use of the following identities:

$$g_L(R_2) = 1, \quad {g'}_{L-1}(R_2) = -{g'}_L(R_2), \qquad \frac{1}{\gamma} = \frac{1}{R_2} - \frac{j^2}{qR_2^2} - \frac{{k_z}^2}{q}$$

There are three boundary conditions in (15) and (16) but they are redundant with the continuity equation at the boundary. Since continuity is built into our expansions, only two conditions (17) and (18) are required.

For the $k_z = 0$ case, the conditions in (15) and (16) are satisfied when:

$$\alpha^-_L = 0 \tag{19}$$

$$\alpha^+_{L-1} - \alpha^+_L\left[1 + \frac{j}{R_2 g'_L(R_2)}\right] = 0 \tag{20}$$

3. Future plans

The only major parts of the code that remain to be implemented are the time integration scheme and the non-linear convective term. To time march the equations, we propose to use the method of Spalart *et al.* (1991) which is a mixed implicit/ explicit scheme. The linear viscous term is time-marched implicitly using a Crank-Nicholson scheme and the non-linear terms are time-marched explicitly using a third order Runge-Kutta scheme. To compute the non-linear term, we revert back to physical space, take the cross product, integrate exactly by Gauss quadratures (doing the integrals exactly takes care of aliasing), and revert back to wave-space.

There are two types of free shear flow simulations that are of interest. First, a fully turbulent jet that can be simulated using a turbulent pipe simulation result as the initial condition (obtained from the same code). This is similar to the turbulent mixing layer and wake simulations of Moser and Rogers (1994). Second, a transitional jet or wake simulated using different initial conditions, similar to the work of Sondergaard *et al.* (1994) on wakes.

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