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COMPUTER AIDED PROCESSING OF POLYHEDRIC CONFIGURATIONS

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INTRODUCTION

The objective of this chapter is to establish a methodology on which computer aided techniques for processing of polyhedric configurations may be based. The term 'polyhedric configuration' is used to refer to any geometric arrangement which is based on polyhedra. In particular, the focus of attention is on polyhedric configurations that are of importance in architectural and structural engineering fields.

The natural medium for processing of polyhedric configurations is a programming language that incorporates the concepts of 'formex algebra'. 'Formian' is such a programming language in which the processing of polyhedric configurations can be carried out using the standard elements of the language [1]. The term 'processing of polyhedric configurations' in the present context simply means 'creation and manipulation of polyhedric configurations'.

The actual usage of the ideas presented in this chapter is envisaged to be through a programming language such as Formian. However, the main body of the material presented is independent of any particular mathematical system or computer software. The emphasis is on the primary concepts that are fundamental for processing of polyhedric configurations in any medium.

The approach used in presenting the material is to begin by exploring the basic classes of polyhedric configurations. This is followed by a review of the properties of two families of polyhedra that are of central importance in relation to polyhedric configurations. The rest of the chapter is devoted to describing the basic procedures for processing of polyhedric configurations.

SOME BASIC POLYHEDRA

Polyhedra have been the subject of fascination and interest since the ancient times. They have been studied throughout the ages by mathematicians, philosophers and artists and they play an important role in a number of branches of science and technology.

The interest in polyhedra in this chapter stems from the fact that they provide a basis for the generation of a number of important classes of structural forms. Examples of polyhedra that are of particular interest in the present chapter are shown in Fig 1. These are called tetrahedron, octahedron, dodecahedron, icosahedron and cuboctahedron, where

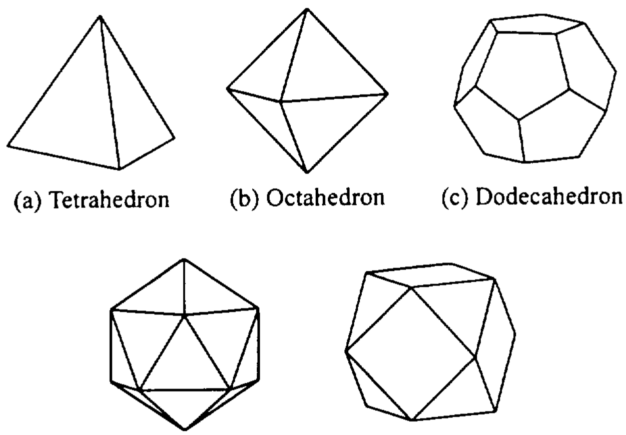
- tetrahedron has four triangular faces,
- octahedron has eight triangular faces,
- dodecahedron has twelve pentagonal faces,
- icosahedron has twenty triangular faces and
- cuboctahedron has eight triangular faces and six square faces.

MAPPING ONTO FACES OF POLYHEDRA

The first class of polyhedric configurations to be considered is obtained by placing objects onto the faces of polyhedra. For example, the configuration shown in Fig 2(d) is a polyhedric configuration that is obtained by placing a triangulated pattern on five faces of an icosahedron. A configuration that is used for mapping onto the faces of a polyhedron is referred to as a 'face-object'. The face-object in the example under consideration is shown in Fig 2(a). Also, the faces of the icosahedron that are to be mapped onto are shown in Fig 2(b). These faces are shown again in Fig 2(c) with one of them having the face-object placed onto it. The complete arrangement with the face-object mapped onto all five faces is shown in Fig 2(d). In the above description of the procedure for obtaining a polyhedric configuration the terms 'mapping' and 'placing' have been used interchangeably. This is appropriate since 'mapping' in the present context simply means 'placing'.

Another example of a polyhedric configuration is shown in Fig 2(f). This configuration is obtained using the same procedure as described above. However, in this case, a different face-object is used for mapping. The new face-object is shown in Fig 2(e) and the boundaries of one of the faces of the icosahedron on which the face-object is mapped are shown in dotted lines in Fig 2(f).

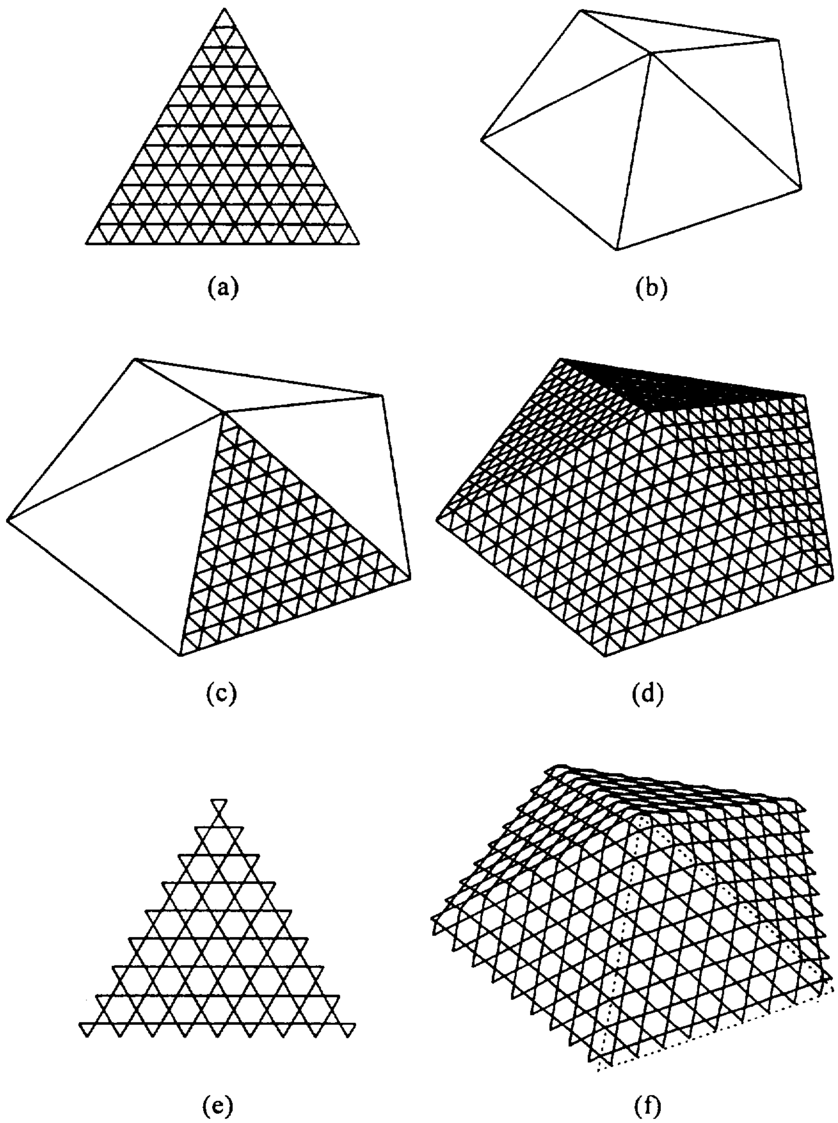
Further examples of polyhedric configurations that are obtained by face mapping are shown in Fig 3. The configurations shown in Figs 3(a) to 3(d) are obtained by mapping different face-objects onto five faces of an icosahedron. The point illustrated by these configurations is that face-objects are not



(a) Tetrahedron (b) Octahedron (c) Dodecahedron

(d) Icosahedron (e) Cuboctahedron

Fig 1. Some basic polyhedra



(a) (b) (c) (d) (e) (f)

Fig 2. Mapping onto faces of an icosahedron

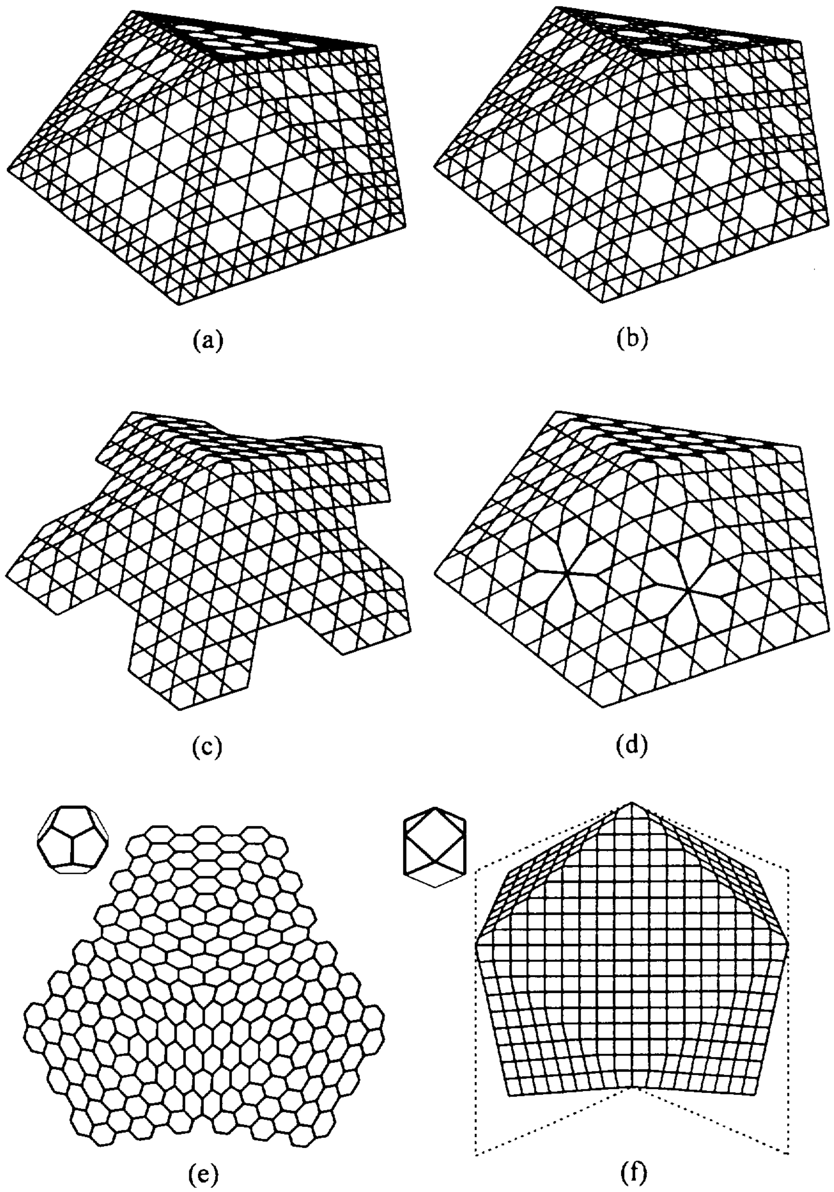


Fig 3. Mapping with different face-objects

limited to simple primary patterns and one is free to choose any required pattern for mapping. The polyhedric configuration shown in Fig 3(c) illustrates the fact that a face-object need not necessarily 'match' the boundaries of the faces onto which it is mapped. Indeed, in general, a face-object may only partially 'fill' a face or may extend beyond a face. The point illustrated by Fig 3(d) is that a polyhedric configuration may involve more than one type of face-object. In the configuration of Fig 3(d), three faces have a face-object with a uniform pattern and two faces have a face-object with openings that create a 'daisy window' effect.

Fig 3(e) shows a polyhedric configuration that is obtained by mapping a face-object onto three neighbouring faces of a dodecahedron. These faces are shown in thick lines on a small sketch at the top left corner of the figure. The face-object has a pentagonal boundary with internal hexagonal subdivisions. The polyhedric configuration of Fig 3(f) is obtained by mapping face-objects onto five faces of a cuboctahedron. These faces are shown in thick lines on a small sketch at the top left corner of the figure. A new situation is encountered here in that the faces are of different types. Namely, there are four triangular faces and one square face. This, however, does not create any problem since one can use different face-objects for different types of faces as required. The face-objects used for the polyhedric configuration of Fig 3(f) are a square shaped face-object for the top face and a triangular face-object for the four side faces. It is to be noted, however, that the triangular face-object used does not fill the side faces. This fact is indicated in Fig 3(f) by showing the actual boundaries of the side faces in dotted lines.

The polyhedric configurations shown in Figs 2 and 3 are samples of a wide variety of configurations that may be created by mapping different face-objects on faces of polyhedra. These polyhedric configurations constitute an important class of structural forms. In addition, they provide the bases for creation of geodesic forms, as will be discussed later.

MAPPING ON EDGES OF POLYHEDRA

The constitution of a polyhedron may be perceived in different ways. A tetrahedron, for example, may be regarded as a solid body with four faces, six edges and four vertices. Alternatively, it may be regarded as a 'stick arrangement' consisting of six line segments (sticks) that meet at the vertices. With this new way of visualising a tetrahedron, one can again recognise four faces, six edges and four vertices. Another way of perceiving a tetrahedron is to think of it as a basis for mapping. Thus, the tetrahedron is regarded as a 'geometric jig' that has four faces, six edges and four vertices and is used for positioning of mapping objects. This way of perceiving a polyhedron is helpful in visualising the process of mapping face-objects as described in the previous section. This point of view is also useful for visualising the mapping of objects on the edges of polyhedra. Mapping on the edges of polyhedra is the production mechanism for a major class of polyhedric configurations. Examples of this kind of configuration are shown in Fig 4.

Fig 4(c) shows a polyhedric configuration that is obtained by mapping (placing) a space truss configuration on the edges of a tetrahedron. A configuration that is used for mapping on the edges of a polyhedron is referred to as an 'edge-object'. The edge-object in the example under consideration is shown in Fig 4(a). Also, the edges of the tetrahedron with the edge-object mapped on one of them is shown in Fig 4(b). In this example, the ends of the edge-object are shaped such that when it is mapped on the edges of the tetrahedron then the ends match with one another at the vertices. Fig 4(d) shows a polyhedric configuration that is obtained by mapping a space truss configuration on the edges of an octahedron. The ends of the edge-object are again suitably shaped such that they match with one another after mapping. Fig 4(e) illustrates the fact that the mapping of an edge-object need not necessarily involve all the edges of a polyhedron. In the case of the polyhedric configuration of Fig 4(e) the edge-object is mapped on eight edges of an octahedron. In Fig 4(f) an icosahedron has been used as the basis for mapping. The edge-object is again a space truss with its ends suitably shaped. The same edge-object is used to produce the polyhedric configuration of Fig 4(g). In this case, a group of ten edges of an icosahedron is used for the operation.

The pioneering work of J F Gabriel involves a number of examples of polyhedric configurations of the type described above [2,3].

MAPPING ON VERTICES OF POLYHEDRA

The idea of mapping objects on vertices of polyhedra is a natural extension of the processes of mapping objects on the faces and edges of polyhedra. Examples of polyhedric configurations that are obtained by mapping objects on vertices of polyhedra are shown in Fig 5.

Fig 5(a) shows a polyhedric configuration that is produced by mapping (placing) a star-like object on the vertices of a tetrahedron. In this figure, the dotted lines indicate the positions of the edges of the tetrahedron. A configuration that is used for mapping on the vertices of a polyhedron is referred to as a 'vertex-object'. The vertex-object used for creation of the polyhedric configuration of Fig 5(a) is shown in Fig 5(b). A similar operation is performed to produce the configuration of Fig 5(d) using an icosahedron as the basis. The vertex-object is shown in Fig 5(c).

Fig 5(e) shows a polyhedric configuration that is obtained by mapping the vertex-object of Fig 5(f) on

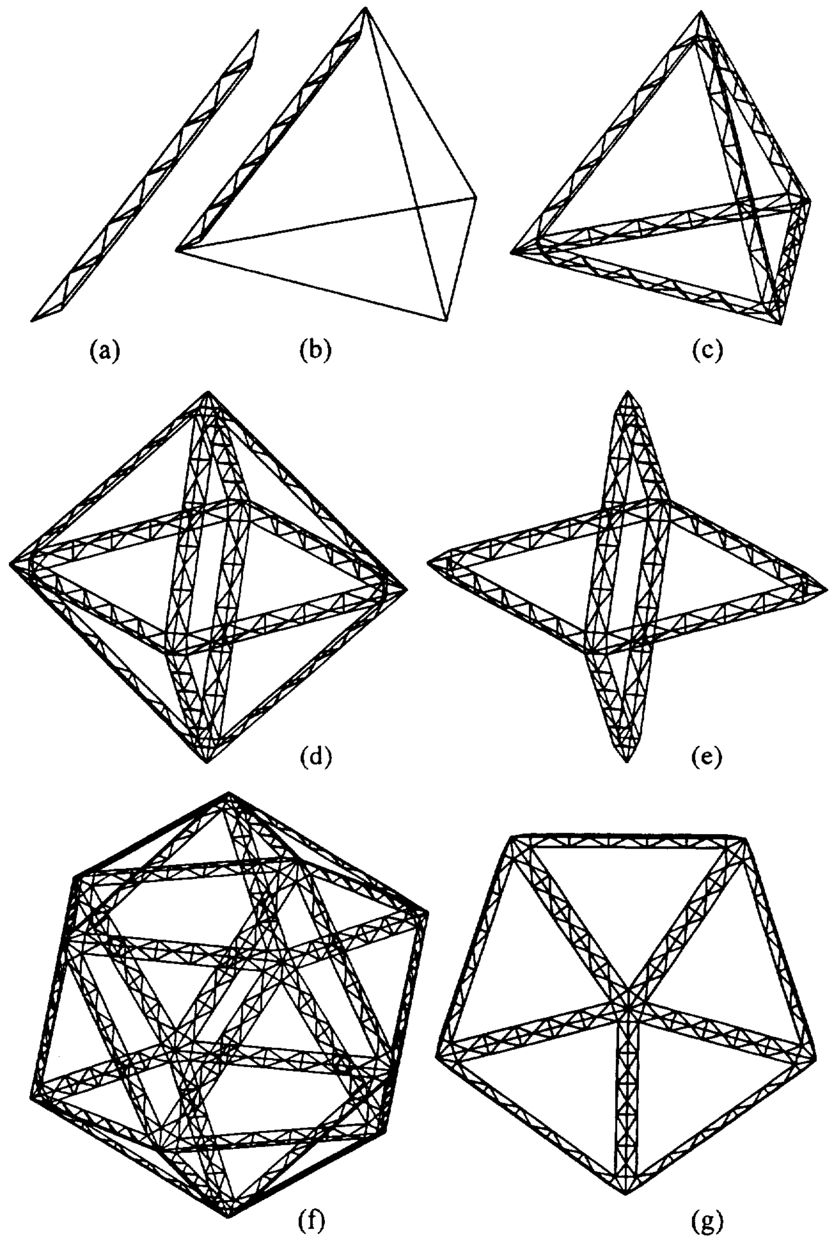


Fig 4. Mapping on edges of polyhedra

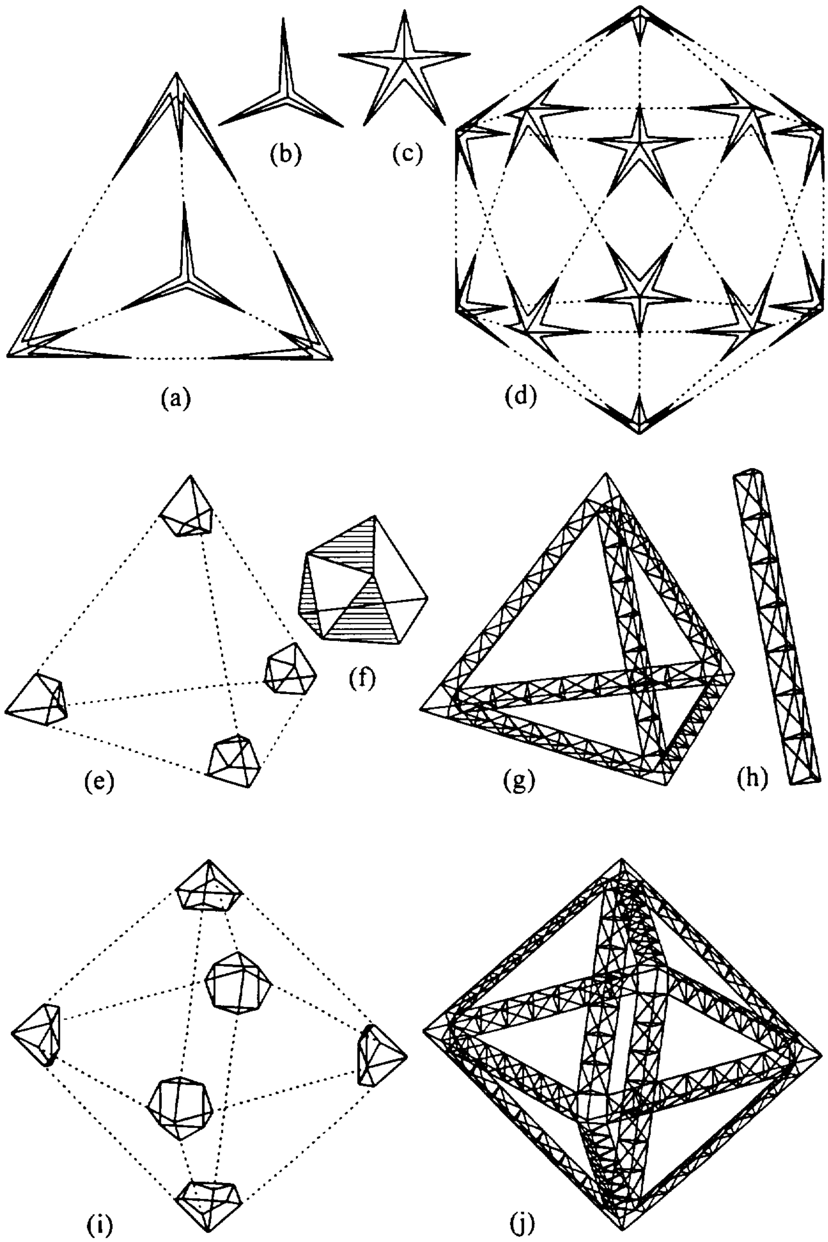


Fig 5. Mapping on vertices of polyhedra

the vertices of a tetrahedron. This vertex-object has an interesting effect. Namely, it creates 'end bases' for the edges of the tetrahedron. The facets of the vertex-object that create the end bases are shown shaded in Fig 5(f) and the significance of these end bases becomes clear in relation to the polyhedric configuration of Fig 5(g). This configuration is obtained by a combination of vertex mapping and edge mapping. To elaborate, a smaller version of the vertex-object of Fig 5(f) is mapped on the vertices of a tetrahedron. This is followed by mapping the space truss configuration of Fig 5(h) on the edges of the tetrahedron. The scale and position of this edge-object are chosen such that the ends of the space trusses fit the triangular bases created by the vertex-object. A similar procedure is followed in producing the polyhedric configuration of Figs 5(i) and 5(j). In this case, an octahedron has been used as the basis for the operation.

The technique employed to create the polyhedric configurations of Figs 5(g) and 5(j) can be of value in some practical applications. The technique provides an alternative way of dealing with the 'end matching' problem. Thus, instead of shaping the ends of the edge-object for matching at the vertices, the vertex-object is designed to act as a connecting medium. This will result in a simpler edge-object since it only requires straightforward ends.

Another point which is illustrated by Figs 5(g) and 5(j) is worth highlighting. Namely, a polyhedric configuration may involve a combination of edge and vertex mappings. Indeed, in general, there is no restriction regarding the mixing of different types of mappings and any combination of face, edge and vertex mappings may be used without any problem.

GEODESIC CONFIGURATIONS

The configuration shown in Fig 6(a) is obtained by projecting the configuration of Fig 2(d) onto the surface of a sphere. The sphere is concentric with the icosahedron on which the configuration of Fig 2(d) is based. This common centre of the sphere and icosahedron is also chosen as the centre of projection. A polyhedric configuration of the type shown in Fig 6(a) is referred to as a 'geodesic form' or 'geodesic configuration'. The same procedure is used to produce the geodesic configurations shown in Figs 6(b), 6(c) and 6(d). These configurations are obtained using the polyhedric configurations of Figs 2(f), 3(b) and 3(c) as the bases for projection.

The surface on which a geodesic form is produced need not necessarily be spherical. Indeed, a variety of different surfaces such as ellipsoids and paraboloids may be used for creation of geodesic forms. Also, the type of projection need not necessarily be central and other kinds of projection, such as parallel projection, may be used instead.

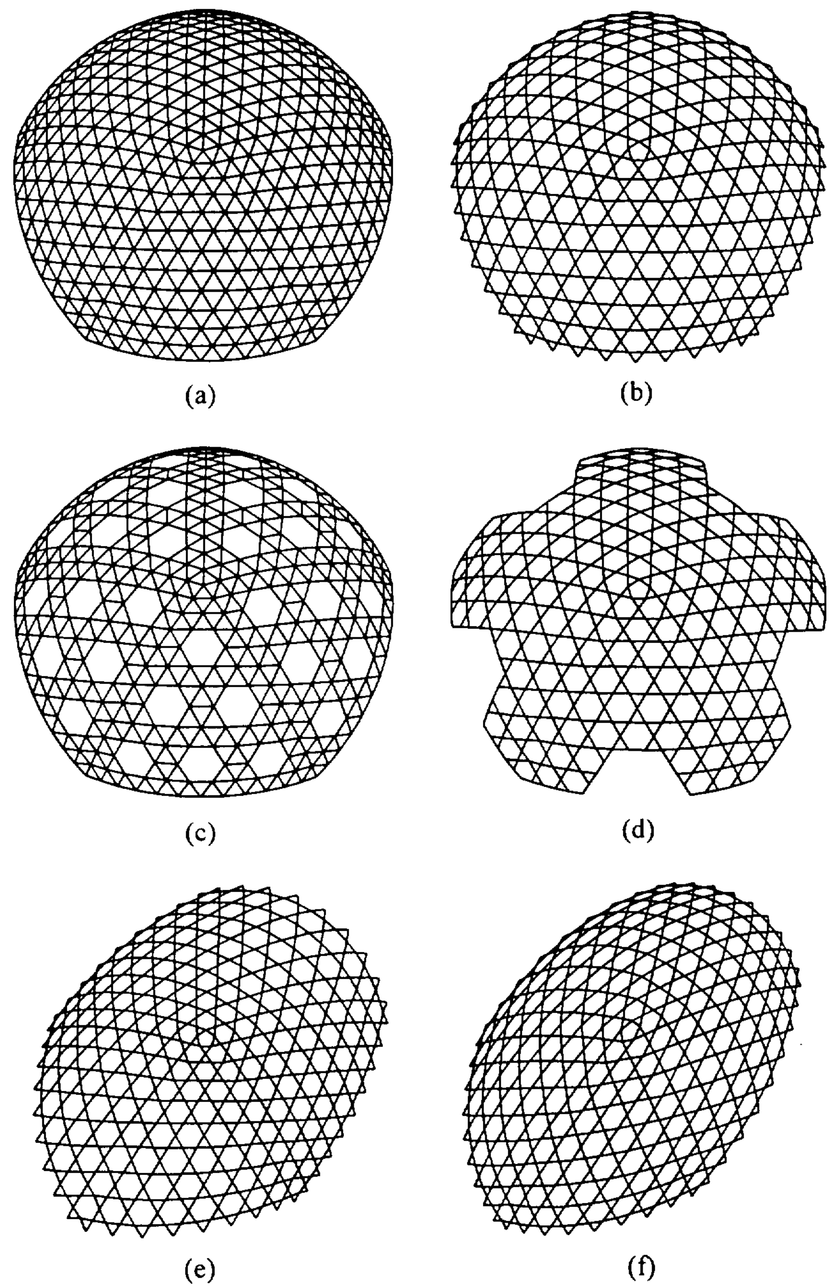


Fig 6. Some geodesic forms

Fig 6(e) shows a geodesic form that is obtained by projecting the polyhedric configuration of Fig 2(f) onto an ellipsoidal surface. A different process is involved in producing the configuration shown in Fig 6(f). This is obtained by stretching the configuration of Fig 6(b) in one direction.

The configuration shown in Fig 6(f) illustrates a point of general importance. Namely, any polyhedric configuration may be subjected to modifications and alterations to suit a particular application. In other words, there is no 'inherent' final stage in the processing of a polyhedric configuration. Like a lump of steel in the hands of a blacksmith, a polyhedric configuration may be worked, in as many stages as required, to turn it into a desired shape.

A geodesic form may involve two or more layers. For example, the geodesic configuration shown in Fig 7(b) has two layers of elements that are interconnected together by intermediate web elements. This double layer geodesic form is based on the configuration shown in Fig 7(a). This is a polyhedric configuration that is obtained by mapping (placing) a double layer face-object on five faces of an icosahedron. The geodesic form of Fig 7(b) is obtained by projecting the two layers of the configuration of Fig 7(a) onto two concentric spheres. A similar procedure is used in obtaining the double layer geodesic forms of Figs 7(d) and 7(f) from the configurations shown in Figs 7(c) and 7(e), respectively.

The projection stage in the creation of a geodesic form involves a relatively simple operation. This is true for projection on a single surface as well as projection on two or more surfaces. The reason for the simplicity of operation is that projection is a straightforward concept and can easily be dealt with through a standard computer based routine [4].

There is an abundance of structures that are constructed all over the world using various forms of geodesic configurations. These begin with the pioneering work of Buckminster Fuller and include many impressive examples [5,6].

PROCESSING OF POLYHEDRIC CONFIGURATIONS

The processing of polyhedric configurations in the pre-computer days was an extremely difficult task. In spite of this, a number of gifted designers managed to deal with the problem and create many beautiful structures based on polyhedric configurations. The constraint of the processing difficulties, however, did not allow the designers to take full advantage of the whole spectrum of possibilities and their scope remained rather limited. Even today, the processing of polyhedric configurations is mainly

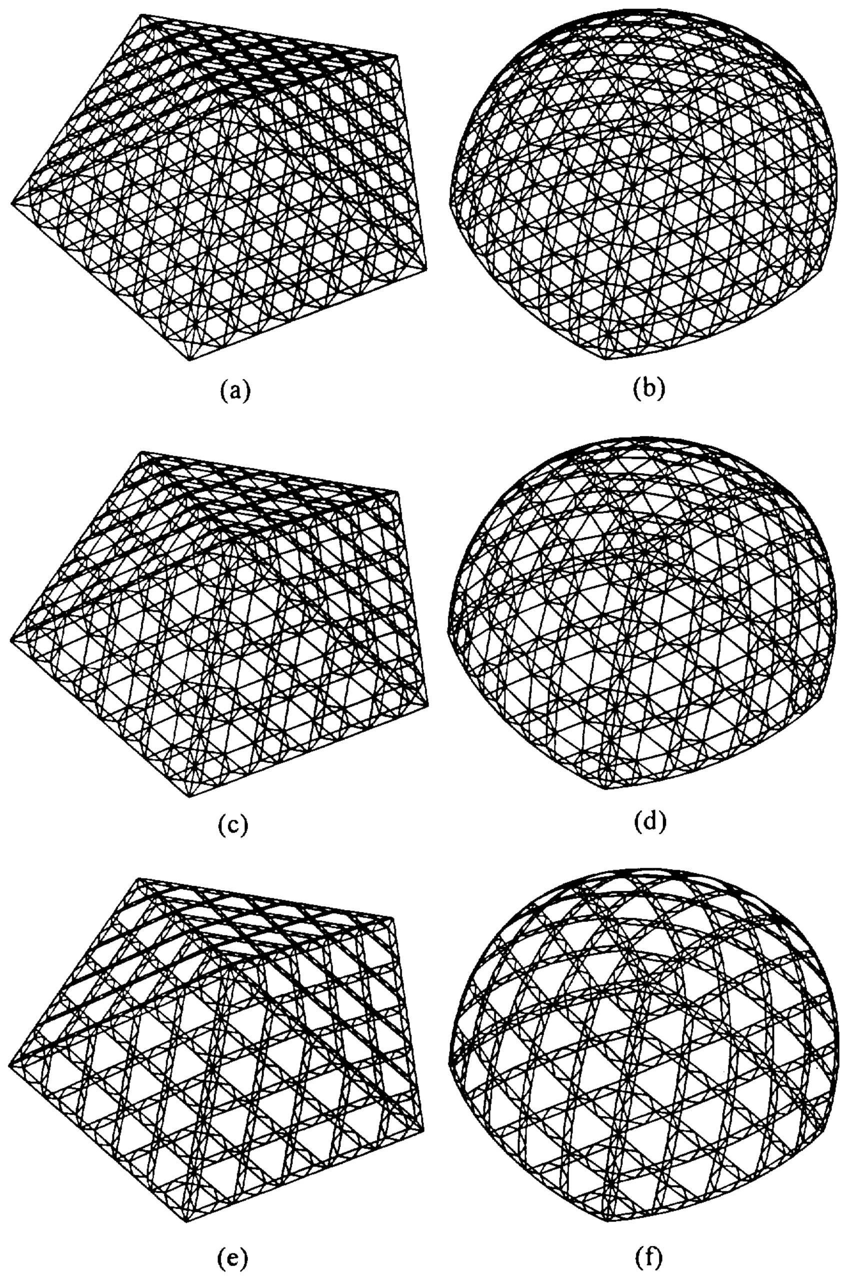


Fig 7. Some double layer polyhedric configurations

carried out using computer programs that lack generality and have many limitations and shortcomings.

In contrast, the conceptual methodology that will be presented in this chapter, combined with a suitable computer software such as Formian, provides a means for dealing with the processing of any kind of polyhedric configuration with relative ease.

One key factor in dealing with the processing of polyhedric configurations is the ability to generate the face-objects, edge-objects and vertex-objects in a convenient manner. The creation of these objects in Formian can be carried out using the concepts of formex algebra. The algebra works through concepts that effect movement, propagation, deformation and curtailment of forms, Fig 8 [1,7].

PLATONIC AND ARCHIMEDEAN POLYHEDRA

In this chapter, the use of polyhedra in the creation of structural forms is discussed in terms of Platonic and Archimedean polyhedra. There are five Platonic polyhedra whose views are shown in Fig 9. These five polyhedra were known to the ancient world before Plato and the designation 'Platonic' is due to the fact that Plato paid special attention to these polyhedra [8]. Each one of the Platonic polyhedra is a convex body with faces that are congruent regular polygons of the same type.

An Archimedean polyhedron is also a convex body with faces that are regular polygons. However, unlike the Platonic polyhedra, the faces of an Archimedean polyhedron are not all of the same type. There are fifteen Archimedean polyhedra whose views are shown in Fig 9. Each of these polyhedra has either two or three different types of faces. Archimedean polyhedra were discovered in ancient Greece and were described by Archimedes. However, the writings of Archimedes in this relation together with the knowledge of these polyhedra were lost and it was not until the Renaissance period when they were gradually rediscovered [8].

The Platonic and Archimedean polyhedra are closely related and a family tree showing the relationships between them is shown in Fig 10. This is a modified version of a family tree which has been produced by R Motro [5]. It is seen from Fig 10 that tetrahedron is the 'mother polyhedron' and all the other Platonic and Archimedean polyhedra may be derived from it. This may be done through five basic transformations that are briefly described in Fig 11. These transformations are referred to as truncation, canting, snubbing, duality and planing. Detailed general description of Platonic and Archimedean polyhedra may be found in many excellent publications, for example, see [8,9,10].

Formex algebra includes:

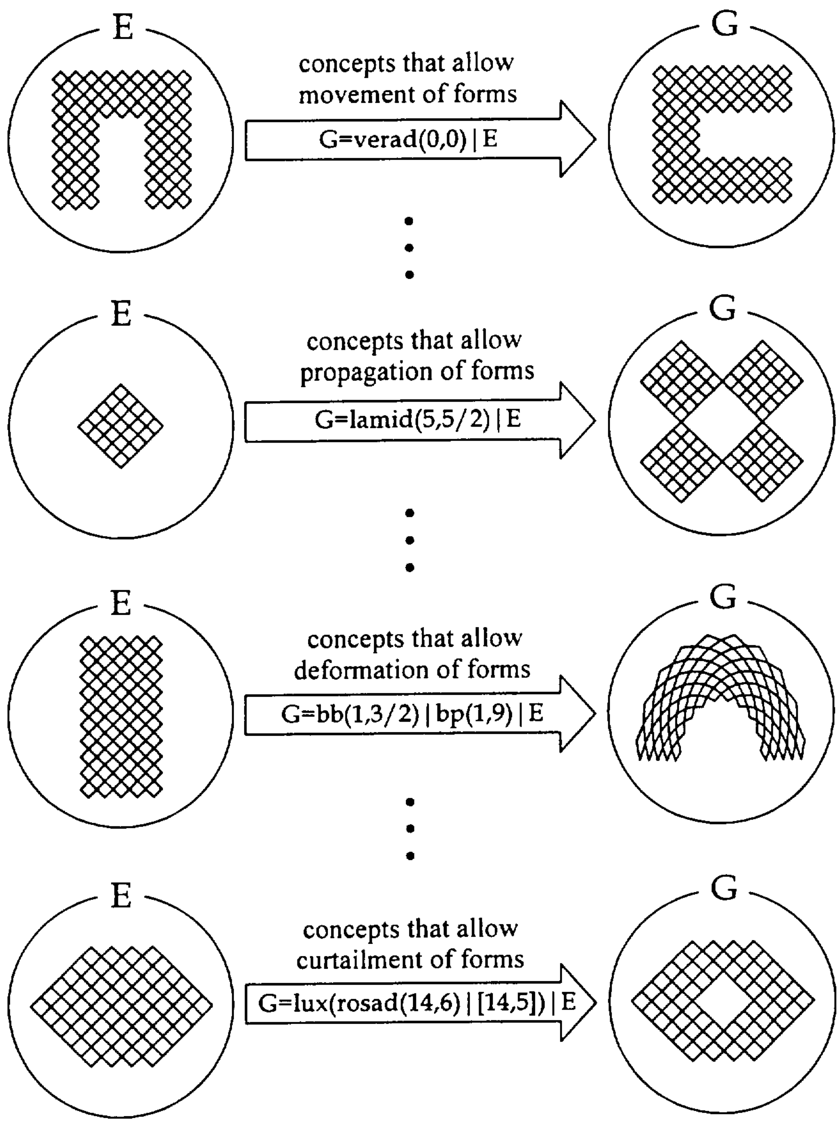


Fig 8. Basic concepts of formex algebra

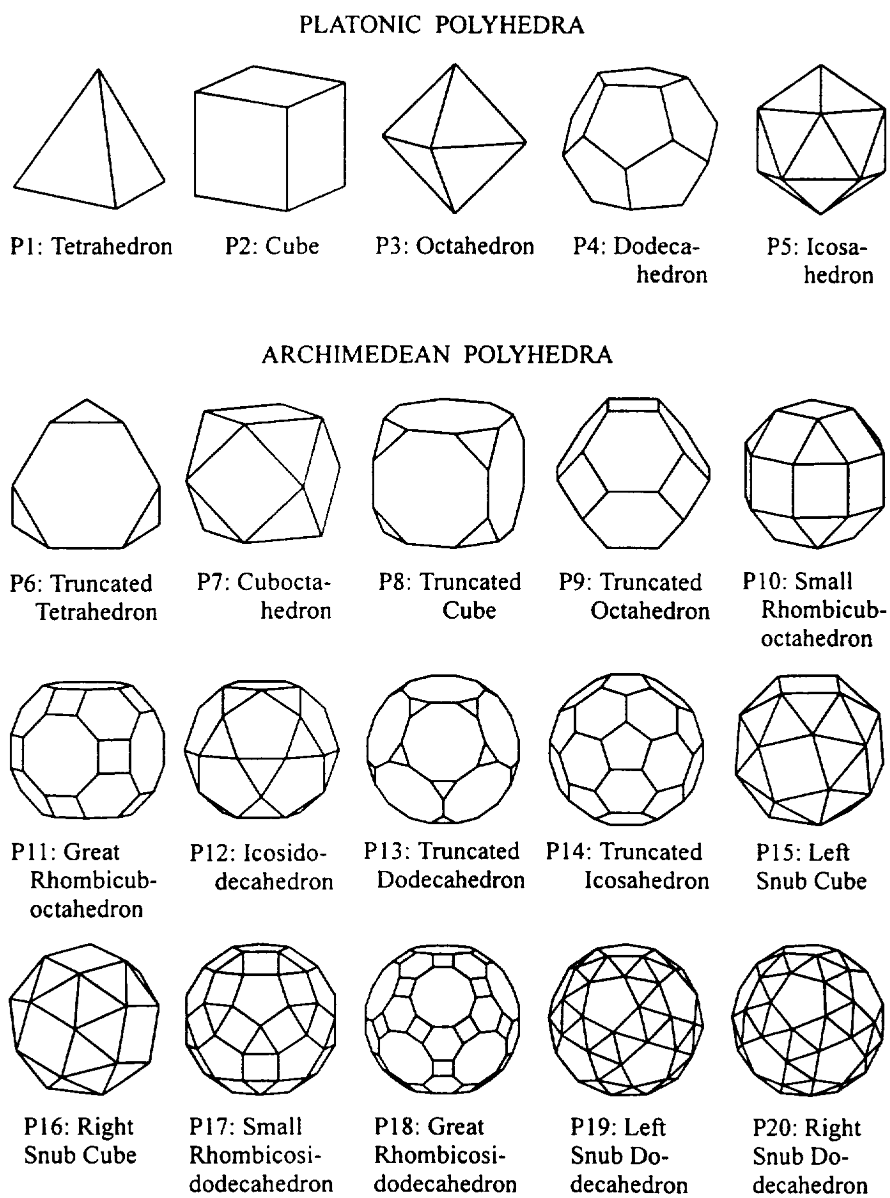


Fig 9. Platonic and Archimedean polyhedra

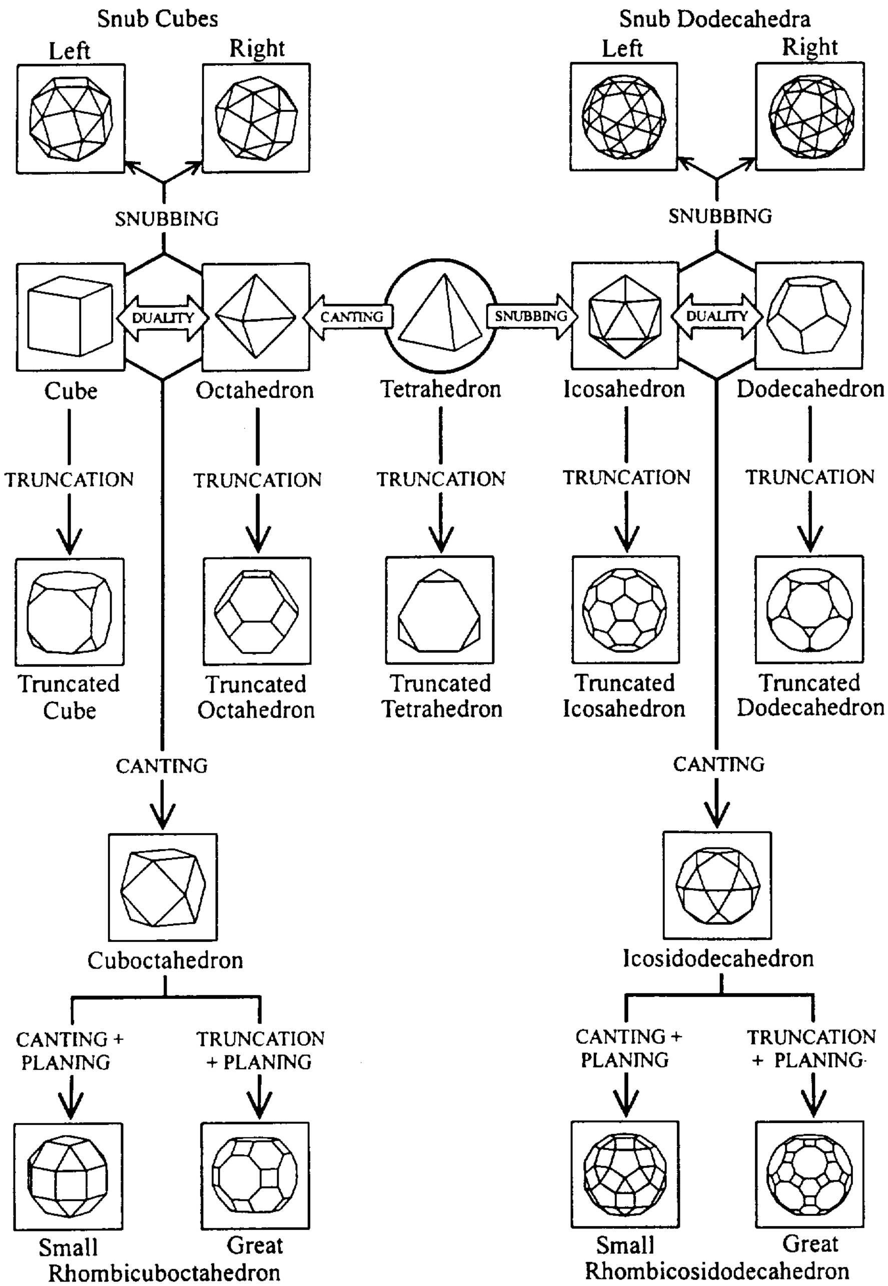
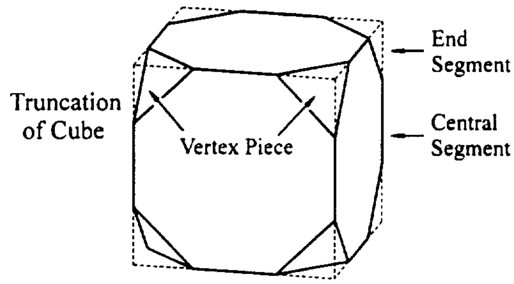
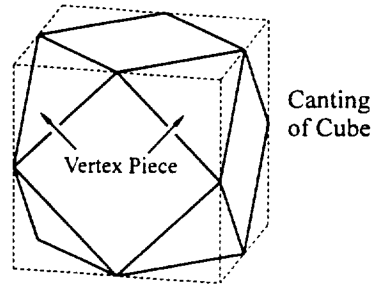


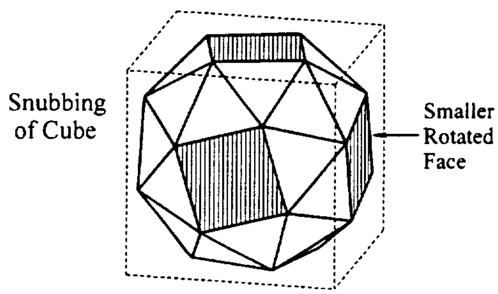
Fig 10. Family tree of Platonic and Archimedean polyhedra

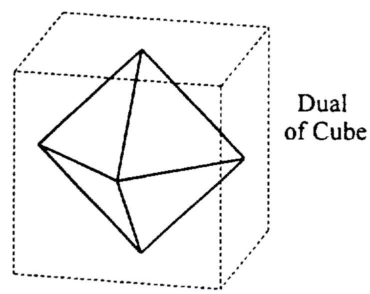




Truncation: Each edge is divided into a central segment and two end segments and the vertex pieces obtained by connecting the division points are cut off, as shown.

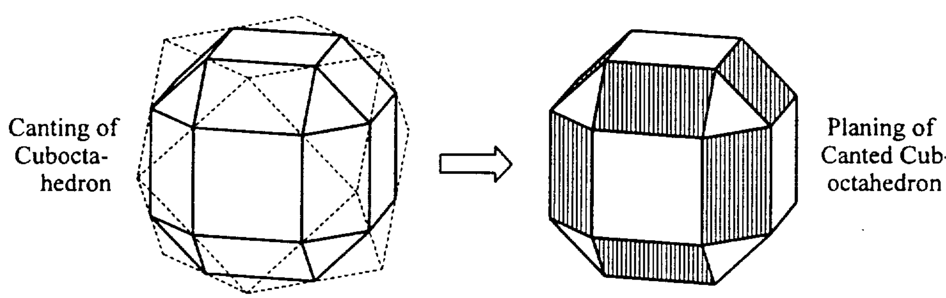
Canting: Each edge is divided into two equal segments and the vertex pieces obtained by connecting the division points are cut off, as shown.





Snubbing: A smaller rotated version of each face is placed centrally on the face (shown shaded) and the regions between the edges of the new faces are trimmed off by faceting.

Duality: The centre of each face is regarded as the vertex of another polyhedron (or each vertex is regarded as the centre of a face of a polyhedron).



Planing: The term 'planing' implies 'planing down' (scraping off) the surface of a polyhedron. For instance, in producing the small rhombicuboctahedron, a cuboctahedron is subjected to canting, as shown above. This will give rise to a polyhedron that is similar to the small rhombicuboctahedron but in which the faces that are shown shaded are rectangular rather than square. To overcome the problem, the shaded rectangular faces together with the triangular faces are planed down to a depth that equalises all the edges.

Fig 11. Truncation, canting, snubbing, duality and planing

POLYHEDRON CODES AND P-NAMES

A numeric code is required for identification of the Platonic and Archimedean polyhedra in computer based procedures for processing of polyhedric configurations. This numeric code is chosen to consist of the integer numbers 1 to 20, associated with the Platonic and Archimedean polyhedra in the order they appear in Fig 9. These identity numbers are referred to as 'polyhedron codes'. For instance, the polyhedron codes for the tetrahedron, cuboctahedron and icosidodecahedron are 1, 7 and 12, respectively. A polyhedron code, preceded by the letter P, is used as an alternative name for the polyhedron. A name of this form is referred to as a 'P-name'. The P-names of the Platonic and Archimedean polyhedra are shown in Fig 9 together with the traditional names of the polyhedra. In the following material the P-names are sometimes used by themselves or together with the traditional names to identify polyhedra.

PROPERTIES OF PLATONIC AND ARCHIMEDEAN POLYHEDRA

The basic particulars of the Platonic and Archimedean polyhedra are given in Table 1. The first column of this table lists the names of the polyhedra together with their P-names. The second column of Table 1 lists the numbers of faces, edges and vertices. For instance, these items for tetrahedron are given as

F3:4

E:6

V:4

Here, the letter F stands for 'face' and the digit that follows F indicates the number of sides of the face. Also, the letter E stands for 'edge' and the letter V stands for 'vertex'. The items given in the second column of Table 1 for tetrahedron indicate that it has 4 triangular faces, 6 edges and 4 vertices. Also, the information given in the second column of the table for P7 (cuboctahedron) indicates that it has 8 triangular faces, 6 square faces, 24 edges and 12 vertices.

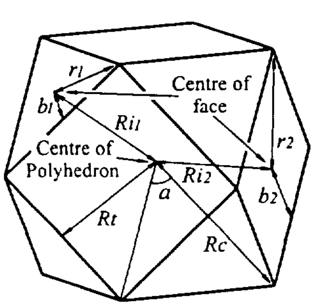
The third column of Table 1 lists the radii of inspheres of the Platonic and Archimedean polyhedra. An insphere is a sphere that is tangent to all the faces of the same type of a polyhedron. A Platonic polyhedron has only one insphere. An Archimedean polyhedron, on the other hand, has either two or three inspheres, depending on whether it has two or three different types of faces. The radius of insphere for an Archimedean polyhedron given in the third column of Table 1 corresponds to the smallest insphere, that is, the insphere that is tangent to the largest faces. Also included at the end of Table 1 are two general formulae for evaluation of radii of inspheres for Platonic and Archimedean polyhedra. Each value in the third column of Table 1 is given in terms of a parameter L that represents the edge length of the polyhedron.

Table 1. Properties of Platonic and Archimedean polyhedra

Polyhedron	Faces, Edges, Vertices	Radius of Insphere	Radius of Intersphere	Radius of Circumsphere	Dihedral Angle
P1: Tetrahedron	F3: 4 E: 6 V: 4	$\frac{\sqrt{6}}{12}L$ $(0.204124145\,L)$	$\frac{\sqrt{2}}{4}L$ $(0.353553390\,L)$	$\frac{\sqrt{6}}{4}L$ $(0.612372435\,L)$	$\operatorname{acos}(\frac{1}{3})$ $(70.5287794°)$
P2: Cube	F4: 6 E: 12 V: 8	$\frac{L}{2}$	$\frac{L}{\sqrt{2}}$ $(0.707106781\,L)$	$\frac{\sqrt{3}}{2}L$ $(0.866025403\,L)$	90°
P3: Octahedron	F3: 8 E: 12 V: 6	$\frac{L}{\sqrt{6}}$ $(0.408248290\,L)$	$\frac{L}{2}$	$\frac{L}{\sqrt{2}}$ $(0.707106781\,L)$	$\operatorname{acos}(\frac{-1}{3})$ $(109.471221°)$
P4: Dodecahedron	F5: 12 E: 30 V: 20	$\frac{\sqrt{25+11\sqrt{5}}}{2\sqrt{10}}L$ $(1.11351636\,L)$	$\frac{3+\sqrt{5}}{4}L$ $(1.30901699\,L)$	$\frac{\sqrt{18+6\sqrt{5}}}{4}L$ $(1.40125854\,L)$	$\operatorname{acos}(\frac{-1}{\sqrt{5}})$ $(116.565051°)$
P5: Icosahedron	F3: 20 E: 30 V: 12	$\frac{3+\sqrt{5}}{4\sqrt{3}}L$ $(0.755761314\,L)$	$\frac{1+\sqrt{5}}{4}L$ $(0.809016994\,L)$	$\frac{\sqrt{10+2\sqrt{5}}}{4}L$ $(0.951056516\,L)$	$\operatorname{acos}(\frac{-\sqrt{5}}{3})$ $(138.189685°)$
P6: Truncated Tetrahedron	F3: 4 F6: 4 E: 18 V: 12	$\frac{\sqrt{6}}{4}L$ $(0.612372435\,L)$	$\frac{3\sqrt{2}}{4}L$ $(1.06066017\,L)$	$\frac{\sqrt{22}}{4}L$ $(1.17260394\,L)$	[6-6] $\operatorname{acos}(1/3)$ $(70.5287794°)$ [6-3] $\operatorname{acos}(-1/3)$ $(109.471221°)$
P7: Cuboctahedron	F3: 8 F4: 6 E: 24 V: 12	$\frac{L}{\sqrt{2}}$ $(0.707106781\,L)$	$\frac{\sqrt{3}}{2}L$ $(0.866025403\,L)$	L	$\operatorname{acos}(\frac{-1}{\sqrt{3}})$ $(125.264390°)$
P8: Truncated Cube	F3: 8 F8: 6 E: 36 V: 24	$\frac{1+\sqrt{2}}{2}L$ $(1.20710678\,L)$	$\frac{2+\sqrt{2}}{2}L$ $(1.70710678\,L)$	$\frac{\sqrt{7+4\sqrt{2}}}{2}L$ $(1.77882365\,L)$	[8-8] 90° [8-3] $\operatorname{acos}(-1/\sqrt{3})$ $(125.264390°)$
P9: Truncated Octahedron	F4: 6 F6: 8 E: 36 V: 24	$\frac{\sqrt{6}}{2}L$ $(1.22474487\,L)$	$\frac{3}{2}L$	$\frac{\sqrt{10}}{2}L$ $(1.58113883\,L)$	[6-6] $\operatorname{acos}(-1/3)$ $(109.471221°)$ [6-4] $\operatorname{acos}(-1/\sqrt{3})$ $(125.264390°)$
P10: Small Rhombicuboctahedron	F3: 8 F4: 18 E: 48 V: 24	$\frac{1+\sqrt{2}}{2}L$ $(1.20710678\,L)$	$\frac{\sqrt{4+2\sqrt{2}}}{2}L$ $(1.30656297\,L)$	$\frac{\sqrt{5+2\sqrt{2}}}{2}L$ $(1.39896633\,L)$	[4-4] 135° [4-3] $\operatorname{acos}(-\sqrt{6}/3)$ $(144.735610°)$
P11: Great Rhombicuboctahedron	F4: 12 F6: 8 F8: 6 E: 72 V: 48	$\frac{1+2\sqrt{2}}{2}L$ $(1.91421356\,L)$	$\frac{\sqrt{12+6\sqrt{2}}}{2}L$ $(2.26303344\,L)$	$\frac{\sqrt{13+6\sqrt{2}}}{2}L$ $(2.31761091\,L)$	[8-6] $\operatorname{acos}(-1/\sqrt{3})$ $(125.264390°)$ [8-4] 135° [6-4] $\operatorname{acos}(-\sqrt{6}/3)$ $(144.735610°)$
P12: Icosidodecahedron	F3: 20 F5: 12 E: 60 V: 30	$\frac{\sqrt{5+2\sqrt{5}}}{\sqrt{5}}L$ $(1.37638192\,L)$	$\frac{\sqrt{5+2\sqrt{5}}}{2}L$ $(1.53884177\,L)$	$\frac{1+\sqrt{5}}{2}L$ $(1.61803399\,L)$	$\operatorname{acos}\left(\frac{-\sqrt{5+2\sqrt{5}}}{\sqrt{15}}\right)$ $(142.622632°)$
P13: Truncated Dodecahedron	F3: 20 F10:12 E: 90 V: 60	$\frac{\sqrt{50+22\sqrt{5}}}{4}L$ $(2.48989829\,L)$	$\frac{5+3\sqrt{5}}{4}L$ $(2.92705098\,L)$	$\frac{\sqrt{74+30\sqrt{5}}}{4}L$ $(2.96944902\,L)$	[10-10] $\operatorname{acos}(-1/\sqrt{5})$ $(116.565051°)$ [10-3] $\operatorname{acos}\left(\frac{-\sqrt{5+2\sqrt{5}}}{\sqrt{15}}\right)$ $(142.622632°)$

Polyhedron	Faces, Edges, Vertices	Radius of Insphere	Radius of Intersphere	Radius of Circumsphere	Dihedral Angle
P14: Truncated Icosahedron	F5: 12 F6: 20 E: 90 V: 60	$\frac{\sqrt{42+18\sqrt{5}}}{4}L$ (2.26728394 L)	$\frac{3+3\sqrt{5}}{4}L$ (2.42705098 L)	$\frac{\sqrt{58+18\sqrt{5}}}{4}L$ (2.47801866 L)	[6-6] $\mathrm{acos}(-\sqrt{5}/3)$ (138.189685°) [6-5] $\mathrm{acos}\left(\frac{-\sqrt{5+2\sqrt{5}}}{\sqrt{15}}\right)$ (142.622632°)
P15: Left Snub Cube **P16:** Right Snub Cube	F3: 32 F4: 6 E: 60 V: 24	$\frac{L}{2k}$ (1.14261351 L)	$\frac{\sqrt{1+k^2}}{2k}L$ (1.24722317 L)	$\frac{\sqrt{1+2k^2}}{2k}L$ (1.34371337 L)	[4-3] 142.983430° [3-3] 153.234588°
	k is equal to 0.437593286 and represents the ratio of the edge length of a snub cube and that of its parent cube. The angle of rotation of a square face of a snub cube with respect to the corresponding face of its parent cube is equal to 16.4675604°.				
P17: Small Rhombicosi-dodecahedron	F3: 20 F4: 30 F5: 12 E: 120 V: 60	$\frac{3\sqrt{5+2\sqrt{5}}}{2\sqrt{5}}L$ (2.06457288 L)	$\frac{\sqrt{10+4\sqrt{5}}}{2}L$ (2.17625090 L)	$\frac{\sqrt{11+4\sqrt{5}}}{2}L$ (2.23295051 L)	[5-4] $\mathrm{acos}\left(\frac{-\sqrt{10+2\sqrt{5}}}{2\sqrt{5}}\right)$ (148.282526°) [4-3] $\mathrm{acos}\left(\frac{-(1+\sqrt{5})}{2\sqrt{3}}\right)$ (159.094843°)
P18: Great Rhombicosi-dodecahedron	F4: 30 F6: 20 F10:12 E: 180 V: 120	$\frac{\sqrt{25+10\sqrt{5}}}{2}L$ (3.44095480 L)	$\frac{\sqrt{30+12\sqrt{5}}}{2}L$ (3.76937713 L)	$\frac{\sqrt{31+12\sqrt{5}}}{2}L$ (3.80239450 L)	[10-6] $\mathrm{acos}\left(\frac{-\sqrt{5+2\sqrt{5}}}{\sqrt{15}}\right)$ (142.622632°) [10-4] $\mathrm{acos}\left(\frac{-\sqrt{10+2\sqrt{5}}}{2\sqrt{5}}\right)$ (148.282526°) [6-4] $\mathrm{acos}\left(\frac{-(1+\sqrt{5})}{2\sqrt{3}}\right)$ (159.094843°)
P19: Left Snub Dodecahedron **P20:** Right Snub Dodecahedron	F3: 80 F5: 12 E: 150 V: 60	$\sqrt{\frac{25+11\sqrt{5}}{40k^2}}L$ (1.98091595 L)	$\sqrt{\frac{3+\sqrt{5}+2k^2}{(40-16\sqrt{5})k^2}}L$ (2.09705384 L)	$\sqrt{\frac{7+3\sqrt{5}+8k^2}{(20-4\sqrt{5})k^2}}L$ (2.15583737 L)	[5-3] 152.929920° [3-3] 164.175366°
	k is equal to 0.562121965 and represents the ratio of the edge length of a snub dodecahedron and that of its parent dodecahedron. The angle of rotation of a pentagonal face of a snub dodecahedron with respect to the corresponding face of its parent dodecahedron is equal to 13.1064034°.				

Some General Relations



(1) $Rc=\sqrt{Rt^2+L^2/4}$ (2) $Rt=\sqrt{Rc^2-L^2/4}$ (3) $Rip=\sqrt{Rc^2-rp^2}$ (4) $Rip=\sqrt{Rt^2-bp^2}$

(5) $L=2\sqrt{Rc^2-Rt^2}$ (6) $a=2\,\mathrm{asin}(L/2Rc)$ (7) $a=2\,\mathrm{acos}(Rt/Rc)$ (8) $a=2\,\mathrm{atan}(L/2Rt)$

The above relations are applicable to every Platonic and Archimedean polyhedron, where: ■ L is the edge length ■ a is the angle subtended by an edge at the centre of the polyhedron ■ Rc is the radius of the circumsphere ■ Rt is the radius of the intersphere ■ Rip is the radius of an insphere, that is, a sphere which is tangent to all the faces of type p ■ rp is the distance between the centre and a corner of a face of type p ■ bp is the distance between the centre and the midpoint of a side of a face of type p ■ The radius of insphere for an Archimedean polyhedron given in the third column of the Table corresponds to the smallest insphere, that is, the insphere which is tangent to the largest faces.

The parameter L, representing the edge length, also appears in columns four and five of Table 1. Columns four and five list the radii of interspheres and circumspheres of the Platonic and Archimedean polyhedra. An intersphere is a sphere that is tangent to all the edges of a polyhedron and a circumsphere is a sphere that passes through all the vertices of a polyhedron. Each Platonic or Archimedean polyhedron has one intersphere and one circumsphere.

The last column of Table 1 lists the dihedral angles of the Platonic and Archimedean polyhedra. A dihedral angle is the angle between two faces of a polyhedron that share an edge. For any Platonic polyhedron, all the dihedral angles are equal. In contrast, an Archimedean polyhedron, may have up to three different dihedral angles, as shown in the last column of Table 1. For an Archimedean polyhedron that has more than one dihedral angle, the faces that correspond to each dihedral angle are specified by the numbers of their sides given in square brackets. For example, in the case of P6 (truncated tetrahedron), the first dihedral angle is preceded by [6-6] indicating that the angle is between two hexagonal faces and the second dihedral angle is preceded by [6-3] indicating that the angle is between a hexagonal face and a triangular face.

The properties of the Platonic and Archimedean polyhedra, as given in Table 1, are incorporated into the part of Formian that deals with the processing of polyhedric configurations. The information is built into Formian in terms of the formulae given in Table 1. The use of formulae will allow the full available accuracy of the hardware to be utilised. High accuracy of the basic polyhedral data is essential in many situations. This is the case, for instance, when dealing with complex polyhedric configurations that consist of many thousands of elements, in particular, when the generated geometric details are to be used as a basis for other operations, such as structural analysis.

In relation to the accuracy of entries in Table 1, it should be noted that for the snub polyhedra (P15, P16, P19 and P20), the accuracy of the entries for radii of insphere, intersphere and circumsphere depends on the accuracy of a parameter k. The value of this parameter in Table 1 is given accurate to nine decimal places. The values of the dihedral angles for the snub polyhedra in Table 1 are also given accurate to nine decimal places.

POLYHEDRAL COORDINATE SYSTEMS

A view of a polyhedric configuration is shown in Fig 12(a). The configuration is obtained by mapping a triangulated pattern onto the faces of a tetrahedron. When a computer aided approach is used in processing such a configuration then the internal computer representation of the configuration will be

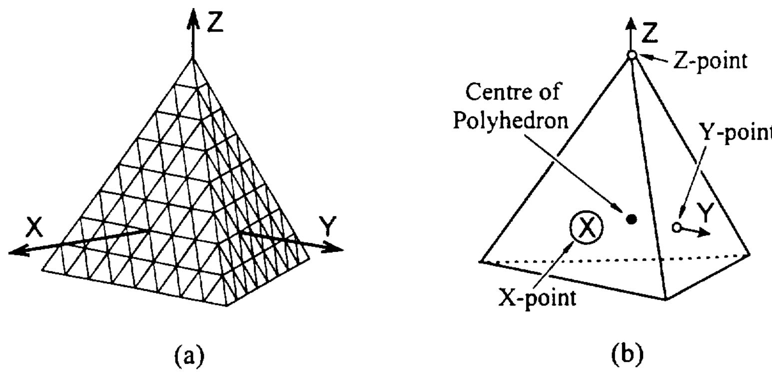


Fig 12. A coordinate system for tetrahedron

a 'numerical model' that describes the configuration in terms of the coordinates of its nodal points. It is therefore necessary to have a 'coordinate system' with respect to which the nodal coordinates are specified. The most convenient approach in this relation is to establish a standard coordinate system for the tetrahedron and use it for all polyhedric configurations that are based on tetrahedron.

The chosen standard coordinate system for tetrahedron is the right-handed Cartesian coordinate system that is shown as X-Y-Z in Fig 12(a). Fig 12(b) illustrates the conventions used in specifying this standard coordinate system. The origin of the coordinate system is at the centre of the polyhedron. This point is indicated by a large dot. The points where the X, Y and Z axes intersect the body of the polyhedron are referred to as X-point, Y-point and Z-point, respectively. The X-point is at the centre of the circle with an X inside it. The Y-point is indicated by a little circle and the Y-axis is shown as an arrow emanating from the Y-point. Similarly, the Z-point is indicated by a little circle and the Z-axis is shown as an arrow emanating from the Z-point.

The standard coordinate system for tetrahedron is shown again in Fig 13 together with the standard coordinate systems for all the other Platonic and Archimedean polyhedra. Some of the polyhedra in this figure have additional sketches shown near them. Tetrahedron, for example, has such a sketch. These sketches provide information about the precise positions of the X-points and are included whenever the positions of X-points are not obvious from the main figures.

A view of a set of cardboard models of Platonic and Archimedean polyhedra is shown in Fig 14. The models of the Platonic polyhedra are placed in the front row and those of the Archimedean polyhedra are arranged in the three back rows. The X-point and Z-point for each model are situated at the centres of the circular spots on the model. The darker spot that appears in front of the model indicates the position of the X-point and the lighter spot that appears on the top indicates the position of the Z-point.

From the point of view of compatibility of the coordinate systems, as given in Fig 13, the Platonic and Archimedean polyhedra may be divided into three groups. Firstly, there is a group consisting of two polyhedra, namely, tetrahedron and truncated tetrahedron. These two polyhedra occupy the central part of the family tree in Fig 10. The coordinate systems for these polyhedra are compatible with each other. By the term 'compatible', in this context, it is meant that when a truncated tetrahedron in produced by cutting off the corners of a tetrahedron then the original coordinate system of the tetrahedron will become the coordinate system for the truncated tetrahedron without any change.

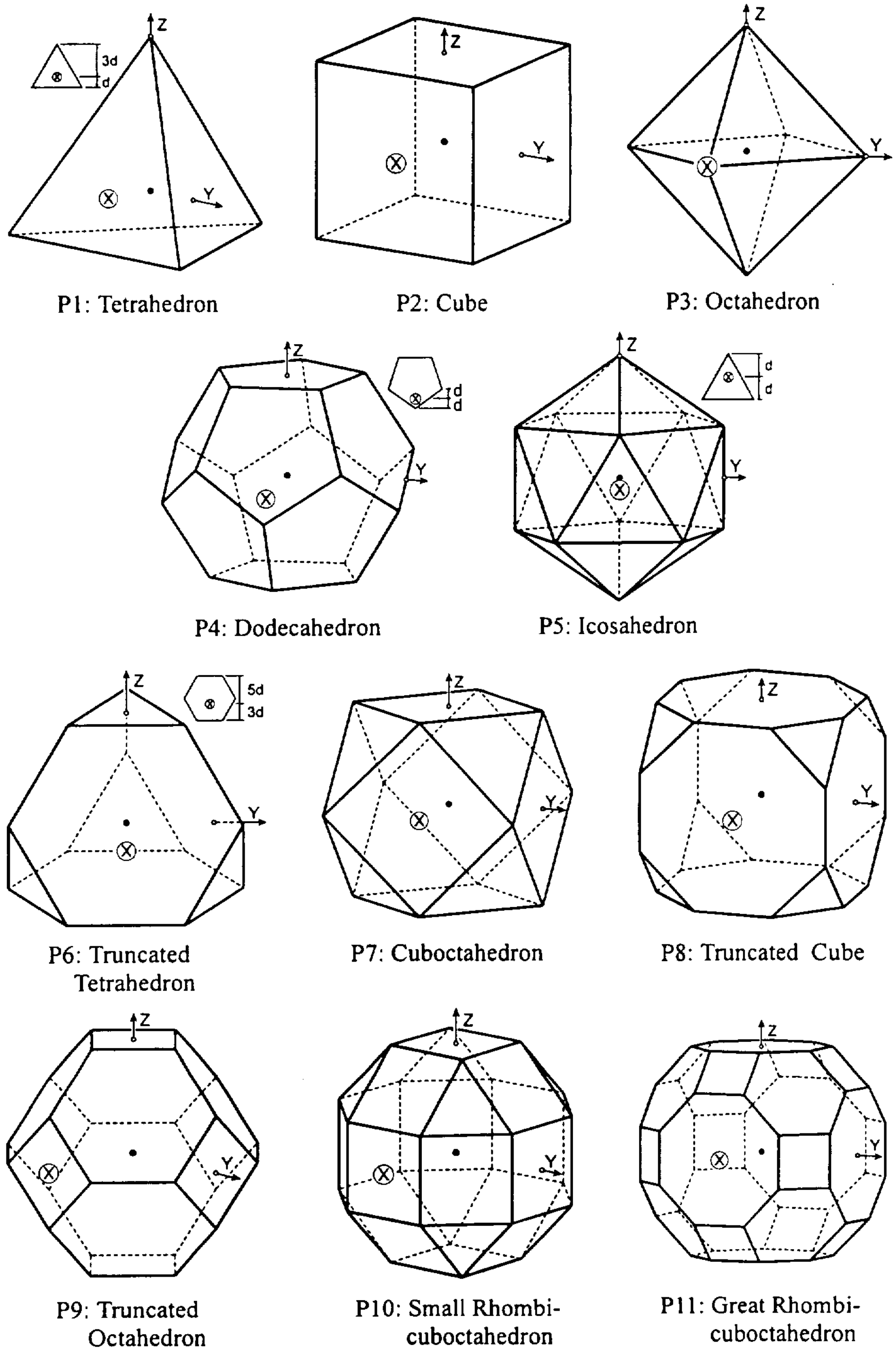
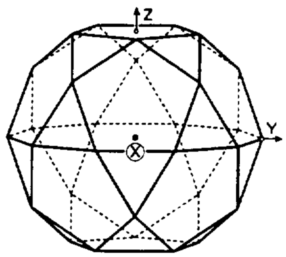
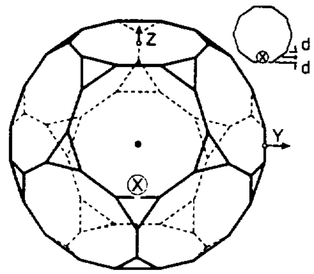


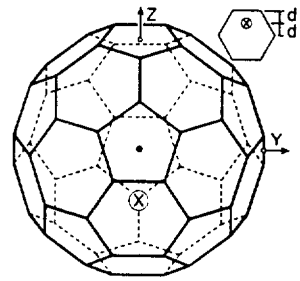
Fig 13 (part 1). Coordinate systems of Platonic and Archimedean polyhedra



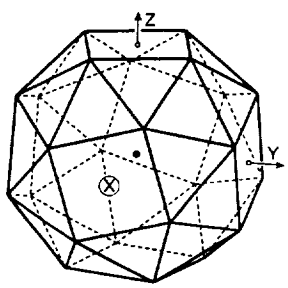
P12: Icosido-
decahedron



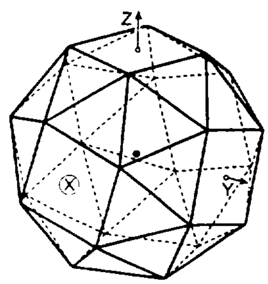
P13: Truncated
Dodecahedron



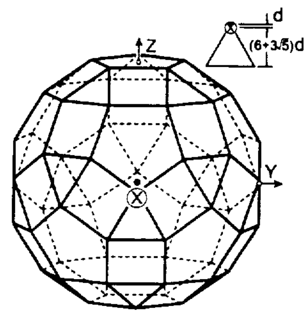
P14: Truncated
Icosahedron



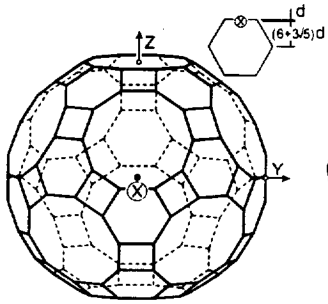
P15: Left Snub Cube



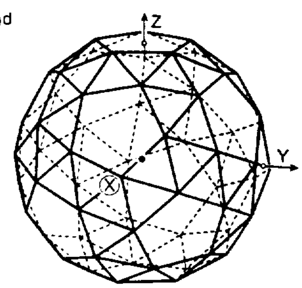
P16: Right Snub Cube



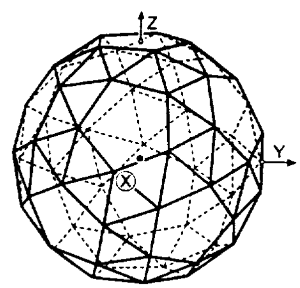
P17: Small Rhomb-
icosidodecahedron



P18: Great Rhomb-
icosidodecahedron



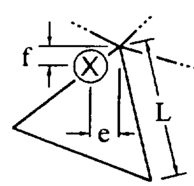
P19: Left Snub
Dodecahedron



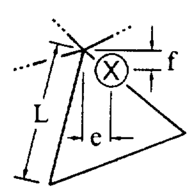
P20: Right Snub
Dodecahedron

e = 0.192893711 L
f = 0.144866867 L

e is parallel to the Y axis and f is parallel to the Z axis.



Position of
X-point for P19



Position of
X-point for P20

Fig 13 (part 2). Coordinate systems of Platonic and Archimedean polyhedra

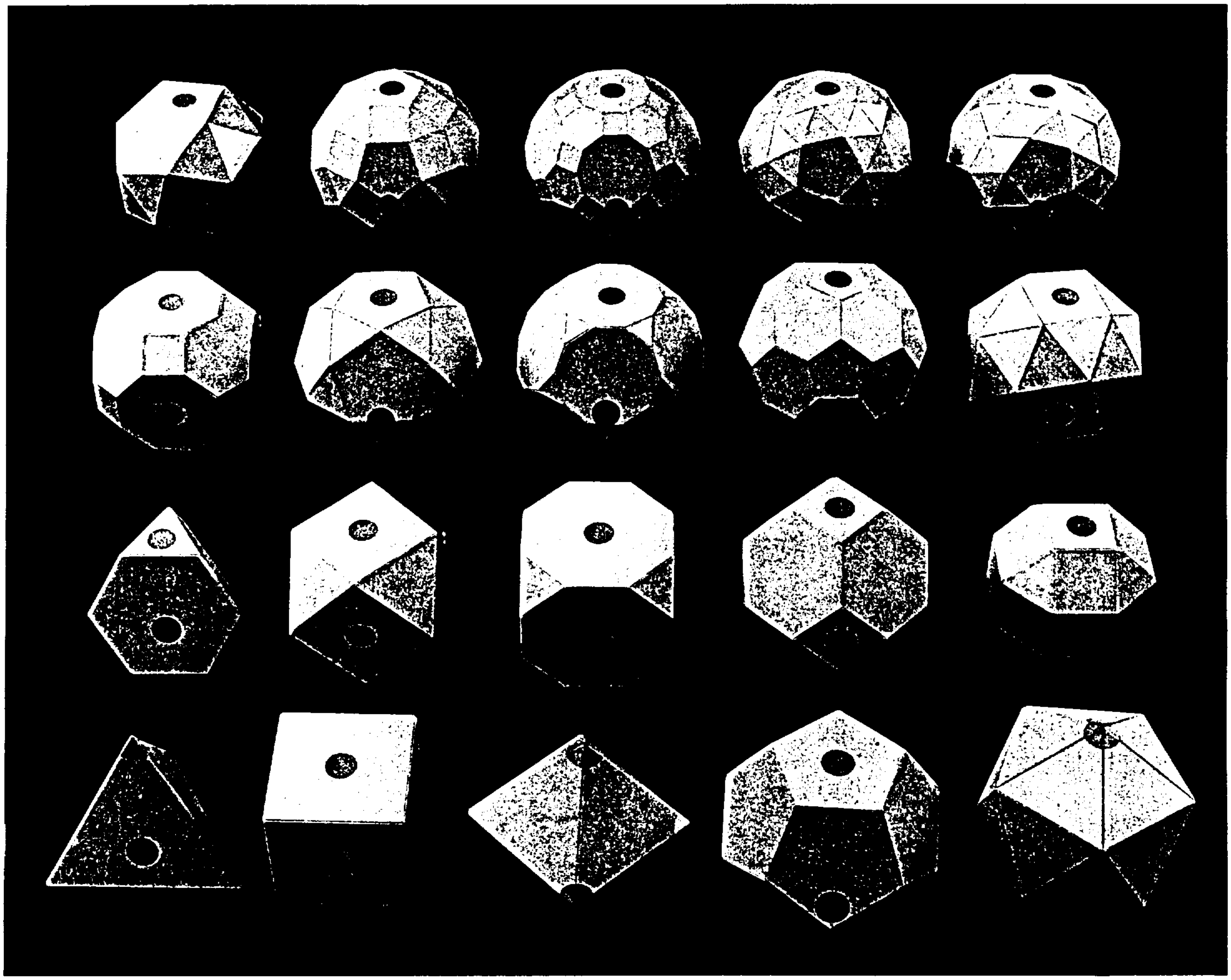


Fig 14. Models of Platonic and Archimedean polyhedra with the darker front spots indicating the X-points and the lighter top spots indicating the Z-points

The second family of polyhedra with compatible coordinate systems consists of nine polyhedra. These are the polyhedra that can be derived from the cube or octahedron and appear to the left of the centre in the family tree of Fig 10. The third family of polyhedra with compatible coordinate systems again has nine members. These are the polyhedra that can be derived from dodecahedron or icosahedron and appear to the right of the centre in the family tree of Fig 10.

The standard coordinate systems shown in Fig 13 are used in Formian as the basis for formulation of transformations that are necessary for creation of numerical models representing polyhedric configurations.

IDENTITY NUMBERS AND BASELINES FOR FACES OF POLYHEDRA

Fig 15(e) shows a polyhedric configuration that is obtained by mapping the face-object of Fig 15(a) onto the top five faces of an icosahedron. If the face-object for mapping onto the faces is chosen to be that of Fig 15(b) then the result will be the polyhedric configuration shown in Fig 15(f). In the case of the polyhedric configuration of Fig 15(g), the face-object of Fig 15(a) is mapped onto one of the top faces of an icosahedron and the face-object of Fig 15(c) is mapped onto the remaining four faces. The polyhedric configuration of Fig 15(h) is obtained by a similar procedure using the face-objects shown in Figs 15(b) and 15(d).

The point that is meant to be illustrated by the above examples is that in most practical cases a face-object is mapped onto a selected number of the faces rather than all the faces of a polyhedron. In the examples of Figs 15(e) to 15(h), the top five faces of an icosahedron have been selected for mapping. Furthermore, in the examples of Figs 15(g) and 15(h), one of the faces has been selected for mapping of a face-object and the other four faces have been selected for mapping of a different face-object. In order to select faces, it is necessary to have a means of identifying the faces of a polyhedron. This is achieved by associating an identity number with each face of a polyhedron as will be described in the sequel.

Another problem that has to be addressed is illustrated in terms of the polyhedric configurations shown in Figs 15(i) and 15(j). Fig 15(i) shows a polyhedric configuration that is obtained by mapping the face-object of Fig 15(c) onto the top five faces of an icosahedron. However, the orientation of the face-object as mapped onto the faces in Fig 15(i) is different from the orientation of the face-object as it appears in Fig 15(c). The polyhedric configuration shown in Fig 15(j) has the face-object of Fig 15(a) mapped on three of the faces and the face-object of Fig 15(c) mapped on two of the faces with

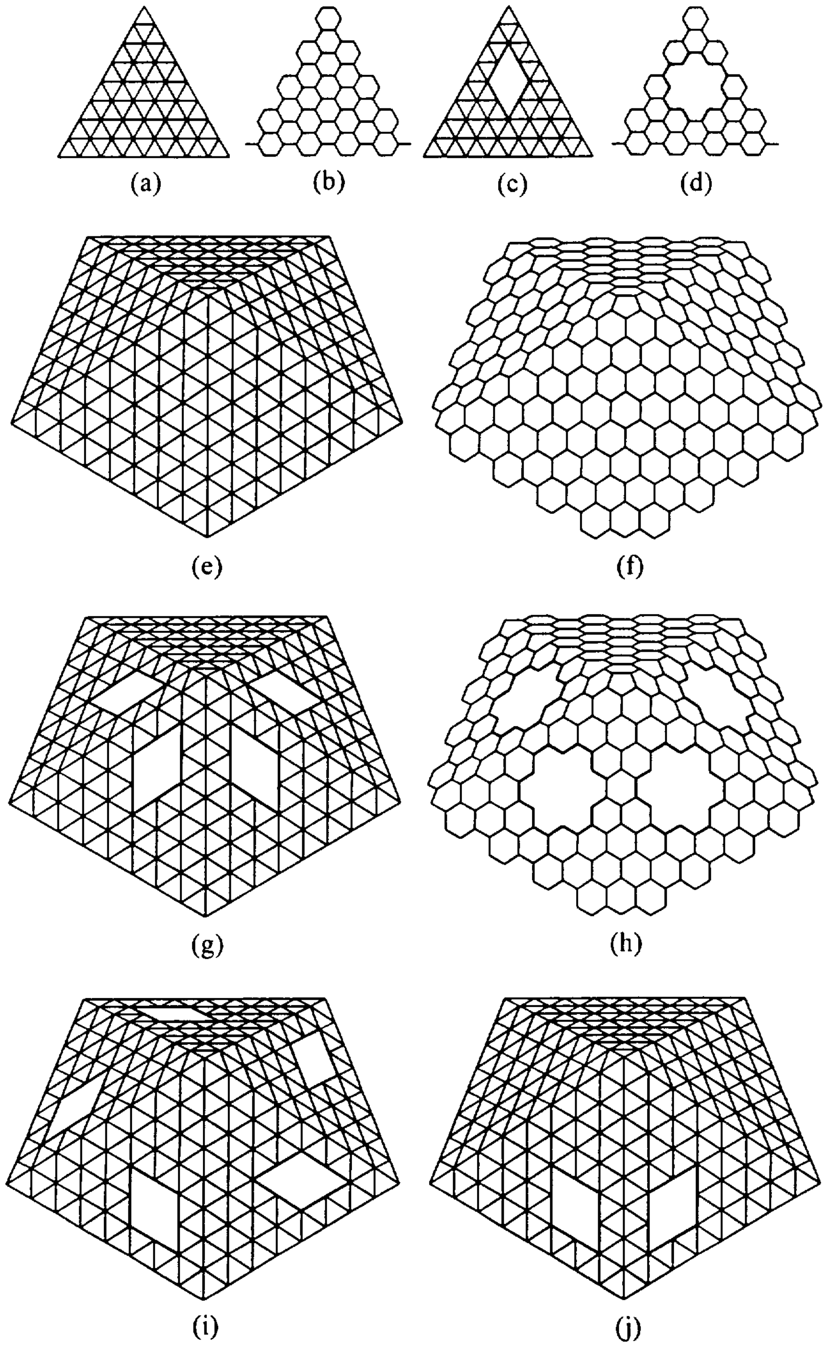


Fig 15. Examples of face mapping

different orientations. These examples show that, in addition to the requirement for the allocation of an identity number to each face of a polyhedron, it is necessary to associate a frame of reference with the face. This would then allow the required position of a face-object for mapping onto the face to be specified unambiguously.

A frame of reference for a face of a polyhedron is established by assigning the status of 'baseline' to one of the sides of the face and by associating the letters A and B to the end points of this baseline, as shown in Figs 16(a) and 16(b). The end of the baseline that is associated with the letter A is referred to as the A-end and the end that is associated with the letter B is referred to as the B-end. The baseline of a face is indicated by a vector. The vector is placed near the baseline with its arrow-head showing the direction from the A-end to the B-end. In addition, the identity number of each face is placed near the baseline vector.

The baseline vectors and the face identity numbers for the top part of an icosahedron are shown in Fig 16(a). The allocation of identity numbers and the selection of baselines for the faces of polyhedra are governed by a number of rules that are described in the Appendix. Also, the face identity numbers together with the baselines for a group of six polyhedra are shown in Fig 17. This group contains all the Platonic polyhedra and one Archimedean polyhedron, namely, cuboctahedron.

MAPPING OF FACE-OBJECTS

The process of mapping a face-object onto a face of a polyhedron involves the following steps:

(1) A face-object is specified by a formex relative to the standard X-Y-Z coordinate system of the polyhedron.

(2) Two points of the face-object are specified by their X-Y-Z coordinates. These points are referred to as the A-point and B-point. The role of the A-point and B-point is to provide information regarding the required position, orientation and size of the face-object in its final mapped position on the face of the polyhedron, as exemplified in Figs 16(c) to 16(f).

(3) The face-object is scaled such that the distance between the A-point and B-point is equal to the edge length of the polyhedron. In this scaling process, the same scale factor is used in the X, Y and Z directions.

(4) The 'mapping plane' is determined. This is the plane of the face-object that is to coincide with the face of the polyhedron. When the line containing the A-point and B-point of the face-object is parallel to (or coincident with) the X axis then the mapping plane is the plane that contains the A-point and B-point and is parallel to (or coincident with) the X-Y plane. This simple case is

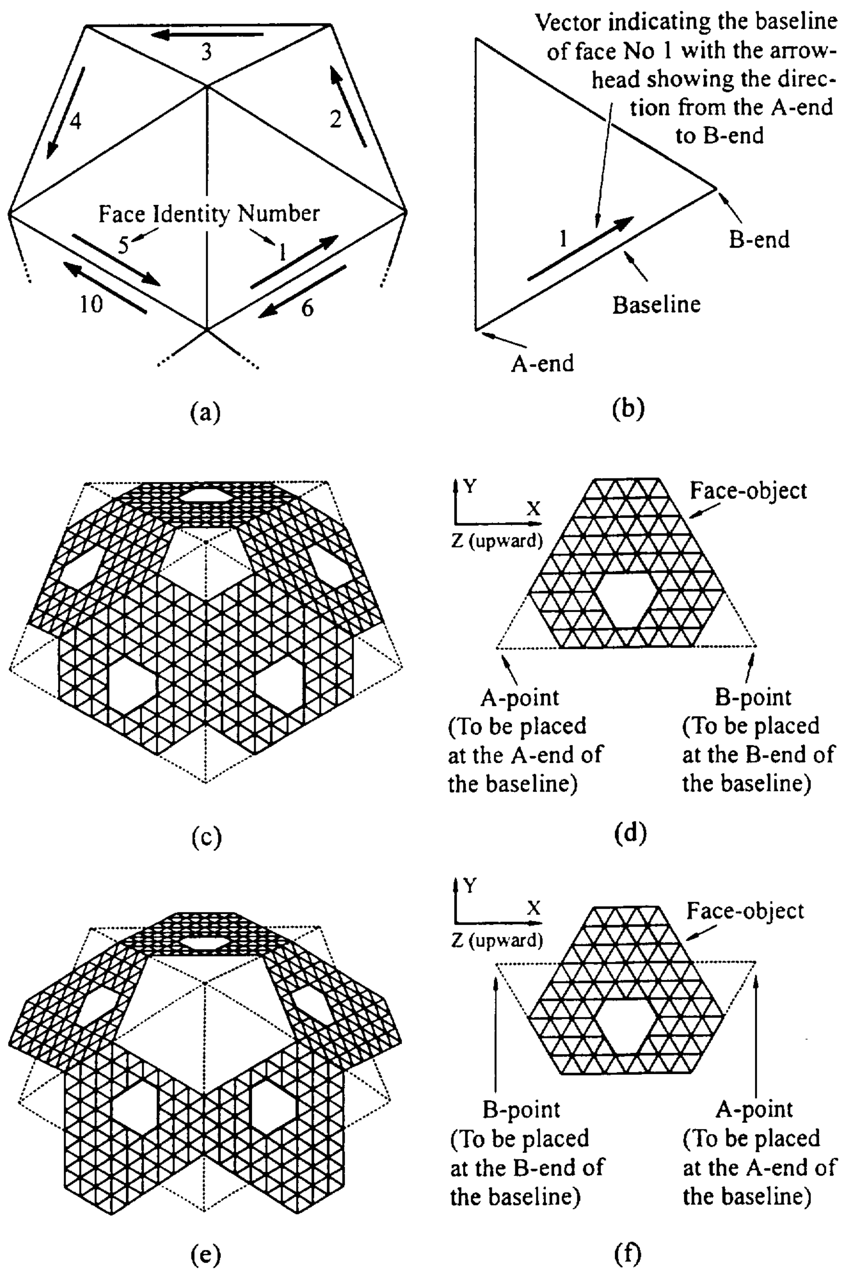
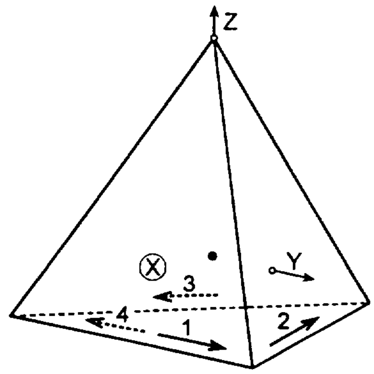
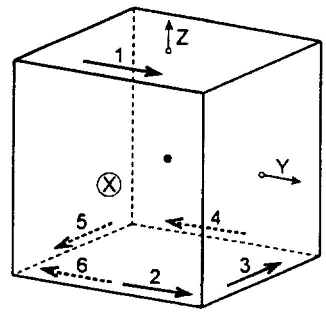


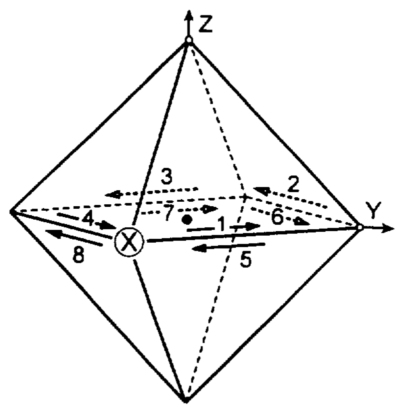
Fig 16. Face mapping process



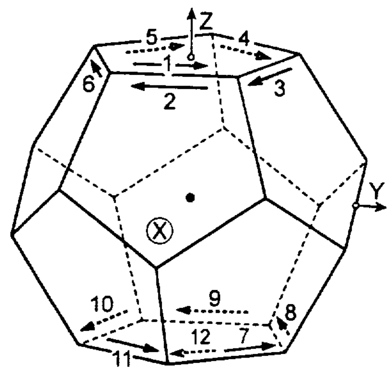
P1: Tetrahedron



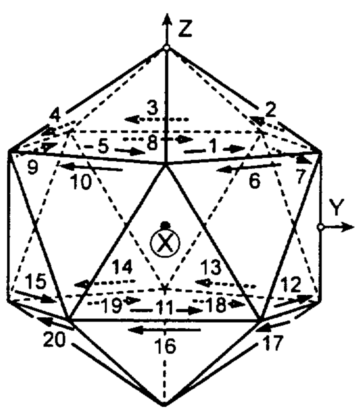
P2: Cube



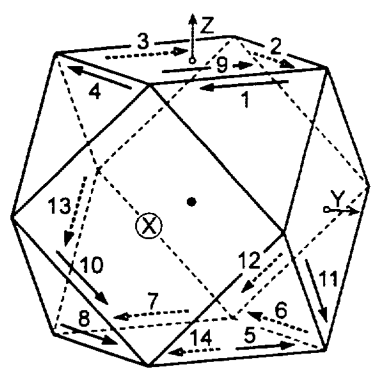
P3: Octahedron



P4: Dodecahedron



P5: Icosahedron



P7: Cuboctahedron

Fig 17. Identity numbers and baselines for
the faces of a selection of polyhedra

applicable in most practical situations and is the only case considered here.

(5) The face-object is subjected to a sequence of rigid body movements (translations and rotations) such that the following conditions are satisfied:

- The A-point of the face-object is coincident with the A-end of the baseline of the face.
- The B-point of the face-object is coincident with the B-end of the baseline of the face.
- The mapping plane is coincident with the face.
- The direction which was initially the positive Z direction of the face-object is pointing to the outside of the polyhedron.

Two examples of the face mapping process are shown in Fig 16. Fig 16(c) shows the top part of an icosahedron with a face-object mapped onto five faces. The boundaries of these faces are shown in dotted lines. The face-object is shown in Fig 16(d) with the line that passes through the A-point and B-point being parallel to the X axis. The mapping is achieved by suitably scaling the face-object and then placing it on each face in a position where the A-point coincides with the A-end of the baseline of the face and the B-point coincides with the B-end of the base line of the face.

It is important to note that the A-point and B-point of the face-object need not necessarily be actual points of the face-object. For example, in the case of the face-object in Fig 16(d), the A-point and B-point are outside the face-object altogether. The dotted lines here are included to indicate the positions of the A-point and B-point. These dotted lines are not supposed to be parts of the face-object.

A second example of face mapping is shown in Figs 16(e) and 16(f). The face-object in this example is the same as that of the previous one. The only difference is in the positions of the A-point and B-point. To be specific, the positions of the A-point and B-point have crossed over as well as being shifted. Consequently, the face-object has been turned round for mapping.

IDENTITY NUMBERS AND DIRECTIONS FOR EDGES OF POLYHEDRA

The upper part of Fig 18(a) shows a polyhedric configuration that is obtained by mapping a truss configuration on the edges of an octahedron. The edge-object is shown in the lower part of Fig 18(a) with the A-point and B-point being assumed to be on the X axis. The mapping is carried out by placing a suitably scaled version of the edge-object on the edges of the octahedron. For each edge, the edge-object is positioned such that the end points of the top chord of the truss coincide with the end points of the edge and the plane of the truss passes through the centre of the octahedron. The angle of the inclined sides of the truss is chosen such that, after mapping on the edges of the octahedron, the

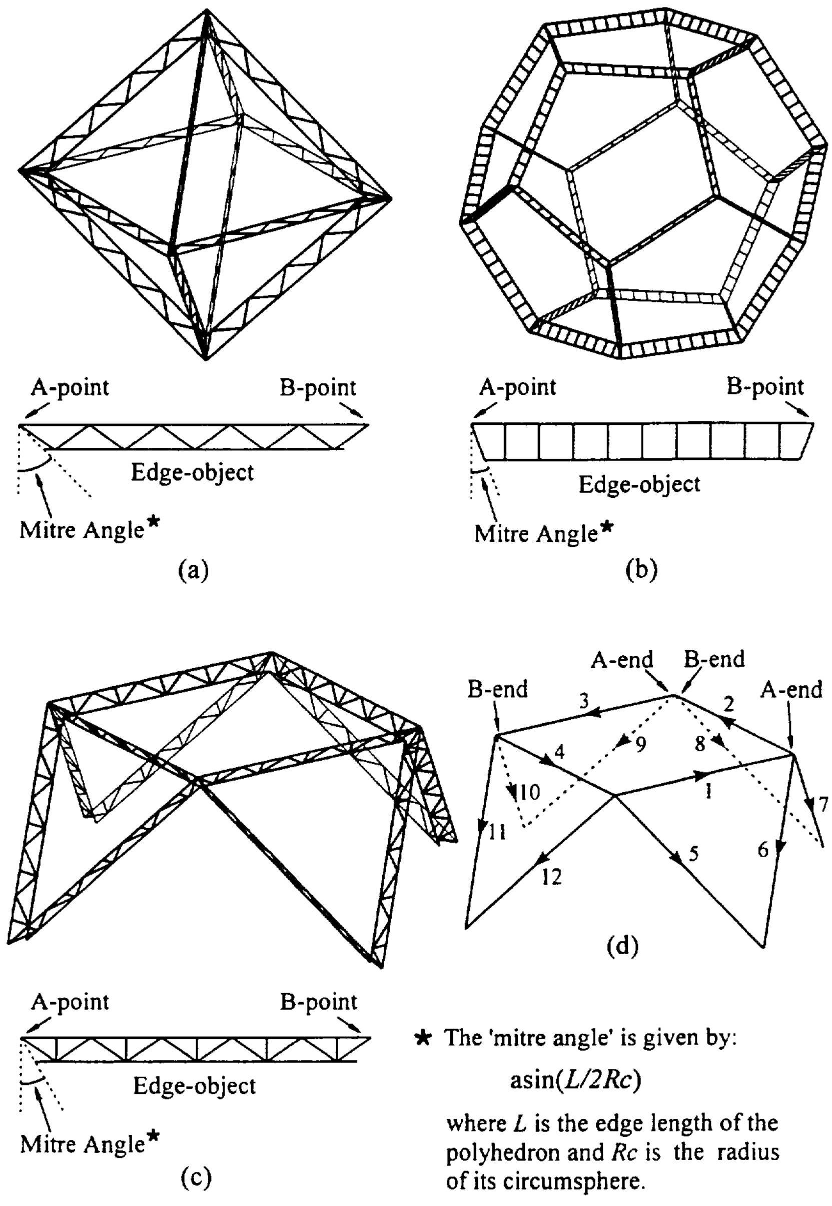


Fig 18. Examples of edge mapping

ends of the bottom chords of the trusses meet without any gaps. The term 'mitre angle' is used to refer to the angle that will allow the ends of the trusses to match after mapping. Fig 18(b) shows the result of mapping a Vierendeel girder type configuration on the edges of a dodecahedron. Also, the result of mapping a truss-like configuration on the upper half of a cuboctahedron is shown in Fig 18(c).

If the edge-object consists of a plane configuration and if this is to be mapped on the edges of a Platonic or Archimedean polyhedron then the mitre angle may be obtained from the general formula given at the right bottom corner of Fig 18.

In order to carry out the mapping of an object on an edge of a polyhedron, it is necessary to identify the edge on which the object is to be mapped and to establish a way of specifying the position, orientation and size of the object at its final mapped form. The identification of the edges of a polyhedron is achieved by allocating an identity number to each edge as exemplified in Fig 18(d) for the upper half of a cuboctahedron. Also, each end of an edge is associated with a letter. One end is associated with the letter A and is referred to as the A-end and the other end is associated with the letter B and is referred to as the B-end. The A-ends and B-ends for two of the edges of a cuboctahedron are shown in Fig 18(d). The convention is adopted that the positions of the A-end and B-end of an edge are indicated by placing an arrow-head on the edge pointing from the A-end to B-end, as shown in Fig 18(d). Thus, the A-end and B-end effectively establish a 'direction' for the edge.

The identity numbers of the edges together with the arrow-heads indicating the A-ends and B-ends for the Platonic polyhedra and a sample of an Archimedean polyhedra (namely, a cuboctahedron) are shown in Fig 19. This figure also includes some information relating to the vertices of the polyhedra, as will be discussed later. The allocation of identity numbers to the edges as well as the choices of the A-ends and B-ends, as shown in Fig 19, are governed by a number of rules that are described in the Appendix.

MAPPING OF EDGE-OBJECTS

The process of mapping an edge-object on an edge of a polyhedron involves the following steps:

(1) An edge-object is specified by a formex relative to the standard X-Y-Z coordinate system of the polyhedron.

(2) Two points of the edge-object are specified by their X-Y-Z coordinates. These points are referred to as the A-point and B-point. The role of the A-point and B-point is to provide information

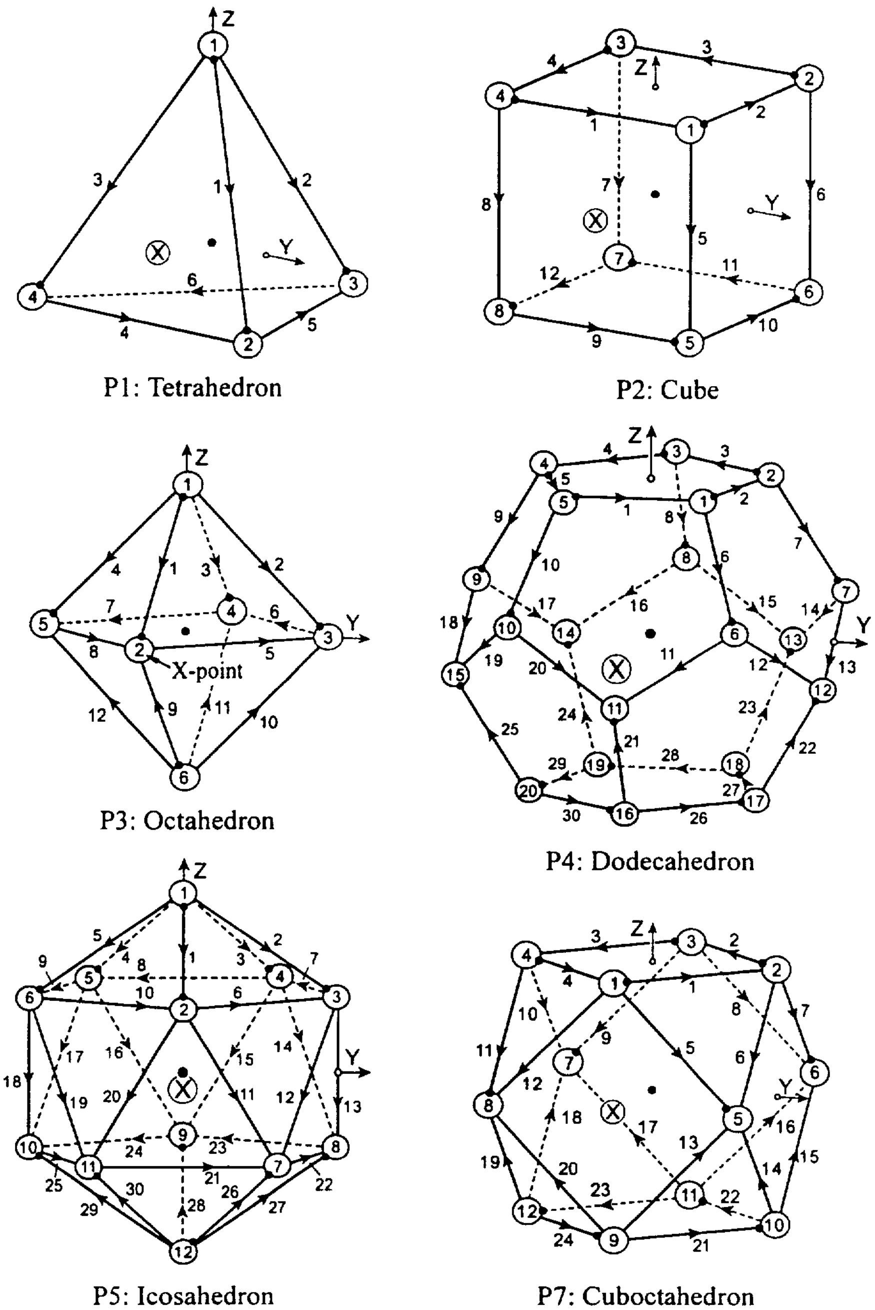


Fig 19. Identity numbers, directions and handles for
edges and vertices of a selection of polyhedra

regarding the required position, orientation and size of the edge-object in its final mapped position, as illustrated in Fig 18.

(3) The edge-object is scaled such that the distance between the A-point and B-point is equal to the edge length of the polyhedron. In this scaling process, the same scale factor is used in the X, Y and Z directions.

(4) The 'mapping plane' is determined. This is the plane of the edge-object that is to coincide with the plane that contains the edge and passes through the centre of the polyhedron. When the line containing the A-point and B-point is parallel to (or coincident with) the X axis then the mapping plane is the plane that contains the A-point and B-point and is parallel to (or coincident with) the X-Z plane. This simple case is applicable in most practical situations and is the only case considered here.

(5) The edge-object is subjected to a sequence of rigid body movements (translations and rotations) such that the following conditions are satisfied:

- The A-point of the edge-object is coincident with the A-end of the edge.
- The B-point of the edge-object is coincident with the B-end of the edge.
- The mapping plane is coincident with the plane that contains the edge and passes through the centre of the polyhedron.
- The direction which was initially the positive Z direction of the edge-object is pointing to the outside of the polyhedron.

IDENTITY NUMBERS AND HANDLES FOR VERTICES OF POLYHEDRA

Examples of mapping of objects on vertices of polyhedra are shown in Fig 20. To begin with, as for the faces and edges, it is necessary to allocate an identity number and a frame of reference to each vertex of a polyhedron. Identity numbers for vertices in the upper half of a cuboctahedron are shown in Fig 20(a). The convention is adopted that a vertex identity number is shown in a circle placed at the vertex. The frame of reference for a vertex is provided by selecting one of its edges to become a base with respect to which vertex-objects may be mapped on the vertex. This edge is referred to as the 'handle' of the vertex. Also, the vertex end of the handle is referred to as the A-end and the other end is referred to as the B-end. The convention is adopted that the handle of a vertex is indicated by placing a dot (referred to as a 'handle dot') at its A-end, as shown in Fig 20(a).

The vertex identity numbers and handles for all the Platonic polyhedra and a sample of Archimedean polyhedra (cuboctahedron) are shown in Fig 19. The rules governing the choices of identity numbers and handles for vertices are given in the Appendix.

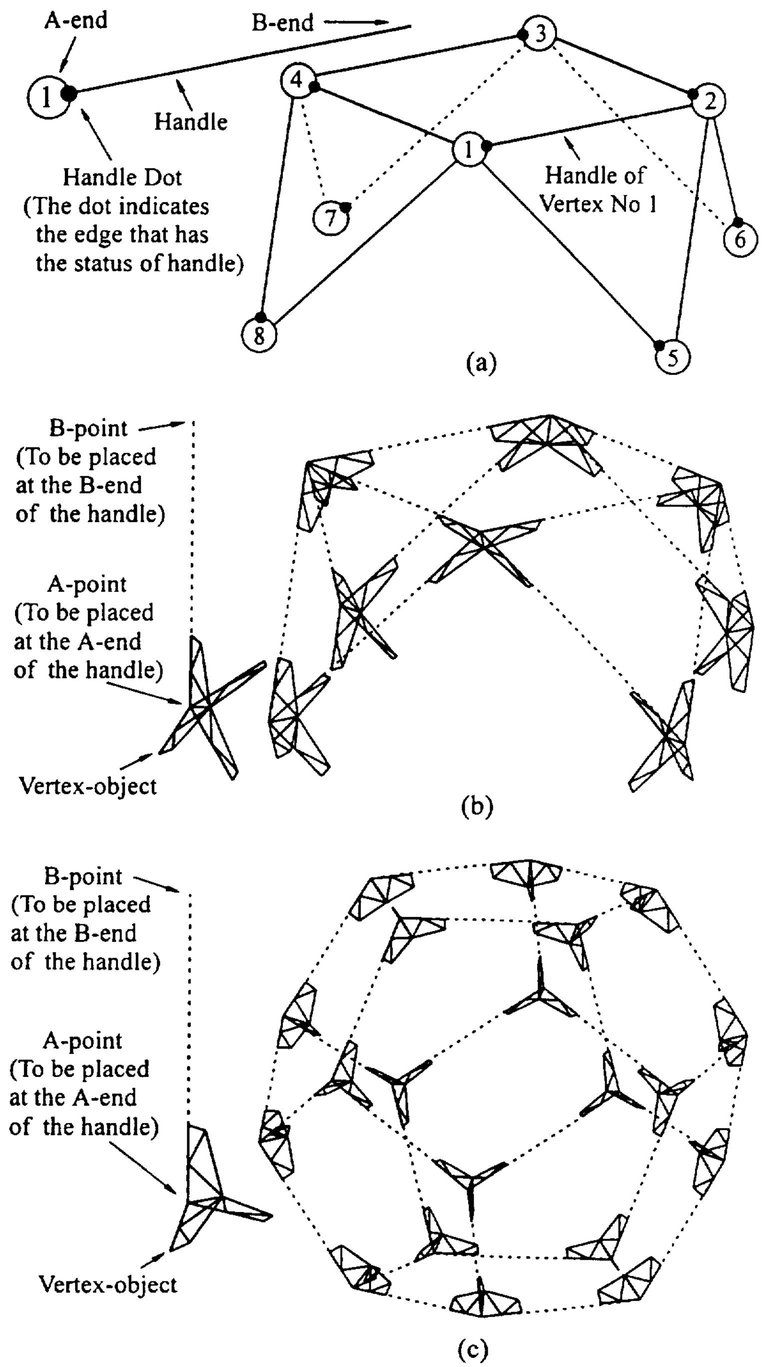


Fig 20. Examples of vertex mapping

MAPPING OF VERTEX-OBJECTS

Fig 20(b) shows the result of mapping a vertex-object on the vertices of the upper half of a cuboctahedron. Another example of vertex mapping is shown in Fig 20(c) where a vertex-object is mapped on the vertices of a dodecahedron.

With one important difference that will be discussed below, the process of mapping a vertex-object is identical to the procedure for mapping an edge-object. To elaborate, when a vertex-object is to be mapped on a vertex of a polyhedron, then the procedure followed will be as though the vertex-object is an edge-object which is to be mapped on the edge that is the handle of the vertex.

The important difference between the vertex mapping as compared with edge mapping (and face mapping) is that in some cases a vertex-object is to be subjected to reflection in the mapping process. To elaborate, the mapping of a face-object or an edge-object is always carried out through a sequence of rigid body movements (translations and rotations) and simple scaling. This fact remains true for a vertex-object in most cases. However, for two Archimedean polyhedra the process of mapping of a vertex-object may require an additional operation of reflection. These two polyhedra are P11 (great rhombicuboctahedron) and P18 (great rhombicosidodecahedron) and the reason for the need for reflection in these cases is discussed in the Appendix.

POLYMATION FUNCTION

The processes involved in mapping objects on faces, edges and vertices of polyhedra are discussed in the previous sections. In Formian, these processes are carried out through the 'polymation function'. For example, a Formian instruction that creates a formex representing the polyhedric configuration of Fig 20(c) may be written as

G = pol(3, 4, '[all]', 1, [0,0; 1,0])| E

where:

- E is a formex representing the vertex-object.
- G is a formex representing the polyhedric configuration of Fig 20(c).
- pol is an abbreviation for the name of the function, that is, polymation.
- The first item in parentheses is the 'operation code' specifying the type of mapping to be performed, where the integer 3 indicates mapping on vertices.
- The second item in parentheses is the 'polyhedron code' specifying the polyhedron to be used as the basis for mapping. The polyhedron code is the integer that follows the letter P in the P-name of a polyhedron. The integer 4 in the above example implies a dodecahedron.
- The third item in parentheses is the 'entity list' specifying the vertices on which mapping is to be

performed. The entity list as given in the above example indicates 'all the vertices'.

- The fourth item in parentheses is the 'radius specifier' that determines the size of the polyhedron by specifying the radius of the circumsphere of the polyhedron.
- The last item in parentheses is the 'locator' specifying the A-point and B-point of the vertex-object. In general, the A-point and B-point are to be specified by giving their X, Y and Z coordinates. However, if the third coordinates are not given then it will be assumed that Z coordinates are equal to zero. In the above example, the X and Y coordinates of the A-point are given as 0,0 and those of the B-point are given as 1,0. The Z coordinates will then be assumed to be equal to zero.

The parenthesised items of a polymation function provide information about the manner in which the mapping is required to be carried out. Further details about these items are given in Fig 21. Information about the locations of all the faces, edges and vertices of Platonic and Archimedean polyhedra, in terms of their identity numbers, is incorporated into Formian. This information is used by the polymation function for determining the locations of the faces, edges or vertices specified by the entity list (third item in parentheses in Fig 21). Also, Formian incorporates complete information about the baselines of faces, directions of edges and handles of vertices for the Platonic and Archimedean polyhedra. This information is used by the polymation function for the correct positioning of the objects to be mapped.

The argument of the polymation function in Fig 21 is represented by E and is separated from the function by the symbol |. Normally, this argument is a formex variable that represents the object to be used for mapping. Examples of formex formulations for creation of formex variables representing the mapping objects are shown in Fig 22. Fig 22(a) shows a face-object together with its formex formulation which is shown in a box. This formex formulation gives rise to the formex variable E that represents the face-object. The face-object has a pattern similar to the one used for the polyhedric configuration of Fig 3(a). A truss-like edge-object together with its formex formulation is shown in Fig 22(b). Also, a vertex-object with its formex formulation is shown in Fig 22(c). This is the vertex-object that has been used for the polyhedric configuration of Fig 20(c).

A reader who is familiar with the concepts of formex algebra will be able to follow the formex formulations of Fig 22. However, a reader who is unfamiliar with formex algebra should not worry about the details of the formulations at this point. In the present discussion, the main aim is to describe the basics of the processes that are involved in creation of polyhedric configurations. The formex formulations in this context may then be seen as 'boxes of instructions' that imply the given

'Operation code' specifies the type of operation to be performed, namely, mapping on faces, mapping on edges or mapping on vertices. The operation code is an integer expression whose value is 1, 2 or 3 specifying mapping on faces, edges or vertices, respectively. In the example shown, the integer 2 specifies mapping on edges.

'Polyhedron code' specifies the type of polyhedron to be used as the basis for the operation. The polyhedron code is an integer expression whose value is in the range 1 to 20 specifying one of the Platonic or Archimedean polyhedra. In the example shown, the integer 7 indicates cuboctahedron.

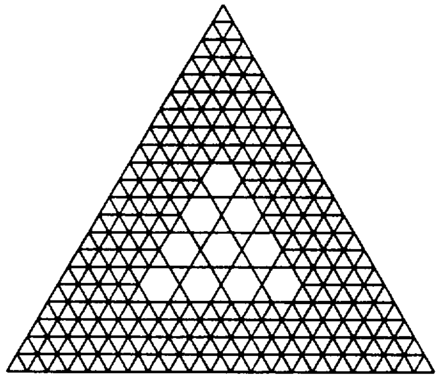
'Radius specifier' determines the size of the polyhedron by specifying the radius of its circumsphere. The radius specifier is a numeric expression whose value is a nonzero positive number. In the example shown, the radius is given as 10. It is possible to specify different radii for different layers of the object to be used for mapping. In this case, the radii are specified in terms of a formex expression.

```
G=pol(2,7,'[1-7,11,12]',10,[0,0;1,0]) | E
```

'Entity list' gives the list of face, edge or vertex numbers on which mapping is to be performed. The entity list is a string expression whose value is a list of items separated by commas. The items in the example shown are 1-7, 11 and 12, where 1-7 is equivalent to 1, 2, 3, 4, 5, 6 and 7. It is possible to use 'all' as an item implying all the faces, edges or vertices, as appropriate, listed in the ascending order. An item may also be a negative integer, like -8, or a negative parenthesised list, like -(6,12-15,9). A negative item has a cancelling effect.

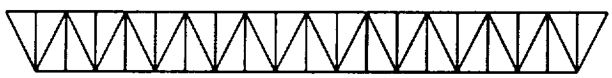
'Locator' is a formex expression whose value specifies the coordinates of the A-point and B-point of the object to be used for mapping. In the example shown, 0,0 and 1,0 are the coordinates of the A-point and B-point of the object, respectively.

Fig 21. Polymation function



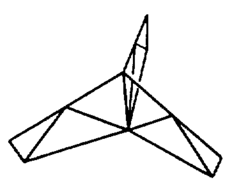
F=bb(1,sqrt | 3) | lux(genid(4,4,4,2,2,-1) | [15,5]) |
genid(21,21,2,1,1,-1) | {[0,0;2,0],[2,0;1,1],[1,1;0,0]}

(a)



E=pan(2,0) | (rin(1,20,1) | [0,0;1,0]#rin(1,10,2) | lam(1,1) |
[0,0;1,-2]#rin(1,18,1) | [1,-2;2,-2]#rin(1,19,1) | [1,0;1,-2])

(b)



a=-0.15; b=0.25; c=-a*tan | asin | (0.5/1.40125854)
V=rosax(0,0,0,c,0,a,3,120) | {[0,0,0;b,0,0],[0,0,0;c,0,a],[c,0,a;
b,0,a/2],[c,0,a;b/2,0,0],[b/2,0,0;b,0,a/2],[b,0,0;b,0,a/2]}

(c)

Fig 22. Formex formulations for a face-object, an edge-object
and a vertex-object

configurations.

SHAPING AND COMPOSING POLYHEDRA

The polymation function may be employed to create a variety of different kinds of polyhedric configurations some of which are outside the categories of configurations discussed so far. Two such classes of polyhedric configurations are discussed below.

The configurations shown in Fig 23 are obtained by cutting away parts of three polyhedra, where the original polyhedra are shown in thin lines and the resulting configurations are shown in thick lines. All three configurations are obtained using the polymation function and the formex formulation for each case is given enclosed in a box.

The configuration shown in thick lines in Fig 23(a) is similar to a P6 (truncated tetrahedron) but its proportions are different from those of P6. The configuration shown in thick lines in Fig 23(b) is a decahedron (a polyhedron with ten faces) which is obtained by cutting away parts of a cube and the configuration shown in Fig 23(c) is again a decahedron which is obtained by cutting away the top and bottom corners of an octahedron.

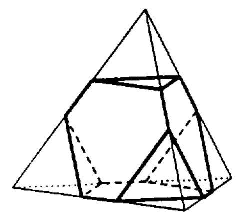
Examples of another class of polyhedric configurations are shown in Fig 24. Here, the polymation function has been used to create polyhedric configurations involving a combination of polyhedra or their parts. The formex formulations for these configurations are shown enclosed in boxes.

Fig 24(a) shows a configuration that is obtained by placing half-cubes on four faces of a cuboctahedron. Fig 24(b) shows a configuration that is obtained by placing two tetrahedra on two opposite faces of an octahedron. Fig 24(c) shows a configuration that is obtained by taking the top part of an icosahedron and combining it with its own reflection.

PROCESSING OF MULTI-LAYER POLYHEDRIC CONFIGURATIONS

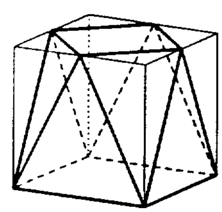
An example of a double layer polyhedric configuration is shown in Fig 25(a). Here, a Vierendeel girder type edge-object is mapped on a number edges of a dodecahedron and a formex formulation for the operation is shown enclosed in a box. The approach employed in handling the process is the same as that described for the example of Fig 18(b).

Fig 25(b) shows a different approach in dealing with the problem. Here, the mapping of both the top layer and bottom layer of the edge-object is controlled by the polymation function. Thus, there are



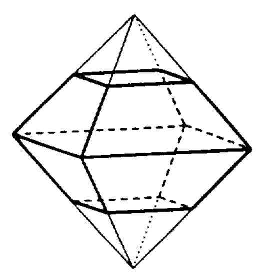
```
G=pex | pol(1,1,'[all]',1,[0,0,0;1,0,0]) | rosad(1/2,sqrt | 3/6,
   3,120) | {[0.4,0,0;0.6,0,0],[0.4,0,0;0.2,sqrt | 3/5,0]}
```

(a)



```
G=pol(1,2,'[1]',1,[0,0,0;2,0,0]) | rosad(1,1) | [0,1,0;1,0,0]#rosad(0,0) |
   pol(1,2,'[2]',1,[0,0,0;2,0,0]) | {[0,0,0;2,0,0],[2,0,0;1,2,0],[1,2,0;0,0,0]}
```

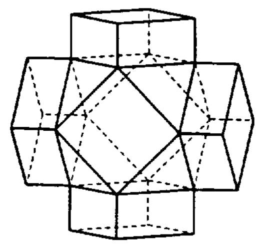
(b)



```
G=rosad(0,0) | pex | lam(3,0) | pol(1,3,'[1]',1,[0,0,0;4,0,0]) |
   {[0,0,0;4,0,0],[0,0,0;1,tan | 60,0],[1,tan | 60,0;3,tan | 60,0]}
```

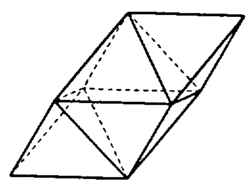
(c)

Fig 23. Examples of polyhedron shaping



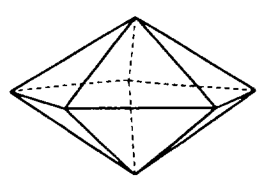
G=pol(2,7,'[all]',1,[0,0,0;1,0,0]) | [0,0,0;1,0,0]#pol(1,7,'[9-14,-(10,12)]',
1,[0,0,0;1,0,0]) | dil(3,0.5) | tranix(0.5,0.5,0.5) | pol(2,2,'[1-8]',
sqrt | 3/2,[0,0,0;1,0,0]) | [0,0,0;1,0,0]

(a)



G=pol(2,3,'[all]',1/sqrt | 2,[0,0,0;1,0,0]) | [0,0,0;1,0,0]#pol(1,3,'[1,7]',
1/sqrt | 2,[0,0,0;1,0,0]) | ver(2,1,0,0) | tranix(-cos | 30/3,0.5,
1/sqrt | 24) | pol(2,1,'[1-3]',sqrt | 6/4,[0,0,0;1,0,0]) | [0,0,0;1,0,0]

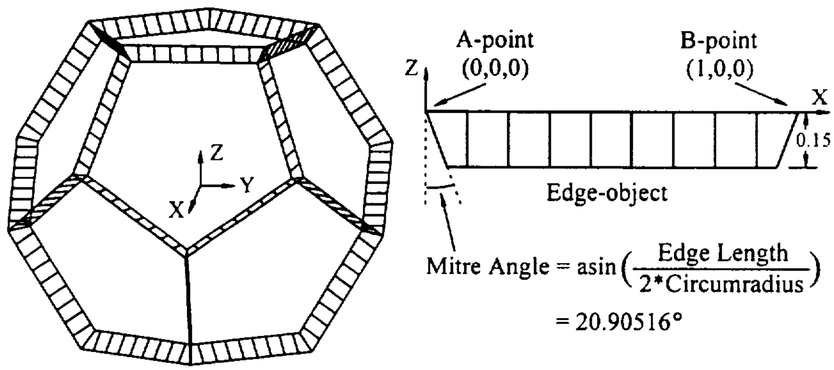
(b)



G=pex | lam(3,cos | (2*asin | (0.5/0.951056516))) |
pol(2,5,'[1-10]',1,[0,0,0;1,0,0]) | [0,0,0;1,0,0]

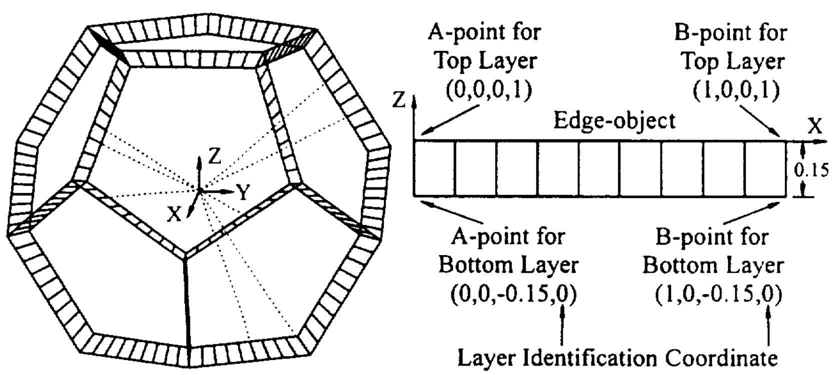
(c)

Fig 24. Examples of polyhedral compositions




```
rc=1.40125854; a=-0.15; b=1/9; c=-a*tan|asin|(0.5/rc)
E=lam(1,0.5)|{[0,0,0;c,0,a],[c,0,a;b,0,a]}#lux([1,0,a])|
   rin(1,9,b)|{[0,0,0;b,0,0],[b,0,0;b,0,a],[b,0,a;2*b,0,a]}
G=pol(2,4,'[1-13,-8,18-26,-(23-24),30]',rc,[0,0,0;1,0,0])|E
```

(a)




```
rc=1.40125854; a=-0.15; b=1/9; d=-a/cos|asin|(0.5/rc)
E=rin(1,9,b)|[0,0,0,1;b,0,0,1]#rin(1,9,b)|
   [0,0,a,0;b,0,a,0]#rin(1,10,b)|[0,0,0,1;0,0,a,0]
G=pol(2,4,'[1-13,-8,18-26,-(23-24),30]',[2,4;1,rc;0,rc-d],
   [0,0,0;1,0,0],[0,0,a;1,0,a])|E
```

(b)

Fig 25. Double layer mapping

two A-points and two B-points with additional 4^{th} coordinates for layer identification. A formex formulation for the operation is shown in a box in Fig 25(b). Also, the set up of the polymation function for the problem is given in Fig 26. The procedure followed in this approach is more elaborate than that used in relation to Fig 25(a). The main advantage in the second approach is that the problem of mitring is sorted out automatically.

The approach employed in creating the polyhedric configuration of Fig 25(b) may also be applied in the cases when there are more than two layers and for the cases involving multi-layer face or vertex mapping.

It should be noted that the polyhedric configurations of Figs 25(a) and 25(b) are not completely identical. The difference is in the orientations of the web elements. To be specific, the web elements in Fig 25(a) remain perpendicular to the top and bottom chords whereas the web elements in Fig 25(b) are along radial lines emanating from the centre, as indicated by dotted lines in the figure. However, this particular feature of the configuration of Fig 25(b) should not be considered as a necessary consequence of the second mapping approach. The situation, in general, may be described as follows:

In the first mapping strategy, as exemplified by Fig 25(a), the geometric proportions of the face-object, edge-object or vertex-object remain unchanged in the process of mapping. Here, the term 'geometric proportions' is used to mean those aspects of a configuration that remain unchanged under photographic enlargement or reduction. In the second mapping strategy, as exemplified by Fig 25(b), the geometric proportions of the face-object, edge-object or vertex-object may change in the process of mapping. However, there are no general rules regarding the manner in which the proportions may change. These changes are governed by the choices of the A-points and B-points.

ACKNOWLEDGEMENTS

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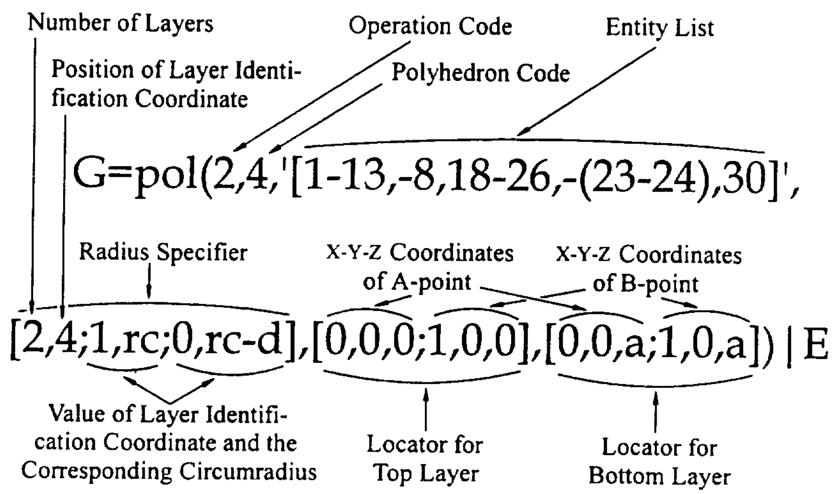

Fig 26. Polymation function for double layer mapping

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APPENDIX

ORDERING RULES

This appendix contains a collection of rules for ordering of the faces, edges and vertices of a Platonic or Archimedean polyhedron, where, the term 'ordering' is used to mean 'putting in a sequence'. The sequencing would then allow identity numbers to be assigned to the faces, edges and vertices of a polyhedron. This is done by taking the serial position number of an entity in the sequence as its identity number. Included in the appendix are also rules that govern the choices of baselines for faces, A-ends and B-ends for edges and handles for vertices. The rules are as follows:

(1) When all the faces of a polyhedron have the same number of sides, then the faces are ordered with respect to the ascending values of the angular spherical coordinates s and t of the centres of the faces, where the value of t is considered first and the value of s is considered only if the centres of the faces compared have the same value of t. The disposition of the s and t spherical coordinates together with the X-Y-Z global coordinate system in relation to a polyhedron (cuboctahedron) are shown in Fig A1.

When the faces of a polyhedron have different numbers of sides, then all the faces that have the same number of sides are considered together for ordering, starting with the faces that have the

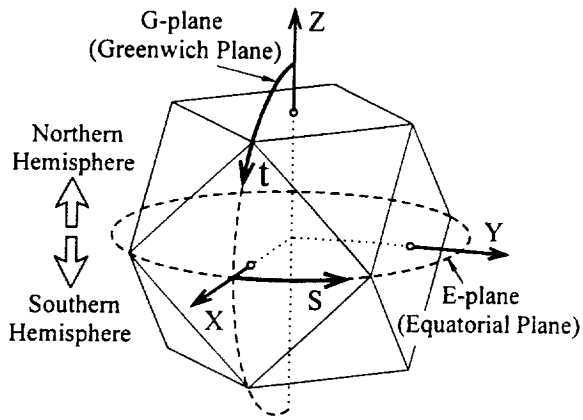

Fig A1. Cartesian and spherical coordinate
systems for a polyhedron

least number of sides and proceeding in the order of increasing number of sides.

(2) For each face of a polyhedron, one of its sides is designated as the 'baseline'. The baseline of a face is chosen in the following manner:

(a) If only one of the sides of the face is parallel to the 't=90° plane', (that is, the X-Y plane), then this side is chosen as the baseline of the face. The 't=90° plane' is referred to as the 'equatorial plane' or the 'E-plane', Fig A1.

(b) If only two of the sides of the face are parallel to the E-plane, then of these two sides, the one that is nearer to the E-plane is chosen as the baseline and if the sides are equidistant from the E-plane then the 'southern' side is chosen as the baseline.

(c) If the face is parallel to the E-plane and one of its sides intersects the 's=0 semi-plane', then this side is chosen as the baseline of the face. The 's=0 semi-plane' is referred to as the 'Greenwich plane' or the 'G-plane'. This is the part of the X-Z plane for which $X \geq 0$, Fig A1.

(d) If the face is parallel to the E-plane and the G-plane passes through a corner of the face, then of the two sides that are connected to this corner, the one whose midpoint has the smaller s coordinate is chosen as the baseline of the face.

(e) If the face does not have a side that is parallel to the E-plane, then the face is imagined to be subjected to a rotation in the 'right-handed screw direction' and, as the angle of rotation increases, the first side that assumes a position satisfying one of the conditions (a) or (b) above is chosen as the baseline of the face. To describe the term 'right-handed screw direction', imagine a right-handed screw whose head is at the centre of the polyhedron and is pointing towards the centre of a face. The direction of rotation that causes the screw to move towards the face is referred to as the 'right-handed screw direction' or the 'RS-direction'.

(3) Each end of the baseline of a face of a polyhedron has an associated letter. To elaborate, one end is associated with the letter A and is referred to as the A-end and the other end is associated with the letter B and is referred to as the B-end. The allocation of the letters A and B to the ends is made such that movement from A to B is in the RS-direction.

(4) The edges of a polyhedron are ordered with respect to the ascending values of the angular

spherical coordinates s and t of their midpoints, where the value of t is considered first and the value of s is considered only if the midpoints of the edges compared have the same value of t.

(5) Each end of an edge of a polyhedron has an associated letter. To elaborate, one end is associated with the letter A and is referred to as the A-end and the other end is associated with the letter B and is referred to as the B-end. The allocation of the letters A and B to the ends is made in the following manner:

(a) If the edge is parallel to the E-plane then the A-end and B-end of the edge are chosen such that movement from A to B is in the positive s direction, Fig A1.

(b) If the edge is not parallel to the E-plane then the A-end and B-end of the edge are chosen such that

- if the midpoint of the edge is in the E-plane or if its midpoint is in the northern hemisphere then movement from A to B is southward and
- if the midpoint of the edge is in the southern hemisphere then movement from A to B is northward.

(6) The vertices of a polyhedron are ordered with respect to the ascending values of their angular spherical coordinates s and t, where the value of t is considered first and the value of s is considered only if the vertices compared have the same value of t.

(7) For each vertex, one of the edges that is connected to it is designated as the 'handle'. The handle of the first vertex of a polyhedron (that is, vertex No 1) is the edge that connects it to vertex No 2. The handle of any other vertex is obtained by mapping the configuration of the first vertex onto the configuration of that vertex and selecting the edge that corresponds to the handle of the first vertex. The configuration of a vertex of a polyhedron can, in most cases, be mapped onto the configuration of any other vertex of the same polyhedron by simple 'rigid motion' (by translation and rotation). However, in some cases the mapping of the configuration of a vertex onto that of another vertex cannot be achieved unless an additional reflectional operation is performed. To elaborate, with two exceptions, for every Platonic or Archimedean polyhedron, all the vertices of the polyhedron are 'directly congruent'. That is, the configuration of each vertex of the polyhedron may be mapped onto that of every other vertex of the polyhedron by simple rigid motion of the configuration. The exceptions are P11 (great rhombicuboctahedron) and P18 (great rhombicosidodecahedron). For each of these two polyhedra, some vertices are directly congruent

to the first vertex of the polyhedron and the other vertices are 'oppositely congruent' to the first vertex. The term 'oppositely congruent' is used to refer to two configurations that cannot be mapped into one another without reflection (in addition to rigid motion). The need for reflection in vertex mapping for P11 and P18 arises as a consequence of the shapes of their vertex figures, as shown in Table A1 (a 'vertex figure' is a polygon obtained by connecting the midpoints of the edges that meet at a vertex). From Table A1 it may be seen that every Platonic or Archimedean polyhedron, other than P11 and P18, has only one vertex figure. On the other hand, in the case of P11 or P18, there are two vertex figures that cannot be mapped into one another without reflection.

The vertex mapping procedure described above will allow the handles to be 'uniquely' determined for all the vertices in all the cases except for P12 (icosidodecahedron), P7 (cuboctahedron) and the Platonic polyhedra. For each of these seven polyhedra, the mapping of the configuration of the first vertex onto that of another vertex can be done in more than one way. This is a consequence of the shapes of the vertex figures for these seven polyhedra. To elaborate, it may be seen from Table A1 that the vertex figure of each of these polyhedra can map into itself in more than one way. In the case of these polyhedra, the handles of vertices are chosen using the following rules:

(a) For a 'ring' of vertices, that is, for a circularly disposed set of vertices that lie in a plane parallel to the E-plane, the handles are chosen such that they constitute a cyclically symmetric configuration.

(b) The disposition of the handles for a southern ring of vertices is obtained by 'turning over' the corresponding northern ring (and rotating it, if necessary).

(c) If there is a vertex whose handle is not uniquely determined by the above rules then, amongst different possible handles, the one that has the smallest vertex number at the other end is chosen.

(8) Each end of the handle of a vertex of a polyhedron has an associated letter. To elaborate, the end that is at the vertex is associated with the letter A and is referred to as the A-end and the other end is associated with the letter B and is referred to as the B-end.

Table A1. Vertex figures of Platonic and Archimedean polyhedra

Polyhedron	Vertex Figure
P1: Tetrahedron	Equilateral Triangle
P2: Cube	Equilateral Triangle
P3: Octahedron	Square
P4: Dodecahedron	Equilateral Triangle
P5: Icosahedron	Regular Pentagon
P6: Truncated Tetrahedron	Isosceles Triangle
P7: Cuboctahedron	Rectangle
P8: Truncated Cube	Isosceles Triangle
P9: Truncated Octahedron	Isosceles Triangle
P10: Small Rhombicuboctahedron	Isosceles Trapezium
P11: Great Rhombicuboctahedron	Irregular Triangles or (reflections)
P12: Icosidodecahedron	Rectangle
P13: Truncated Dodecahedron	Isosceles Triangle
P14: Truncated Icosahedron	Isosceles Triangle
P15: Left Snub Cube **P16:** Right Snub Cube	Pentagon (with a single axis of symmetry)
P17: Small Rhombicosidodecahedron	Isosceles Trapezium
P18: Great Rhombicosidodecahedron	Irregular Triangles or (reflections)
P19: Left Snub Dodecahedron **P20:** Right Snub Dodecahedron	Pentagon (with a single axis of symmetry)