MIGRATIONAL INSTABILITIES IN PARTICLE SUSPENSIONS

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ABSTRACT

This work deals with an instability arising from the shear-induced migration of particles in dense suspensions coupled with a dependence of viscosity on particle concentration. The analysis summarized here treats the inertialess ($Re = 0$) linear stability of homogeneous simple shear flows for a Stokesian suspension model of the type proposed by Leighton and Acrivos (1987). Depending on the importance of shear-induced migration relative to concentration-driven diffusion, this model admits short-wave instability arising from wave-vector stretching by the base flow and evolving into particle-depleted shear bands. Moreover, this instability in the time-dependent problem corresponds to loss of ellipticity in the associated static problem ($Re = 0, Pe = 0$).

While the isotropic version of the Leighton-Acrivos model is found to be stable with their experimentally determined parameters for simple shear, it is known that the stable model does not give a good quantitative description of particle clustering in the core of pipe flow (Nott and Brady 1994). This leads to the conjecture that an appropriate variant on the above model could explain such clustering as a two-phase bifurcation in the base flow.

INTRODUCTION

The general theme of the present work is *material instability*, whose recent history, from the 1950s onwards, mainly resides in solid mechanics. There one encounters a wide variety of "strain-softening" phenomena giving rise to spatial localization of strain in the form of "shear bands" or "necks". By comparison, there is a much smaller literature dealing with related instabilities in fluid-like materials, such as non-monotone viscous stress in particulate suspensions and polymeric fluids or dissipative clustering in rapid granular flows.

A recent report (Goddard 1996) contains a brief review and fairly extensive bibliography whose common thread is the "short-wavelength" instability of homogeneous deformations and the associated "change of type" in the underlying field equations, *e.g.* loss of static ellipticity or of dynamic hyperbolicity (Joseph 1990).

The present paper deals with a class of material instabilities in which diffusional scalar transport plays a crucial rôle. Prior related studies involve the effects of heat conduction on thermally-softening media, where mechanical dissipation (having no direct counterpart here) gives rise to phenomena such as bifurcation in liquid flows or "adiabatic" shear bands in solids.

MODEL AND ANALYSIS

We consider a neutrally-buoyant "Stokesian" suspension (*cf.* Leighton and Acrivos, 1987) with stress:

$$\mathbf{T} = 2\eta(\phi)\mathbf{D} - p\mathbf{1} \tag{1}$$

and particle flux:

$$\mathbf{j} = -\{\kappa(\phi,\dot\gamma)\nabla\phi + \nu(\phi,\dot\gamma)\nabla\dot\gamma\} \tag{2}$$

where ϕ denotes particle volume fraction,

$$\mathbf{D} := \frac{1}{2}\{\nabla\mathbf{v} + (\nabla\mathbf{v})^T\}, \quad \dot\gamma := \left\{2tr\left(\mathbf{D}^2\right)\right\}^{\frac{1}{2}} \tag{3}$$

and $\mathbf{v}$ is mixture velocity. The quasi-static mechanics ($Re = 0, Pe \geq 0$) is governed by the above constitutive equations, the balances

$$\nabla\cdot\mathbf{T} = \nabla p \quad \text{with} \quad \nabla\cdot\mathbf{v} = 0, \tag{4}$$

and

$$\partial_t\phi + \mathbf{v}\cdot\nabla\phi + \nabla\cdot\mathbf{j} = 0, \tag{5}$$

together with suitable boundary conditions (compatibility with the base state and/or regularity of perturbed states in unbounded domains).

For planar infinitesimal perturbations $\mathbf{v}^{(1)}, \phi^{(1)}$ of the uniform base state:

$$v_x^{(0)} = \dot\gamma^{(0)}y \quad \text{and} \quad \{v_y^{(0)}, v_z^{(0)}, \nabla\dot\gamma^{(0)}, \nabla\phi^{(0)}\} = 0, \tag{6}$$

the standard stream-function *cum* Fourier representation in the x-y plane, $\Psi(k_x, k_y, t)$, $\Phi(k_x, k_y, t)$, must satisfy

$$k^4\Psi + \alpha(k_x^2 - k_y^2)\Phi = 0 \tag{7}$$

and

$$\partial_t\Phi - k_x\partial_{k_y}\Phi = -k^2\{\nu(k_x^2 - k_y^2)\Psi + \kappa\Phi\} \tag{8}$$

where we let

$$k^2 = k_x^2 + k_y^2 \text{ and } \alpha = \dot\gamma^{(0)}\left(\frac{dlog\eta}{d\phi}\right)^{(0)} \tag{9}$$

and adopt the change of notation

$$\dot\gamma^{(0)}t \to t, \ \frac{\kappa^{(0)}}{\dot\gamma^{(0)}} \to \kappa, \ \frac{\nu^{(0)}}{\dot\gamma^{(0)}} \to \nu \tag{10}$$

The stream function Ψ is readily eliminated from (7)-(8), which are easily seen to be elliptic for $\kappa > \nu\alpha$ and otherwise hyperbolic, with the possibility of singular surfaces.

Wave-Vector Shearing

With Ψ eliminated, (7)-(8) become

$$\partial_t\Phi - k_x\partial_{k_y}\Phi = \sigma\{\mathbf{k}\}\Phi \tag{11}$$

where

$$\sigma\{\mathbf{k}\} = -k^2\{\kappa - \frac{\nu\alpha}{k^4}(k_x^2 - k_y^2)^2\} \tag{12}$$

Were it not for the term $k_x\partial_{k_y}$, representing a wave-vector shearing (Thomson 1887, Savage 1992, Wang *et al.* 1996) that becomes dominant for large time, the preceding relation would immediately provide the dispersion relation for mode $\mathbf{k}$ which, incidentally, is identical with that given by the simplified 1-D treatment in the concluding section of Nott and Brady (1994).

We note that wave-vector stretching in an arbitrary isochoric homogeneous flow, with $\nabla\mathbf{v} = \mathbf{L}^T$ =const. and $tr\{\mathbf{L}\} \equiv \nabla\cdot\mathbf{v} = 0$, can be concisely represented by the respective transformations between material ("embedded") coordinates $(\hat{\mathbf{x}}, \hat{\mathbf{k}})$ in physical space and "reciprocal" space:

$$\hat{\mathbf{x}} = \mathbf{F}^{-1}\mathbf{x}, \quad \text{with } (\partial_t)_{\hat{\mathbf{x}}} = (\partial_t)_{\mathbf{x}} + (\mathbf{L}\mathbf{x})\cdot\partial_{\mathbf{x}} \tag{13}$$

and

$$\hat{\mathbf{k}} = \mathbf{F}^T\mathbf{k}, \quad \text{with } (\partial_t)_{\hat{\mathbf{k}}} = (\partial_t)_{\mathbf{k}} - (\mathbf{L}^T\mathbf{k})\cdot\partial_{\mathbf{k}} \tag{14}$$

where

$$\mathbf{F}(t) = e^{\mathbf{L}t} \tag{15}$$

is the physcial-space deformation gradient.

For the simple shear at hand, the transformation (14) gives $\mathbf{k}(\hat{\mathbf{k}}, t)$ as

$$k_x \equiv \hat{k}_x \text{ and } k_y = \hat{k}_y - \hat{k}_x t \tag{16}$$

and converts (11) into the o.d.e.

$$\frac{dlog\Phi}{dt} = \sigma\{\mathbf{k}(\hat{\mathbf{k}}, t)\} \tag{17}$$

where $\sigma\{\mathbf{k}\}$ is given by (12).

While (17) can be integrated subject to (16), asymptotic stability is determined by its limiting form for large t, and there are two cases to consider, $\hat{k}_x \neq 0$ and $\hat{k}_x = 0$, for which, respectively, one easily finds from (16)-(17) that

$$log\Phi \sim (\nu\alpha - \kappa)\frac{\hat{k}_x^2 t^3}{3} \sim (\nu\alpha - \kappa)\frac{k_y^2 t}{3} \tag{18}$$

and

$$log\Phi \equiv (\nu\alpha - \kappa)k_y^2 t \tag{19}$$

for $t \to \infty$. Thus, $\kappa < \nu\alpha$ implies short-wavelength instability in the form of a ("Kelvin-mode") shear-band structure, with $k_y >> k_x$, representing particle-depleted strata lying perpendicular to the y-direction.

A Special Case

To describe the particle distribution in a fully-developed rectilinear shear flow, Leighton and Acrivos (1987) employ (1) together with the 1-D form for particle flux :

$$j_y = -\phi^2\left\{K_1\frac{d\dot{\gamma}}{dy} + K_2\alpha\frac{d\phi}{dy}\right\} \tag{20}$$

where

$$\alpha = \dot{\gamma}\frac{dlog\eta}{d\phi} \geq 0 \tag{21}$$

and where the coefficients $K_1 \geq 0$ and $K_2 \geq 0$ are constants, or only weakly dependent on ϕ (and proportional to particle diameter squared).

Taking (2) as the obvious extension of (20) to 3D (*i.e.* ignoring shear-induced anisotropy), one can easily deduce from the above stability analysis that linear stability (against planar disturbances) requires that

$$K_2 > K_1, \tag{22}$$

plainly satisfied by the empirically assigned values of Leighton and Acrivos (1987), for which $K_2 \approx 2K_1$ over a relatively broad range of ϕ.

Recently Nott and Brady (1994) have made the interesting observation that the stable form of (20) does not give a particularly good quantitative description of the the dense "plug-flow" clustering of particles in the core of pipe flow. While they propose a more elaborate constitutive theory, this author wonders whether a mere adjustment of the coefficients K_1, K_2, to impart instability near the state of maximum packing ϕ_{max}, might not suffice to explain the above clustering as a flow bifurcation with two-phase, core-annular structure .

CONCLUSIONS

Depending on its coefficients, the above particle-migration model can exhibit instability in simple shear, the dominant linear mode being particle-depleted shear bands. While the empirical form proposed by Leighton and Acrivos is linearly stable, an unstable version might allow for a better description of particle clustering in pipe flow.

It is plausible that the migrational instability considered here would be enhanced by non-Newtonian rheology, as in shear-thinning suspensions of colloidal or deformable particles, thereby accentuating the tendency towards clustering or phase separation.

For the simple constitutive model of the present work, loss of ellipticity in the static problem is directly connected with time-dependent instability. This broaches an interesting and unresolved mathematical question as to the precise relation between material stability and static ellipticity for materals with long-range memory.

Several of the above issues are the subject of further work in progress.

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