A Variational Formulation for the Finite Element Analysis of Sound Wave Propagation in a Spherical Shell

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Abstract

Development of design tools to furnish optimal acoustic environments for lightweight aircraft demands the ability to simulate the acoustic system on a workstation. In order to form an effective mathematical model of the phenomena at hand, we have begun by studying the propagation of acoustic waves inside closed spherical shells. Using a fully-coupled fluid-structure interaction model based upon variational principles, we have written a finite element analysis program and are in the process of examining several test cases. Future investigations are planned to increase model accuracy by incorporating non-linear and viscous effects.

Introduction

An important problem in the design of commercial aircraft is noise reduction in both the cabin and the outside environment. The need for a low level of noise both inside and outside the airplane must be balanced with the need to maintain a light structure. We have undertaken a program of computational modeling of this system in order to develop the modeling tools necessary for design optimization of future aircraft.

A favorite computational scheme for studying problems of this type is the finite element method. The finite element method was first used for multi-disciplinary applications in the 1960's, when it was found to be appropriate for issues including heat conduction, hydrodynamic lubrication, and eigenvalue solutions of the Helmholtz equation for fluids vibrating in closed spaces [1], p. 274. One of the earliest instances of the employment of finite elements in structural-acoustic problems was by the automotive industry. In 1982, Nefske et al. [2] performed this kind of analysis for passenger car and truck cabins. In this work they present representations of the acoustic modes within the compartments as well as frequency response computations using modal synthesis.

Much other research has been done to establish a method of coupling the behavior of an enclosed fluid to the dynamics of the surrounding structure. Usually this is simply done by requiring that no fluid enters the solid surface. This implies that the surface normal component of the relative velocity between the fluid and the structure must equal zero. The equations of the solid and the fluid are thus solved seperately, and the matching of the normal velocities at the surface forms the coupling. It is done this way because to start with the equations of motion for the whole system and simply try to apply a finite element method (such as a Galerkin method) results in difficult, asymmetric matrices. However, the problem with separate solution of the equations is that certain terms drop out which in fact would not drop out if a more complicated model of the interaction between fluid and structure was used. These terms are the mass terms, and their inclusion would make for more precise mathematical predictions.

Instead of starting from the equations of motion, we have started from a variational formulation of Hamilton's principle, and included the fluid-structure interaction as a constraint on the system. This formulation is based upon the work of Kock and Olson [3]. In our first case we have ignored gravitational and viscous effects. Though this is not an improvement upon previous methods, we wanted to use this simplest case as a method of comparison. It should produce exactly the same result as solving the equations separately and matching velocities. If it does so, the stage is set for the inclusion of additional terms into the statement of Hamilton's principle which will improve the accuracy of the model.

Variational Formulation

Hamilton's principle is a general statement about the conservation of energy of a system of particles (see, e.g. Bedford [4]). It states that the value of the integral over time of the kinetic minus potential energy of the system takes a minimum value. Using the common notation of variational calculus (see e.g. Courant and Hilbert [5]), this can be written as

$$\delta \int_{t_0}^{t_1} (T - U)\, dt = 0$$

in the case of a conservative system, where T is the kinetic energy and U is the internal energy. By using the principle of virtual work, Hamilton's principle is easily extended to apply to nonconservative systems (in which some force adds energy to the system), and also to systems where the generalized coordinates are not independent. Physically, this corresponds to the case where the quantities in question are related by some equations, and those equations are known as constraints. With the addition of constraints C and virtual work terms W, Hamilton's principle becomes

$$\delta \int_{t_0}^{t_1} (T - U + W + C)\, dt = 0. \tag{1}$$

For the problem at hand, we will use equation 1, with a virtual work term supplied by a surface force on the sphere, a constraint that enforces velocity matching on the boundary, and constraints that enforce local and global conservation of mass.

The solid shell contributes kinetic energy, potential energy and a surface force term.

$$T_s = \int_{\Omega_s} \frac{1}{2}\, \rho_s \left(\frac{\partial \overline{U}}{\partial t} \cdot \frac{\partial \overline{U}}{\partial t} \right) d\Omega_s\,, \qquad U_s = \int_{\Omega_s} \frac{1}{2}\, \overline{\varepsilon}\, \mathbf{C}\, \overline{\varepsilon}\, d\Omega_s$$

$$W_s = \int_{\Gamma_s} \overline{U} \cdot \overline{f}_s\, d\Gamma_s\,. \tag{2}$$

Here Ω_s is the solid volume, Γ_s, the solid boundary, $\overline{U}$, the displacement vector, $\overline{\varepsilon}$, the strain vector, $\mathbf{C}$, the material stiffness tensor, and $\overline{f}_s$ is the surface force acting on the solid.

By taking into account the assumption that the shell is thin, we can change the volume integrals to surface integrals. This is due to the fact that we assume the displacements are linearly distributed throughout the thickness of the shell, so that the normal to the element is preserved – it remains normal and does not change in length under the deformation. Then the displacement compoments $\overline{U} = (U_1, U_2, W)$ can be written as

$$U_1(\alpha_1, \alpha_2, \zeta) = u_1(\alpha_1, \alpha_2) + \zeta \beta_1(\alpha_1, \alpha_2, 0)\,,$$
$$U_2(\alpha_1, \alpha_2, \zeta) = u_2(\alpha_1, \alpha_2) + \zeta \beta_2(\alpha_1, \alpha_2, 0)\,,$$
$$W(\alpha_1, \alpha_2, \zeta) = w(\alpha_1, \alpha_2)\,,$$

where $\alpha_1, \alpha_2, \zeta$ are general curvilinear coordinates, u_1, u_2, w are the components of the displacement of a point on the reference surface and β_1, β_2 are rotations. The thin shell assumptions also allow us to write the rotations in terms of u_1, u_2, and w. We can write the strain vector $\overline{\varepsilon}$ in terms of the displacements $\overline{u}$ by using the strain-displacement relations in general curvilinear coordinates. These relations will not be reproduced here as they are lengthy and can be found in many references, see e.g. Kraus [6], p. 31. We will write them as $\overline{\varepsilon} = \mathbf{A}\overline{u}$, where $\mathbf{A}$ is a matrix operator.

We now substitute all of these relations into equations 2 and integrate over the normal component ζ. Then the contributions of the solid become

$$T_s = \int_{\Gamma_s} \frac{h\rho_s}{2} \left(\frac{\partial \overline{u}}{\partial t} \cdot \frac{\partial \overline{u}}{\partial t} \right) d\Gamma_s\,, \qquad U_s = \int_{\Gamma_s} (\mathbf{A}\overline{u})^{\mathrm{T}} \mathbf{Q} \mathbf{A}\overline{u}\, d\Gamma_s\,.$$

400

$\mathbf{Q}$ is the matrix $\mathbf{C}$ after the integration – it is multipied by some combinations of powers of h. This h is the thickness of the sphere, and W_s, of course, is unchanged. The solid functional, then, is

$$\Pi_s = \int_{t_0}^{t_1} \left[\int_{\Gamma_s} (\mathbf{A}\overline{u})^{\mathrm{T}} \mathbf{Q} \mathbf{A} \overline{u}\, d\Gamma_s - \int_{\Gamma_s} \frac{h\rho_s}{2} \left(\frac{\partial t\overline{u}}{\partial t} \cdot \frac{\partial \overline{u}}{\partial t} \right) d\Gamma_s - \int_{\Gamma_s} \overline{u} \cdot \overline{f}_s\, d\Gamma_s \right] dt.$$

Taking the first variational, and integrating by parts in the T_s term gives

$$\delta\Pi_s = \int_{\Gamma_s} (\mathbf{A}\overline{u})^{\mathrm{T}} \mathbf{Q} \mathbf{A} \delta\overline{u}\, d\Gamma_s + \int_{\Gamma_s} \frac{h\rho_s}{2} \frac{\partial^2 \overline{u}}{\partial t^2} \cdot \delta\overline{u}\, d\Gamma_s - \int_{\Gamma_s} \overline{f}_s \cdot \delta\overline{u}\, d\Gamma_s = 0. \qquad (3)$$

The fluid contributes the following terms:

$$T_f = \int_{\Omega_f} \frac{1}{2}\rho_f \overline{v} \cdot \overline{v}\, d\Omega_f, \qquad U_f = \int_{\Omega_f} \rho_f e\, d\Omega_f,$$

where Ω_f is the volume of the fluid, and e is the specific internal energy of the fluid. The constraints of local and global conservation of mass, as well as the fluid-structure interaction , are part of the fluid functional. Hence, this functional is of the form

$$\Pi_f = \int_{t_0}^{t_1} \left[\int_{\Omega_f} \left\{ \rho_f e - \frac{1}{2}\rho_f \overline{v} \cdot \overline{v} - \lambda_1 \left(\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \overline{v}) \right) - \lambda_2 \rho_f \right\} d\Omega_f \right.$$
$$\left. + \int_{\Gamma_s} \lambda_3 \left(-\frac{\partial u_n}{\partial t} + v_n \right) d\Gamma_s + \lambda_2 M_f \right] dt$$

where $\lambda_1, \lambda_2, \lambda_3$ are Lagrange multipliers, u_n is the normal displacement of the fluid boundary, v_n is the normal velocity of the fluid boundary, and M_f is the mass of the fluid.

Taking the first variation with respect to $\overline{v}$ and setting each integral to zero gives a series of equations. These equations reveal the true meanings of the Lagrange multipliers. This has been done in detail by Kock and Olson [3], p. 467. We will reproduce only their results. The Lagrange multiplier $\lambda_1 = \varphi$, where φ is the scalar potential (that is, $\overline{v} = \nabla\varphi$). In addition, $\lambda_3 = \rho_f \varphi$, and λ_2 is still unknown, so it shall be renamed λ. After integration by parts, the fluid functional becomes

$$\Pi_f = \int_{t_0}^{t_1} \left[\int_{\Omega_f} \left\{ \rho_f e - \frac{1}{2}\rho_f (\nabla\varphi)^2 - \varphi \left(\frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \nabla\varphi) \right) \right. \right.$$
$$\left. \left. - \lambda\rho_f + \rho_f \frac{\partial\varphi}{\partial t} \right\} d\Omega_f + \lambda M_f \right] dt.$$

In the linearized case it is assumed that the change in fluid density is only a small pertubation from the inital density ρ_0. By taking $\rho_f = \rho_0 + \rho$ and linearizing about the initial domain, that is, $\Omega_f \Rightarrow \Omega_{f_0}$ and $\Gamma_f \Rightarrow \Gamma_{f_0}$, the functional simplifies further. Following Kock and Olson ([3], p. 485), the expression becomes

$$\Pi_f = \int_{t_0}^{t_1} \left[\int_{\Omega_{f_0}} \left\{ -\frac{1}{2}\frac{\rho_0}{c_0} \left(\lambda - \frac{\partial\varphi}{\partial t} \right)^2 + \frac{1}{2}\rho_0 (\nabla\varphi)^2 \right\} d\Omega_{f_0} \right.$$
$$\left. + \int_{\Gamma_{f_0}} \rho_0 \left(\frac{\partial\varphi}{\partial t} - \lambda \right) u_n\, d\Gamma_{f_0} \right] dt,$$

where c_0 is the speed of sound. In this case taking the first variation and setting each part to zero gives

$$\int_{\Omega_{f_0}} \left[\frac{\rho_0}{c_0^2} \left(\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial \lambda}{\partial t} \right) - \nabla \cdot (\rho_0 \nabla \varphi) \right] \delta\varphi \, d\Omega_{f_0} + \int_{\Gamma_{f_0}} \left[\rho_0 \left(-\frac{\partial u_n}{\partial t} + \frac{\partial \varphi}{\partial n} \right) \right] \delta\varphi \, d\Gamma_{f_0} = 0 \, ,$$

$$\int_{\Gamma_{f_0}} \rho_0 \left(\frac{\partial \varphi}{\partial t} - \lambda \right) \delta u_n \, d\Gamma_{f_0} = 0 \, ,$$

$$\left[\int_{\Omega_{f_0}} -\frac{\rho_0}{c_0^2} \left(\lambda - \frac{\partial \varphi}{\partial t} \right) d\Omega_{f_0} - \int_{\Gamma_{f_0}} \rho_0 \, u_n \, d\Gamma_{f_0} \right] \delta\lambda = 0 \, . \tag{4}$$

Finite Element Analysis

We implement the finite element approximation using standard isoparametric elements (see, e.g., Huebner et al. [1], Ch. 5) to discretize the domain. Since the boundaries of each element will, of necessity, be curved, the boundary itself must be approximated. For each coordinate we will write an expression of the form

$$x(\eta, \xi, \zeta, t) = \sum_{i=1}^{m} N_i(\eta, \xi, \zeta) x_i(t) \, . \tag{5}$$

Here η, ξ, ζ are the natural coordinates of the element. N_i are the interpolation functions, in natural coordinates, and m is the number of nodes per element. We approximate all of the coordinates in this way. In addition, we use the same interpolation functions N_i to approximate the variables in the problem, that is, $u, v, w, \varphi,$ and λ. Displacement coordinates are now (u, v, w) instead of (u_1, u_2, w) to prevent confusion.

We will use two-dimensional elements to model the sphere, and three-dimensional elements for the fluid. The approximations are of the form

$$\overline{u}(\eta, \xi, t) = \sum_{i,j=1}^{m} N_{ij}(\eta, \xi) u_j(t) \, ,$$

$$\varphi(\eta, \xi, \zeta, t) = \sum_{i=1}^{m} N_i(\eta, \xi, \zeta) \varphi_i(t) \, ,$$

$$\lambda(t) = \sum_{i=1}^{m} \lambda_i(t) \, . \tag{6}$$

where N_{ij} is the $3 \times 3m$ matrix of interpolation functions

$$\mathbf{N} = \begin{pmatrix} N_1 & 0 & 0 & \ldots & N_m & 0 & 0 \\ 0 & N_1 & 0 & \ldots & 0 & N_m & 0 \\ 0 & 0 & N_1 & \ldots & 0 & 0 & N_m \end{pmatrix}$$

and $\overline{u}$ is the vector

$$\overline{u} = \begin{bmatrix} u_1 & v_1 & w_1 & \ldots & u_m & v_m & w_m \end{bmatrix} \, .$$

The choice of Lagrange polynomials as interpolation functions allows the nodal values $u_i, v_i, w_i, \varphi_i, \lambda_i$ to have the physical meaning of being the value of the function at that node. In our problem the nodal values are functions of time. Since we will be forcing the sphere sinusoidally, we will use Fourier transforms to transform out time and solve the probelm in the frequency domain. This will, of course, have no effect on the spatial integrals.

To substitute the approximations into the variational principles derived, it is necessary to address the issue of differing coordinate frames. Ths interpolation functions are in local coordinates η, ξ, ζ , whereas the variational form is in global coordinates – cartesian x, y, z in the case of the fluid, and a generalized curvilinear α_1, α_2 for the solid. To compound the problem, in our case all of the nodal coordinates and element information is obtained from the neutral file output of the commercial finite element code PATRAN. The PATRAN output has the coordinates of all nodes listed in a global cartesian system. So it is necessary to undertake various coordinate transformations before the integrals in the variational expressions can be evaluated.

The fluid expressions give no difficulty in this area. Equation 5 gives the form of the tranformation from cartesian to natural coordinates. It is known from calculus that $dx\,dy\,dz = |J|\,d\eta\,d\xi\,d\zeta$, where $|J|$ is the determinant of the Jacobian matrix of partial derivatives. Thus the integrals will transform in the following manner:

$$\int_V f(x,y,z)\,dx\,dy\,dz = \int_{V'} f'(\eta,\xi,\zeta)\,|J|\,d\eta\,d\xi\,d\zeta\ ,$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \zeta} \end{pmatrix}\ .$$

where the primes denotate the trasformed function and volume. Substituting equations 6 into the fluid equations 4 yields the following expressions.

$$\sum_{i,j=1}^{m} \left[\frac{d^2\varphi_i}{dt^2} \int_V \frac{\rho_0}{c_0^2} N_i\,N_j\,|J|\,d\eta\,d\xi\,d\zeta + \varphi_i \int_V \rho_0\,\nabla N_i\,\nabla N_j\,|J|\,d\eta\,d\xi\,d\zeta \right.$$
$$\left. - \frac{d\lambda_i}{dt} \int_V \frac{\rho_0}{c_0^2}\,N_i\,|J|\,d\eta\,d\xi\,d\zeta - \frac{du_i^n}{dt} \int_S \rho_0\,N_i(\eta,\xi)\,N_j(\eta,\xi)\,|J|\,d\eta\,d\xi \right] \delta\varphi_j = 0\ ,$$

$$\sum_{i,j=1}^{m} \left[\frac{d\varphi_i}{dt} \int_S \rho_0\,N_i(\eta,\xi)\,N_j(\eta,\xi)\,|J|\,d\eta\,d\xi - \lambda_i \int_S \rho_0\,N_i(\eta,\xi)\,|J|\,d\eta\,d\xi \right] \delta u_j^n = 0\ ,$$

$$\sum_{i,j=1}^{m} \left[\lambda_i \int_V -\frac{\rho_0}{c_0^2}\,|J|\,d\eta\,d\xi\,d\zeta + \frac{d\varphi_i}{dt} \int_V \frac{\rho_0}{c_0^2}\,N_i\,|J|\,d\eta\,d\xi\,d\zeta \right.$$
$$\left. - u_i^n \int_S \rho_0\,N_i(\eta,\xi)\,|J|\,d\eta\,d\xi \right] \delta\lambda_j = 0\ .$$

Note that this integration is over the area or volume of the element in natural coordinates, so that the range of each variable is $(-1, 1)$. Following Kock and Olson [3], we will write these integrals in shorthand, where subscripts denote vector and matrix components:

$$\sum_{i,j=1}^{m} \left[\frac{d^2\varphi_i}{dt^2} M_{ij}^{\varphi} + \varphi_i K_{ij}^{\varphi} - \frac{d\lambda_i}{dt} C_i^{\varphi\lambda} - \frac{du_i^n}{dt} C_{ij}^{\varphi u} \right] = 0 \,,$$

$$\sum_{i,j=1}^{m} \left[\frac{d\varphi_i}{dt} C_{ij}^{\varphi u} - \lambda_i K_i^{\lambda u} \right] = 0 \,,$$

$$\sum_{i,j=1}^{m} \left[-\lambda_i K^{\lambda} + \frac{d\varphi_i}{dt} C_i^{\varphi\lambda} - u_i^n K_i^{\lambda u} \right] = 0 \,.$$

The solid elements demand an additional coordinate transformation. This will not be written out in detail here. The strain-displacement relations written in generalized curvilinear coordinates are first transformed to a local arc length coordinate system. Then they are fairly simple to transform into natural coordinates using a Jacobian matrix. The final result is that equation 3 written with indicial notation in terms of a finite element approximation becomes

$$\sum_{i,j=1}^{m} \left[u_i \int_S N_{ki} A_{pk} Q_{pq} A_{qr} N_{rj} |J| \, d\eta \, d\xi + \frac{d^2 u_i}{dt^2} \int_S \frac{h \rho_s}{2} N_{ki} N_{kj} |J| \, d\eta \, d\xi \right.$$

$$\left. - \int_S f_i N_{ij} |J| \, d\eta \, d\xi \right] \delta u_j = 0 \,.$$

where $\mathbf{A}$ is the matrix of strain-displacement relations in natural coordinates. We can write this as

$$\sum_{i,j=1}^{m} \left[u_i K_{ij}^u + \frac{d^2 u_i}{dt^2} M_{ij}^u - F_i \right] = 0 \,.$$

Then, putting all of these equations together and taking the Fourier transform of both sides, the linear system to solve becomes

$$-\omega^2 \begin{pmatrix} \mathbf{K}^u & 0 & -\mathbf{K}^{\lambda u^T} \\ 0 & -\mathbf{K}^{\varphi} & 0 \\ -\mathbf{K}^{\lambda u} & 0 & -K^{\lambda} \end{pmatrix} \begin{pmatrix} \overline{u} \\ \varphi \\ \lambda \end{pmatrix} + i\omega \begin{pmatrix} 0 & \mathbf{C}^{\varphi u^T} & 0 \\ \mathbf{C}^{\varphi u} & 0 & \mathbf{C}^{\varphi\lambda^T} \\ 0 & \mathbf{C}^{\varphi\lambda} & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \varphi \\ \lambda \end{pmatrix}$$

$$+ \begin{pmatrix} \mathbf{M}^u & 0 & 0 \\ 0 & -\mathbf{M}^{\varphi} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \varphi \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ 0 \\ 0 \end{pmatrix}$$

where the vector $\begin{bmatrix} \overline{u} & \varphi & \lambda \end{bmatrix}$ is the vector of nodal values. The next step is to use a complex solver suitable to this system of equations to solve for the nodal variables. We will use a linear system solver taken from the fortran subroutine library LAPACK.

Summary and Conclusions

We have formuated and begun to implement a finite element analysis program based on variational principles for the problem of sound wave propagation in a spherical shell. At the time of the writing of this report results were not yet available. The program has yet to be completely debugged, and several more weeks of work are necessary before accurate, complete solutions can be obtained. This work will continue over the course of the next year as we incorporate additional effects into our model in our effort to develop an improved computational method for structural acoustics problems.

References

[1] Huebner, K.H. and Thornton, E.A., <u>The Finite Element Method For Engineers</u>, Third Edition, John Wiley, New York.

[2] Nefske, D.J., Wolf, J.a., Jr. and Howell, L.J., "Structural-Acoustic Analysis Using Finite Element Methods," Chapter 1, in <u>Modern Automotive Structural Analysis</u>, Kamal, M.M. and Wolf, J.A., Jr. (Editors), Von Nostrand, New York, 1982.

[3] Kock, E. and L. Olson, "Fluid-Structure Interaction Analysis by the Finite Element Method – A Variational Approach", *International Journal for Numerical Methods in Engineering*, Vol. 31, p. 463-491, 1991.

[4] Bedford, A., <u>Hamilton's Principle in Continuum Mechanics</u>. Research Notes in Mathematics, Vol. 139. Pitman Advanced Publishing Program, London, 1985.

[5] Courant, R. and D. Hilbert, <u>Methods of Mathematical Physics</u>, Vols. I and II. First English Edition, John Wiley, New York, 1953.