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LEAKAGE THROUGH A CHANNEL FORMED BY A GASKET, A SEALING SURFACE, AND A FILAMENT TRAPPED BETWEEN THEM

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ABSTRACT

Much critical equipment at Kennedy Space Center is plumbing for the transport of liquid Hydrogen or liquid Oxygen. Every piece of hardware for handling such a *hazardous cryogen* is subject to testing prior to installation and use.

Safe, realistic testing of all such hardware is prohibitively expensive, which leads, perforce, to expedients, such as: (1) Leak testing with non-flammable tracer fluids (*e.g.* liquid Nitrogen); and (2) Leak testing with room temperature tracer fluids (*e.g* gaseous Helium).

Such expedients undermine the realism of the tests. If, however, one could apply rational fluid dynamics methods to derive an general analytical expression with which one could relate the throughput of gaseous Helium through a given leak channel to the throughput of liquid Hydrogen through the same channel then one could recover much of the information that one would otherwise forfeit through these expedients. These facts lead to following questions: (1) What would be an example of a generic flaw in a gasket?; and (2) How can one calculate the flow of fluid it?.

The following report addresses these questions. It considers a particular leak geometry, namely one formed by a gasket, a sealing surface, and a filament trapped between them (so that the cross section of the leak channel is a *flat bottomed curvilinear triangle*, two sides of which are circular arcs and which has cusps on all three corners).

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JOHN M. RUSSELL

1. INTRODUCTION

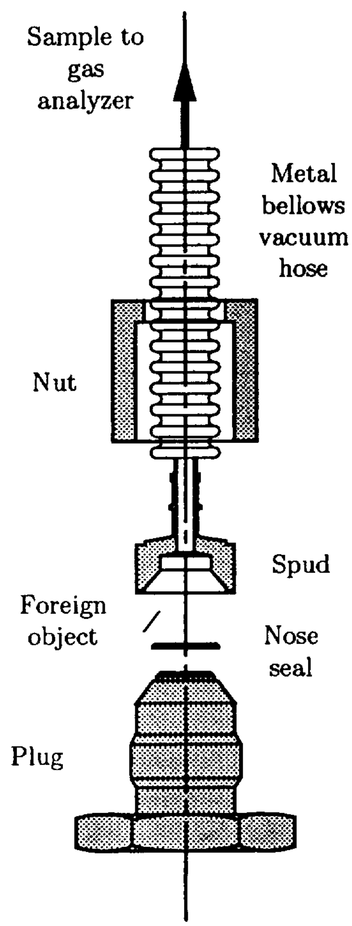

Figure 1

Consider an apparatus of the form illustrated in Figure 1, at left. One end of the metal bellows hose is plugged and the other end is attached to a residual gas analyzer capable of measuring small partial pressures of tracer fluid. The hose is evacuated and the plugged end is subject to any of a variety of test conditions, such as: (1) Immersion in a bath of liquid Nitrogen; (2) Exposure to cold gaseous Nitrogen by suspension above the liquid level in a liquid Nitrogen Dewar; (3) Exposure to gaseous Nitrogen at room temperature; (4) Exposure to gaseous Helium at room temperature. In this way, one may measure the leakage of tracer fluid that leaks from the outside of the plugged end of the hose to the inside.

If the seal on the plug were flawless no tracer fluid would leak. One may, however, simulate the kind of flaw that may arise in practice by draping a fine wire or other hard filament across the nose seal prior to assembly (see Figure 1, where the filament is labeled "foreign object").

One would like to be able to model the flow through a leak channel of the above sort, but to do so, one must describe the shape of the resulting leak channel in detail. To this end, consider what a hypothetical observer might see if he or she could observe the filament along a line of sight parallel to the filament itself during assembly of the apparatus (see Figure 2 on the next page).

I have illustrated the spud and the plug as jaws of a vise. I have illustrated the nose seal in light shading. As the soft jaws close over the assembly, the soft seal material gives way to the hard filament as illustrated in Panel (b). One may surmise that the wire crevice will be effectively flat bottomed and that the arc of the crevice corresponding to the wire will be circular. There is no physical reason to assume that the arc corresponding to the gasket is also circular. The analysis simplifies considerably under this assumption, however, so I will make it in the remainder of this work. I will also use the term *wire crevice* to denote the kind of curvilinear triangle (with a flat bottom, two circular arcs, and cusps in the corners) illustrated in Panel (c) at the bottom of Figure 2.

The remainder of this report is an analysis of the flow through a single wire crevice. Section 2 presents the necessary scientific background on the flow of liquids and gases through capillaries. Section 3 formulates the theory appropriate to the wire crevice. Section 4 presents the main result, namely a set of contours of

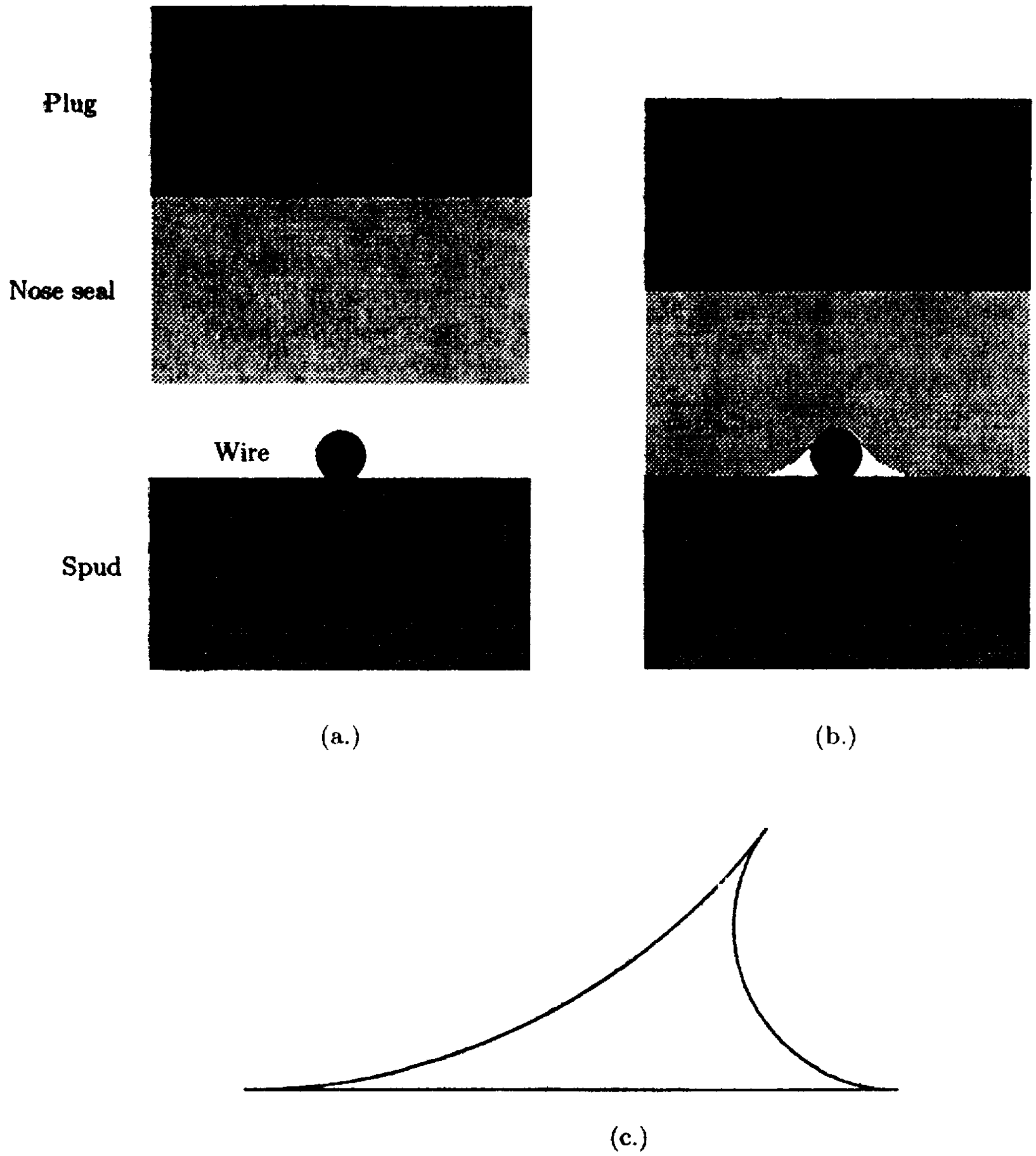

Figure 2. Generation of a wire crevice. Panel (a) represents the placement of the plug, nose seal, wire, and spud prior to assembly. Panel (b) represents these components after assembly. Panel (c) is a detail of the left member of the pair of leak channels illustrated in white in Panel (b).

constant streamwise velocity (in nondimensional variables) for the flow in question, and Section 5 presents the conclusions and recommendations.

2. SCIENTIFIC BACKGROUND

Authors of textbooks on fluid mechanics assign pride of place to the classical problem of ***fully-developed laminar flow*** through a pipe or channel and this problem applies to the present case when the fluid is a liquid. Many of the assumptions and conventions one introduces in the liquid-flow problem apply equally to the flow of a gas, so I will state them here with all due ceremony.

Assumption of a prismatic boundary, orientation of coordinates and notation for velocity components and fluid parameters. One assumes that the boundary of the region occupied by the fluid has straight-line generators. Let $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$ be cartesian coordinates (each of which has the dimensions of length) with the positive Z-axis parallel to the generators of the boundary surface and in the direction of the flow of fluid. Let $(\mathrm{U}, \mathrm{V}, \mathrm{W})$ be the components of the fluid velocity vector (each of which has the dimensions of velocity) belonging to the coordinates $(\mathrm{X}, \mathrm{Y}, \mathrm{Z})$. Let P be the fluid pressure (a positive valued parameter with the dimensions of force-per-unit-area). Let ρ be the fluid density (a positive valued parameter with the dimensions of mass-per-unit-volume). Let μ denote the dynamic viscosity of the fluid (a positive valued parameter with the dimensions of stress divided by strain rate, or, equivalently, force times time divided by area).

Notation for typical lengths. Let L denote the diameter of the smallest circle into which the cross section of the channel can be fit. One may align the X-axis so that the two most widely separated points on a typical cross section are points on the X-axis and I will assume that the coordinate axes have this feature. Let δ denote a typical dimension of the cross section in the direction of the Y-axis. The assumptions I have introduced thus far ensure that $\delta \leq L$. Finally, let ℓ denote the *streamwise length* of the capillary.

Slenderness assumption. In capillary flow theory, one assumes that $L/\ell \equiv \epsilon \ll 1$. I will refer to the parameter ϵ as the *slenderness parameter* in what follows.

On the neglect of gravitational terms. One may interpret the differential equation for the rate of change of translational momentum of a fluid in the form of the statement "The acceleration of a fluid particle equals the resultant force-per-unit-mass (from all causes) to which that particle is subject." One of the force-per-unit mass terms in the streamwise component of the momentum equation is the expression $-(1/\rho)(dp/d\mathrm{Z})$, which represents direct driving by the unequal pressures at the inlet and the outlet. The presence of the minus sign reflects the fact that the fluid in a capillary flows in the direction of *decreasing* pressure. Another of the force-per-unit-mass term is $-g \sin \gamma$, in which $g > 0$ is the acceleration due to gravity and γ is the angle of inclination of the long axis of the capillary above the horizontal. Here γ is positive or negative according to whether the fluid flows uphill or downhill, respectively.

The neglect of the gravitational term, $-g \sin \gamma$, in comparison with the term due to direct driving by the pressure gradient, $-(1/\rho)(dp/d\mathrm{Z})$, thus amounts to the assumption that the ratio of the orders of magnitude of these terms is very much less than one, *viz.*

$$\frac{g}{(1/\rho)|dp/d\mathrm{Z}|} \ll 1 . \tag{1}$$

In the case of gas flow, we have the equation of state of an ideal gas, namely $p = \rho RT$, in which R is the gas constant particular to the gas in question (a parameter with the dimensions of energy per unit mass per unit temperature) and T is the absolute temperature. If one eliminates ρ from (1) by means of $p = \rho RT$, one obtains a restatement of (1) in the form

$$\frac{p/|dp/d\mathrm{Z}|}{RT/g} \ll 1 . \tag{2}$$

One may estimate the numerator of the fraction in the left member of (2) by the capillary length ℓ. The denominator RT/g has the dimensions of length and represents the change in elevation over which the density of a stationary isothermal atmosphere decreases by a power of e. This so called ***scale height of the atmosphere*** is on the order of kilometers (about ten kilometers for air at 15° C). For a capillary whose length is on the order of millimeters, therefore, one may apply the assumption (2) with complete confidence.

For a liquid, the assumption (1) amounts to the statement that the pressure difference due to the hydrostatic head of the working fluid between the inlet and the outlet is small compared to the applied pressure difference. If the applied pressure is on the order of atmospheres and the capillary length is on the order of millimeters, then this one may again apply (2) with confidence.

Steadiness. I will restrict attention to the case when the velocity and pressure fields are independent of time.

On the neglect of inertial reaction to streamwise acceleration and the exclusion of turbulent flow. A standard exercise in classical fluid mechanics is to estimate the ratio so called ***inertial*** to ***viscous*** terms in the momentum equation. If there were only one representative length scale λ in the problem, then this ratio would be the REYNOLDS number $Re = \rho W \lambda/\mu$, where W is a typical streamwise velocity. In capillary flow, however, there are two relevant length scales, namely the capillary length, ℓ, and the smallest cross-stream length, δ, of the channel. A measure of the inertial force-per-unit-mass is thus W^2/ℓ while a measure of the viscous force per unit mass is $\mu W/(\rho\delta^2)$. One thus arrives at the following estimate of the ratio of the inertial to the viscous terms:

$$\left(\frac{\delta}{\ell}\right)\left(\frac{\rho W \delta}{\mu}\right) \ll 1\,. \tag{3}$$

The first factor in the right member is the slenderness factor, ϵ, and the second factor is the REYNOLDS number, Re, based on the smallest cross-stream length of the channel. One may therefore restate the condition for the neglect of inertial terms in the momentum equation as follows:

$$\epsilon\, Re \ll 1\,. \tag{4}$$

If ℓ is on the order of millimeters and δ is on the order of microns then ϵ will be on the order of 10^{-3}. In this case, Re need not be small and may even be in the hundreds without violating assumption (4). I will, however, confine attention to the case of non-tubulent flow, which generally requires that Re be less than 2,000.

Assumption of constant temperature. I will assume that the temperature of the fluid (whether gaseous or liquid) is of uniform temperature. Since the fluid viscosity, μ, is primarily a fuction of temperature, the constant temperature assumptions effectively guarantees that the viscosity is uniform throughout the fluid.

The Poisson equation for the streamwise velocity. Every viscous term in the equation of motion of a fluid involves a second derivative of a velocity component with respect to some combination of the spatial coordinates. In view of the slenderness assumption $\epsilon \ll 1$, one may separate the various viscous terms into classes distinguished by their respective orders of magnitude. The largest such terms are those in which both spatial derivatives are with respect to the cross-stream coordinates (X, Y). Keeping these terms and making a few additional uses of the assumption $\epsilon \ll 1$, one arrives at the following dominant balance of terms in the streamwise momentum equation:

$$\frac{\partial^2 \mathrm{W}}{\partial \mathrm{X}^2} + \frac{\partial^2 \mathrm{W}}{\partial \mathrm{Y}^2} \approx \frac{1}{\mu}\frac{dp}{d\mathrm{Z}}\,, \tag{5}$$

in which P is a function of Z only.

For a particular cross section Z = constant, equation (5) takes the form of a POISSON equation for the distibution of streamwise velocity W across that station. The boundary condition for (5) is the condition of no-slip between a viscous fluid and an adjacent solid wall:

$$\mathrm{W}\big|_{\mathrm{wall}} = 0\,. \tag{6}$$

Introduction of nondimensional variables. One may introduce nondimensional cross-stream position coordinates (x, y) by the definitions

$$x \equiv \mathrm{X}/L \quad , \quad y \equiv \mathrm{Y}/L \ . \tag{7 a,b}$$

In the foregoing discussion, I have used the symbol W for a measure of the typical streamwise velocity across the cross section of the channel. One may make this definition more specific by taking

$$W \equiv -\frac{L^2}{\mu}\frac{dp}{d\mathrm{Z}} \ . \tag{8}$$

Note that fluid flows from greater to lesser pressure and the positive Z-axis in the direction of the flow. It follows that $dp/d\mathrm{Z} < 0$, so the right member of (8) is positive as expected. Having a scale for the streamwise velocity, one may define the nondimensional velocity component w by

$$w \equiv \mathrm{w}/W \ . \tag{10}$$

If one multiplies the POISSON equation (5) by L^2/W, one obtains, after some reductions, the following normalized form:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -1 \ . \tag{11}$$

whose solution is subject to a boundary condition equivalent to (6), *viz.*

$$w\big|_{\text{wall}} = 0 \ . \tag{12}$$

The solution of the boundary-value defined by (11) and (12) will be a function $(x, y) \mapsto w$. Note, in particular, that w *does not depend upon anything other than* x *and* y.

Conservation of mass along the channel and its consequences. Let $\mathcal{S}'$ denote a surface of the X-Y plane corresponding to the channel cross section. By the same token, let the unpunctuated symbol $\mathcal{S}$ denote the corresponding surface with respect to the nondimensional variables (x, y). One may write an expression for the rate of transport of mass, $\dot{m}$, across a typical cross section of the channel by

$$\dot{m} = \iint_{\mathcal{S}'} \rho \mathrm{w} \, d\mathrm{X} d\mathrm{Y} \ . \tag{13}$$

Under the assumptions stated thus far, the mass density, ρ, is, at worst, a function of the streamwise coordinate Z. One may therefore move the coefficient ρ outside of the integral sign in (13). If one carries out this step and also introduces the nondimensional coordinates (x, y) and the nondimensional streamwise velocity w by means of the definitions (7 a, b) and (10), one obtains

$$\dot{m} = \rho W L^2 \iint_{\mathcal{S}} w \, dx dy \ . \tag{14}$$

If one eliminates W by means of (8) and rearranges slightly, one obtains

$$\frac{\dot{m}\mu}{L^4 \iint_{\mathcal{S}} w \, dx dy} = -\rho \frac{dp}{d\mathrm{Z}} \ . \tag{15}$$

The assumption of steadiness and the law of conservation of mass imply that $\dot{m}$ is independent of z. All of the other terms in the left member of (15) are also independent of z under assumptions already stated. One concludes that for either liquids or gases

$$\rho \frac{dp}{dz} = \text{independent of } z\,. \tag{16}$$

In the particular case of liquid flow, ρ is constant, so constancy of $\rho\, dp/dz$ with respect to z implies constancy of dp/dz. It follows that p varies linearly with respect to z and we have

$$\rho \frac{dp}{dz} = \rho \frac{p_2 - p_1}{z_2 - z_1} = \rho \frac{p_2 - p_1}{\ell} \tag{17}$$

for liquid flow. Here, the subscripts "1" and "2" denotes conditions at the inlet to and the outlet from the capillary, respectively.

In the particular case of gas flow, ρ varies according to the equation of state $p = \rho RT$, so

$$\rho \frac{dp}{dz} = \frac{p}{RT}\frac{dp}{dz} = \frac{1}{RT}\frac{d(p^2/2)}{dz}\,. \tag{18}$$

Since the leftmost member of (18) is independent of z (by (16)), one concludes from (18) and the assumption of constant temperature that p^2 varies linearly with respect to z. It follows that

$$\rho \frac{dp}{dz} = \frac{1}{2RT}\frac{p_2^2 - p_1^2}{z_2 - z_1} = \frac{1}{2RT}\frac{p_2^2 - p_1^2}{\ell} \tag{19}$$

for gas flow.

One may combine the results expressed by (15), (17), and (19) in the following form

$$\dot{m} = \begin{cases} \dfrac{L^4}{\mu} \displaystyle\iint_S w\, dx dy\, \rho \frac{p_1 - p_2}{\ell}, & \text{for liquid flow;} \\ \dfrac{L^4}{\mu} \displaystyle\iint_S w\, dx dy\, \frac{1}{2RT}\frac{p_1^2 - p_2^2}{\ell}, & \text{for gas flow;} \end{cases} \tag{20}$$

The two cases of equation (20) furnish an analytical correlation between the throughputs of liquid and gases through the same capillary.

Authors of works on leak detection sometimes employ the symbol S to denote the rate of transport of fluid volume across a cross section of a fluid conduit. These same authors also sometimes write Q for the product pS and refer to the resulting quantity as the *gas throughput*. In view of the equation of state of an ideal gas, one may relate Q to $\dot{m}$ by the following identities

$$\dot{m} = \rho S = \frac{p}{RT} S = \frac{Q}{RT}\,. \tag{21}$$

If one multiplies (20) (in the case of gas flow) by RT, one obtains, in view of the outermost equality in (21), the identity

$$Q = \frac{L^4}{\mu} \iint_S w\, dx dy\, \frac{p_1^2 - p_2^2}{2\ell}\,. \tag{22}$$

The quantity Q so defined has the dimensions of pressure times volume transport rate. Thus, the expression atm·cm^3/s (also known as ***standard cubic centimeters per second***) are suitable units for the measurement of Q. Of course, the outermost identity in (21) allows one to relate Q to the mass transport rate $\dot{m}$.

The case of a circular cross section; Classical formulas. The practical implementation of the foregoing formulas (which apply to all capillaries regardless of the cross sectional shape) depends upon the evaluation of the integral

$$\iint_S w\,dxdy\,, \tag{23}$$

which, in turn, depends upon the solution of the normalized boundary value problem defined by (11) and (12). The simplest example of a solution of this system is that of the capillary whose cross section is a circle of diameter L. In nondimensional coordinates, the diameter becomes one and, thus the radius becomes 1/2. The equation of the boundary shape (*i.e.* a circle of radius 1/2 centered on the origin) thus becomes $(1/2)^2 - x^2 - y^2 = 0$. If one notes that the the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{24}$$

takes the expression $(1/2)^2 - x^2 - y^2$ to a constant (in this case, -4) in the interior of the circle and that the expression $(1/2)^2 - x^2 - y^2$ vanishes on the boundary, it becomes plain that multiplication of $(1/2)^2 - x^2 - y^2$ by a suitable constant (in this case, 1/4) yields a solution of the boundary value problem (11), (12) for the circular geometry. One concludes that

$$w = \frac{(1/2)^2 - x^2 - y^2}{4}\,. \tag{25}$$

in circular geometry.

If one introduces (r, θ) for polar coordinates in the (x, y) plane, then (25) becomes

$$w = \frac{(1/2)^2 - r^2}{4} \tag{26}$$

and the integral (23) is equivalent to

$$\iint_S w\,dxdy = \int_0^{1/2} w(r)\,2\pi r\,dr = \frac{\pi}{8}\frac{1}{2^4}\,. \tag{27}$$

If one substitutes this result into (20) and notes that $p_1^2 - p_2^2 = (p_1 + p_2)(p_1 - p_2)$, one obtains

$$\dot{m} = \begin{cases} \dfrac{\pi}{8\mu}\left(\dfrac{L}{2}\right)^4 \rho\dfrac{p_1 - p_2}{\ell}, & \text{for liquid flow;} \\[2ex] \dfrac{\pi}{8\mu}\left(\dfrac{L}{2}\right)^4 \left(\dfrac{(p_1+p_2)/2}{RT}\right)\dfrac{p_1 - p_2}{\ell}, & \text{for gas flow.} \end{cases} \tag{28}$$

The liquid-flow form of (28) is associated with the names of HAGEN and POISEUILLE, both of whom studied the flow experimentally in the nineteenth century. The result was derived analytically by GEORGE GABRIEL STOKES. The earliest analytical derivation of the gas-flow form of (28) appears to be that of OSKAR EMIL MEYER in 1866 (Ref. 1, page 269).

In view of the ideal gas law, the constant valued expression

$$\frac{(p_1+p_2)/2}{RT}$$

has the dimensions of density, and one may regard it as an *effective constant density* for the gas in a capillary (based on the ideal gas law and the arithemetic mean of the inlet and outlet pressures). While this interpretation is convenient as a mnemonic device, it is important to realize that the derivation of the MEYER formula does not assume constancy of the density even as an approximation. Indeed, the MEYER formula holds for *any* expansion ratio. The foregoing derivation supports the last assertion nothwithstanding contrary statements by modern authors in the literature on leak detection.

3. FLOW THROUGH A WIRE CREVICE

Transformation to a Dirichlet problem. Let $(x, y) \mapsto w_p$ be a function defined by the equation

$$w_p = -y^2/2. \tag{29}$$

It follows that

$$\frac{\partial^2 w_p}{\partial x^2} + \frac{\partial^2 w_p}{\partial y^2} = -1\,, \tag{30}$$

One concludes that w_p is a particular solution of the differential equation (11) satisfied by w. Let $(x, y) \mapsto \phi$ be a function defined by the equation

$$w = w_p + \phi\,. \tag{31}$$

If one subtracts (30) from (11), and uses the identity $w - w_p = \phi$ (which follows from (31)), one obtains

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0\,. \tag{32}$$

If, now one substitites (31) into the boundary condition (12), one obtains

$$\phi\big|_{\text{wall}} = -w_p\big|_{\text{wall}} = -(-y^2/2)\big|_{\text{wall}}\,,$$

or

$$\phi\big|_{\text{wall}} = \frac{(y_{\text{w}})^2}{2}\,, \tag{33}$$

in which the subscript $(\;)_{\text{w}}$ denotes evaluation on the wall.

The partial differential equation (32) that ϕ satisfies is LAPLACE's equation. A function that satisfies LAPLACE's equation is called a *harmonic* function. The boundary-value problem consisting of LAPLACE's equation and a statement of the boundary values of the unknown (as distinguished from the normal derivative of the unknown or some combination of the unknown and its normal derivative) is called the DIRICHLET *problem.* The pair of equations (32) and (33) thus constitutes a DIRICHLET problem for ϕ.

Construction of harmonic functions from differentiable functions of a complex variable. Experience has shown that one may expedite the solution of the DIRICHLET problem in two space dimensions considerably through the use of complex variables. There are two reasons why this is so. To illustrate the first reason, let $\zeta = \xi + i\eta$ be a complex number. Here $i = \sqrt{-1}$ and the pair of real numbers (ξ, η) represent a point in a plane (the complex ζ-plane). If $\zeta \mapsto F$ is an *analytic* (*i.e.* differentiable) function that maps ζ to the complex number F (with the expansion, say, $F = \phi + i\psi$), then the real and imaginary parts of F

(namely ϕ and ψ, respectively) are related to the real and imaginary parts of ζ (namely ξ and η, respectively) by the CAUCHY-RIEMANN equations

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \psi}{\partial \eta} \quad , \quad \frac{\partial \phi}{\partial \eta} = -\frac{\partial \psi}{\partial \xi} \, . \tag{34 a, b}$$

If one eliminates ψ from the (34 a, b) by cross differentiation, one obtains, immediately,

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0 \, . \tag{35}$$

In this way, any analytic function of a complex variable (*e.g.* $\zeta \mapsto F$) is related in a systematic way to a harmonic function (*e.g.* $(\xi, \eta) \mapsto \phi$).

A class of changes of independent variable that leaves Laplace's equation invariant. A second reason why complex varibles are useful in solving the DIRICHLET problem in two dimension concerns the invariance of LAPLACE's equation under a class of transformations of the independent variables. Suppose, for example, that $z \mapsto m$ is an analytic function that takes the complex number z to the complex number m and that the expansions of z and m into real and imaginary parts are

$$z = x + iy \quad , \quad m = m_r + i m_i \, . \tag{36}$$

Then the function $z \mapsto m$ induces a transformation $(x, y) \mapsto (m_r, m_i)$ (called a *conformal transformation*) between points in the complex z-plane and points in the complex m-plane. A basic exercise in the theory of functions of a complex variable is the demonstration that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left| \frac{dm}{dz} \right|^2 \left(\frac{\partial^2}{\partial m_r^2} + \frac{\partial^2}{\partial m_i^2} \right) \tag{37}$$

(*cf.* JEFFREYS & JEFFREYS, p534) (Ref. 2). If, for example, $(x, y) \mapsto \phi$ is harmonic, then

$$0 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \left| \frac{dm}{dz} \right|^2 \left(\frac{\partial^2 \phi}{\partial m_r^2} + \frac{\partial^2 \phi}{\partial m_i^2} \right) \, . \tag{38}$$

Division of the outermost equality by $|dm/dz|^2$ yields

$$\frac{\partial^2 \phi}{\partial m_r^2} + \frac{\partial^2 \phi}{\partial m_i^2} = 0 \, . \tag{39}$$

Thus, if ϕ is a harmonic function of one set of independent variables, and those variables are related to a new set of variables by a conformal transformation then ϕ will also be harmonic with respect to the new variables.

Transformation of the wire crevice to the interior of the unit circle. The book *Analytic function theory* by EINAR HILLE (Ref. 3) gives several examples of conformal transformations, of which two are particularly germane to the present problem. Figure 3 illustrates regions in the planes of four distinct complex variables. The transformation between what I have called the m-plane and the τ-plane appears on

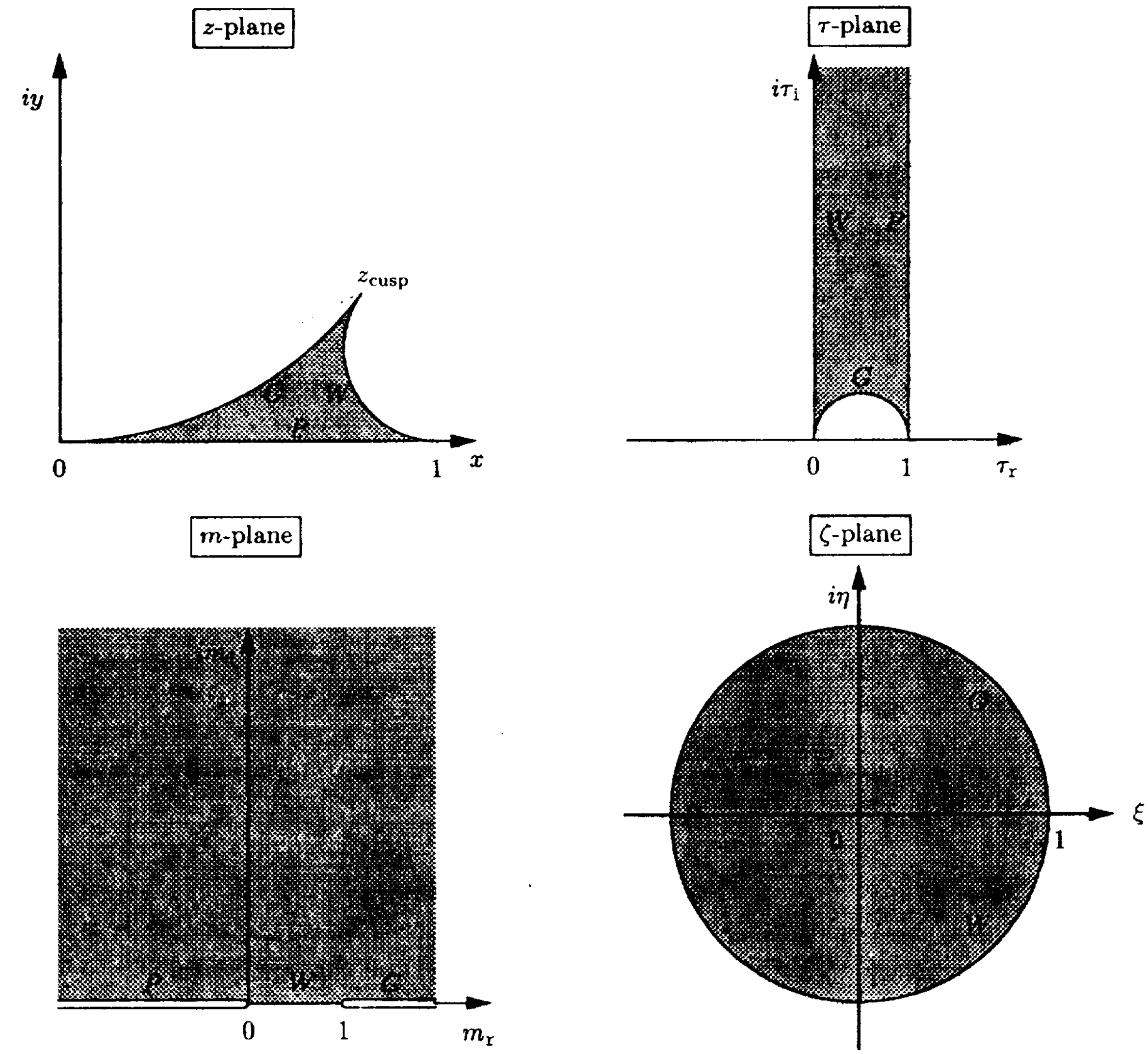

Figure 3. Three step conformal transformation of a wire crevice (in the z-plane) to the interior of a unit circle (in the ζ-plane).

(and near) page 177 of HILLE's book and the transformation between what I have called the τ-plane and the ζ plane appear on (and near) page 370 of that source.

HILLE's formula for the transformation $m \mapsto \tau$ is equivalent to

$$\tau = \frac{iK(1-m)}{K(m)}, \tag{40}$$

in which $m \mapsto K(m)$ is a so-called *complete elliptic integral of the first kind* (*ibid.*, p150). The inverse of (40) is

$$m = \left(\frac{\vartheta_2(0|\tau)}{\vartheta_3(0|\tau)}\right), \tag{41}$$

in which $(z, \tau) \mapsto \vartheta_i(z|\tau)$ $(i \in \{1, 2, 3, 4\})$ are the so-called *theta functions*, (*ibid.*, p156). Some modern software packages, such as *Mathematica*, enable evaluation of $m \mapsto K(m)$ and $(z, \tau) \mapsto \vartheta_i(z|\tau)$ $(i \in \{1, 2, 3, 4\})$ for complex arguments as easily as the evaluation of sines and cosines. The use of such higher functions is therefore not a practical limitation on a the development of a computational algorithm for the present problem.

HILLE's formula for the transformation $\zeta \mapsto m$ (or a trivial modification therof) is equivalent to

$$m = i\left(\frac{1 - i\zeta}{1 + i\zeta}\right) , \tag{42}$$

whose inverse is

$$\zeta = i\left(\frac{m - i}{m + i}\right) . \tag{43}$$

HILLE (*ibid.*, p 46, *et seq.*) discusses a class of conformal transformations which he associates with the name MÖBIUS. In these transformations, two complex variables are related by a formula which equates the first variable to a ratio of linear functions of the second (The inverse of such a transformation equates the second variable to a ratio of linear functions of the first). HILLE calls attention (*ibid.*, page 51) to the class of geometric figures consisting of circles and straight lines. A basic feature of the MÖBIUS transformation is that it takes a circle or line in the plane of the first complex variable to a circle or line in the plane of the second. In this sense, the class of circles and lines is invariant under MÖBIUS transformations. With this theorem in mind, it is not difficult to find (by trial and error) a MÖBIUS transformation that takes the wire crevice in the z-plane to the interior of the shaded figure in the τ-plane. Such a MÖBIUS transformation is given by

$$\tau = \frac{1 - z/z_{\text{cusp}}}{1 - z} , \tag{44}$$

whose inverse is

$$z = \frac{\tau - 1}{\tau - (1/z_{\text{cusp}})} . \tag{45}$$

Here z_{cusp} is the intersection of the gasket and the wire in the wire crevice problem.

The condition that the gasket and the wire be tangent to each other restricts values of z_{cusp} to points on a semicircle of diameter one centered at the point 1/2 on the real axis. One may write a formula which encompasses all possible values of z_{cusp} that fulfill this condition in the form

$$z_{\text{cusp}} = \frac{1}{1 - iA} , \tag{46}$$

in which $A > 0$ is a real parameter. In fact $A^2 = R_w/R_g$, where R_w and R_g are the radii of curvature of the wire arc and the gasket arc, respectively.

Boundary value of ϕ in the ζ-plane. The foregoing conformal transformation formulas enable one to associate any point (x, y) in the in or on the wire crevice in the z-plane (where $z = x + iy$) to one and only one point (ξ, η) in or on the unit disk in the ζ-plane (where $\zeta = \xi + i\eta$). Thus, the boundary values (33) for the unknown harmonic function $(x, y) \mapsto \phi$ in (32) carry over to corresponding boundary values of ϕ in the ζ-plane. Figure 4 illustrates the corresponding distribution of these boundary values expressed in terms of the polar coordinate angle θ in the ζ-plane (The reader should note that my use of (r, θ) as polar coordinates in the ζ-plane here is a new use of these symbols, not to be confused with my earlier use of them in the text between (25) and (26) above).

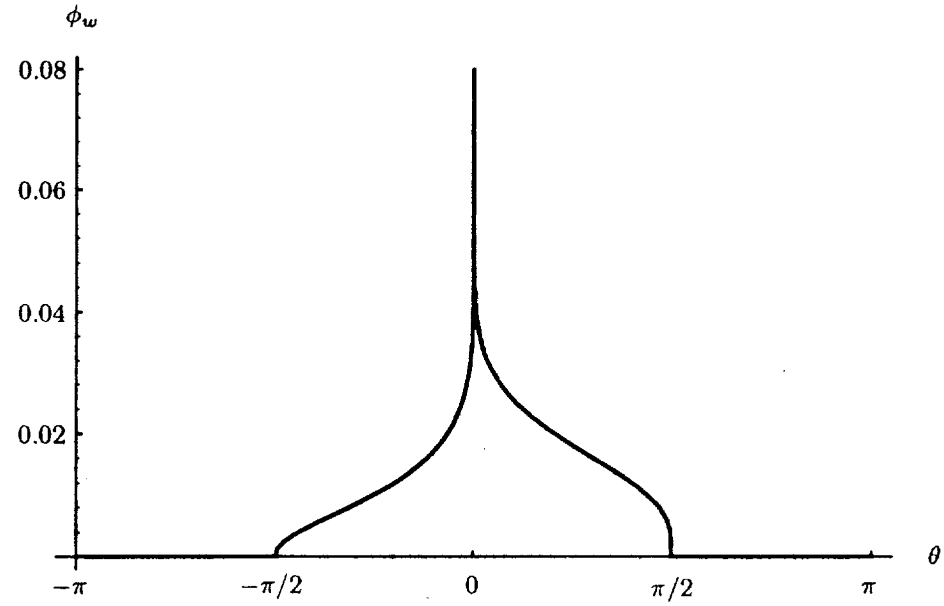

Figure 4. Distribution of the boundary value of the harmonic function ϕ as a function of polar angle in the ζ-plane

Solution for ϕ in the ζ-plane as an infinite series. One may relate the LAPLACE operator in cartesian coordinates (ξ, η) in the ζ-plane to polar coordinates (r, θ) in that plane by the identity

$$\frac{\partial^2(\;)}{\partial \xi^2} + \frac{\partial^2(\;)}{\partial \eta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(\;)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2(\;)}{\partial \theta^2} \,. \tag{47}$$

One may verify that for any integer n, the operator in the right member of (47) takes expression $r^n \cos(n\theta)$ to zero. The same operator takes $r^n \sin(n\theta)$ to zero and one arrives at the conclusion that any function of the form

$$\phi = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n [a_n \cos(n\theta) + b_n \sin(n\theta)] \,, \tag{48}$$

whose constant, real-valued coefficients (a_n, b_n) are subject to the condition that the series (48) converge, is a solution of LAPLACE's equation in the ζ-plane. In view of the foregoing discussion under the heading "A class of changes of independent variable that leaves LAPLACE's equation invariant" such a representation for ϕ must also satisfy LAPLACE's equation in the z-plane.

In view of the complex variable identities $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$, one may arrange (48) in the following equivalent form

$$\phi = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n \left[\left(\frac{a_n - ib_n}{2}\right) e^{in\theta} + \left(\frac{a_n + ib_n}{2}\right) e^{-in\theta}\right] \,. \tag{49}$$

One may shorten the notation by defining values of a_n and b_n for zero and negative subscript as follows (*e.g.* LIGHTHILL (1959, p3), Ref 4):

$$a_{-n} \equiv a_n \quad , \quad b_{-n} \equiv -b_n \tag{50}$$

(so $b_0 = 0$, by construction). Moreover, for any integer n, we write

$$\frac{a_n - ib_n}{2} \equiv C_n \; . \tag{51}$$

With these conventions, equation (49) takes the form

$$\phi = \sum_{n=-\infty}^{\infty} C_n r^{|n|} e^{in\theta} \; . \tag{52}$$

If one substitutes $r = 1$, one obtains the boundary value of ϕ, namley ϕ_w, so

$$\phi_w = \sum_{n=-\infty}^{\infty} C_n e^{in\theta} \; . \tag{53}$$

I have already remarked that the boundary-value function $\theta \mapsto \phi_w$ is known (*cf.* Figure 4 above). Let k be an arbitrary integer. If one multiplies (53) by $e^{-ik\theta}$ and integrates with respect to θ from zero to 2π, one obtains

$$\int_0^{2\pi} \phi_w e^{-ik\theta} d\theta = \sum_{n=-\infty}^{\infty} C_n \int_0^{2\pi} e^{i(n-k)\theta} d\theta \; . \tag{54}$$

But

$$e^{i(n-k)\theta} = \begin{cases} \frac{d}{d\theta}(\theta) & \text{if } k = n; \\ \frac{d}{d\theta}\left(\frac{e^{i(n-k)\theta}}{i(k-n)}\right) & \text{if } k \neq n. \end{cases} \tag{55}$$

It follows that

$$\int_0^{2\pi} e^{i(n-k)\theta} d\theta = \begin{cases} 2\pi & \text{if } k = n; \\ 0 & \text{if } k \neq n. \end{cases} \tag{56}$$

If one substitutes this last result into (54), one obtains

$$\int_0^{2\pi} \phi_w e^{-ik\theta} d\theta = 2\pi C_k \; , \tag{57}$$

or, equivalently,

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} \phi_w e^{-in\theta} d\theta \; , \tag{58}$$

(which holds for all integer n. Equation (58) thus determines the coefficients in the series representation (52).

Solution for w in the z-plane. Knowing ϕ at any point in the ζ-plane, one may associate the same value of ϕ with the corresponding point in the z-plane (*via* the conformal transformations given in equations (40)–(45)). One concludes that ϕ is now known through the whole wire crevice. Since w_p is specified by (29), one concludes that the whole right member of (31) for w is known throughout the wire crevice.

4. RESULTS

Figure 5 gives a plot of the contours of constant w obtained from a numerical evaluation. The slight raggedness of the outermost contour is probably due to an overly course discretization of θ in the calculation.

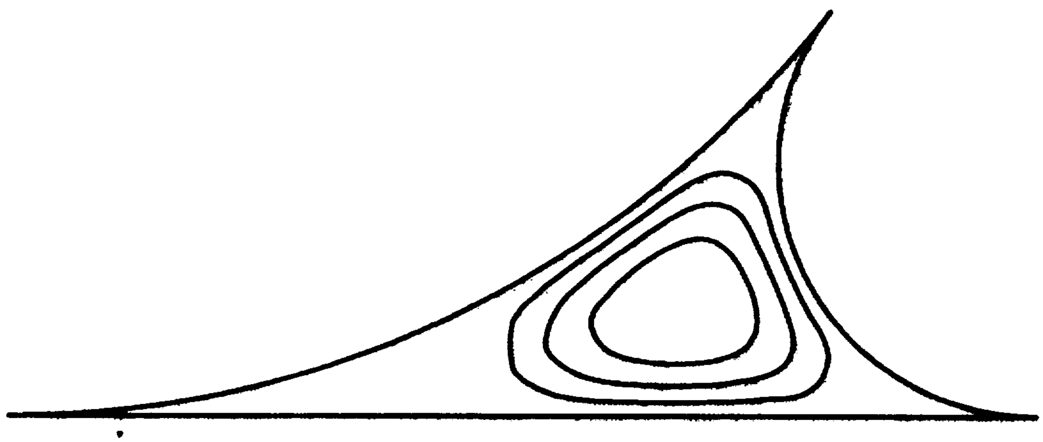

Figure 5. Contours of constant streamwise velocity in the flow through a wire crevice. The innermost contour represents 3/4 of the maximum value of w, the next innermost contour represent 1/2 the maximum, and so on.

5. CONCLUSIONS AND RECOMMENDATION

The foregoing work supports the following conclusions:

1. The liquid-gas flowrate correlation embodied by (20) (which holds for arbitrary shaped cross sections) and (28) (which holds for circular cross sections) are on a firm scientific basis and are suitable for immediate application.
2. One may solve the boundary-value problem for the flow through a wire crevice analytically and the solution is expressible in terms of a single infinite series.
3. The solution for wire-crevice geometry is computationally feasable on a small computer.

I had hoped to refine the calculations so as to eliminate the raggedness of the w-contours before completion of the summer term, but ran into some snags during the last two weeks. The recommendation, of course, is to continue the effort including the calculation of the integral (23), and thence, the throughput.

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