FLUID PHYSICS IN A FLUCTUATING ACCELERATION ENVIRONMENT

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1 Introduction

Our program of research aims at developing a stochastic description of the residual acceleration field onboard spacecraft (g-jitter) [1, 2] to describe in quantitative detail its effect on fluid motion [3, 4, 5]. Our main premise is that such a statistical description is necessary in those cases in which the characteristic time scales of the process under investigation are long compared with the correlation time of g-jitter. Although a clear separation between time scales makes this approach feasible, there remain several difficulties of practical nature: (i), g-jitter time series are not statistically stationary but rather show definite dependences on factors such as active or rest crew periods; (ii), it is very difficult to extract reliably the low frequency range of the power spectrum of the acceleration field. This range controls the magnitude of diffusive processes; and (iii), models used to date are Gaussian, but there is evidence that large amplitude disturbances occur much more frequently than a Gaussian distribution would predict. The lack of stationarity does not constitute a severe limitation in practice, since the intensity of the stochastic components changes very slowly during space missions (perhaps over times of the order of hours). A separate analysis of large amplitude disturbances has not been undertaken yet, but it does not seem difficult a priori to devise models that may describe this range better than a Gaussian distribution. The effect of low frequency components, on the other hand, is more difficult to ascertain, partly due to the difficulty associated with measuring them, and partly because they may be indistinguishable from slowly changing averages. This latter effect is further complicated by the lack of statistical stationarity of the time series.

Recent work has focused on the effect of stochastic modulation on the onset of oscillatory instabilities [6] as an example of resonant interaction between the driving acceleration and normal modes of the system, and on cavity flow [7] as an example of how an oscillatory response under periodic driving becomes diffusive if the forcing is random instead. This paper describes three different topics that illustrate behavior that is peculiar to a stochastic acceleration field. In the first case, we show that g-jitter can induce effective attractive or repulsive forces between a pair of spherical particles that are suspended in an incompressible fluid of different density provided that the momentum diffusion length is larger than the inter particle separation (as in the case in most colloidal suspensions). Second, a stochastic modulation of the control parameter in the vicinity of a pitchfork or supercritical bifurcation is known not to affect the location of the threshold. We show, however, that resonance between the modulation and linearly stable modes close to onset can lead to a shift in threshold. Finally, we discuss the classical problem of vorticity diffusion away from a plane boundary that is being vibrated along its own plane. Periodic motion with zero average vorticity production results in an exponential decay of the vorticity away from the boundary. Random vibration, on the other hand, results in power law decay away from the boundary even if vorticity production averages to zero.

2 Acceleration induced interactions between pairs of particles

Consider an ensemble of spherical particles of radii R_i and density ρ_p suspended in an incompressible fluid of density ρ_f and shear viscosity μ. If the fluid is enclosed by perfectly rigid boundaries, the buoyancy force acting on each suspended particle is $\vec{F} = \frac{4}{3}\pi(\rho_p - \rho_f)R^3\vec{g}(t)$, where $\vec{g}(t)$ is the effective acceleration field. In the frame of reference co-moving with the container enclosing the fluid, $\vec{g}(t)$ is a body force, with intensity equal to the value of the acceleration of the container. For containers of reasonable size in a microgravity environment, $\vec{g}$ can be assumed to be spatially uniform. The hydrodynamic interaction between two such particles is given in the overdamped limit of Stokes flow by [8, 9],

$$\frac{d\vec{r}}{dt} = (\boldsymbol{\omega}_{21} - \boldsymbol{\omega}_{11})\cdot\vec{F}_1 + (\boldsymbol{\omega}_{22} - \boldsymbol{\omega}_{21})\cdot\vec{F}_2, \qquad (1)$$

where $\vec{r}$ is the relative position of particle 2 with respect to particle 1, $\vec{F}_i$ is the force acting on the i-th particle, and ω_{ij} are hydrodynamic mobility tensors, given, e.g., in references [8, 9]. After some straightforward algebra, the leading contribution at distances large compared to the particle radii is given by,

$$\frac{d\vec{r}}{dt} = \frac{2(\rho_p - \rho_f)}{9\mu}\left(R_2^2 - R_1^2\right)\vec{g}(t) +$$

$$\frac{(\rho_p - \rho_f)\left(R_1^3 - R_2^3\right)}{3\mu}\frac{1}{r}\left[\frac{\vec{r}\vec{r}}{r^2} + \frac{1}{2}\left(\mathcal{I} - \frac{\vec{r}\vec{r}}{r^2}\right)\right]\cdot\vec{g}(t), \quad (2)$$

where $\mathcal{I}$ is the identity tensor. The first term in the right hand side describes the relative motion of two *independent* particles of different size. Both the longitudinal and transverse components of the second term in the right hand side of Eq. (2) are of the form,

$$\frac{dr}{dt} = \frac{A}{r}g(t), \quad (3)$$

where, for the longitudinal component, $A = (\rho_p - \rho_f)(R_1^3 - R_2^3)/3\mu$.

Consider an initial interparticle separation $r_0 \gg R_i$. In this case, and for times shorter than the average time needed for the two particles to coalesce, the quantity $y = r^2/2A$ is a Wiener process if $g(t)$ is Gaussian and white ($\langle g \rangle = 0$, $\langle g(t)g(t') \rangle = 2D\delta(t-t')$), and therefore the conditional probability for r is,

$$P(r,t|r_0,t_0) = \frac{r}{|A|\sqrt{4\pi D(t-t_0)}}e^{-\frac{(r^2-r_0^2)^2}{16DA^2(t-t_0)}}. \quad (4)$$

The ensemble average of r, $\langle r \rangle$ can be computed analytically,

$$\langle r \rangle = \sqrt{\frac{|A|}{4}}\left[2D(t-t_0)\right]e^{-\frac{r_0^4}{32A^2D(t-t_0)}}$$

$$D_{-3/2}\left(-\frac{r_0^2}{2|A|\sqrt{2D(t-t_0)}}\right), \quad (5)$$

where $D_p(z)$ is a parabolic cylinder function [10] (formula 9.240). For short times, the asymptotic form of $D_p(z)$ for large z allows the computation of $\langle r \rangle$

$$\langle r \rangle = r_0\left(1 - \frac{A^2D(t-t_0)}{r_0^4}\right), \quad (6)$$

which decreases in time regardless of the sign of A. Therefore g-jitter induces an effective hydrodynamic *attraction* between pairs of particles. The attractive interaction is not confined to short times, but it arises directly from the $1/r$ dependence in Eq. (3). By taking the average of Eq. (3), using the Furutsu-Novikov theorem [11] and the fact that the noise is Gaussian and white, one finds,

$$\frac{d\langle r \rangle}{dt} = AD\left\langle \frac{\delta 1/r(t)}{\delta g(t)} \right\rangle, \quad (7)$$

where $\delta/\delta g(t)$ stands for functional derivative with respect to g. Directly from Eq. (3), we find that $\delta(1/r(t))/\delta g(t) = -A/r^3$, and therefore,

$$\frac{d\langle r \rangle}{dt} = -A^2D\left\langle \frac{1}{r^3} \right\rangle, \quad (8)$$

identical to Eq. (6) with $1/r_0^3$ replaced by $\langle 1/r^3 \rangle$. Since r is a positive quantity, $d\langle r \rangle/dt < 0$ for all values of r. It is also interesting to note that the effective attractive interaction is not confined to the term proportional to $1/r$ in the hydrodynamic mobility, but that attractive contributions arise from higher powers of $1/r$ as well. In fact, this attraction is generic for over damped motion and multiplicative noise provided that the mobility is a decaying function of the interparticle separation [12].

The question naturally arises as to the behavior of pairs of particles near contact, or of particles near a solid wall. In either case, lubrication theory allows the calculation of the mobility tensor. The longitudinal component vanishes linearly with interparticle distance whereas the transverse component becomes nonanalytic (diverges logarithmically at short distances) [9]. In both cases, the mobility *increases* with interparticle separation leading to an average repulsion ($d\langle r \rangle/dt > 0$) following the same arguments given above.

3 Stochastic resonance and bifurcations

Consider the normal form appropriate for a pitchfork bifurcation in which A is the linearly unstable mode and B is some linearly stable mode. Then,

$$\frac{d}{dt}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \epsilon & 0 \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} + \begin{bmatrix} -cA^3 \\ 0 \end{bmatrix} +$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} g(t), \quad (9)$$

with $\epsilon \ll 1$ and the remaining coefficients of order one. We further assume that $D \sim \mathcal{O}(\epsilon)$. Close to threshold, A changes in a slow scale $T = \epsilon t$, compared with either the relaxation of B (of order one) or the noise.

If the coupling to the unstable mode B is neglected, it is known that the threshold in the stochastic case remains at $\epsilon = 0$ [13, 14]. This can be seen by averaging Eq. (9) over the fast time scale so that $\langle A(T)g(t)\rangle \approx$

$A(T)\langle g(t)\rangle = 0$. The resulting equation for $A(T)$ no longer depends on the noise. If, on the other hand, both equations are averaged over the fast time scale, we find that,

$$\langle Bg\rangle = \langle B\rangle\langle g\rangle + D\left\langle\frac{\delta B}{\delta g}\right\rangle = Dm_{21}A + \dots \qquad (10)$$

where we have used the Furutsu-Novikov theorem [15, 16]. Therefore, the correlation of $B(t)g(t)$ itself evolves over the slow time scale T as a consequence of the fact that the equations for both A and B contain exactly the same stochastic process. The coefficient of the linear term in the equation for $A(T)$ is now $\epsilon + Dm_{21}m_{12}$ and hence the bifurcation point will occur at $\epsilon = -Dm_{12}m_{21} < 0$.

4 Flow due to a randomly vibrating plane boundary

Consider a semi-infinite fluid layer occupying the region $x > 0$ and a solid boundary at $x = 0$ which is being displaced along the y direction with a prescribed, time-dependent velocity $v_0(t)$. For an incompressible, Newtonian fluid, the y component of the velocity field in the fluid v satisfies,

$$\frac{\partial v}{\partial t} = \nu\frac{\partial^2 v}{\partial x^2}, \qquad (11)$$

with boundary conditions $v(x=0,t) = v_0(t)$ and $v(x\to\infty, t) = 0$. We consider a function $v_0(t)$ which is a stochastic process in time, and hence proceed to solve the initial value problem (11) with a stochastic boundary condition. Let $v_0(t)$ be a Gaussian, white stochastic process with mean $\langle v_0(t)\rangle = 0$ and variance $\langle v_0(t)v_0(t')\rangle = 2D\delta(t-t')$. We find that,

$$\left\langle v^2(x,t)\right\rangle = \frac{2D\nu}{\pi x^2}\left(1+\frac{x^2}{2\nu t}\right)e^{-x^2/2\nu t}. \qquad (12)$$

The velocity disturbance propagates into the fluid diffusively, with a diffusion coefficient ν. At long times, however, even though the vorticity produced at the wall averages to zero, the velocity does not decay exponentially away from the wall, but rather as a power law,

$$\left\langle v^2(x,t\to\infty)\right\rangle = \frac{2D\nu}{\pi x^2}. \qquad (13)$$

We now turn to the case in which the velocity of the boundary is not white, but a narrow band noise instead. As discussed elsewhere [2], narrow band noise provides an approximate description of the spectral components of the residual acceleration field in microgravity, as well as a convenient way to interpolate between the white noise and monochromatic noise limits. Narrow band noise is a Gaussian process that satisfies, [17]

$$\langle v_0(t)\rangle = 0, \quad \left\langle v_0(t)v_0(t')\right\rangle = v_0^2 e^{-|t-t'|/\tau}\cos\Omega(t-t'), \qquad (14)$$

where $\langle\,\rangle$ denotes an ensemble average. Ω is a characteristic angular frequency that corresponds to the peak in the spectral density of the process, and τ is a correlation or coherence time determining the width of that peak. In essence, this process describes a periodic signal of characteristic frequency Ω, but that only remains coherent for a time τ on average. The amplitudes are distributed gaussianly with variance v_0^2. The white noise limit is obtained when $\Omega\tau \to 0$ while $v_0^2\tau = D$ remains finite, whereas the monochromatic noise limit corresponds to $\Omega\tau\to\infty$ with v_0^2 finite. We do not attempt to find a general solution of Eq. (11) for an initially quiescent fluid and narrow band forcing. We focus instead on long time or statistically stationary averages. We find,

$$\left\langle v^2(x)\right\rangle = \int_{-\infty}^{\infty} d\omega P(\omega)e^{-x\sqrt{\frac{2|\omega|}{\nu}}}. \qquad (15)$$

where $P(\omega)$ is the power spectrum that corresponds to the autocorrelation function (14). The integral can be carried out explicitly in the limits of small and large τ. In the vicinity of the white noise limit, we find,

$$\left\langle v^2(x)\right\rangle = \frac{2\nu v_0^2\tau}{\pi x^2} - \frac{1}{2}\frac{v_0^2(120+4\Omega^2x^4/\nu^2)\nu^3}{\pi x^6}\tau^2 + \mathcal{O}(\tau^3). \qquad (16)$$

The first term in the right hand side is the white noise limit already given in Eq. (13). The first correction term is also a power law decaying as x^{-6} away from the wall. The low frequency part of the power spectrum dominates the decay of the velocity field at long distances and leads to a very slow rate of decay. In the opposite limit of $\tau rightarrow \infty$, we find,

$$\begin{aligned}\left\langle v^2(x)\right\rangle &= \left\langle v_0^2\right\rangle e^{-x\sqrt{\frac{2\Omega}{\nu}}} + \\ \frac{v_0^2\nu}{\pi x^2}&\left\{\left[\frac{x^2}{\nu}\left(1-\frac{\pi}{2}\right) + \tau x\sqrt{\frac{2(\Omega+1/\tau)}{\nu}}\right]e^{-x\sqrt{\frac{2(\Omega+1/\tau)}{\nu}}}\right. \\ &\left.+\left[\frac{x^2}{\nu}\left(1+\frac{\pi}{2}\right) - \tau x\sqrt{\frac{2(\Omega-1/\tau)}{\nu}}\right]e^{-x\sqrt{\frac{2(\Omega-1/\tau)}{\nu}}}\right\}\end{aligned} \qquad (17)$$

In the limit $\tau\to\infty$ the term within braces vanishes and one recovers the classical result that the characteristic velocity $\sqrt{\langle v^2\rangle}$ decays exponentially into the fluid with a characteristic decay length given by $\sqrt{2\nu/\Omega}$. If, on the other hand, the spectrum of the forcing function has a finite width, the velocity field still decays exponentially, but the decay length increases to $\sqrt{2\nu/(\Omega - 1/\tau)}$.

Acknowledgments

This work has been supported by the Microgravity Science and Applications Division of the NASA under contract No. NAG3-1885. This work is also supported in part by the Supercomputer Computations Research Institute, which is partially funded by the U.S. Department of Energy, contract No. DE-FC05-85ER25000.

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[12] It is important to emphasize the relevance of overdamped motion to the results presented. If the correlation time of the fluctuating force is smaller than the viscous decay time (as is the case in thermal Brownian motion), the fluctuating force enters additively in the equations of motion for the particles and the effects discussed do not arise.

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