Interlaminar Stresses by Refined Beam Theories and the Sinc Method Based on Interpolation of Highest Derivative

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Abstract

Computation of interlaminar stresses from the higher-order shear and normal deformable beam theory and the refined zigzag theory was performed using the Sinc method based on Interpolation of Highest Derivative. The Sinc method based on Interpolation of Highest Derivative was proposed as an efficient method for determining through-the-thickness variations of interlaminar stresses from one- and two-dimensional analysis by integration of the equilibrium equations of three-dimensional elasticity. However, the use of traditional equivalent single layer theories often results in inaccuracies near the boundaries and when the lamina have extremely large differences in material properties. Interlaminar stresses in symmetric cross-ply laminated beams were obtained by solving the higher-order shear and normal deformable beam theory and the refined zigzag theory with the Sinc method based on Interpolation of Highest Derivative. Interlaminar stresses and bending stresses from the present approach were compared with a detailed finite element solution obtained by ABAQUS/Standard. The results illustrate the ease with which the Sinc method based on Interpolation of Highest Derivative can be used to obtain the through-the-thickness distributions of interlaminar stresses from the beam theories. Moreover, the results indicate that the refined zigzag theory is a substantial improvement over the Timoshenko beam theory due to the piecewise continuous displacement field which more accurately represents interlaminar discontinuities in the strain field. The higher-order shear and normal deformable beam theory more accurately captures the interlaminar stresses at the ends of the beam because it allows transverse normal strain. However, the continuous nature of the displacement field requires a large number of monomial terms before the interlaminar stresses are computed as accurately as the refined zigzag theory.

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1 Introdction

Composite materials remain a topic of considerable interest to researchers due to their vulnerability to impact loading. Because of interlaminar bonding imperfections, delamination can be initiated by interlaminar stresses. This effect is amplified in sandwich composites subjected to compressive loads because the facesheets tend to buckle, accentuating the risk of delamination growth. The delamination can eventually result in ultimate failure of the composite. Modern composite failure criteria incorporate the interlaminar stresses. Therefore, accurate determination of interlaminar stresses is a topic of considerable interest in the research community. We will recapitulate some of the significant developments in this area. The state of interlaminar stress computation has been reviewed extensively by Kapania and Raciti [1] and Kant and Swaminathan [2].

Reddy [3] provided an alternative to the use of constitutive relations to obtain transverse normal and shear stresses by integrating the equilibrium equations of 3D elasticity directly. However, this technique requires in-plane derivatives of in-plane strains, a feature not explicitly available from equivalent single layer (ESL) theories such as classical lamination or first-order shear deformable theories in finite element implementation. For the transverse shear stress components, the equilibrium integration approach necessitates the first derivatives of in-plane strains. For the transverse normal stress component, the second derivatives of in-plane strains are required.

A significant amount of research has been conducted in the area of developing post-processing schemes to be used with commercial finite element software to obtain the transverse stresses. Lajczok [4] used a finite difference scheme to compute the higher derivatives of in-plane strains necessary within the equilibrium equation approach. Byun and Kapania [5] used Chebyshev and other orthogonal polynomials to interpolate the displacements and compute the higher derivatives of in-plane strain. Lee and Lee [6] developed a post-processing scheme for determining the transverse stresses in geometrically nonlinear composites for cross-ply, symmetric composites and sandwich composites with laminated facesheets and isotropic core material. Roos, Kress, and Ermanni [7] developed a post-processing method for determining the transverse normal stress component in doubly-curved, laminated, shell elements in commercial finite element codes. Noor and Malik [8] used a two-stage procedure in which first-order shear deformable theory (FSDT) was used to compute in-plane stresses in stage one. In stage two, integration of the equilibrium equations of 3D elasticity is preformed to determine through-the-thickness distributions of stress. An elasticity model is used to update the in-plane stresses. Stage two is repeated until a convergence criteria on the transverse normal stress is satisfied. Tessler and Riggs [9] developed a variational smoothing algorithm to accurately obtain the higher order derivatives

and achieve the interlaminar stresses. Tessler *et al.* [10] used the "Smoothing Element Analysis" for improving the accuracy of finite element stresses for general application and not just for interlaminar stresses; however, the approach is essentially a second finite element analysis.

Recently, Slemp and Kapania [11, 12] proposed using the Sinc Method based on Interpolation of Highest Derivative (SIHD) for problems in which the interlaminar stresses are desired. SIHD was developed by Li and Wu [13] as a numerical method for solving of one- and two-dimensional boundary value problems. The method is substantially different from traditional methods such as Ritz method or finite element method in which approximate the unknown function using some global or local basis and the basis is differentiated to obtain the derivatives. With the SIHD method, the highest derivative of interest is approximated using a Sinc basis. The method utilizes the numerical indefinite integration based on the double exponential transformation proposed by Muhammad and Mori [14] to obtain the lower derivatives and unknown function. The concept is that by using the process of integration as appose to differentiation, the highest-order derivatives would be very accurate.

Li and Wu [13] illustrated that the method is very accurate and has very good convergence properties for one- and two-dimensional boundary value problems involving only Dirichlet boundary conditions. In [11], Slemp and Kapania assessed the accuracy of the method was examined for the analysis of interlaminar stresses in composite beams and plates. Numerical results were compared against the analytic solution for the problem of a simply-supported Timoshenko beam and for a simply-supported classical laminated composite plate. The method was shown to exhibit excellent accuracy and convergence properties for the displacements, stresses, and interlaminar stresses by using the through-the-thickness integration approach.

In the next study, Slemp and Kapania used the SIHD method as a tool to solve the one-dimensional beam equations for the Timoshenko (FSDT) and the Bickford beam composite ESL theories. The numerical results for the interlaminar stresses were compared against a 3D finite element solution to assess the accuracy of the one-dimensional theory as compared with elasticity. The results for symmetric cross-ply and functionally graded composites under uniform loading indicated that the stresses compare well against 3D elasticity through most of the length of the beam; however, in the vicinity of the boundary, the results were poor. This error was attributed to the three-dimensional effects in the vicinity of the boundary that may not be captured by the one-dimensional ESL theories. The present study attempts to examine two refined beam theories in order address the viability of these methods as a low cost option for accurately obtaining interlaminar stresses. The present study also utilizes the SIHD method to solve the boundary value problem to once again illustrate the success with which this approach can be used to obtain the interlaminar stresses without post processing to

compute the necessary strain derivatives.

There are two philosophies for improving results from ESL beam theories. First, the order of approximation for the displacement fields may be increased. Both the in-plane displacement and the transverse deflection fields could take a higher-order polynomial approximation. Such an approach is typically referred to as a higher-order shear and normal deformable theory (HOSNDT). HOSNDTs have been used by many authors to analyze composite laminates and functionally graded materials. The higher-order shear and normal deformable theories as derived by Batra [15] allows users the ability to specify the order of the assumed displacement field. Xiao *et al.* [16, 17] used MLPG with radial basis functions to analyze both thick isotropic and thick composite plates. Their results showed that the HOSNDT produces excellent through-the-thickness longitudinal stress variations for thick laminates. Transverse normal and transverse shear stresses in some cross-ply laminates computed by the constitutive equations were accurate using as low as a fifth-order theory. These theories have also been used to study functionally graded materials. Qian, Batra, and Chen [18] used HOSNDT to study functionally graded plates comprised of aluminum and zirconia or aluminum and a another ceramic such as SiC.

A central drawback of the HOSNDT is the inability of continuous functions to approximate discontinuities. It is this philosophy that spurred the development of the so called layer-wise theories [19]. In layer-wise theories, the displacement is piecewise linear through the thickness. A new displacement degree of freedom is introduced for each layer and continuity of tractions between each layer is imposed. While the methods are very accurate, even comparable to three-dimensional elasticity [19, 20], the increased number of kinematic degrees-of-freedom becomes a significant drawback when a laminate has many layers. The "zigzag" theories attempt to introduce the piecewise linear displacement while keeping the number of degrees-of-freedom constant with increasing number of layers. Because the slope of the displacements are discontinuous, the strains need not be continuous through the thickness allowing the stresses, if desired, to satisfy the traction continuity between adhesively bonded layers. Early work by Di Sciuva [21] and Murakami [22] suggested displacements that satisfied the interlaminar traction continuity. However, the Di Sciuva theory suffers from the inability to find physical significance in the computed shear stress. The area integral of shear stress does not equal the shear force applied.

Averill [23] modified Di Sciuva's theory by introducing a penalty term in the variational principal to enforce interlaminar shear continuity. However, the theory suffers in much the same way as Di Sciuva. Averill resolves the issue at the cost of variational consistency of the boundary conditions. Recently, Tessler, Di Sciuva, and Gherlone [24, 20] presented a variationally consistent refined zigzag theory in which interlaminar shear continuity was not enforced. However, the theory serves as a substantial

improvement over the Timoshenko beam theory or the FSDT plate theory. Tessler, Di Sciuva, and Gherlone [20] compares transverse shear stresses from the constitutive relation with Pagano's elasticity solution [25]. Tessler, Di Sciuva, and Gherlone also obtained interlaminar stresses by integration of the equations of three dimensional elasticity and showed that they very accurately approximated the three-dimensional elasticity solution [26].

In this paper, the cross-ply laminated beams are analyzed by SIHD using the HOSNDT and Tessler's refined zigzag theory. The interlaminar stresses are computed using the equilibrium integration technique and compared with a finite element solution. The remainder of this paper is arranged as follows: The higher-oder shear and normal deformable theory for laminated beams and the refined zigzag theory are presented in Sections 2 and 3 respectively with details provided for computing interlaminar stresses from the two theories. Two numerical examples are presented in Section 4. The paper concludes with a brief conclusion.

2 Higher-Order Shear and Normal Deformable Theory for Beams

The development of the higher-order shear and normal deformable beam theory (HOSNDT) following the general approach of Batra [15] is reviewed below. First, consider a beam with rectangular cross-section as shown in Fig. 1. Because the has small width ($b << a$), the problem can be reduced to consider only the displacement field U and W. Begin by assuming a displacement field of the form,

$$U(x,z) = \sum_{i=0}^{n_u} z^i u_i(x),$$
$$W(x,z) = \sum_{i=0}^{n_w} z^i w_i(x),$$

(1)

where U and W are the longitudinal displacement and the transverse deflection respectively, and where the domain is $x \in [0, a]$ and $z \in [-h/2, h/2]$. Accordingly, the strains are

$$\epsilon_{xx} = \frac{\partial U}{\partial x} = \sum_{i=0}^{n_u} z^i u_{i,x},$$
$$\epsilon_{zz} = \frac{\partial W}{\partial z} = \sum_{i=0}^{n_w} (i) z^{i-1} w_i,$$
$$\gamma_{xz} = \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = \sum_{i=0}^{n_u} (i) z^{i-1} u_i + \sum_{i=0}^{n_w} z^i w_{i,x}.$$

(2)

Note that $(i)z^{i-1}$ should be zero for all z when $i = 0$; however, to avoid further complication by defining these in a piecewise fashion, the inconsistency is noted textually.
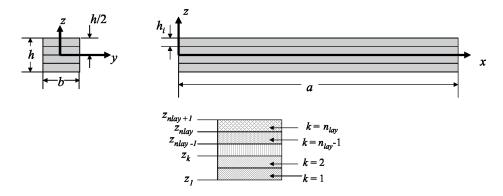

Figure 1: Definition of beam geometry and coordinate system.

The constitutive law is reduced assuming plane stress in the y direction. Note that for the traditional ESL beam theories studied in Slemp and Kapania [12], the stress resultants N_{yy}, N_{xy}, M_{yy}, M_{xy}, P_{yy}, P_{xy}, and Q_{yz} were assumed to be zero where these were defined by:

$$\{N_{yy}, N_{xy}, M_{yy}, M_{xy}, P_{yy}, P_{xy}, Q_{yz}\} = \int_{-h/2}^{h/2} \{\sigma_{yy}, \sigma_{xy}, z\sigma_{yy}, z\sigma_{xy}, z^3\sigma_{yy}, z^3\sigma_{xy}, \sigma_{yz}\} dz = 0$$

For this HOSNDT each lamina is assumed to experience plane stress ($b << a$) with stress components $\sigma_{yy} = \sigma_{xy} = \sigma_{yz} = 0$. Accordingly, the reduced constitutive relation is given by:

$$\sigma_{xx} = \bar{Q}_{11}^{(k)}\epsilon_{xx} + \bar{Q}_{13}^{(k)}\epsilon_{zz},$$

$$\sigma_{zz} = \bar{Q}_{13}^{(k)}\epsilon_{xx} + \bar{Q}_{33}^{(k)}\epsilon_{zz}$$

$$\sigma_{xz} = \bar{Q}_{55}^{(k)}\gamma_{xz}$$

(3)

where $\bar{Q}_{ij}^{(k)}$ is the plane stress reduced stiffness matrix for the kth lamina (See Fig. 1). To obtain $\bar{Q}_{ij}^{(k)}$, in terms of the lamina engineering constants, please refer Reddy [19]. Note, the stiffnesses were reduced for plan stress about the y axis and not the z axis in this sense.

The governing equations are derived using the principle of virtual work. The internal virtual work is written as:

$$\delta\mathcal{W}_{int} = \int_{\Omega} \sigma_{ij}^{(k)}\delta\epsilon_{ij}^{(k)} d\Omega$$

(4)

where repeated index implies summation (where appropriate) and Ω is the beams volume. Substituting

Eqs. (2) and (3) into the internal virtual work and integrating by parts where appropriate yields

$$
\delta \mathcal{W}_{int} = b \int_0^a \Big[-\left(T_{11ij} u_{j,xx} + T_{13ij} w_{j,x} \right) \delta u_i + \left(T_{31ij} u_{j,x} + T_{33ij} w_j \right) \delta w_i
$$

$$
+ \left(T_{51ij} u_j + T_{53ij} w_{j,x} \right) \delta u_i - \left(T_{61ij} u_{j,x} + T_{63ij} w_{j,xx} \right) \delta w_i \Big] dx
$$

$$
+ b \left(T_{11ij} u_{j,x} + T_{13ij} w_j \right) \delta u_j \Big|_0^a + b \left(T_{61ij} u_j + T_{63ij} w_{j,x} \right) \delta w_i \Big|_0^a , \quad (5)
$$

where

$$
T_{11ij} = \sum_{k=1}^{n_{lay}} \bar{Q}_{11}^{(k)} \left[\frac{z_{k+1}^{i+j+1} - z_k^{i+j+1}}{i+j+1} \right] , \qquad T_{13ij} = \sum_{k=1}^{n_{lay}} \bar{Q}_{13}^{(k)}(j) \left[\frac{z_{k+1}^{i+j-1} - z_k^{i+j-1}}{i+j} \right] ,
$$

$$
T_{31ij} = \sum_{k=1}^{n_{lay}} \bar{Q}_{13}^{(k)}(i) \left[\frac{z_{k+1}^{i+j} - z_k^{i+j}}{i+j} \right] , \qquad T_{33ij} = \sum_{k=1}^{n_{lay}} \bar{Q}_{33}^{(k)}(ij) \left[\frac{z_{k+1}^{i+j-1} - z_k^{i+j-1}}{i+j-1} \right] ,
$$

$$
T_{51ij} = \sum_{k=1}^{n_{lay}} \bar{Q}_{55}^{(k)}(ij) \left[\frac{z_{k+1}^{i+j-1} - z_k^{i+j-1}}{i+j-1} \right] , \qquad T_{53ij} = \sum_{k=1}^{n_{lay}} \bar{Q}_{55}^{(k)}(i) \left[\frac{z_{k+1}^{i+j} - z_k^{i+j}}{i+j} \right] ,
$$

$$
T_{61ij} = \sum_{k=1}^{n_{lay}} \bar{Q}_{55}^{(k)}(j) \left[\frac{z_{k+1}^{i+j} - z_k^{i+j}}{i+j} \right] , \qquad T_{63ij} = \sum_{k=1}^{n_{lay}} \bar{Q}_{55}^{(k)} \left[\frac{z_{k+1}^{i+j+1} - z_k^{i+j+1}}{i+j+1} \right] ,
$$

n_{lay} is the total number of composite layers, and repeated index of i and j implies summation over the terms in that index. For T_{11ij}, T_{13ij}, T_{51ij}, and T_{53ij}, $i = 0, 1, 2, ... n_u$; for T_{31ij}, T_{33ij}, T_{61ij}, T_{63ij}, $i = 0, 1, 2, ... n_w$; for T_{11ij}, T_{31ij}, T_{51ij}, and T_{61ij}, $j = 0, 1, 2, ... n_u$; for T_{13ij}, T_{33ij}, T_{53ij}, and T_{63ij}, $j = 0, 1, 2, ... n_w$. Also, note that if a zero-denominator is encountered in any T_{ijkl}, these terms are set to zero. The zero-denominator only results from differentiation of a z^i when $i = 0$. Therefore, these terms are zero and no denominator is computed.

The external virtual work for a beam under the simply-supported condition with transverse pressure loading $q(x)$ applied on the top surface of the beam ($z = h/2$) has the form:

$$
\delta \mathcal{W}_{ext} = \int_0^a q(x) \delta W(x, h/2) dx = \int_0^a -q(x) \sum_{i=0}^{n_w} (h/2)^i \delta w_i dx
$$

The variational statement,

$$
b \int_0^a \Big[- \left(-T_{11ij} u_{j,xx} - T_{13ij} w_{j,x} + T_{51ij} u_j + T_{53ij} w_{j,x} \right) \delta u_i
$$

$$
- \left(+T_{31ij} u_j, + T_{33ij} w_j - T_{61ij} u_{j,x} - T_{63ij} w_{j,xx} + q(x)(h/2)^i \right) \delta w_i \Big] dx
$$

$$
- b \left(T_{11ij} u_{j,x} + T_{13ij} w_j \right) \delta u_i \Big|_0^a - b \left(T_{61ij} u_j + T_{63ij} w_{j,x} \right) \delta w_i \Big|_0^a = 0, \quad (6)
$$

governs equilibrium and boundary conditions. Because the variations, δu_i and δw_i, are arbitrary the boundary value problem may be written by the following Euler equations of equilibrium:

$$
\delta u_i : \quad T_{11ij} u_{j,xx} + T_{13ij} w_{j,x} - T_{51ij} u_j - T_{53ij} w_{j,x} = 0 \tag{7}
$$

$$
\delta w_i : \quad - T_{31ij} u_{j,x} - T_{33ij} w_j + T_{61ij} u_{j,x} + T_{63ij} w_{j,xx} = q(x)(h/2)^i \tag{8}
$$

For the simply support condition, the kinematics require $W(0,z) = 0$ and $W(a,z) = 0$. This is imposed by constraining all w_i at the edges. At the edge $x = 0$, a point constraint is applied to the U degree of freedom ($U(0,0) = 0$). This is satisfied by setting $u_i(0) = 0$ for $i = 0, 2, 4, \dots$. No constraints on the u_i for $i = 1, 3, 5, \dots$ were imposed. Thus the complete set of essential and natural boundary conditions used for the analysis are:

$$
w_i(0) = 0 \quad (i = 0, 1, 2, \dots, n_w)
$$

$$
u_i(0) = 0 \quad (i = 0, 2, 4, \dots)
$$

$$
\left(T_{11ij} u_{j,x} + T_{13ij} w_j \right) \Big|_{x=0} = 0 \quad (i = 1, 3, 5, \dots) \tag{9}
$$

$$
w_i(a) = 0 \quad (i = 0, 1, 2, \dots, n_w)
$$

$$
\left(T_{11ij} u_{j,x} + T_{13ij} w_j \right) \Big|_{x=a} = 0 \quad (i = 0, 1, 2, \dots, n_u)
$$

where repeated index implies summation over all elements in that index.

Note that by selecting the order of polynomial expressions for U and W, n_u and n_w respectively, the governing equations and boundary conditions are given by Eqs. (7) and (8). The boundary value problem may be solved using the SIHD method in an identical fashion to Slemp and Kapania [12]. While we omit the details, it is important to note that by using SIHD to solve the boundary value problems all derivatives up to $u_{i,xxx}(x)$ and $w_{i,xx}$ are known as a by-product of the analysis and no additional post-processing algorithm needs to be used to find these higher order derivatives.

The interlaminar stresses can be found by integrating the equations of 2D elasticity through the

thickness of the kth lamina layer ($z_k \leq z \leq z_{k+1}$).

$$\sigma_{xz}^{(k)} = -\int_{z_k}^{z} \left(\frac{\partial \sigma_{xx}}{\partial x}^{(k)} \right) dz + G^{(k)} \tag{10}$$

$$\sigma_{zz}^{(k)} = -\int_{z_k}^{z} \left(\frac{\partial \sigma_{xz}}{\partial x}^{(k)} \right) dz + H^{(k)} \tag{11}$$

Note that $G^{(k)}$, $F^{(k)}$, and $H^{(k)}$ are functions of x that may be determined from traction equilibrium between the layer boundaries. First, the transverse shear stresses are considered. Substituting the Hooke's law and strain-displacement relations into Eq. (9), the transverse shear stress, $\sigma_{xz}^{(k)}$, is expressed by:

$$\sigma_{xz}^{(k)} = -\int_{z_k}^{z} \left[\bar{Q}_{11}^{(k)} \sum_{i=0}^{n_u} \left(z^i u_{i,xx} \right) + \bar{Q}_{13}^{(k)} \sum_{i=1}^{n_w} \left(i z^{i-1} w_{i,x} \right) \right] dz + G^{(k)} \tag{12}$$

Performing the integration through the layer thickness, this is expressed by:

$$\sigma_{xz}^{(k)} = -\left[\bar{Q}_{11}^{(k)} \sum_{i=0}^{n_u} \left(\frac{z^{i+1} - z_k^{i+1}}{i+1} u_{i,xx} \right) + \bar{Q}_{13}^{(k)} \sum_{i=1}^{n_w} \left((z^i - z_k^i) w_{i,x} \right) \right] + \sigma_{xz}^{(k)}(x, z_k) \tag{13}$$

Differentiating with respect to x and substituting into Eq. (10) yields:

$$\sigma_{zz}^{(k)} = -\int_{z_k}^{z} \left[-\bar{Q}_{11} \sum_{i=0}^{n_u} \left(\frac{z^{i+1} - z_k^{i+1}}{i+1} u_{i,xxx} \right) - \bar{Q}_{13} \sum_{i=1}^{n_w} \left((z^i - z_k^i) w_{i,xx} \right) + \sigma_{xz,x}^{(k)}(x, z_k) \right] dz + H^{(k)} \tag{14}$$

Evaluating the integration on a layer-by-layer basis through the lamina thickness yields:

$$\sigma_{zz}^{(k)} = \bar{Q}_{11} \sum_{i=0}^{n_u} \left[\left(\frac{z^{i+2} - z_k^{i+2}}{(i+1)(i+2)} - \frac{z_k^{i+1}(z - z_k)}{i+1} \right) u_{i,xxx} \right] + \bar{Q}_{13} \sum_{i=1}^{n_w} \left[\left(\frac{z^{i+1} - z_k^{i+1}}{i+1} - z_k^i(z - z_k) \right) w_{i,xx} \right]$$
$$- \sigma_{xz}^{(k)}(x, z_k)(z - z_k) + \sigma_{zz}^{(k)}(x, z_k) \tag{15}$$

Thus the transverse shear and transverse normal stresses can be found from the displacement variables. The terms $\sigma_{xz}(x, z_k)$, $\sigma_{xz,x}(x, z_k)$, and $\sigma_{zz}(x, z_k)$ are determined by imposing continuity of the tractions in a layer-by-layer fashion, starting from the bottom free surface.

3   Refined Zigzag Theory for Beam Analysis

The development of the refined zigzag theory for beam analysis, following the approach of Tessler, Di Sciuva, and Gherlone [20], is reviewed below. Consider the same beam geometry shown in Fig. 1.

Begin by assuming a displacement field of the form,

$$U(x, z) = u_0(x) + z\theta(x) + \phi^{(k)}(z)\psi(x),$$

$$W(x, z) = w_0(x),$$

$$(16)$$

where U and W are the longitudinal displacement and the transverse deflections respectively, $\phi^{(k)}(z)$ is a piecewise linear function of the thickness coordinate, and the domain is $x \in [0, a]$ and $z \in [-h/2, h/2]$. Accordingly, the strains are

$$\epsilon_{xx}(x, z) = u_{0,x}(x) + z\theta_{,x}(x) + \phi^{(k)}(z)\psi_{,x}(x),$$

$$\gamma_{xz}(x, z) = \theta(x) + \beta^{(k)}(z)\psi(x) + w_{0,x}(x),$$

$$(17)$$

where $\beta^{(k)} = \phi_{,z}^{(k)}$. Note that $\beta^{(k)}$ is a constant within a given layer.

The piecewise linear variations, $\phi^{(k)}$ is set to zero on the bottom and top surfaces of the beam and is C^0 continuous at the interfaces. Therefore, the distribution over the kth lamina may be written as:

$$\phi^{(k)}(z) = \beta^{(k)}(z - z_k) + \phi^{(k-1)}(z_k)$$

$$(18)$$

Note that $\phi^{(k-1)}(z_k)$ implies evaluation at z_k and not multiplication. Thus determining $\beta^{(k)}$, fully prescribes the displacement field. Rather than prescribing traction continuity between layers, Tessler, Di Sciuva, and Gherlone [20] requires that the following equation holds:

$$\bar{Q}_{55}^{(k)}(1 + \beta^{(k)}) = \left[\frac{1}{h} \int_{-h/2}^{h/2} \frac{dz}{\bar{Q}_{55}^{(k)}} \right]^{-1}$$

$$(19)$$

The assumption is made that each beam layer is in a state of plane stress in the z and y directions. Thus the bending stiffness, $\bar{Q}_{11}^{(k)}$, and shear stiffnesses, $\bar{Q}_{55}^{(k)}$, reduce to the elastic moduli, $E_{xx}^{(k)}$ and $G_{xz}^{(k)}$, respectively. The governing equation and boundary conditions are derived from the variational principle (see Tessler, Di Sciuva, and Gherlone [20]). The governing equations are:

$$N_{xx,x} = 0$$

$$M_{xx,x} - V_x = 0$$

$$V_{x,x} + q = 0$$

$$M_{\phi,x} - V_\phi = 0$$

$$(20)$$

10

The kinematic variables are u_0, θ, w, and ψ. The stress resultants are defined by:

$$\{N_{xx}, M_{xx}, M_\phi, V_x, V_\phi\} = \int_{-h/2}^{h/2} \{\sigma_{xx}^{(k)}, \sigma_{xx}^{(k)} z, \sigma_{xx}^{(k)} \phi^{(k)}(z), \sigma_{xz}^{(k)}, \sigma_{xz}^{(k)} \beta^{(k)}\} dz. \tag{21}$$

To impose the simply support condition in a similar manner to the HOSNDT formulation, the kinematic conditions require $w_0(0) = 0$, $w_0(a) = 0$, and $u_0(0) = 0$ with no constraints on $u_0(a)$, $\psi(0)$, $\psi(a)$, $\theta(0)$, and $\theta(a)$. Thus the complete set of essential and natural boundary conditions are:

$$
\begin{aligned}
w_0(0) &= 0, \quad w_0(a) = 0, \\
u_0(0) &= 0, \quad N_{xx}(a) = 0, \\
M_{xx}(0) &= 0, \quad M_{xx}(a) = 0, \\
V_x(0) &= 0, \quad V_x(a) = 0, \\
M_\phi(0) &= 0, \quad M_\phi(a) = 0.
\end{aligned}
\tag{22}
$$

For the present study, the SIHD method was used to approximately solve the boundary value problem. Again the details are omitted (see Slemp and Kapania [12]); however, it should be noted that doing so provides all derivatives up to $u_{0,xxx}$, $\theta_{,xxx}$, and $\psi_{,xxx}$ so that $\epsilon_{xx,xx}$ and its lower derivatives are known across the beam without post-processing.

The interlaminar stresses can be derived from integration of 2D elasticity in a similar fashion to the HOSNDT. The following relations for transverse shear and transverse normal stresses were obtained.

$$\sigma_{xz}^{(k)} = -(z - z_k) \left[u_{0,xx} - \psi_{,xx} \left(\beta^{(k)} z_k + \phi^{(k-1)}(z_k) \right) \right] + \frac{z^2 - z_k^2}{2} \left(\theta_{,xx} + \psi_{,xx} \beta^{(k)} \right) + \sigma_{xz}^{(k)}(x, z_k) \tag{23}$$

$$
\begin{aligned}
\sigma_{zz}^{(k)} &= \left(\frac{z^2 - z_k^2}{2} - z_k(z - z_k) \right) \left[u_{0,xxx} - \psi_{,xxx} \left(\beta^{(k)} z_k - \phi^{(k-1)}(z_k) \right) \right] \\
&\quad + \left(\frac{z^3 - z_k^3}{6} - \frac{z_k^2(z - z_k)}{2} \right) \left(\theta_{,xxx} + \psi_{,xxx} \beta^{(k)} \right) - \sigma_{xz,x}^{(k)}(x, z_k)(z - z_k) + \sigma_{zz}^{(k)}(x, z_k) \tag{24}
\end{aligned}
$$

The transverse shear and transverse normal stresses can be found from the displacement variables. The terms $\sigma_{xz}(x, z_k)$, $\sigma_{xz,x}(x, z_k)$, and $\sigma_{zz}(x, z_k)$ determined from traction equilibrium between the layer boundaries.

4 Numerical solution of HOSNDT and Refined Zigzag Theory with SIHD

The three-layered, [0/90/0], graphite/epoxy laminated beams of Slemp and Kapania [12] was revisited using the HOSNDT and the refined zigzag. This material configuration was referred to as Material A. The properties for this beam are give Table 1. A second beam was taken with Material B, a fictitious lamina with extremely different shear moduli for G_{13} and G_{23}. The properties are given in Table 1. For each case the length-to-thickness ratio was $a/h = 5$ where a and h are the length and thickness of the beam respectively.

Table 1: Material properties for a graphite/epoxy lamina (A) and a fictitious orthotropic lamina with extremely different shear moduli (B).

	A	B
E_1 [Msi/GPa]	24.9 / 172	24.9 / 172
E_2 [Msi/GPa]	1.00 / 6.89	1.00 / 6.89
G_{12}, G_{13} [Msi / GPa]	0.50 / 3.45	14.5 / 100
G_{23} [Msi / GPa]	0.20 / 1.38	0.145 / 1
ν_{12}	0.25	0.25

4.1 Material A

Eight-node, linear, reduced-integration, brick elements (C3D8R) in ABAQUS/Standard were used to obtain an approximate three-dimensional elasticity solution for the beams. The element size was reduced until a similar solution from two successively fine meshes was obtained. The converged mesh used a total of 140 elements along the length dimension, 49 elements through the thickness and 26 elements across the width of the beam. The elements were biased toward the ends of the beam to obtain a high level of mesh density near the boundaries. The simply-supported boundary condition was enforced at both edges by constraining all nodes on the transverse faces from u_3 translation. The mid-plane's intersection with the transverse face was constrained in the u_1 direction, allowing rotation of the simply-supported faces. The central node on a transverse edge was constrained against u_2 translation, preventing a rigid body mode while allowing Poisson contraction everywhere so no artificial loads are introduced by the boundary conditions. A uniformly distributed load was applied to the top surface of the beam.

An approximate solution to the HOSNDT and the refined zigzag beam theories was obtained using the SIHD method as implemented in Slemp and Kapania [12]. The HOSNDT was solved using SIHD implemented in Fortran 90. The Sinc mesh size was set by specifying the Sinc span to be $\alpha = 2.0$ (see

Slemp and Kapania [12]). A total of 81 (N = 40) Sinc points were used to obtain a converged solution. The zigzag theory was implemented in MATLAB using 201 (N = 100) Sinc points. In this case the Sinc mesh size was also set by specifying the Sinc span to be $\alpha = 2.0$.

In Fig. 2, the longitudinal and transverse normal stress components obtained by Slemp and Kapania [12] using the Bickford and Timoshenko beam theories, the HOSNDT with $n_u = 5$ and 10 and $n_w = 3$, and the refined zigzag theory of Tessler, Di Sciuva, and Gherlone [20] were compared with the 3D FEM solution.
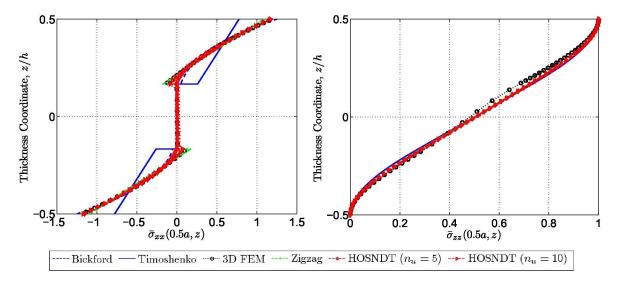

Figure 2: Longitudinal and transverse normal stress components at $x = 0.5$ for simply supported laminated beam with Material A. The Timoshenko, Bickford, and FEM results are from Slemp and Kapania [12].

The transverse normal stress was very accurately approximated by each of the approaches and there is little benefit to the additional complexity of the HOSNDT or the zigzag theory. However, the benefit is apparent with the in-plane longitudinal stress. The Bickford theory performs very well at capturing the trend of the curve; however, at the material interface it is erroneous. The zigzag theory, while it predicts only linear through lamina distributions of in-plane stress, it matches very well with the FEA for the lamina interfacial stresses. The zigzag theory is superior to the Timoshenko beam theory and performs better than the Bickford theory even though it lacks the ability to produce the curved through-the-lamina-thickness stress distribution present in this thick laminate.

For the HOSNDT, the increasing order of u displacement appears to push the interfacial longitudinal normal stress toward the FEA results; however, it is only with significant additional unknowns that similar accuracy as the zigzag theory is achieved.

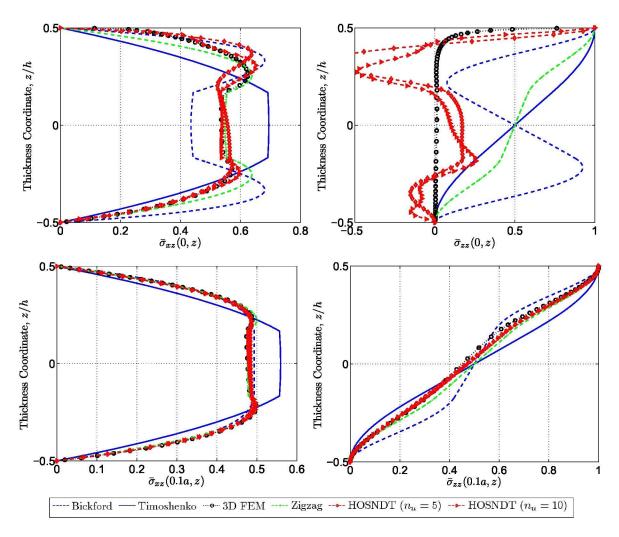If Fig. 3, the transverse shear and normal stresses are plotted at the end and near the end of the

Figure 3: Transverse shear and normal stress components at $x = 0, 0.1a$ for simply supported laminated beam with Material A. The Timoshenko, Bickford, and FEM results are from Slemp and Kapania [12].

beam. In the previous study [12], it was noted that the edge effects cannot be captured by the ESL theories for this problem. The refined theories appear to improve substantially the transverse shear stress at the edge of the beam. The interfacial stress is approximated very well by the HOSNDTs and the zigzag theory; however, the zigzag theory is unable to capture the stress asymmetry about the mid-plane (higher transverse normal stress in the top lamina). The HOSNDTs appear to account for the stress asymmetry quite well; though neither $n_u = 5$ nor $n_u = 10$ ($n_w = 3$ for both) is sufficient to exactly capture the stress distribution.

While the refined theories do a better job particularly for the transverse shear stress at the edge, none of the present theories capture the transverse normal stress at the boundary very well. The HOSNDTs improve upon this stress component significantly; however, they appear to oscillate about the FEA solution. Further increasing the order of the in-plane and transverse deflection profiles do not significantly improve the results. The HOSNDT suffers from the so called Gibbs phenomenon. Continuous functions are never able to approximate discontinuities very well.

At $0.1a$, the refined theories each perform well at capturing both the transverse normal and transverse shear stresses. The refined theories are a substantial improvement over the ESL theories for these stresses.

If the normal deformability is neglected (i.e. taking $n_w = 0$), HOSNDT produces results very similar to the zigzag theory. It should be noted that the present problem is quite a difficult one in that the loading is uniform. The edge experiences a sudden drop in applied load. Even the 3D FEM solution is not plausible because the boundary conditions are not exactly satisfied. The stress should equal the applied load on the top surface and the derivative of the transverse normal stress with respect to z should be zero (by 3D equilibrium); however, neither of these are exactly satisfied by the FEA solution.

4.2 Material B

The fictitious lamina (Material B) was chosen to demonstrate a key short coming of using ESL theories to calculate interlaminar stresses. Namely, computation of interlaminar stresses from the three-dimensional equilibrium equations of elasticity relies on the accuracy of the in-plane derivatives of in-plane stresses. However, there are some lamina for which the material properties are so discontinuous that obtaining accurate derivatives of in-plane stresses is not possible. The present example clearly demonstrates the refined zigzag theories superiority to the Timoshenko beam theory and the HOSNDT.

Analysis was performed using SIHD with the Timoshenko beam theory, the refined zigzag theory, and the HOSNDT with $n_u = 5$, 10, and 15 and with $n_w = 3$. The Timoshenko beam theory and the zigzag theory were implemented in MATLAB while the HOSNDT was implemented in Fortran 90. The results were compared with a two-dimensional plane stress finite element analysis. For the FEA, the mesh size was decreased until a similar solution was obtained from two successively fine meshes. The converged mesh size had a total of 200 elements along the length. The elements were biased toward the ends with a bias ratio of 50. Through the thickness of the top and bottom layer 36 elements were used which were biased toward the top and bottom respectively. Though the thickness of the middle lamina, 12 elements were used in a uniformly distributed manner.

The longitudinal and transverse normal stress components for material configuration B were plotted in Fig. 4. The bending stresses are very accurately predicted by the zigzag theory, nearly indistinguishable from the FEM results. Note that the Timoshenko beam theory fails to estimate the bending stresses. The Timoshenko beam theory prediction for bending stress at the interface is wrong by about 175 % and about 40 % at the top and bottom surface. The HOSNDT with using ten monomials approximating the in-plane displacement performs somewhat better; however, at a substantial computational cost. With fifteen monomials, the HOSNDT results approach the accuracy of the zigzag theory.
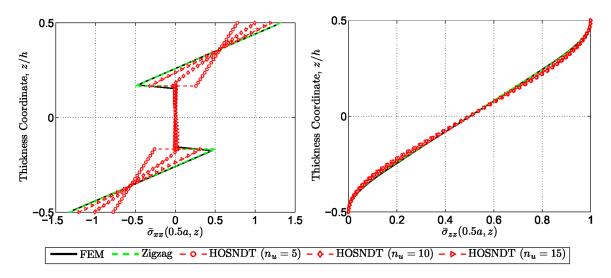

Figure 4: Longitudinal and transverse normal stress components at $x = 0.5$ for simply supported laminated beam with Material B.

The transverse shear and normal stresses were plotted in Fig. 5 at the edge of the beam and one-tenth of the length of the beam. The transverse shear stress is also very well approximated using the refined theories; however, the zigzag theories advantage is somewhat less convincing. While the
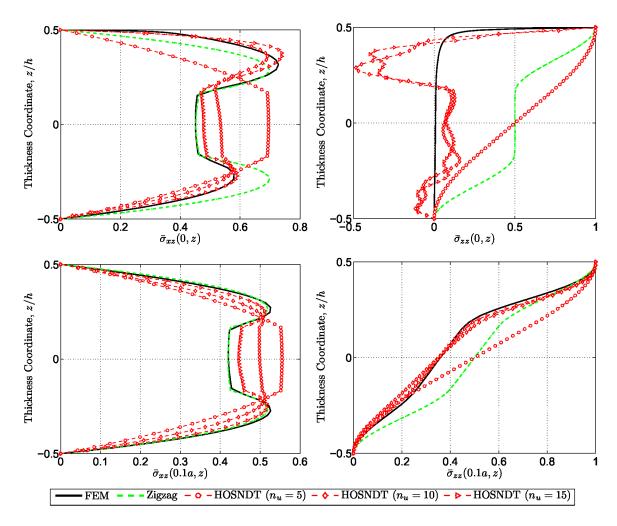
Figure 5: Transverse shear and normal stress components at $x = 0, 0.1a$ for simply supported laminated beam with Material B.

interfacial shear stress is accurately approximated with the zigzag theory, the asymmetry of shear through the thickness cannot be captured without normal deformability. However, including normal deformability alone does not grant accuracy for these stress components. Note that with fifteen monomials approximating the U displacement, the transverse shear component at the lamina interfaces is still not as accurately computed using the HOSNDT than with the zigzag theory - a piecewise discontinuous linear theory.

The transverse normal stress component is again inaccurate at the boundary. While introducing normal deformability is necessary to very accurately capture the stress at one-tenth of the length of the beam, the transverse normal stress at the edge is not captured very well at all by the theories studied.

4.3   Summary and Conclusions

In this paper, computation of transverse shear and transverse normal stresses or interlaminar stresses from refined beam theories was performed using the Sinc method based on Interpolation of Highest Derivative (SIHD). The higher-order shear and normal deformable beam theory (HOSNDT) and the refined zigzag theory were introduced. Integration of the equilibrium equations of three-dimensional elasticity was performed to obtain the transverse normal and transverse shear stress components for each theory.

The aims of this paper were to determine if the erroneous nature of interlaminar stresses computed by the Timoshenko and Bickford beam theories could be improved by refining the assumed displacement field. Like the previous study [12], the results illustrate that the SIHD method can be beneficial for this type of problem because the higher-order derivatives of the strain needed to perform through the thickness integration are accurately obtained without post processing. Two material configurations were considered. First, comparison was made with the material configuration of the previous study [12] using SIHD and the Timoshenko and Bickford beam theories. A second material configuration was examined which demonstrated the serious limitations of the traditional ESL theories when the shear moduli of lamina varies too substantially.

The HOSNDTs were used to illustrate the effects of increasing the order of the assumed displacement field and including the transverse normal strain. However, the present implementation of the HOSNDT does not allow for discontinuities of the shear strains through the thickness. The numerical results provide very compelling evidence that the refined zigzag theory provides an excellent way to introduce a discontinuous displacement field without significant computational costs, and its accuracy

benefits substantially. The present results have shown that the refined zigzag theory provides excellent accuracy of the interfacial bending and transverse shear stresses. While the transverse normal stress was not improved substantially and cannot be accurately predicted without allowing a nonzero transverse normal strain component, the refined zigzag theory should be used over simple increasing the order of the in-plane displacement field.

Acknowledgement

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