

Reduced Order Modeling of Incompressible Flows

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POD-based Reduced Order Modeling

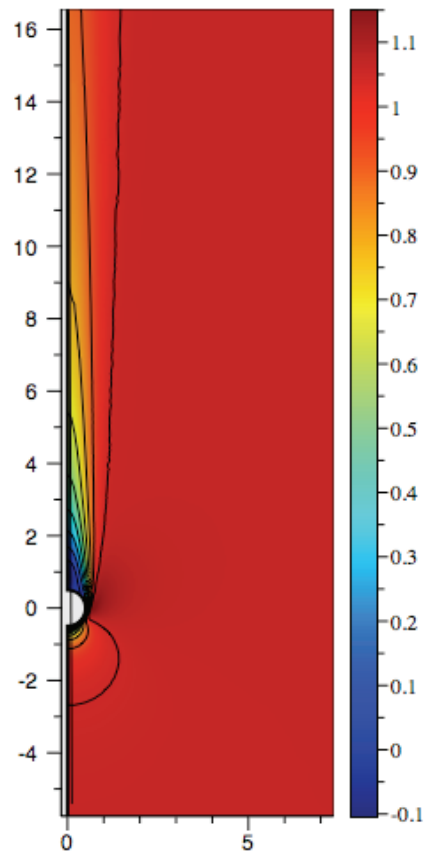
- ▶ Develop low-dimensional model that can represent the dynamics of a higher dimensional system
- ▶ Uses:
 - ▶ Models for real-time control
 - ▶ Sub-model generation
- ▶ Difficulties:
 - ▶ Round-off sensitive
 - ▶ Unstable / poorly conditioned
 - ▶ Units inconsistency for systems

Particle Modeling

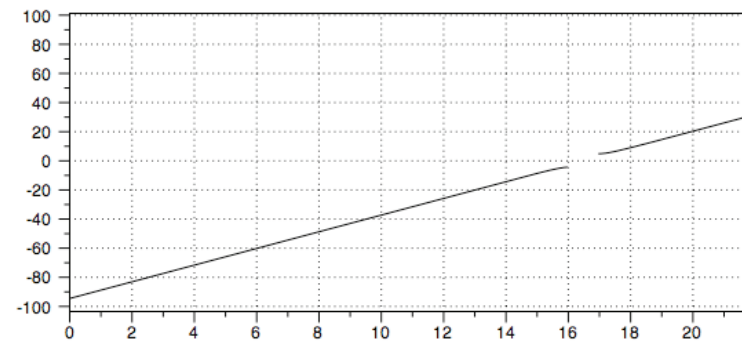
- ▶ Incompressible, flow over a sphere with variable inlet velocity

$$u = u_{\infty} (1 + A \sin(2\pi t / T))$$

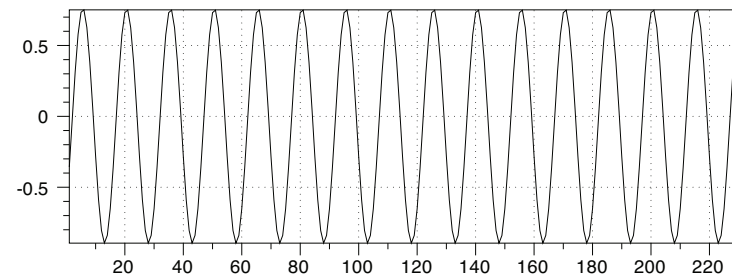
- ▶ Axial Velocity



- ▶ Centerline pressure



- ▶ Force



Generate Modes

- ▶ Find decomposition $\sum_j a_j(t)\phi_j(x)$ to represent solution $u(x, t)$
- ▶ Choose optimal functions by maximizing mean square projection:

$$\Pi(\phi) = \frac{\langle(\phi, u)^2\rangle}{(\phi, \phi)}$$

- ▶ Some trouble:

$$\vec{u} = \begin{bmatrix} u_r \\ u_z \\ p \end{bmatrix}$$

$$\Pi(\phi) = \frac{\langle(\vec{\phi}, \vec{u})^2\rangle}{(\vec{\phi}, \vec{\phi})}$$

Dimensionally inconsistent?

Possible Choices for \vec{u}

- ▶ Depends on non-dimensionalization:

$$\vec{u} = (u_r(\vec{x}, t), u_z(\vec{x}, t), p(\vec{x}, t)/(\rho u_\infty))$$

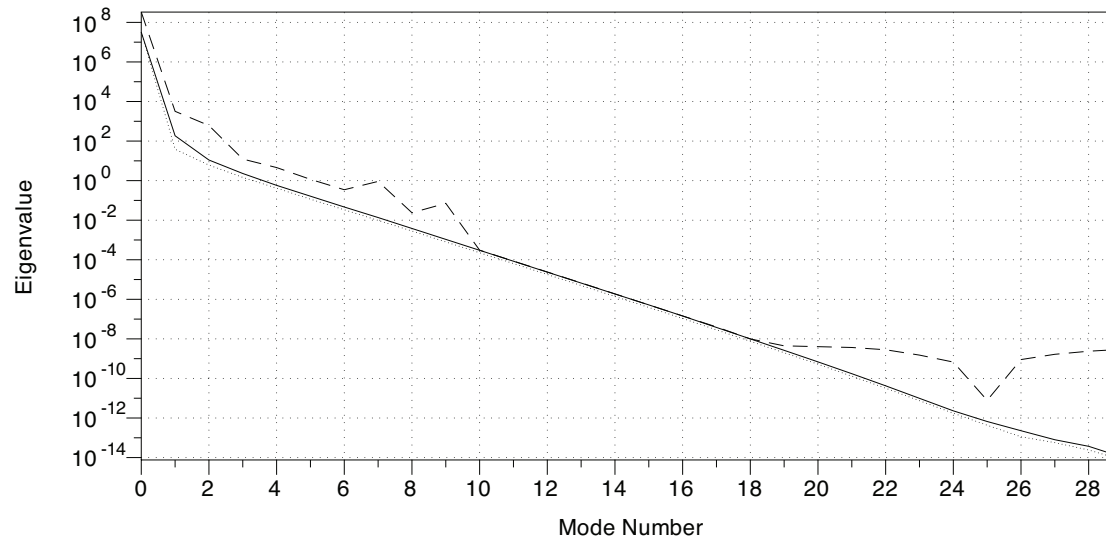
- ▶ Could be imaginary:

$$\vec{u} = (u_r(\vec{x}, t), u_z(\vec{x}, t), \sqrt{2(p(\vec{x}, t))/\rho})$$

- ▶ Also has arbitrary constant:

$$\vec{u} = (u_r(\vec{x}, t), u_z(\vec{x}, t), \sqrt{2(p(\vec{x}, t) + p_0)/\rho})$$

Eigenvalues of Steady Problem



- ▶ 31 steady snapshots from $Re = 0.1$ to 100
- ▶ Eigenvalues exponentially decay
- ▶ Small number of modes can capture most of the energy
- ▶ Last five modes of Lapack DGESVD are negative

Generation of Reduced Order Model

- ▶ Failure of Galerkin Projection

$$\int_{\Omega} [\phi_r, \phi_z, \phi_p] \left[\begin{array}{l} \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \\ \nabla \cdot \vec{v} \end{array} = -\nabla p + \mu \nabla^2 \vec{v} \right] d\Omega = 0$$

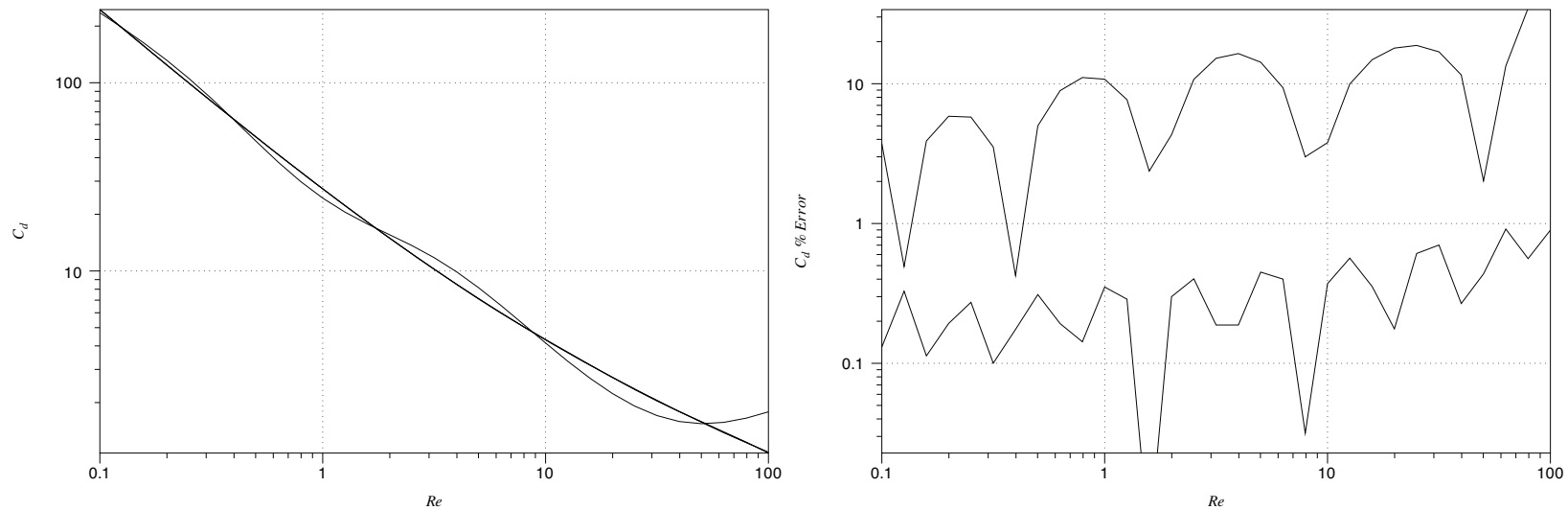
- ▶ Modes are all incompressible
- ▶ Continuity is always zero
- ▶ ∇p term integrated by parts is zero
- ▶ Pressure modes may be undetermined

SUPG Projection

$$\sum_{e=1}^{n_{el}} \left\{ \int \int_{\Omega} \left[-\frac{\partial \vec{\phi}^T}{\partial \xi} \vec{e} - \frac{\partial \vec{\phi}^T}{\partial \eta} \vec{f} \right] d\Omega + \int_{\Gamma} \vec{\phi}^T (\vec{e}, \vec{f}) \cdot \vec{n}_{\Gamma} d\Gamma \right. \\ \left. + \int \int_{\Omega} \left[\frac{\partial \vec{\phi}^T}{\partial \xi} \frac{\partial \vec{e}}{\partial \vec{u}} + \frac{\partial \vec{\phi}^T}{\partial \eta} \frac{\partial \vec{f}}{\partial \vec{u}} \right] \mathbf{T} \left[\frac{\partial}{\partial \xi} \vec{e} + \frac{\partial}{\partial \eta} \vec{f} \right] d\Omega \right\} = 0 \quad \forall \vec{\phi}$$

- ▶ Streamwise-upwind-Petrov-Galerkin variational approach
- ▶ Allows us to seek u_r, u_z, p with no pressure decoupling
- ▶ Upwinds all the terms in the governing equations consistently
- ▶ Results in a system of M ODE's (Solved using DIRK & Newton-Rhapson)

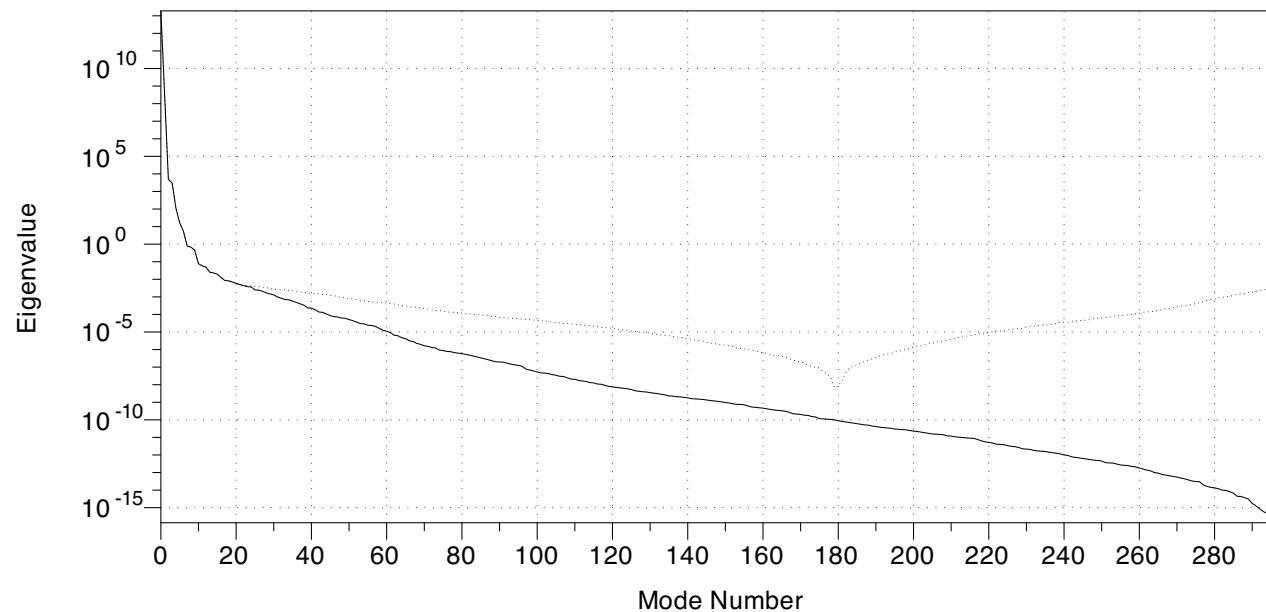
Drag Results - Steady



- ▶ Accuracy of drag prediction versus Re for 5 and 10 modes
- ▶ Significant improvement over DNS (10 vs. 21,000 degrees of freedom)
- ▶ Close to empirical correlations in performance
- ▶ All three sets of modes perform similarly

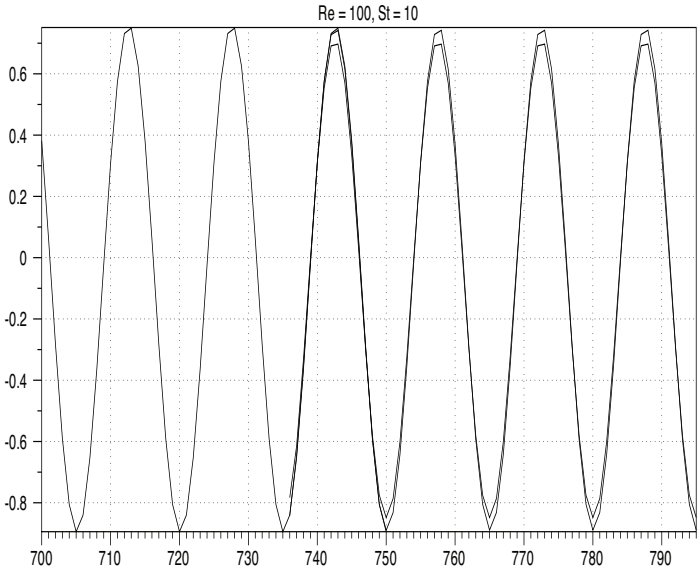
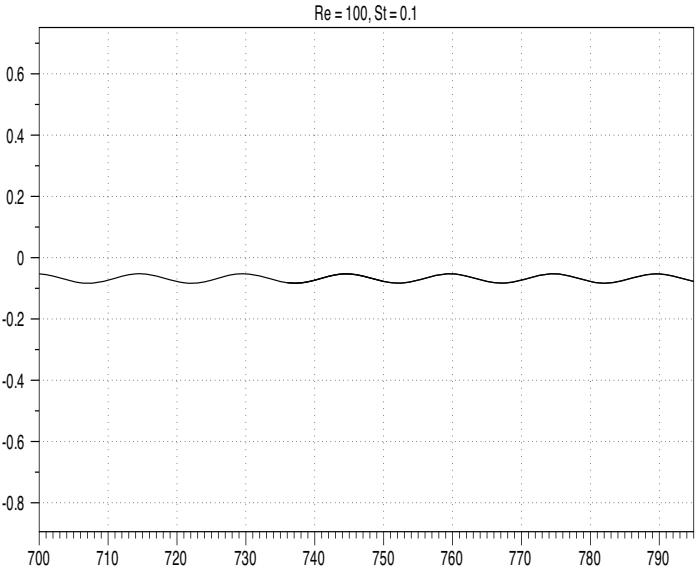
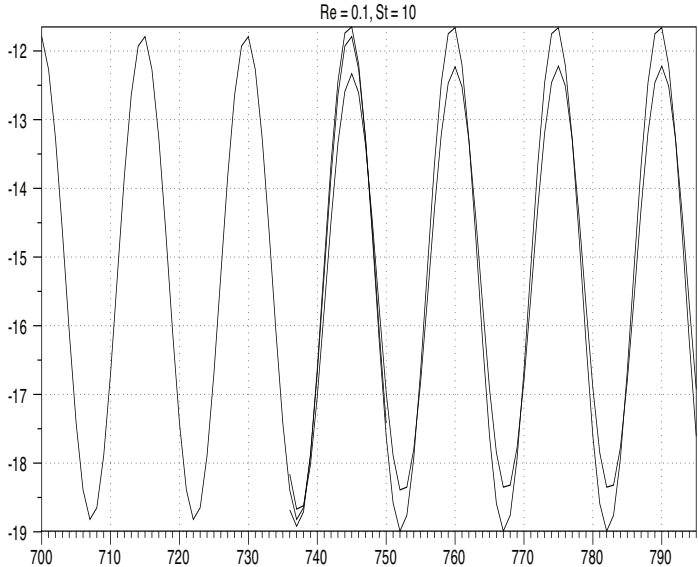
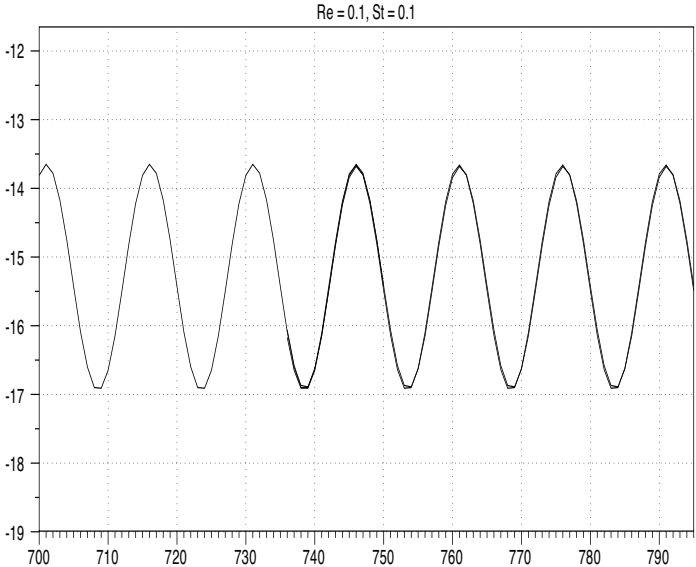
Unsteady Data

- ▶ $Re = 0.1, 1, 10, 100 \times St = 0.1, 0.5, 1, 2, 10$
- ▶ 15 time-steps/period, 20x15 snapshots

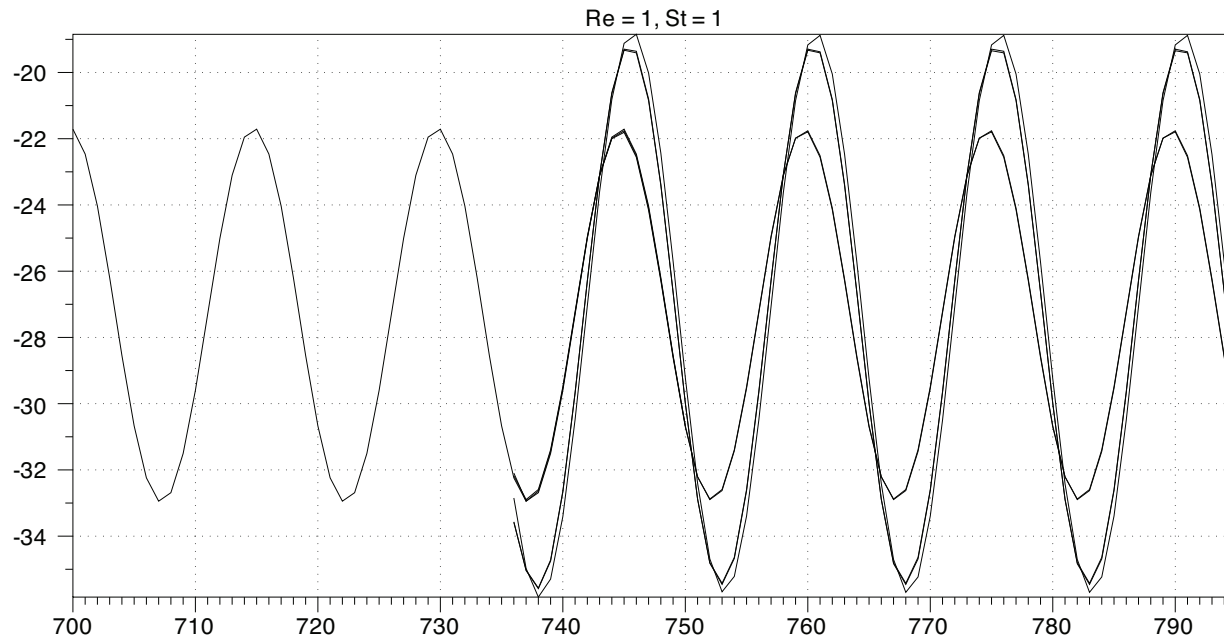


- ▶ All eigenvalues after ≈ 60 are probably round-off dominated.
- ▶ Using Lapack DGESVD, all modes beyond 180 have negative eigenvalues

Unsteady Results



Comparison of Modes



- ▶ $Re = 1$ and $St = 1$.
- ▶ SVD modes seem ok in spite of eigenvalue distribution.
- ▶ Low Mach \neq modes give similar results with 20 modes, but for 60 modes Newton diverges.

Conclusions

- ▶ The devil is in the details!!!
 - ▶ Need stable numerical methods
 - ▶ Round off error can be considerable
 - ▶ Not convinced modes are correct for incompressible flow
- ▶ Nonetheless, can derive compact and accurate reduced-order models.
- ▶ Can be used to generate actuator models or full flow-field models