2.18 Adaptive multidimensional modeling with applications in scientific computing

Adaptive multidimensional modeling
with applications in scientific computing

- network problems (energy distribution, telecom networks, queueing problems, ...)
- vision/graphics (shape reconstruction, reflectance distribution, video signal filtering, ...)
- metamodelling (microwave devices, material design, computational finance, ...)
- ...

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1 / 25

Adaptive multidimensional modeling

$$(x_1^{(\ell)}, \ldots, x_d^{(\ell)}) \longrightarrow \boxed{\quad f \quad} \longrightarrow f^{(\ell)} \in F^{(\ell)} = [f_<^{(\ell)}, f_>^{(\ell)}]$$
$$\ell = 0, \ldots, s$$

$$r_{n,m}(x_1, \ldots, x_d) = \frac{\sum_{k=0}^{n} a_k g_k(x_1, \ldots, x_d)}{\sum_{k=0}^{m} b_k g_k(x_1, \ldots, x_d)}$$

such that $r(x_1^{(\ell)}, \ldots, x_d^{(\ell)}) \in F^{(\ell)}$

2 / 25

Adaptive multidimensional modeling

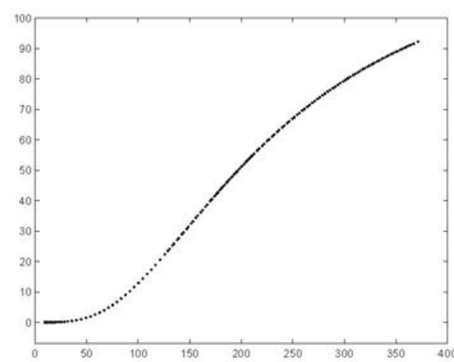
$$r_{n,m}(x_1,\ldots,x_d) = \frac{p_{n,m}(x_1,\ldots,x_d)}{q_{n,m}(x_1,\ldots,x_d)}$$

$$r_{n,m}(x_1^{(\ell)},\ldots,x_d^{(\ell)}) \in F^{(\ell)} \underset{q_{n,m}(x_1^{(\ell)},\ldots,x_d^{(\ell)})>0}{\Leftrightarrow} \begin{cases} -p_{n,m}^{(\ell)} + f_{>}^{(\ell)} q_{n,m}^{(\ell)} \geqslant 0 \\ p_{n,m}^{(\ell)} - f_{<}^{(\ell)} q_{n,m}^{(\ell)} \geqslant 0 \end{cases}$$

$$\underset{\text{nonempty interior}}{\Leftrightarrow} \quad \text{strictly convex QP}$$

Benchmark

- National Institute of Standards and Technology (NIST) reference dataset
- 151 observations

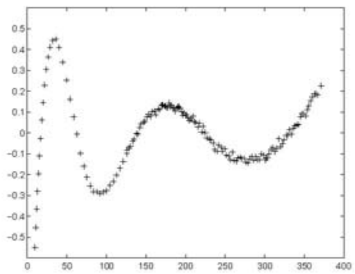


$d = 1,\ g_k(x) = x^k,\ s = 150$

best ℓ_2-approximation

$$\sum_{\ell=0}^{s}\left(r^*_{2,2}(x_1^{(\ell)},\ldots,x_d^{(\ell)})-f^{(\ell)}\right)^2 \text{ minimal}$$

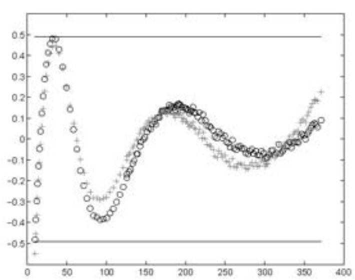
$$r^*_{2,2}=\frac{1.6745-0.13927x+0.00260x^2}{1-0.00172x+0.00002x^2}$$



residuals, $\sigma = 0.16355$

$f^{(\ell)}_{>} - f^{(\ell)}_{<} = 2(3\sigma) = 0.9813 \Rightarrow n = 2, m = 2$

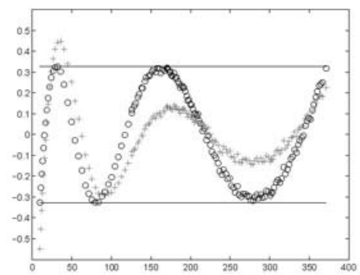
$$r_{2,2}=\frac{1.56271-0.13713x+0.00261x^2}{1-0.00173x+0.00002x^2}$$



residuals

$$f_{>}^{(\ell)} - f_{<}^{(\ell)} = 2(2\sigma) = 0.6542 \Rightarrow n = 2, m = 2$$

$$r_{2,2} = \frac{1.16217 - 0.1080x + 0.00224x^2}{1 - 0.00223x + 0.00002x^2}$$

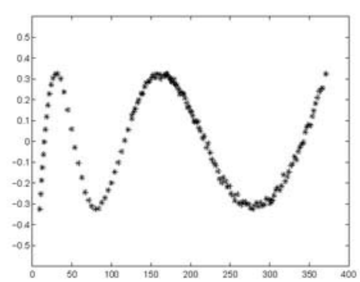


residuals

compare to best ℓ_∞-approximation

$$\max_{\ell=0,\dots,s} \left| r_{2,2}^{\infty}(x_1^{(\ell)}, \dots, x_d^{(\ell)}) - f^{(\ell)} \right| \text{ minimal}$$

$$r_{2,2}^{\infty} = \frac{1.15538 - 0.10751x + 0.00223x^2}{1 - 0.00223x + 0.00002x^2}$$

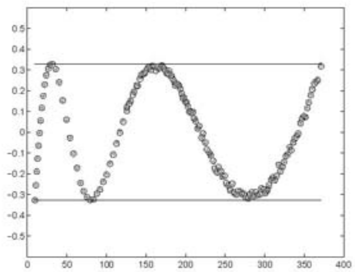


residuals, max $= 0.3244$

compare to best ℓ_∞-approximation

$$\max_{\ell=0,\ldots,s} \left| r_{2,2}^{\infty}(x_1^{(\ell)},\ldots,x_d^{(\ell)}) - f^{(\ell)} \right| \text{ minimal}$$

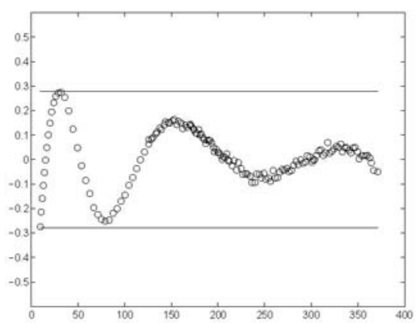
$$r_{2,2}^{\infty} = \frac{1.15538 - 0.10751x + 0.00223x^2}{1 - 0.00223x + 0.00002x^2}$$



residuals

$f_{>}^{(\ell)} - f_{<}^{(\ell)} = 2(1.75\sigma) = 0.5724 \Rightarrow r_{2,2}^{*}$ and $r_{2,2}^{\infty}$ do not satisfy

$\Rightarrow n = 3,\ m = 2$



residuals

Grid structured data

Ideal lowpass filter:

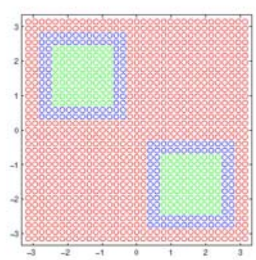
$$H\left(e^{it_1}, e^{it_2}\right) = \begin{cases} 1, & (t_1, t_2) \in P \subset [-\pi, \pi] \times [-\pi, \pi], \\ 0, & (t_1, t_2) \notin P. \end{cases}$$

In practice:

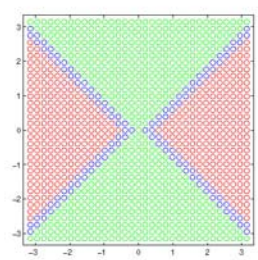
- passband $[1-\delta_1, 1+\delta_1]$, $\quad (t_1, t_2) \in P$
- stopband $[-\delta_2, \delta_2]$, $\quad (t_1, t_2) \notin P \cup T$
- transition band $[-\delta_2, 1+\delta_1]$, $\quad (t_1, t_2) \in T$

Grid structured data

Examples of passband: $s+1 = 33 \times 33$



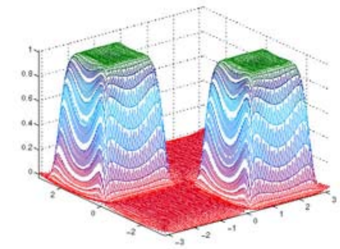
centro symmetric filter



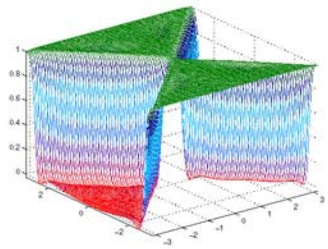
fan filter

Grid structured data

Rational models for the parameters $\delta_1 = 0.01$ and $\delta_2 = 0.02$



$r_{19,20}(t_1, t_2)$



$r_{12,14}(t_1, t_2)$

Point valued data

parameters $\rightarrow$ physical model $\rightarrow$ behaviour

Point valued data

parameters → physical model → behaviour

↓ simplify ↓

Point valued data

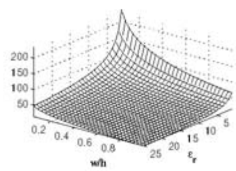
parameters → physical model → behaviour

↓ simplify ↓

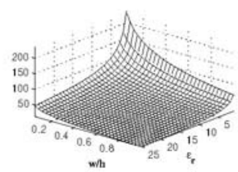
parameters → metamodel → behaviour

Point valued data

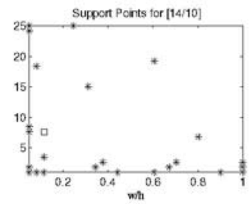
Model of the stripline characteristic impedance $Z_0(q, \epsilon_r)$



$Z_0(q, \epsilon_r)$



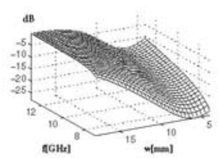
$r_{14,10}(q, \epsilon_r)$



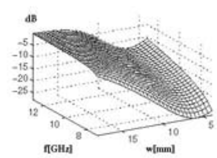
$s + 1 = 25$ data points

Point valued data

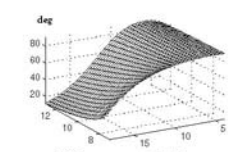
Model of the transmission coefficient $S_{21}(f, w)$ of two inductive posts in rectangular waveguide



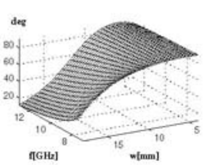
$|S_{21}(f, w)|$



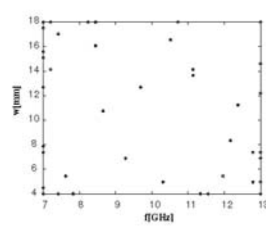
$|r_{19,20}(f, w)|$



$\arg(S_{21}(f, w))$



$\arg(r_{19,20}(f, w))$

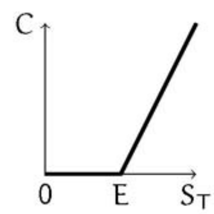


$s + 1 = 40$ data points

Interval valued data

A European call option gives its holder the right (but not the obligation) to purchase from the writer a prescribed asset for a prescribed price at a prescribed time in the future.

T	expiry date ($0 \leqslant t \leqslant T$)
E	strike or exercise price
S	asset price $S_t \geqslant 0$
r	annual interest rate (constant)
σ	market volatility



Interval valued data

Black-Scholes PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

$$\begin{aligned} C(S,T) &= \max(S-E,0) \\ C(0,t) &= 0, \qquad 0 \leqslant t \leqslant T \\ C(S,t) &\approx S, \qquad \text{large } S \end{aligned}$$

Interval valued data

Typically a few million values computed:

$$\left(E^{(i)}, r^{(i)}, \sigma^{(i)}\right), \quad i = O(10^1)$$
$$\left(S^{(i,j)}, t^{(i,k)}\right), \quad (j,k) = O(10^4)$$

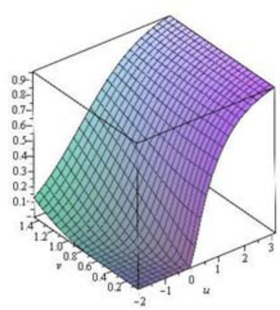
- fit model through subset of $s+1$ data points
- check model on all available values
- increase s and update model

Interval valued data

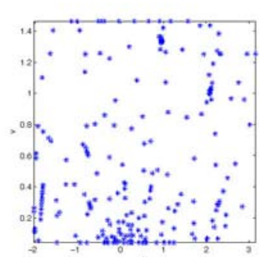
Graph in (u, v): $u = \ln(S) - \ln(E) + rt, \quad v = \sigma\sqrt{t}$

$$C - Sr_{n,m}(u, v)$$

$$n = 16, \quad m = 20, \quad f_{>}^{(\ell)} - f_{<}^{(\ell)} = 0.005$$

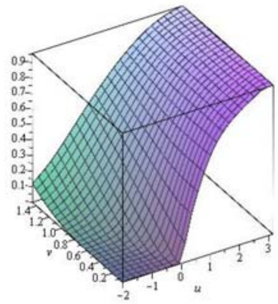


$C(u, v)/S$

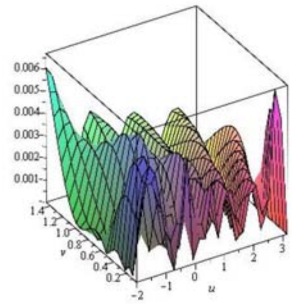


$s + 1 = 212$ data points

Interval valued data



$r_{16,20}(u, v)$



relative error

$$r_{n,m}(u, v) = \frac{\text{degree 4 } + a_{15}u^5 + a_{16}v^5}{\text{degree 5}}$$

Scattered interval data

The Bidirectional Reflectance Distribution Function $\rho(\theta_l, \phi_l, \theta_v, \phi_v)$ describes how a material reflects light from surfaces.

l	lighting direction
v	viewing direction
θ	zenithal angle
ϕ	azimuthal angle

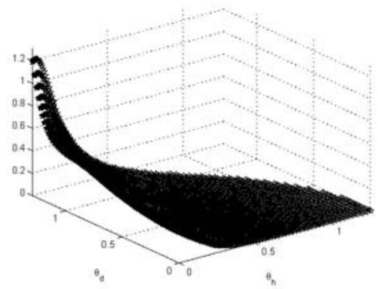
chrome steel

fabric beige

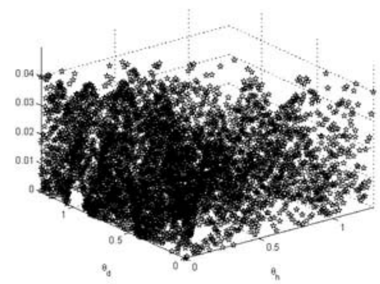
Scattered interval data

For isotropic materials $90 \times 90 \times 180 \approx 1.45$ million measured BRDF samples (RGB values):

- $\rho(\theta_l, \phi_l, \theta_v, \phi_v) \approx r_{n,m}(\theta_h, \theta_d)$
- a priori error control (3–5%) on all data points (≈ 1.12 Mb)
- a posteriori error control on all measured samples



$r_{25,18}(\theta_h, \theta_d)$, $s + 1 = 205$



relative error

Scattered interval data

Rendered example: blue-metallic paint

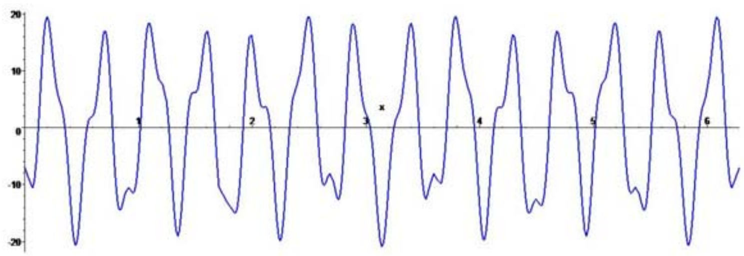
Original (33MB)

Approximation (1.15KB)

Sparsity

Compressive sensing recovers a K–sparse signal from only $M \approx K$ measurements without loss of information.

$$\begin{aligned} x(t) = 2\cos(5t) - 15\cos(14t) + \cos(26t) \qquad 0 \leqslant t \leqslant 2\pi \\ + 5\cos(35t) + \text{noise}([-0.1, 0.1]), \end{aligned}$$



Sparsity

$$t_j = j\frac{2\pi}{71}, \quad j = 0, \ldots, 7, \qquad s + 1 = 8$$

Choice of datapoints allows to recover the frequencies first,

$$5 \quad 14 \quad 26 \quad 35$$

and afterwards the coefficients by fitting the same data,

$$2.004 \quad -14.93 \quad 0.9668 \quad 5.007$$

$$x(t) = 2\cos(5t) - 15\cos(14t) + \cos(26t) + 5\cos(35t)$$