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An In-Depth Tutorial on Constitutive Equations for Elastic Anisotropic Materials

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Space Administration

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CONTENTS

SUMMARY

An in-depth tutorial on the thermoelastic constitutive equations for elastic, anisotropic materials is presented. First, basic concepts are introduced that are used to characterize materials, and then notions about how anisotropic material deform are presented. Next, a common notation used to describe stresses and strains is given, followed by the rules of indicial notation used herein. Based on this notation, Hooke's law and the Duhamel-Neuman law for isotropic materials are presented and discussed.

After discussing isotropic materials, the most general form of Hooke's law for elastic anisotropic materials is presented and symmetry requirements that are based on symmetry of the stress and strain tensors are given. Additional symmetry requirements are then identified based on the reversible nature of the strain energy and complimentary strain energy densities of elastic materials. A similar presentation is then given for the generalized Duhamel-Neuman law for elastic, anisotropic materials that includes thermal effects. Next, a common abridged notation for the constitutive equations is introduced and physical meanings of the elastic constants are discussed.

SUMMARY - CONCLUDED

As a prelude to establishing various material symmetries, the transformation equations for stress and strains are presented, the most general form of the transformation equations for the constitutive matrices are presented. Then, specialized transformation equations are presented for dextral rotations about the coordinate axes. Next, the concepts of material symmetry are introduced, the mathematical process used to describe symmetries is discussed, and examples are given. After describing the mathematics of symmetry, the criteria for the existence of material symmetries are presented and the classes of material symmetries are given. Then, the invariance conditions and simplifications to the constitutive equations are presented for monoclinic, orthotropic, trigonal, tetragonal, transversely isotropic, and completely isotropic materials.

After establishing a broad range of material symmetries, the engineering constants of fully anisotropic, elastic materials are derived from first principles and then specialized to several cases of practical importance. Lastly, reduced constitutive equations are derived for states of plane stress, generalized plane stress, plane strain and generalized plane strain. Transformation equations are also derived for these special cases.

PREFATORY COMMENTS

MOTIVATION AND APPROACH

- Knowledge of anisotropic materials has become prominent in the last few decades because of the applications of advanced, lightweight fiber-reinforced composite materials to aircraft and spacecraft

- The material presented herein is redundant in several sections, by design

    - First, to reinforce concepts and enhance learning

    - Second, to provide stand-alone sections that can be used independently for various reasons

    - Third, to serve as a comprehensive reference document

DEDICATION

- To **Manuel Stein** - Wisdom, knowledge, humility, and kindness incarnate

- To **James H. Starnes, Jr.** - The embodiment of scientific thirst and curiosity, leadership, and professional excellence

- To **Harold G. Bush** - The epitome of common sense and sound engineering practice, and a constant source of advice and entertainment

- To **The Men and Women of the NACA** - The premier examples of government researchers

BASIC CONCEPTS AND NOTATIONS

BASIC CONCEPTS

- **The macroscopic physical, or material, properties of a body are specified by *constitutive equations***
  - **For example, a relationship between stress, strain, and temperature is commonly specified for solid materials**
- **The material properties of a solid, regardless of its shape, are generally functions of the coordinates of the material particles**
  - **Solids for which the material properties vary pointwise are described as *inhomogeneous* (e.g., a bi-metallic strip)**
  - **For *homogeneous* solids, the material properties are the same for every particle of the solid**
- **The material properties of a homogeneous solid are described mathematically as invariant with respect to coordinate-frame translations**

BASIC CONCEPTS - CONCLUDED

- A body is described as ***isotropic*** at a point if its properties at that point are *independent of direction*

  - A body that is not isotropic is described, in the most general case, as ***anisotropic***

- A body that is isotropic at a given point is described mathematically as invariant with respect to coordinate-frame rotations (for that point)

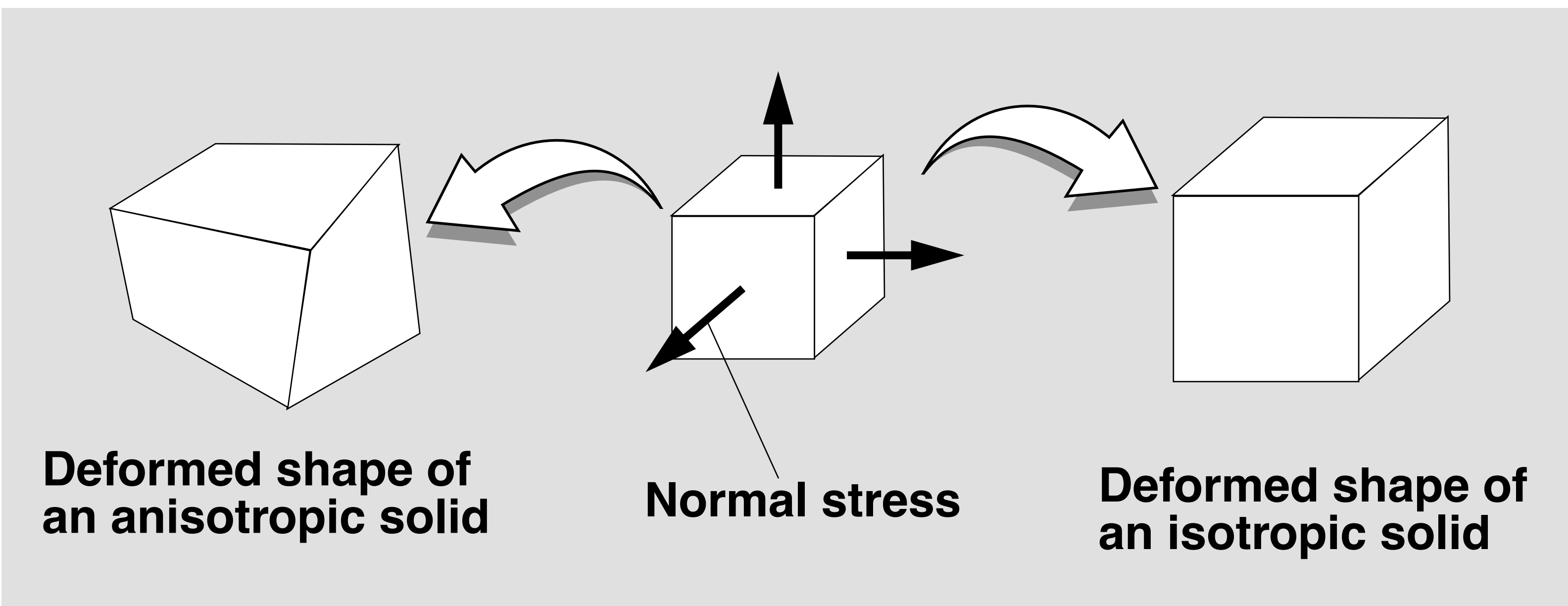
- A body is described as *homogeneous and isotropic* if its properties are independent of direction, and identical, at every point of the body

- ***Distortion*** is defined as deformation that consists of a change in shape without a change in volume (pure shearing deformation)

- ***Dilatation*** is defined as deformation that consists of a change in volume without a change in shape (pure expansion-contraction-type deformation)

BASIC NOTIONS OF DEFORMATION

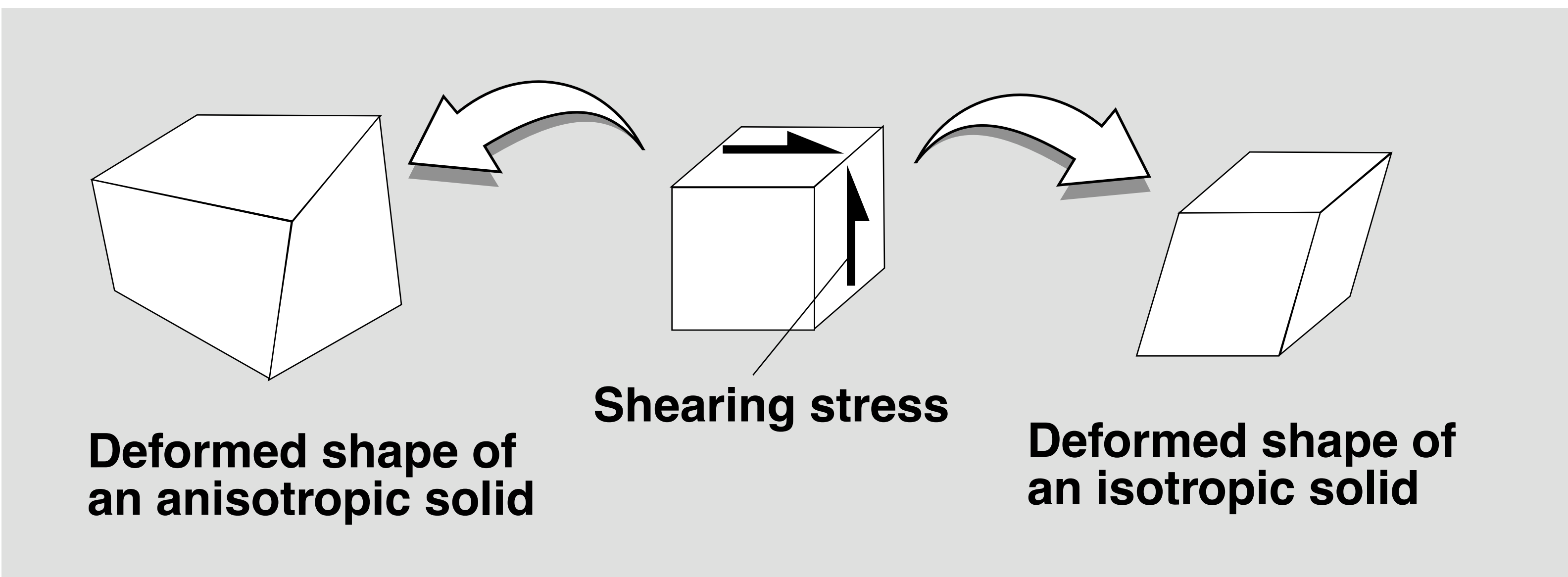
- **Pure *normal stresses* acting within a homogeneous, isotropic solid produce only volumetric, extensional (dilatational) deformations**
  - **The angle between every pair of intersecting material line elements, that lie in the planes that are perpendicular to the normal stresses in the solid, remains unchanged during deformation (no shearing)**



BASIC NOTIONS OF DEFORMATION - CONTINUED

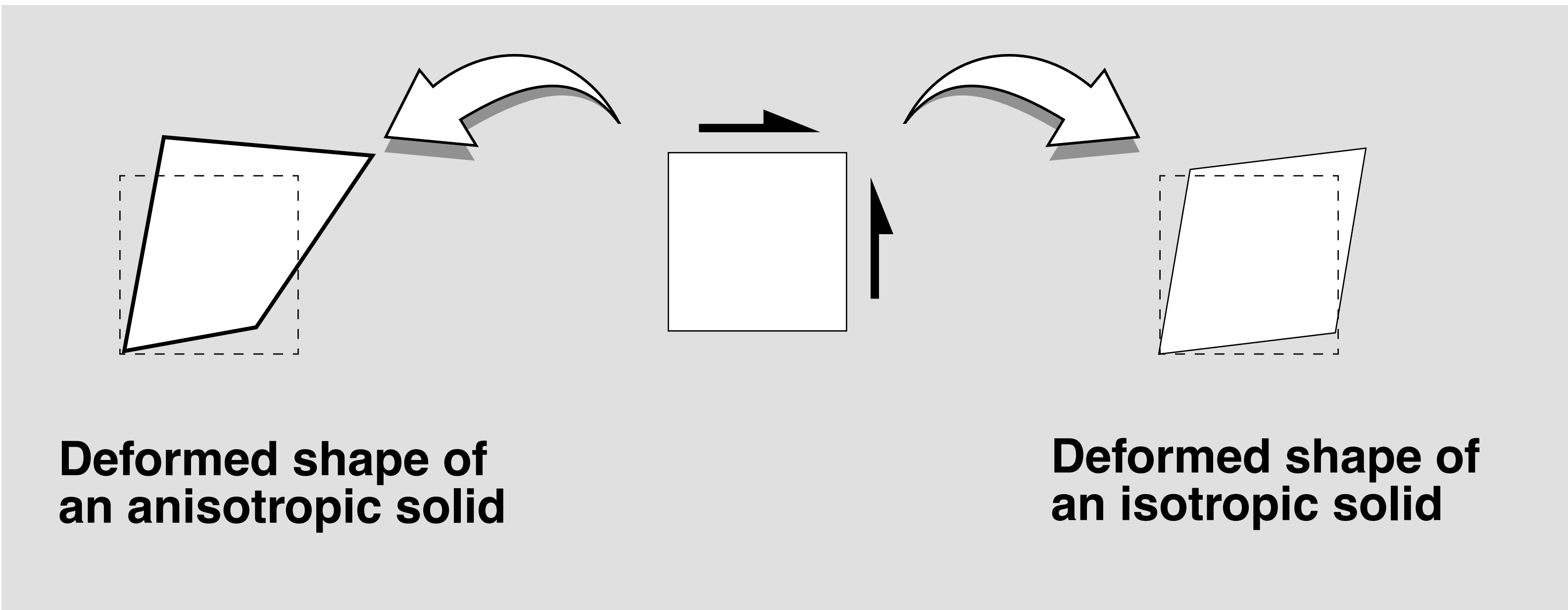
- Pure ***shearing stresses*** acting within a **homogeneous, isotropic solid** produce only distortional, shearing (deviatoric) deformations

    - The angle between every pair of intersecting material line elements that lie in the planes of the shearing stresses in the solid change during deformation, but the length of the line elements does not change (no dilatation)



BASIC NOTIONS OF DEFORMATION - CONTINUED

- Pure ***shearing stresses*** acting within a **homogeneous, isotropic solid** produce only distortional, shearing (deviatoric) deformations that are only in the plane of the shearing stresses



- All unrestrained ***thermal expansion*** is **volumetric** and **uniform** within a **homogeneous, isotropic solid**; not so in a homogeneous, generally anisotropic solid

BASIC NOTIONS OF DEFORMATION - CONCLUDED

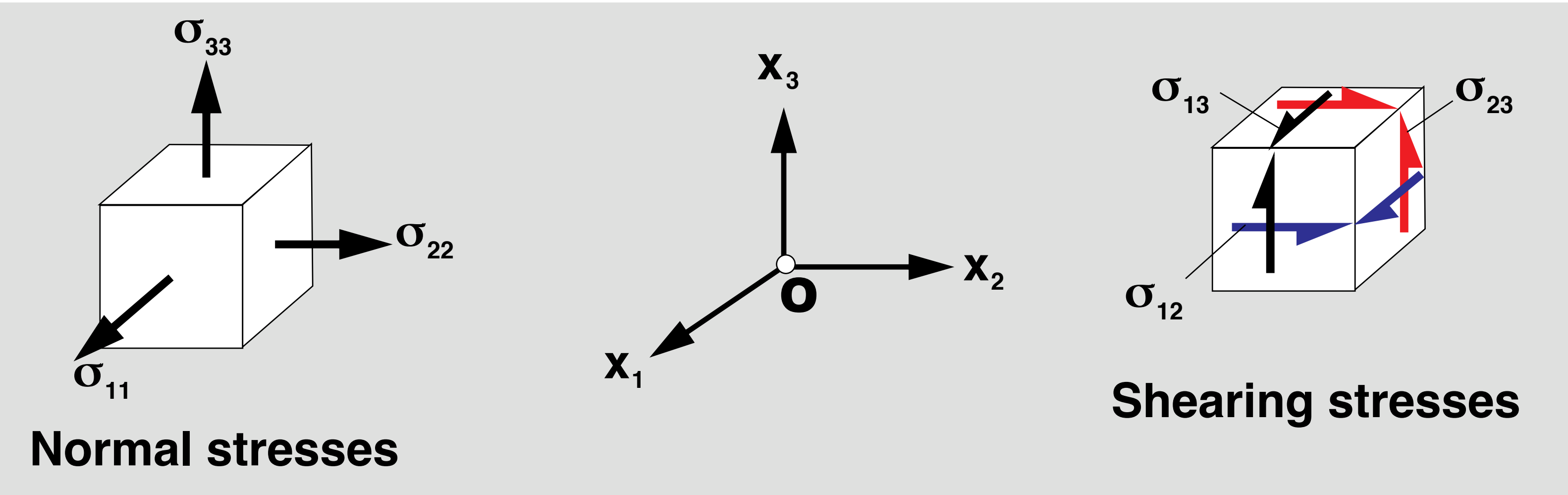
- **Strains that are caused by unconstrained thermal expansions, that do not produce stresses, are defined as *free thermal strains***

- **A *solid* is described as *ideally elastic* (usually just called elastic) when it recovers to its initial, stress- and strain-free configuration upon removal of the applied loads or temperature field**

    - **For this case, there exists a one-to-one (unique) mathematical relationship between the stresses and strains that act within the loaded solid**

NOTATION FOR STRESSES AND STRAINS

- In the development that follows, stresses and strains are defined relative to standard rectangular Cartesian coordinates (x_1, x_2, x_3)



- The normal strains ε_{11}, ε_{22}, and ε_{33} correspond to the normal stresses σ_{11}, σ_{22}, and σ_{33}, respectively

- The shearing strains $2\varepsilon_{12} = \gamma_{12}$, $2\varepsilon_{13} = \gamma_{13}$, and $2\varepsilon_{23} = \gamma_{23}$ correspond to the shearing stresses σ_{12}, σ_{13}, and σ_{23}, respectively

INDICIAL NOTATION

- The rules of indicial notation associated with **Cartesian tensors** are used herein

- In particular, all indices appear as subscripts, unless noted otherwise

  - For example, $\varepsilon_{ij} = \frac{1}{E}\left[\left(1 + \nu\right)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right]$

- Latin indices take on the values $\{1, 2, 3\}$, and repeated latin indices imply summation over this set; e. g.; $\sigma_{kk} = \sum_{k=1}^{3} \sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$

- Indices that are not summed in an equation are called free indices and take on the complete set of possible values

- The symbol δ_{jk} is known as the **Kronecker Delta Symbol** and is equal to one when j = k and is equal to zero otherwise

CONSTITUTIVE EQUATIONS FOR ISOTROPIC MATERIALS

HOOKE'S LAW

HOMOGENEOUS, ISOTROPIC, LINEAR-ELASTIC SOLID

- In the 17th century, ***Robert Hooke*** began developing a constitutive law for elastic, isotropic solids
  - The concept of **elastic deformation** was introduced by Hooke in 1676
- Hooke's work led to the following equations that are in use today

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right]$$

$$2\varepsilon_{12} = \gamma_{12} = \frac{2(1+\nu)}{E}\sigma_{12}$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right]$$

$$2\varepsilon_{13} = \gamma_{13} = \frac{2(1+\nu)}{E}\sigma_{13}$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right]$$

$$2\varepsilon_{23} = \gamma_{23} = \frac{2(1+\nu)}{E}\sigma_{23}$$

or in indicial notation $\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right]$

HOOKE'S LAW - CONCLUDED

HOMOGENEOUS, ISOTROPIC, LINEAR-ELASTIC SOLID

- The ***inverted form*** of Hooke's law is given by

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{11} + \nu(\varepsilon_{22}+\varepsilon_{33})\right] \qquad \sigma_{12} = \frac{E}{2(1+\nu)}\gamma_{12} = \frac{E}{(1+\nu)}\varepsilon_{12}$$

$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{22} + \nu(\varepsilon_{11}+\varepsilon_{33})\right] \qquad \sigma_{13} = \frac{E}{2(1+\nu)}\gamma_{13} = \frac{E}{(1+\nu)}\varepsilon_{13}$$

$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{33} + \nu(\varepsilon_{11}+\varepsilon_{22})\right] \qquad \sigma_{23} = \frac{E}{2(1+\nu)}\gamma_{23} = \frac{E}{(1+\nu)}\varepsilon_{23}$$

or in indicial notation

$$\sigma_{ij} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_{ij} + \nu\delta_{ij}\varepsilon_{kk}\right]$$

- **For these equations, it is important to remember that the strains are caused by the externally applied loads and displacements**
  - **Strains of this type are called (stress-induced) *mechanical strains* and are the result of the internal stresses**

THE DUHAMEL-NEUMANN LAW

HOMOGENEOUS, ISOTROPIC, LINEAR-THERMOELASTIC SOLID

- **Hooke's law was extended by *J. M. C. Duhamel* (circa 1838) and *F. E. Neumann* (circa 1888) to include the first-order effects of thermal loading**

- **This law is based, in part, on the premise that the *total strain* ε_{ij} at a point of a solid, subjected to thermomechanical loading, consists of mechanical strain $\varepsilon_{ij}^{\sigma}$ and strain caused by free thermal expansion ε_{ij}^{T}**

  - **The mechanical strain $\varepsilon_{ij}^{\sigma}$ is the *stress-induced* strain caused by the externally applied loads and displacements, and the *stress-induced* strain caused by nonuniformity in the temperature field or in the thermal expansion properties of the material**

  - **$\varepsilon_{ij} = \varepsilon_{ij}^{\sigma} + \varepsilon_{ij}^{T}$ where $\varepsilon_{ij}^{\sigma} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right]$, $\varepsilon_{ij}^{T} = \alpha\delta_{ij}(T - T_{ref})$, T is the temperature field, and T_{ref} is the temperature field at which the body is deemed stress and strain free (or negligible)**

THE DUHAMEL-NEUMANN LAW - CONTINUED

HOMOGENEOUS, ISOTROPIC, LINEAR-THERMOELASTIC SOLID

- **The temperature fields T and T_{ref} are, in general, functions of position within the body; that is, $T = T(x_1, x_2, x_3)$ and $T_{ref} = T_{ref}(x_1, x_2, x_3)$**
  - **T is, in general, also time dependent and T_{ref} is typically uniform, with a value equal to a nominal ambient temperature**

- **Thermal stresses are caused by two effects:**
  - **The spatial nonuniformity in the field $\alpha(T - T_{ref})$ and**
  - **Geometric restraints that prevent stress-free thermal expansion**

- **When a solid is subjected to a nonuniform temperature field or its thermal expansion properties vary, there arises a mismatch in the thermal expansion of neighboring material particles**
  - **Internal, "thermal stresses" develop to maintain continuity of the material body, which induces mechanical strains**

THE DUHAMEL-NEUMANN LAW - CONTINUED
HOMOGENEOUS, ISOTROPIC, LINEAR-THERMOELASTIC SOLID

- **The work of Hooke, Duhamel, and Neumann led to the following thermoelastic constitutive equations that are used today**

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu\left(\sigma_{22} + \sigma_{33}\right)\right] + \alpha\left(T - T_{ref}\right)$$

$$2\varepsilon_{12} = \gamma_{12} = \frac{2(1 + \nu)}{E}\sigma_{12}$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{33}\right)\right] + \alpha\left(T - T_{ref}\right)$$

$$2\varepsilon_{13} = \gamma_{13} = \frac{2(1 + \nu)}{E}\sigma_{13}$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22}\right)\right] + \alpha\left(T - T_{ref}\right)$$

$$2\varepsilon_{23} = \gamma_{23} = \frac{2(1 + \nu)}{E}\sigma_{23}$$

or in indicial notation

$$\varepsilon_{ij} = \frac{1}{E}\left[\left(1 + \nu\right)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right] + \alpha\delta_{ij}\left(T - T_{ref}\right)$$

THE DUHAMEL-NEUMANN LAW - CONTINUED
HOMOGENEOUS, ISOTROPIC, LINEAR-THERMOELASTIC SOLID

- The ***inverted form*** of the Duhamel-Neumann law is given by

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{11} + \nu(\varepsilon_{22} + \varepsilon_{33})\right] - \frac{E\alpha(T - T_{ref})}{(1-2\nu)}$$

$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{22} + \nu(\varepsilon_{11} + \varepsilon_{33})\right] - \frac{E\alpha(T - T_{ref})}{(1-2\nu)}$$

$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{33} + \nu(\varepsilon_{11} + \varepsilon_{22})\right] - \frac{E\alpha(T - T_{ref})}{(1-2\nu)}$$

$$\sigma_{12} = \frac{E}{2(1+\nu)}\gamma_{12} = \frac{E}{(1+\nu)}\varepsilon_{12} \qquad \sigma_{13} = \frac{E}{2(1+\nu)}\gamma_{13} = \frac{E}{(1+\nu)}\varepsilon_{13}$$

$$\sigma_{23} = \frac{E}{2(1+\nu)}\gamma_{23} = \frac{E}{(1+\nu)}\varepsilon_{23}$$

or in indicial notation

$$\sigma_{ij} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_{ij} + \nu\delta_{ij}\varepsilon_{kk}\right] - \delta_{ij}\frac{E\alpha(T - T_{ref})}{(1-2\nu)}$$

THE DUHAMEL-NEUMANN LAW - CONCLUDED
HOMOGENEOUS, ISOTROPIC, LINEAR-THERMOELASTIC SOLID

- **The constitutive equations show that an isotropic material is characterized fully by two independent *elastic constants* E and ν, and by one *thermal expansion coefficient* α**

  - **E is the modulus of elasticity, which is also called Young's modulus and the elastic modulus**

  - **ν is Poisson's ratio and α is the coefficient of linear thermal expansion**

  - **E and ν are related by $G = \dfrac{E}{2(1+\nu)}$, where G is called the shear modulus or the modulus of rigidity**

GENERALIZED HOOKE'S LAW FOR HOMOGENEOUS, ANISOTROPIC, LINEAR-ELASTIC SOLIDS

GENERAL FORM OF HOOKE'S LAW

- The generalization of *Hooke's law* to anisotropic materials is attributed to **Cauchy** (in 1829) and postulates that every component of the stress tensor is coupled linearly with every component of the strain tensor; i.e.,

$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{Bmatrix} =
\begin{bmatrix}
S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1113} & S_{1112} & S_{1132} & S_{1131} & S_{1121} \\
S_{2211} & S_{2222} & S_{2233} & S_{2223} & S_{2213} & S_{2212} & S_{2232} & S_{2231} & S_{2221} \\
S_{3311} & S_{3322} & S_{3333} & S_{3323} & S_{3313} & S_{3312} & S_{3332} & S_{3331} & S_{3321} \\
S_{2311} & S_{2322} & S_{2333} & S_{2323} & S_{2313} & S_{2312} & S_{2332} & S_{2331} & S_{2321} \\
S_{1311} & S_{1322} & S_{1333} & S_{1323} & S_{1313} & S_{1312} & S_{1332} & S_{1331} & S_{1321} \\
S_{1211} & S_{1222} & S_{1233} & S_{1223} & S_{1213} & S_{1212} & S_{1232} & S_{1231} & S_{1221} \\
S_{3211} & S_{3222} & S_{3233} & S_{3223} & S_{3213} & S_{3212} & S_{3232} & S_{3231} & S_{3221} \\
S_{3111} & S_{3122} & S_{3133} & S_{3123} & S_{3113} & S_{3112} & S_{3132} & S_{3131} & S_{3121} \\
S_{2111} & S_{2122} & S_{2133} & S_{2123} & S_{2113} & S_{2112} & S_{2132} & S_{2131} & S_{2121}
\end{bmatrix}
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{Bmatrix}
$$

or in indicial notation by $\boxed{\varepsilon_{ij} = S_{ijkl}\sigma_{kl}}$

- Note that S_{ijkl} have units of stress^{-1}; e.g., in^2/lb

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- S_{ijkl} are called the components of the (4th-order) compliance tensor and are often called **compliances or compliance coefficients**

  - Without further simplication, there are 3^4 (or 81) independent compliance coefficients that must be determined from experiments, to fully characterize a given homogeneous material

- The previous equation indicates that each normal-stress component produces shearing strains in all three coordinate planes, in addition to three extensional strains

- Similarly, each shearing-stress component produces extensional strains along all three coordinate directions and shearing strains in the two planes perpendicular to the plane of the shearing stress, in addition to a shearing strain in the plane of the shearing stress

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- Thus, ***dilatational deformation*** (expansion-contraction) and ***distortional deformation*** (shearing) are fully coupled in an anisotropic material, unlike common isotropic materials

- The *inverted form* of generalized Hooke's law is given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ \hdashline \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{Bmatrix} = \left[\begin{array}{cccccc:ccc} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} & C_{1132} & C_{1131} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} & C_{2232} & C_{2231} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} & C_{3332} & C_{3331} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} & C_{2332} & C_{2331} & C_{2321} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} & C_{1332} & C_{1331} & C_{1321} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} & C_{1232} & C_{1231} & C_{1221} \\ \hdashline C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3213} & C_{3212} & C_{3232} & C_{3231} & C_{3221} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3113} & C_{3112} & C_{3132} & C_{3131} & C_{3121} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2113} & C_{2112} & C_{2132} & C_{2131} & C_{2121} \end{array}\right] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \hdashline \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{Bmatrix}$$

or in indicial notation by $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

- Note that C_{ijkl} have units of stress; e.g., lb/in^2

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- C_{ijkl} are called the components of the (4th-order) elasticity or stiffness tensor and are often called ***stiffness coefficients***

- Note that S_{ijkl} and C_{ijkl} are constants for a homogeneous material

- The number of independent coefficients in $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$ can be reduced by enforcing ***symmetry*** of the stress and strain tensors

  - $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$ is the same as $\varepsilon_{ji} = S_{jikl}\sigma_{kl}$

  - $\varepsilon_{ij} = \varepsilon_{ji}$ gives $S_{ijkl}\sigma_{kl} = S_{jikl}\sigma_{kl}$, which implies $S_{ijkl} = S_{jikl}$ for a general stress state at a point in a body

  - $\sigma_{kl} = \sigma_{lk}$ can be used to show that $S_{ijkl} = S_{ijlk}$

- $S_{ijkl} = S_{jikl}$ and $S_{ijkl} = S_{ijlk}$ yield 36 <u>independent</u> compliance coefficients

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- The proof that $S_{ijkl} = S_{ijlk}$ is given as follows

- The constitutive equation $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$ can be expressed as $\varepsilon_{ij} = S_{ijlk}\sigma_{lk}$ because l and k are *summation indices* and interchanging them doesn't alter the content of the equation

- Equating $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$ and $\varepsilon_{ij} = S_{ijlk}\sigma_{lk}$ gives $S_{ijkl}\sigma_{kl} = S_{ijlk}\sigma_{lk}$

- Next, enforcing $\sigma_{kl} = \sigma_{lk}$ gives $S_{ijkl}\sigma_{kl} = S_{ijlk}\sigma_{kl}$, which implies

$$S_{ijkl} = S_{ijlk}$$

for a *general stress state* at a point in a body

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- The number of ***independent coefficients*** in $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ can also be reduced directly by enforcing *symmetry* of the stress and strain tensors
  - $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ is the same as $\sigma_{ji} = C_{jikl}\varepsilon_{kl}$
  - $\sigma_{ij} = \sigma_{ji}$ gives $C_{ijkl}\varepsilon_{kl} = C_{jikl}\varepsilon_{kl}$, which implies $C_{ijkl} = C_{jikl}$ for a *general strain state* at a point in a body
  - $\varepsilon_{kl} = \varepsilon_{lk}$ can be used to show that $C_{ijkl} = C_{ijlk}$
- $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{ijlk}$ yield 36 *independent* stiffness coefficients
- Cauchy's generalized form of Hooke's Law ends up with 36 *independent* compliance or stiffness coefficients

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- The *expanded form* of Cauchy's generalized Hooke's law $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$ is obtained as follows

- First, expanding the last summation index gives

$$\varepsilon_{ij} = \left[S_{ijk1}\sigma_{k1}\right] + \left[S_{ijk2}\sigma_{k2}\right] + \left[S_{ijk3}\sigma_{k3}\right]$$

- Then, expanding the summation index k gives

$$\varepsilon_{ij} = \left[S_{ij11}\sigma_{11} + S_{ij21}\sigma_{21} + S_{ij31}\sigma_{31}\right] + \left[S_{ij12}\sigma_{12} + S_{ij22}\sigma_{22} + S_{ij32}\sigma_{32}\right] + \left[S_{ij13}\sigma_{13} + S_{ij23}\sigma_{23} + S_{ij33}\sigma_{33}\right]$$

- Next, enforcing $\sigma_{kl} = \sigma_{lk}$ yields

$$\varepsilon_{ij} = S_{ij11}\sigma_{11} + S_{ij22}\sigma_{22} + S_{ij33}\sigma_{33} + \left(S_{ij23} + S_{ij32}\right)\sigma_{23} + \left(S_{ij13} + S_{ij31}\right)\sigma_{13} + \left(S_{ij12} + S_{ij21}\right)\sigma_{12}$$

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- Then, enforcing the conditions $S_{ijkl} = S_{ijlk}$ give the result

$$\varepsilon_{ij} = S_{ij11}\sigma_{11} + S_{ij22}\sigma_{22} + S_{ij33}\sigma_{33} + 2S_{ij23}\sigma_{23} + 2S_{ij13}\sigma_{13} + 2S_{ij12}\sigma_{12}$$

- Applying this equation for all, independent values of the free indices i and j results in the following matrix representation of Cauchy's generalized Hooke's law $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 2S_{1123} & 2S_{1113} & 2S_{1112} \\ S_{2211} & S_{2222} & S_{2233} & 2S_{2223} & 2S_{2213} & 2S_{2212} \\ S_{3311} & S_{3322} & S_{3333} & 2S_{3323} & 2S_{3313} & 2S_{3312} \\ 2S_{2311} & 2S_{2322} & 2S_{2333} & 4S_{2323} & 4S_{2313} & 4S_{2312} \\ 2S_{1311} & 2S_{1322} & 2S_{1333} & 4S_{1323} & 4S_{1313} & 4S_{1312} \\ 2S_{1211} & 2S_{1222} & 2S_{1233} & 4S_{1223} & 4S_{1213} & 4S_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- Similarly, the *expanded form* of Cauchy's generalized Hooke's law $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ is obtained as follows

- First, expanding the last summation index gives

$$\sigma_{ij} = \left[C_{ijk1}\varepsilon_{k1}\right] + \left[C_{ijk2}\varepsilon_{k2}\right] + \left[C_{ijk3}\varepsilon_{k3}\right]$$

- Then, expanding the summmation index k gives

$$\sigma_{ij} = \left[C_{ij11}\varepsilon_{11} + C_{ij21}\varepsilon_{21} + C_{ij31}\varepsilon_{31}\right] + \left[C_{ij12}\varepsilon_{12} + C_{ij22}\varepsilon_{22} + C_{ij32}\varepsilon_{32}\right] + \left[C_{ij13}\varepsilon_{13} + C_{ij23}\varepsilon_{23} + C_{ij33}\varepsilon_{33}\right]$$

- Next, enforcing $\varepsilon_{kl} = \varepsilon_{lk}$ yields

$$\sigma_{ij} = C_{ij11}\varepsilon_{11} + C_{ij22}\varepsilon_{22} + C_{ij33}\varepsilon_{33} + \left(C_{ij23} + C_{ij32}\right)\varepsilon_{23} + \left(C_{ij13} + C_{ij31}\right)\varepsilon_{13} + \left(C_{ij12} + C_{ij21}\right)\varepsilon_{12}$$

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- Then, enforcing the conditions $C_{ijkl} = C_{ijlk}$ give the result

$$\sigma_{ij} = C_{ij11}\varepsilon_{11} + C_{ij22}\varepsilon_{22} + C_{ij33}\varepsilon_{33} + 2C_{ij23}\varepsilon_{23} + 2C_{ij13}\varepsilon_{13} + 2C_{ij12}\varepsilon_{12}$$

- Applying this equation for all, independent values of the free indices i and j results in the following matrix representation of Cauchy's generalized Hooke's law $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

GENERAL FORM OF HOOKE'S LAW - CONTINUED

- The **compliance** coefficients S_{ijkl} and the **stiffness** coefficients C_{ijkl} are described as components of a *fourth-order tensor (field)* because each are the components of a **linear transformation** that relates components of the second-order stress tensor (field) to components of the second-order strain tensor (field)

REDUCTION TO 21 INDEPENDENT CONSTANTS

- **The number of independent elastic, compliance and stiffness coefficients is reduced from 36 to 21 by enforcing the *thermodynamic properties* of reversible, elastic deformations**
    - **The key quantity to be examined is the strain-energy density of an elastic solid**
    - **The reduction to 21 is attributed to *George Green* (1793-1841)**

- **The strain-energy density $\mathcal{U}$ of a generally elastic solid is defined as the work of the internal stresses, done through *stress-induced* mechanical deformations, that is stored in a loaded body**

- **In an *ideally elastic* solid, experimental evidence indicates that all of the work done by external forces is converted into elastic-strain energy that can be recovered upon unloading, thus a loaded body has the potential to perform work**

REDUCTION TO 21 INDEPENDENT CONSTANTS
CONCLUDED

- **The *existence* of a strain-energy density function for linear- and nonlinear-elastic materials can be shown directly by using the *first and second laws of thermodynamics***
  - **The term *"density"* is used herein to indicate that the strain energy is defined per unit volume of material**
- **The expression for the strain-energy density function $\mathcal{U}$ is obtained by determining the strain-energy-density increment $d\mathcal{U}$ associated with an infinitesimal change in the deformation of a body**
  - **$d\mathcal{U}$ can be obtained *directly* from the laws of thermodynamics or by determining the work done by the internal forces of a body on a differential volume element of material**

STRAIN-ENERGY DENSITY

- The strain-energy-density increment $d\mathcal{U}$ is given by

$$d\mathcal{U} = \sigma_{11}d\varepsilon_{11} + \sigma_{22}d\varepsilon_{22} + \sigma_{33}d\varepsilon_{33} + 2\sigma_{23}d\varepsilon_{23} + 2\sigma_{13}d\varepsilon_{13} + 2\sigma_{12}d\varepsilon_{12}$$

where the stresses depend on the mechanical strains; that is,

$$\sigma_{ij} = \sigma_{ij}\left(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12}\right)$$

- This expression is written compactly in indicial form as

$$d\mathcal{U} = \sigma_{ij}\left(\varepsilon_{pq}\right)\, d\varepsilon_{ij}$$

- The strain-energy density $\mathcal{U}$ is obtained by integrating $d\mathcal{U}$ over the deformation associated with a loading process, that starts at a strain-free state and ends at a particular strain state; that is,

$$\mathcal{U} = \int_{0}^{\varepsilon_{pq}} \sigma_{ij}\left(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12}\right)\, d\varepsilon_{ij} = \mathcal{U}\left(\varepsilon_{pq}\right)$$

STRAIN-ENERGY DENSITY - CONTINUED

- For an arbitrary ***process*** that involves loading followed by total unloading, the strain-energy density $\mathcal{U}$ is given by the circuit integral

$$\mathcal{U} = \oint \sigma_{ij}(\varepsilon_{pq})\, d\varepsilon_{ij}$$

- In addition, because strain-energy density is not lost during an arbitrary elastic loading-unloading process (**conservation of energy** - first law of thermodynamics), it follows that

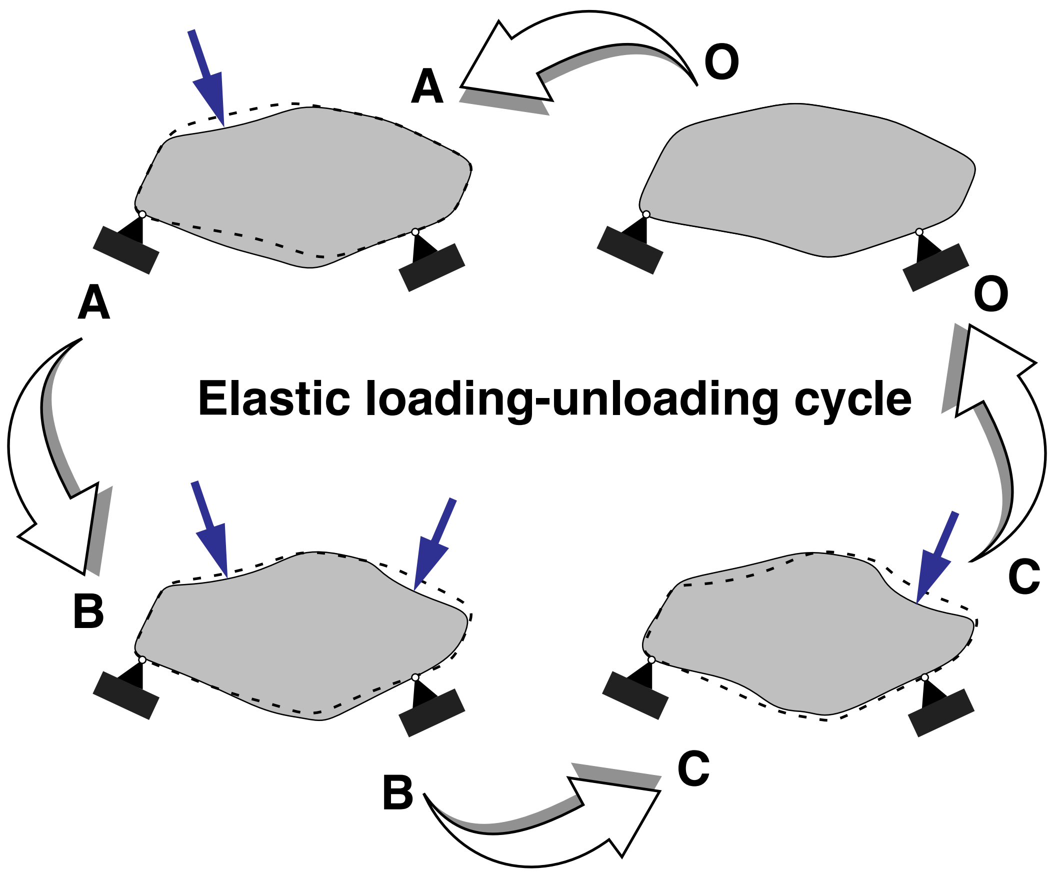
$$\mathcal{U} = \oint \sigma_{ij}(\varepsilon_{pq})\, d\varepsilon_{ij} = 0$$

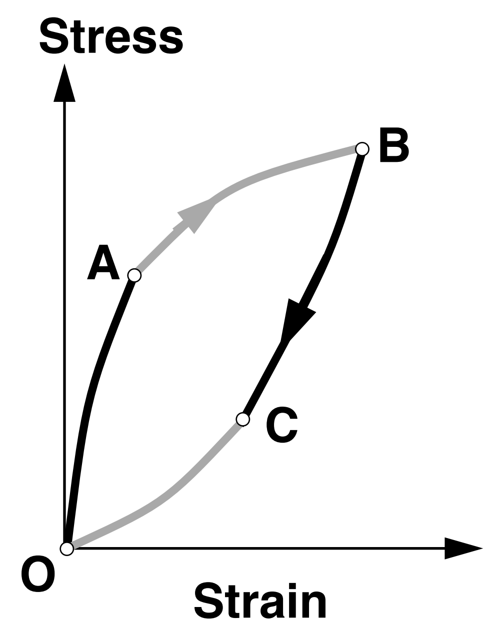
- For the condition that $\mathcal{U} = 0$ for an elastic loading-unloading process to be true, it requires that there must *exist* a strain-energy density function $\mathcal{U}$ for which $d\mathcal{U}$ is an **exact differential**; that is, $\mathcal{U} = \oint d\mathcal{U} = 0$

STRAIN-ENERGY DENSITY - CONTINUED

ILLUSTRATION OF ELASTIC LOADING-UNLOADING PROCESSES

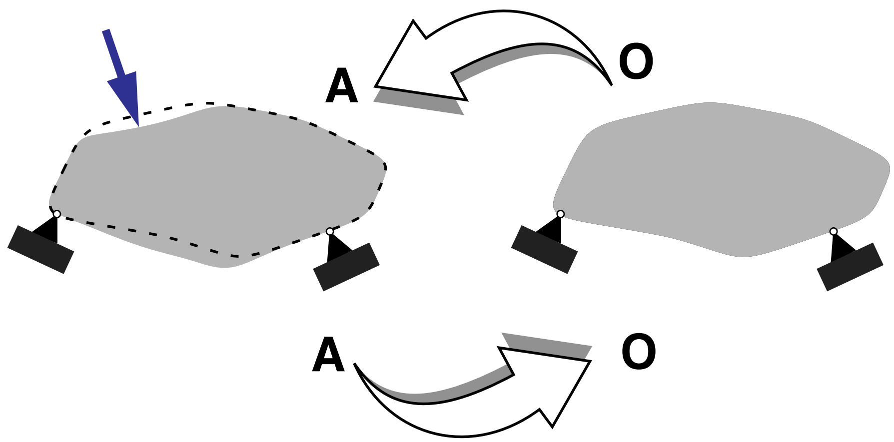
Two Independent Loading Systems



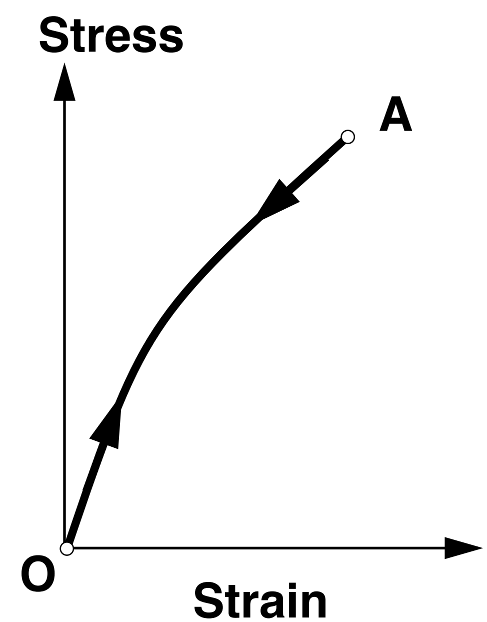


STRAIN-ENERGY DENSITY - CONTINUED
ILLUSTRATION OF ELASTIC LOADING-UNLOADING PROCESSES

One Loading System



Elastic loading-unloading cycle



STRAIN-ENERGY DENSITY - CONTINUED

- Because $\mathcal{U} = \mathcal{U}(\varepsilon_{pq})$, it follows mathematically that an *exact differential* has the property that $d\mathcal{U} = \dfrac{\partial \mathcal{U}}{\partial \varepsilon_{ij}} d\varepsilon_{ij}$
  - A function with this property is described in mathematics as a *potential function*, thus $\mathcal{U}$ is sometimes referred to as the **elastic potential**

- Equating $d\mathcal{U} = \sigma_{ij}(\varepsilon_{pq})\, d\varepsilon_{ij}$ with $d\mathcal{U} = \dfrac{\partial \mathcal{U}}{\partial \varepsilon_{ij}} d\varepsilon_{ij}$ gives $\dfrac{\partial \mathcal{U}}{\partial \varepsilon_{ij}} = \sigma_{ij}$
  - The last equation on the right indicates that the stress-strain relations are derivable from a potential function when the deformation process is elastic
  - A material of this type is called a **hyperelastic** or a **Green-elastic** material

STRAIN-ENERGY DENSITY - CONCLUDED

- The statement $\mathcal{U} = \oint d\mathcal{U} = 0$ also indicates that an arbitrary elastic loading-unloading process is a **path-independent process**
  - This result arises because the integral of an *exact differential* depends only on the limits of integration (end points of the process), according to the **fundamental theorem of calculus**
- A *necessary condition* for a function $\mathcal{U} = \mathcal{U}(\varepsilon_{pq})$ to be path independent is for the following condition to be valid: $$\frac{\partial^2 \mathcal{U}}{\partial \varepsilon_{ij}\,\partial \varepsilon_{kl}} = \frac{\partial^2 \mathcal{U}}{\partial \varepsilon_{kl}\,\partial \varepsilon_{ij}}$$
  - This condition arises from the connection of the path integral with ***Stokes' integral theorem***

PROOF THAT $C_{ijkl} = C_{klij}$

- First, note that $\dfrac{\partial^2 \mathcal{U}}{\partial \varepsilon_{ij}\, \partial \varepsilon_{kl}} = \dfrac{\partial^2 \mathcal{U}}{\partial \varepsilon_{kl}\, \partial \varepsilon_{ij}}$ and $\dfrac{\partial \mathcal{U}}{\partial \varepsilon_{ij}} = \sigma_{ij}$ give $\dfrac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \dfrac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}}$

- Then, $\sigma_{ij} = C_{ijrs}\varepsilon_{rs}$ gives $\dfrac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \dfrac{\partial}{\partial \varepsilon_{kl}}\left[C_{ijrs}\varepsilon_{rs}\right] = C_{ijrs}\dfrac{\partial \varepsilon_{rs}}{\partial \varepsilon_{kl}} = C_{ijrs}\delta_{rk}\delta_{sl} = C_{ijkl}$

- Also, $\sigma_{kl} = C_{klpq}\varepsilon_{pq}$ gives $\dfrac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}} = \dfrac{\partial}{\partial \varepsilon_{ij}}\left[C_{klpq}\varepsilon_{pq}\right] = C_{klpq}\dfrac{\partial \varepsilon_{pq}}{\partial \varepsilon_{ij}} = C_{klpq}\delta_{pi}\delta_{qj} = C_{klij}$

- Thus, $\dfrac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \dfrac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}}$ yields $C_{ijkl} = C_{klij}$, which reduces the number of **independent** stiffness coefficients to **21**

ILLUSTRATION OF THE PATH-INDEPENDENCE CONDITION

- The function $\mathcal{U} = \mathcal{U}(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12})$ can be viewed as an ordinary, simply connected, continuous, smooth function of six independent variables

- To enable visualization of the path-independence condition, consider the case of a similar function of two independent variables, $\mathcal{F}(x_1, x_2)$

- The chain rule of differentiation gives $d\mathcal{F} = \dfrac{\partial \mathcal{F}}{\partial x_1} dx_1 + \dfrac{\partial \mathcal{F}}{\partial x_2} dx_2$

- The vector form of $d\mathcal{F}$ is given by

$$d\mathcal{F} = \left(\hat{i}_1 \frac{\partial \mathcal{F}}{\partial x_1} + \hat{i}_2 \frac{\partial \mathcal{F}}{\partial x_2} \right) \bullet \left(dx_1 \hat{i}_1 + dx_2 \hat{i}_2 \right) \equiv \overrightarrow{\nabla \mathcal{F}} \bullet d\vec{x}$$

ILLUSTRATION OF THE PATH-INDEPENDENCE CONDITION
CONTINUED

- Let $\mathcal{P}$ denote a path traversed in a **loading-unloading** **cycle**, then

$$\oint_{\mathcal{P}} d\mathcal{F} = 0 \quad \text{becomes} \quad \oint_{\mathcal{P}} \overrightarrow{\nabla \mathcal{F}} \cdot d\vec{x} = 0$$

- Recall that *Stokes' Theorem* is given by $\oint_{\partial \mathcal{S}} \vec{g} \cdot d\vec{x} = \int\int_{\mathcal{S}} \hat{n} \cdot \left[\vec{\nabla} \times \vec{g}\right] dA$

  where

  - $\vec{g}(x_1, x_2)$ is an arbitrary vector field with continuous first derivatives

  - $\hat{n}(x_1, x_2)$ is the unit-magnitude normal-vector field for any smooth surface $\mathcal{S}$ enclosed by the curve $\partial \mathcal{S}$

ILLUSTRATION OF THE PATH-INDEPENDENCE CONDITION
CONTINUED

- Applying ***Stokes' theorem*** to $\oint_{\mathcal{P}} \overrightarrow{\nabla \mathcal{F}} \cdot d\vec{x} = 0$ gives

$$\oint_{\mathcal{P}} \overrightarrow{\nabla \mathcal{F}} \cdot d\vec{x} = \int\int_{\mathcal{S}(\mathcal{P})} \hat{n} \cdot \left[\vec{\nabla} \times \overrightarrow{\nabla \mathcal{F}} \right] dA = 0$$

where $\mathcal{S}(\mathcal{P})$ is the surface enclosed by the path $\mathcal{P}$

- For a simply connected region, the <u>necessary</u> and <u>sufficient</u> conditions for the line integral to vanish are given by the requirement that the integrand in the double integral vanish; that is,

$$\hat{n} \cdot \left[\vec{\nabla} \times \overrightarrow{\nabla \mathcal{F}} \right] = 0$$

ILLUSTRATION OF THE PATH-INDEPENDENCE CONDITION
CONTINUED

- Because the ***unit-magnitude normal-vector field*** for an arbitrary smooth surface $\mathcal{S}(\mathcal{P})$ is generally nonzero, the **necessary and sufficient conditions** for the line integral to vanish become

$$\vec{\nabla} \times \overrightarrow{\nabla \mathcal{F}} = \vec{0}$$

- Expanding $\vec{\nabla} \times \overrightarrow{\nabla \mathcal{F}} = \vec{0}$ gives $\left(\hat{i}_1 \frac{\partial}{\partial x_1} + \hat{i}_2 \frac{\partial}{\partial x_2} \right) \times \left(\hat{i}_1 \frac{\partial \mathcal{F}}{\partial x_1} + \hat{i}_2 \frac{\partial \mathcal{F}}{\partial x_2} \right) = \vec{0}$

  which simplifies to $\frac{\partial^2 \mathcal{F}}{\partial x_1 \partial x_2} \left(\hat{i}_1 \times \hat{i}_2 \right) + \frac{\partial^2 \mathcal{F}}{\partial x_2 \partial x_1} \left(\hat{i}_2 \times \hat{i}_1 \right) = \vec{0}$

- Simplifying further gives $\left(\frac{\partial^2 \mathcal{F}}{\partial x_1 \partial x_2} - \frac{\partial^2 \mathcal{F}}{\partial x_2 \partial x_1} \right) \left(\hat{i}_1 \times \hat{i}_2 \right) = \vec{0}$ which yields

  the condition $\frac{\partial^2 \mathcal{F}}{\partial x_1 \partial x_2} = \frac{\partial^2 \mathcal{F}}{\partial x_2 \partial x_1}$

ILLUSTRATION OF THE PATH-INDEPENDENCE CONDITION
CONTINUED

- The condition $\frac{\partial^2 \mathcal{F}}{\partial x_1 \partial x_2} = \frac{\partial^2 \mathcal{F}}{\partial x_2 \partial x_1}$ is, in fact, a statement of path independence at the local level, which is illustrated in the following figure

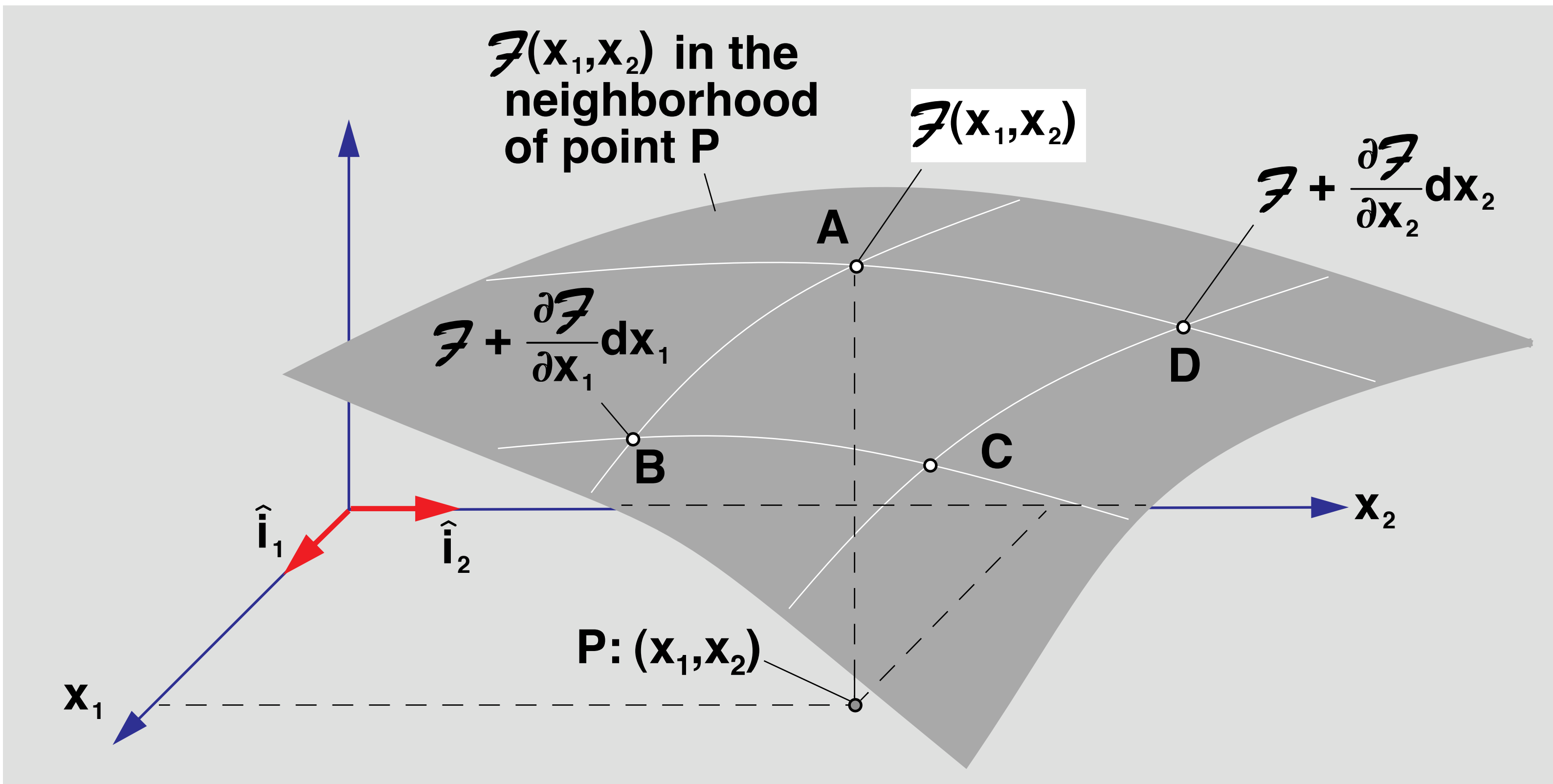


ILLUSTRATION OF THE PATH-INDEPENDENCE CONDITION
CONCLUDED

- By following path ABC, the value of $\mathcal{F}$ at point C is given by

$$\mathcal{F} + \frac{\partial \mathcal{F}}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}\left[\mathcal{F} + \frac{\partial \mathcal{F}}{\partial x_1}dx_1\right]dx_2 = \mathcal{F} + \frac{\partial \mathcal{F}}{\partial x_1}dx_1 + \frac{\partial \mathcal{F}}{\partial x_2}dx_2 + \frac{\partial^2 \mathcal{F}}{\partial x_2 \partial x_1}dx_1 dx_2$$

- By following path ADC, the value of $\mathcal{F}$ at point C is given by

$$\mathcal{F} + \frac{\partial \mathcal{F}}{\partial x_2}dx_2 + \frac{\partial}{\partial x_1}\left[\mathcal{F} + \frac{\partial \mathcal{F}}{\partial x_2}dx_2\right]dx_1 = \mathcal{F} + \frac{\partial \mathcal{F}}{\partial x_2}dx_2 + \frac{\partial \mathcal{F}}{\partial x_1}dx_1 + \frac{\partial^2 \mathcal{F}}{\partial x_1 \partial x_2}dx_2 dx_1$$

- For path independence, it follows that these two expressions must be equal, hence

$$\frac{\partial^2 \mathcal{F}}{\partial x_1 \partial x_2} = \frac{\partial^2 \mathcal{F}}{\partial x_2 \partial x_1}$$

COMPLEMENTARY STRAIN-ENERGY DENSITY

- The ***symmetry condition*** $S_{ijkl} = S_{klij}$ is obtained by examining the complementary strain-energy density functional $\mathcal{U}^*$

- The strain-energy density functional $\mathcal{U}$ was obtained by expressing the stresses in terms of the strains and integrating $d\mathcal{U} = \sigma_{ij} d\varepsilon_{ij}$ from the initial stress- and strain- free state to the current strain state

- An expression for $\mathcal{U}^*$ is obtained by first requiring that a *one-to-one relationship* exists between the stresses and strains, and by using the **product rule** of differentiation to get $d(\sigma_{ij}\varepsilon_{ij}) = \sigma_{ij} d\varepsilon_{ij} + \varepsilon_{ij} d\sigma_{ij}$

  - In the part $\sigma_{ij} d\varepsilon_{ij}$, the strains are taken as the independent variables

  - In the part $\varepsilon_{ij} d\sigma_{ij}$, the stresses are taken as the independent variables

COMPLEMENTARY STRAIN-ENERGY DENSITY

CONTINUED

- **Next, the expression $d(\sigma_{ij}\varepsilon_{ij}) = \sigma_{ij}d\varepsilon_{ij} + \varepsilon_{ij}d\sigma_{ij}$ is integrated from the initial stress- and strain- free state to the current stress and strain state; i.e.,**

$$\int_0^{\varepsilon_{pq}} d(\sigma_{ij}\varepsilon_{ij}) = \int_0^{\varepsilon_{pq}} \sigma_{ij}(\varepsilon_{pq})\, d\varepsilon_{ij} + \int_0^{\sigma_{pq}} \varepsilon_{ij}(\sigma_{pq})\, d\sigma_{ij}$$

- **In the term $\int_0^{\varepsilon_{pq}} d(\sigma_{ij}\varepsilon_{ij})$, it is presumed that the stresses are known as functions of the strains**
    - **This term can also be expressed as $\int_0^{\sigma_{pq}} d(\sigma_{ij}\varepsilon_{ij})$, where it is presumed that the strains are known as functions of the stresses**
    - **Both terms yield $\sigma_{ij}\varepsilon_{ij}$, the product of the current values of the stresses and strains**

COMPLEMENTARY STRAIN-ENERGY DENSITY
CONCLUDED

- Using the previous expression and the definition of the strain-energy density function $\mathcal{U}$ gives $\sigma_{ij}\varepsilon_{ij} = \mathcal{U}(\varepsilon_{pq}) + \int_0^{\sigma_{pq}} \varepsilon_{ij}(\sigma_{pq})\, d\sigma_{ij}$

- The **complementary** (or conjugate) **strain-energy density function** $\mathcal{U}^*$ is defined as $\mathcal{U}^* = \int_0^{\sigma_{pq}} \varepsilon_{ij}(\sigma_{pq})\, d\sigma_{ij}$ such that $\sigma_{ij}\varepsilon_{ij} = \mathcal{U}(\varepsilon_{pq}) + \mathcal{U}^*(\sigma_{pq})$

  - Note that $d\mathcal{U}^* = \varepsilon_{ij}(\sigma_{pq})\, d\sigma_{ij}$

  - The form $\mathcal{U}^*(\sigma_{pq}) = \sigma_{ij}\varepsilon_{ij} - \mathcal{U}(\varepsilon_{pq})$ is known as the ***Legendre transformation***

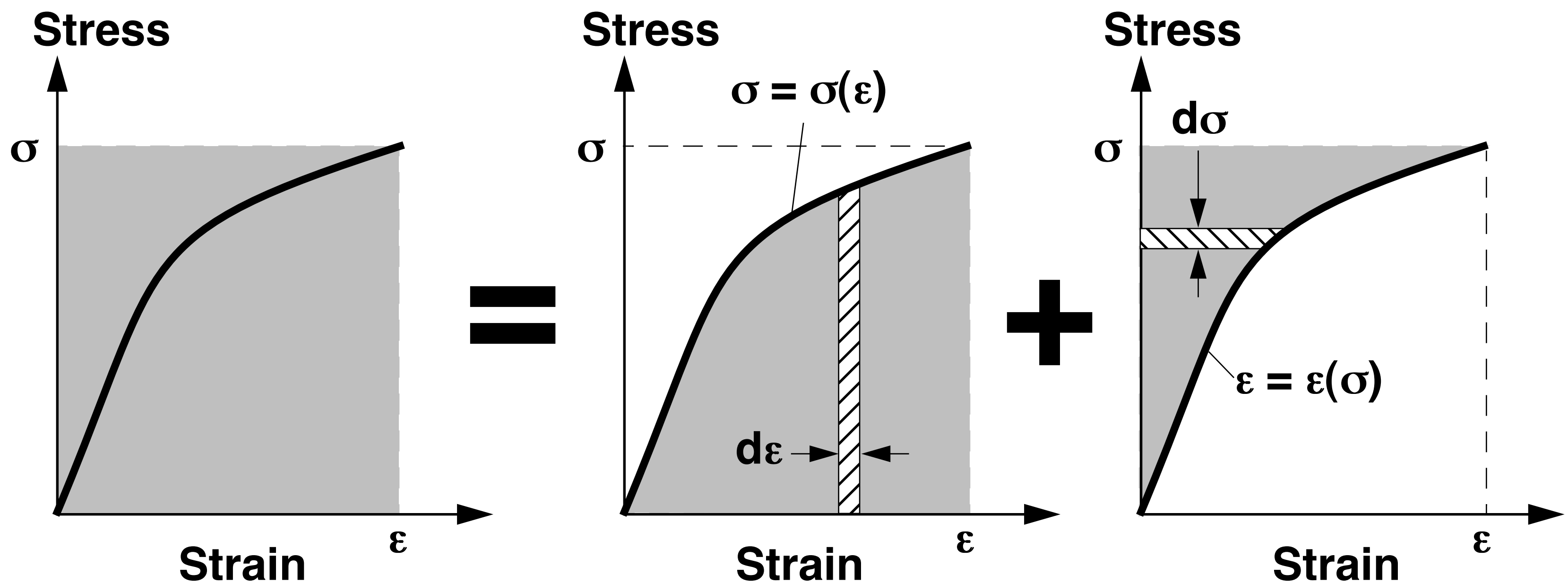
- The complementary or *conjugate* relationship of the strain-energy density function and the complementary strain-energy density function are illustrated on the next chart for a one-dimensional case

ILLUSTRATION OF ENERGY DENSITY FUNCTIONALS

ONE-DIMENSIONAL CASE

$$\sigma\varepsilon = \mathcal{U}(\varepsilon) + \mathcal{U}^*(\sigma)$$

Single-parameter loading system



Area = $\sigma\varepsilon$

Area = $\int_0^{\varepsilon} \sigma(\varepsilon)\, d\varepsilon = \mathcal{U}(\varepsilon)$

Area = $\int_0^{\sigma} \varepsilon(\sigma)\, d\sigma = \mathcal{U}^*(\sigma)$

ILLUSTRATION OF ENERGY DENSITY FUNCTIONALS
ONE-DIMENSIONAL CASE - CONCLUDED

- The previous figure indicates that because strain-energy density is not lost in an **arbitrary elastic loading process**, neither is the complementary strain-energy density
  - Thus, the complementary strain-energy density function is also path independent and conserved in an elastic loading-unloading process
  - Thus, $\mathscr{U}^* = \oint d\mathscr{U}^* = 0$ and $d\mathscr{U}^* = \dfrac{\partial \mathscr{U}^*}{\partial \sigma_{ij}} d\sigma_{ij}$
- Equating $d\mathscr{U}^* = \varepsilon_{ij}(\sigma_{pq})\, d\sigma_{ij}$ with $d\mathscr{U}^* = \dfrac{\partial \mathscr{U}^*}{\partial \sigma_{ij}} d\sigma_{ij}$ gives $\boxed{\dfrac{\partial \mathscr{U}^*}{\partial \sigma_{ij}} = \varepsilon_{ij}}$
  - The equations given above indicate that the strain-stress relations are derivable from a **potential function** when the deformation process is elastic (hyperelastic material)

PROOF THAT $S_{ijkl} = S_{klij}$

- The *necessary and sufficient conditions* for $\mathcal{U}^* = \mathcal{U}^*(\sigma_{pq})$ to be path independent are for the conditions $\dfrac{\partial^2 \mathcal{U}^*}{\partial \sigma_{ij}\, \partial \sigma_{kl}} = \dfrac{\partial^2 \mathcal{U}^*}{\partial \sigma_{kl}\, \partial \sigma_{ij}}$ to be valid

- First note that $\dfrac{\partial^2 \mathcal{U}^*}{\partial \sigma_{ij}\, \partial \sigma_{kl}} = \dfrac{\partial^2 \mathcal{U}^*}{\partial \sigma_{kl}\, \partial \sigma_{ij}}$ and $\dfrac{\partial \mathcal{U}^*}{\partial \sigma_{ij}} = \varepsilon_{ij}$ give $\dfrac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \dfrac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}}$

- Then, $\varepsilon_{ij} = S_{ijrs}\sigma_{rs}$ gives $\dfrac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \dfrac{\partial}{\partial \sigma_{kl}}\left[S_{ijrs}\sigma_{rs}\right] = S_{ijrs}\dfrac{\partial \sigma_{rs}}{\partial \sigma_{kl}} = S_{ijrs}\delta_{rk}\delta_{sl} = S_{ijkl}$

- And, $\varepsilon_{kl} = S_{klpq}\sigma_{pq}$ gives $\dfrac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}} = \dfrac{\partial}{\partial \sigma_{ij}}\left[S_{klpq}\sigma_{pq}\right] = S_{klpq}\dfrac{\partial \sigma_{pq}}{\partial \sigma_{ij}} = S_{klpq}\delta_{pi}\delta_{qj} = S_{klij}$

- Thus, $\dfrac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \dfrac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}}$ yields $S_{ijkl} = S_{klij}$, which reduces the number of independent compliance coefficients to 21

STANDARD FORMS FOR GENERALIZED HOOKE'S LAW

- The **standard forms** of the generalized Hooke's law are now given by

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 2S_{1123} & 2S_{1113} & 2S_{1112} \\ S_{1122} & S_{2222} & S_{2233} & 2S_{2223} & 2S_{2213} & 2S_{2212} \\ S_{1133} & S_{2233} & S_{3333} & 2S_{3323} & 2S_{3313} & 2S_{3312} \\ 2S_{1123} & 2S_{2223} & 2S_{3323} & 4S_{2323} & 4S_{2313} & 4S_{2312} \\ 2S_{1113} & 2S_{2213} & 2S_{3313} & 4S_{2313} & 4S_{1313} & 4S_{1312} \\ 2S_{1112} & 2S_{2212} & 2S_{3312} & 4S_{2312} & 4S_{1312} & 4S_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{1133} & C_{2233} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{1123} & C_{2223} & C_{3323} & C_{2323} & C_{2313} & C_{2312} \\ C_{1113} & C_{2213} & C_{3313} & C_{2313} & C_{1313} & C_{1312} \\ C_{1112} & C_{2212} & C_{3312} & C_{2312} & C_{1312} & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

CLAPEYRON'S FORMULA

- For a linear-elastic solid, strain-energy-density increment $d\mathcal{U} = \sigma_{ij}(\varepsilon_{pq})\, d\varepsilon_{ij}$ is combined with $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ to get $d\mathcal{U} = C_{ijkl}\varepsilon_{kl}d\varepsilon_{ij}$

- Now consider, $\frac{1}{2}\, d\left[C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\right] = \frac{1}{2}\left[C_{ijkl}d\varepsilon_{ij}\varepsilon_{kl} + C_{ijkl}\varepsilon_{ij}d\varepsilon_{kl}\right]$

- Because all indices are summation indices, this expression can be expressed as

$$\frac{1}{2}\, d\left[C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\right] = \frac{1}{2}\left[C_{ijkl}\varepsilon_{kl}d\varepsilon_{ij} + C_{klij}\varepsilon_{kl}d\varepsilon_{ij}\right] = \frac{1}{2}\left[C_{ijkl} + C_{klij}\right]\varepsilon_{kl}d\varepsilon_{ij}$$

- By using the path-independence condition $C_{ijkl} = C_{klij}$, it follows that

$$\frac{1}{2}\, d\left[C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\right] = C_{ijkl}\varepsilon_{kl}d\varepsilon_{ij} \quad \text{and that} \quad d\mathcal{U} = \frac{1}{2}\, d\left[C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\right]$$

CLAPEYRON'S FORMULA - CONCLUDED

- Integrating the last expression gives $\mathcal{U} = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + K$ where K is a constant of integration

- Noting that $\mathcal{U} = 0$ when the strain field is zero-valued gives $K = 0$

- Next, using $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$ gives the desired result, $\mathcal{U} = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$

- This expression for the strain-energy density of a homogeneous, linear-elastic, anisotropic solid is attributed to B. P. E. Clapeyron (1799-1864)

- A similar procedure can be followed to show that $\mathcal{U}^* = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} = \mathcal{U}$ for a homogeneous, linear-elastic, anisotropic solid

POSITIVE-DEFINITENESS OF THE STRAIN-ENERGY DENSITY FUNCTION

- The **strain-energy density** of a solid in its stress- and strain-free state is defined to be zero-valued

- As a solid *deforms* under load, **it stores strain energy** and develops the potential to perform work upon removal of the loads

  - Thus, it follows that the strain-energy density function is a non-negative-valued function for all physically admissible elastic strain states

  - Hence, $\mathcal{U} = \int_0^{\varepsilon_{pq}} \sigma_{ij}(\varepsilon_{pq})\, d\varepsilon_{ij} \geq 0$ for a ***nonlinear-elastic material***

- For a linear-elastic material, $\mathcal{U} = \frac{1}{2} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq 0$ must hold, which places some ***thermodynamic restrictions*** on the stiffness coefficients that must hold for reversible (elastic) loading-unloading processes

POSITIVE-DEFINITENESS OF THE STRAIN-ENERGY DENSITY FUNCTION - CONTINUED

- The ***strain-energy density*** of a linear-elastic material can be expressed in matrix form by

$$\mathcal{U} = \frac{1}{2}\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}^T \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{1133} & C_{2233} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{1123} & C_{2223} & C_{3323} & C_{2323} & C_{2313} & C_{2312} \\ C_{1113} & C_{2213} & C_{3313} & C_{2313} & C_{1313} & C_{1312} \\ C_{1112} & C_{2212} & C_{3312} & C_{2312} & C_{1312} & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

- **Positive-definiteness** of the *strain-energy density* is satisfied by positive-definiteness of the matrix containing the stiffness coefficients
  - Enforcing positive-definiteness defines relationships that the stiffness coefficients must obey; e.g., all the diagonal elements of the matrix must be positive-valued

POSITIVE-DEFINITENESS OF THE STRAIN-ENERGY DENSITY FUNCTION - CONCLUDED

- **Positive-definiteness of the *strain-energy density* is used in the linear theory of elasticity to establish:**
  - **Uniqueness of solutions**
  - **The theorem of minimum potential energy**
  - **The theorem of minimum complementary energy**
  - **Some aspects of *St. Venant's principle***

GENERALIZED DUHAMEL-NEUMANN LAW FOR HOMOGENEOUS, ANISOTROPIC, LINEAR-ELASTIC SOLIDS

THE GENERALIZED DUHAMEL-NEUMANN LAW

- In general, when an elastic solid is subjected to heating or cooling, the equations of **elasticity** are *coupled* with the equations of **thermodynamics and heat transfer**
  - When the **heat generated by deformations** is negligible, the equations uncouple and the *temperature field* can be solved for independently of the structural deformations
  - The temperature field becomes a known quantity (loading) in the solution of the ***linear thermoelasticity*** equations
- In general, when an elastic solid is subjected to heating or cooling, the stress-strain relations depend on the temperature of the body
  - The extent of the temperature dependence depends on the extent of the heating or cooling

THE GENERALIZED DUHAMEL-NEUMANN LAW
CONTINUED

- To obtain a *simple working theory* that is **linear** and that includes thermal effects, a ***constitutive law*** was developed with the following attributes:
  - **Thermal expansion** effects are included
  - Variations in the elastic constants and coefficients of thermal expansion with temperature are neglected
  - Inertial effects associated with heating rates are neglected
- A relatively **simple extension of Hooke's law** that predicted accurately experimentally observed phenomenon was the desired result
  - The resulting equations are typically referred to as the **linear-thermoelastic constitutive equations**

THE GENERALIZED DUHAMEL-NEUMANN LAW
CONTINUED

- The generalized Hooke's law was extended by ***J. M. C. Duhamel*** (1797- 1872) and ***F. E. Neumann*** (1798-1895) to include the first-order, linear effects of thermal loading

- This law states, in part, that the total strain ε_{ij} at a point of a solid, subjected to thermomechanical loading, consists of stress-induced *mechanical strain* $\varepsilon_{ij}^{\sigma}$ and strain caused by *free thermal expansion* ε_{ij}^{T}

  - The mechanical strain $\varepsilon_{ij}^{\sigma}$ is the strain caused by the externally applied loads and displacements, and the strain caused by ***nonuniformity** in the temperature field* or in the thermal expansion properties of the material, or both

  - $\varepsilon_{ij} = \varepsilon_{ij}^{\sigma} + \varepsilon_{ij}^{T}$ where $\varepsilon_{ij}^{\sigma} = S_{ijrs}\sigma_{rs}$, $\varepsilon_{ij}^{T} = \alpha_{ij}(T - T_{ref})$, T is the temperature field, and T_{ref} is the temperature field at which the body is stress and strain free

THE GENERALIZED DUHAMEL-NEUMANN LAW

CONTINUED

- The general form of the **Duhamel-Neumann law** is given in expanded form by

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \hdashline \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{Bmatrix} = \left[\begin{array}{cccccc:ccc} S_{1111} & S_{1122} & S_{1133} & S_{1123} & S_{1113} & S_{1112} & S_{1132} & S_{1131} & S_{1121} \\ S_{2211} & S_{2222} & S_{2233} & S_{2223} & S_{2213} & S_{2212} & S_{2232} & S_{2231} & S_{2221} \\ S_{3311} & S_{3322} & S_{3333} & S_{3323} & S_{3313} & S_{3312} & S_{3332} & S_{3331} & S_{3321} \\ S_{2311} & S_{2322} & S_{2333} & S_{2323} & S_{2313} & S_{2312} & S_{2332} & S_{2331} & S_{2321} \\ S_{1311} & S_{1322} & S_{1333} & S_{1323} & S_{1313} & S_{1312} & S_{1332} & S_{1331} & S_{1321} \\ S_{1211} & S_{1222} & S_{1233} & S_{1223} & S_{1213} & S_{1212} & S_{1232} & S_{1231} & S_{1221} \\ \hdashline S_{3211} & S_{3222} & S_{3233} & S_{3223} & S_{3213} & S_{3212} & S_{3232} & S_{3231} & S_{3221} \\ S_{3111} & S_{3122} & S_{3133} & S_{3123} & S_{3113} & S_{3112} & S_{3132} & S_{3131} & S_{3121} \\ S_{2111} & S_{2122} & S_{2133} & S_{2123} & S_{2113} & S_{2112} & S_{2132} & S_{2131} & S_{2121} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ \hdashline \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ \alpha_{23} \\ \alpha_{13} \\ \alpha_{12} \\ \hdashline \alpha_{32} \\ \alpha_{31} \\ \alpha_{21} \end{Bmatrix} \left(T - T_{ref}\right)$$

and in indicial form by $\varepsilon_{ij} = S_{ijkl}\sigma_{kl} + \alpha_{ij}\left(T - T_{ref}\right)$

- S_{ijkl} are the components of the (4th-order) **compliance tensor**, at $T = T_{ref}$, that appear in the generalized Hooke's law and α_{ij} are the **coefficients of linear thermal expansion** (with units of temperature^{-1})

THE GENERALIZED DUHAMEL-NEUMANN LAW
CONTINUED

- The ***inverted form*** of the **Duhamel-Neumann law** is given in expanded form by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} & C_{1132} & C_{1131} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} & C_{2232} & C_{2231} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} & C_{3332} & C_{3331} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} & C_{2332} & C_{2331} & C_{2321} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} & C_{1332} & C_{1331} & C_{1321} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} & C_{1232} & C_{1231} & C_{1221} \\ C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3213} & C_{3212} & C_{3232} & C_{3231} & C_{3221} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3113} & C_{3112} & C_{3132} & C_{3131} & C_{3121} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2113} & C_{2112} & C_{2132} & C_{2131} & C_{2121} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ \alpha_{23} \\ \alpha_{13} \\ \alpha_{12} \\ \alpha_{32} \\ \alpha_{31} \\ \alpha_{21} \end{Bmatrix} (T - T_{ref}) \right\}$$

and in indicial form by $\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \alpha_{kl}\,(T - T_{ref})\right) = C_{ijkl}\varepsilon^{\sigma}_{kl}$

- C_{ijkl} are the components of the (4th-order) **stiffness tensor**, at $T = T_{ref}$, that appear in the generalized Hooke's law and the column vector on right-hand side of the matrix equation contains the mechanical strains

THE GENERALIZED DUHAMEL-NEUMANN LAW
CONTINUED

- The *inverted form* of the Duhamel-Neumann law is also expressed often in matrix form by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \\ \hdashline \sigma_{32} \\ \sigma_{31} \\ \sigma_{21} \end{Bmatrix} = \left[\begin{array}{cccccc:ccc} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} & C_{1132} & C_{1131} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} & C_{2232} & C_{2231} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} & C_{3332} & C_{3331} & C_{3321} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} & C_{2332} & C_{2331} & C_{2321} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} & C_{1332} & C_{1331} & C_{1321} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} & C_{1232} & C_{1231} & C_{1221} \\ \hdashline C_{3211} & C_{3222} & C_{3233} & C_{3223} & C_{3213} & C_{3212} & C_{3232} & C_{3231} & C_{3221} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3113} & C_{3112} & C_{3132} & C_{3131} & C_{3121} \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2113} & C_{2112} & C_{2132} & C_{2131} & C_{2121} \end{array}\right] \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \\ \hdashline \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{21} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \\ \hdashline \beta_{32} \\ \beta_{31} \\ \beta_{21} \end{Bmatrix} (T - T_{ref})$$

and in indicial form by $\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + \beta_{ij}\,(T - T_{ref})$

- β_{ij} are called the ***thermal moduli***

THE GENERALIZED DUHAMEL-NEUMANN LAW
CONTINUED

- By noting that the **Duhamel-Neumann law** becomes the generalized Hooke's law when $T = T_{ref}$, the following *symmetry conditions* must hold

$$S_{ijkl} = S_{jikl} \qquad S_{ijkl} = S_{ijlk} \qquad S_{ijkl} = S_{klij}$$

$$C_{ijkl} = C_{jikl} \qquad C_{ijkl} = C_{ijlk} \qquad C_{ijkl} = C_{klij}$$

which indicates **21 independent** compliance or stiffness coefficients

- Symmetry of the strain tensor also yields $\alpha_{ij} = \alpha_{ji}$, and reduces the number of *independent coefficients of linear thermal expansion* from 9 to 6

- Likewise, symmetry of the stress tensor also yields $\beta_{ij} = \beta_{ji}$, and reduces the number of *independent thermal moduli* from 9 to 6

THE GENERALIZED DUHAMEL-NEUMANN LAW
CONCLUDED

- The expanded forms of the **Duhamel-Neumann law** are now given by

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 2S_{1123} & 2S_{1113} & 2S_{1112} \\ S_{1122} & S_{2222} & S_{2233} & 2S_{2223} & 2S_{2213} & 2S_{2212} \\ S_{1133} & S_{2233} & S_{3333} & 2S_{3323} & 2S_{3313} & 2S_{3312} \\ 2S_{1123} & 2S_{2223} & 2S_{3323} & 4S_{2323} & 4S_{2313} & 4S_{2312} \\ 2S_{1113} & 2S_{2213} & 2S_{3313} & 4S_{2313} & 4S_{1313} & 4S_{1312} \\ 2S_{1112} & 2S_{2212} & 2S_{3312} & 4S_{2312} & 4S_{1312} & 4S_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{1133} & C_{2233} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{1123} & C_{2223} & C_{3323} & C_{2323} & C_{2313} & C_{2312} \\ C_{1113} & C_{2213} & C_{3313} & C_{2313} & C_{1313} & C_{1312} \\ C_{1112} & C_{2212} & C_{3312} & C_{2312} & C_{1312} & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix} (T - T_{ref})$$

EQUATIONS FOR THE THERMAL MODULI

- The **thermal moduli** β_{ij} are related to the coefficients of linear thermal expansion by $\beta_{ij} = -C_{ijkl}\alpha_{kl}$ or by

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix} = -\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{1133} & C_{2233} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{1123} & C_{2223} & C_{3323} & C_{2323} & C_{2313} & C_{2312} \\ C_{1113} & C_{2213} & C_{3313} & C_{2313} & C_{1313} & C_{1312} \\ C_{1112} & C_{2212} & C_{3312} & C_{2312} & C_{1312} & C_{1212} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix}$$

- Note that β_{ij} have units of stress/ temperature; e.g., lb/in^2-°F

- Similarly, α_{ij} have units of temperature^{-1}

STRAIN-ENERGY DENSITY FOR THERMAL LOADING

- The ***symmetry*** relations $C_{ijkl} = C_{klij}$ for a thermoelastic solid can also be obtained from first principles by enforcing path independence of the strain-energy density function $\mathcal{U}$

- The **strain-energy density** $\mathcal{U}$ of a generally *thermoelastic solid* is defined as the **work of the internal stresses** done through *mechanical deformations*

- The strain-energy-density increment $d\mathcal{U}$ is given for this case by

$$d\mathcal{U} = \sigma_{11} d\varepsilon^{\sigma}_{11} + \sigma_{22} d\varepsilon^{\sigma}_{22} + \sigma_{33} d\varepsilon^{\sigma}_{33} + 2\sigma_{23} d\varepsilon^{\sigma}_{23} + 2\sigma_{13} d\varepsilon^{\sigma}_{13} + 2\sigma_{12} d\varepsilon^{\sigma}_{12}$$

where the stress-induced, mechanical strains are given by $\varepsilon^{\sigma}_{ij} = \varepsilon_{ij} - \varepsilon^{T}_{ij}$ and the stresses depend on the **mechanical strains**; that is,

$$\sigma_{ij} = \sigma_{ij}\left(\varepsilon^{\sigma}_{11},\ \varepsilon^{\sigma}_{22},\ \varepsilon^{\sigma}_{33},\ \varepsilon^{\sigma}_{23},\ \varepsilon^{\sigma}_{13},\ \varepsilon^{\sigma}_{12}\right)$$

STRAIN-ENERGY DENSITY FOR THERMAL LOADING
CONTINUED

- It is important to emphasize that the **mechanical strains** $\varepsilon_{ij}^{\sigma} = \varepsilon_{ij} - \varepsilon_{ij}^{T}$ include strains generated by thermal stresses associated with a *nonuniform temperature field* or spatial variations in the coefficients of thermal expansion

- The expression

$$d\mathcal{U} = \sigma_{11} d\varepsilon_{11}^{\sigma} + \sigma_{22} d\varepsilon_{22}^{\sigma} + \sigma_{33} d\varepsilon_{33}^{\sigma} + 2\sigma_{23} d\varepsilon_{23}^{\sigma} + 2\sigma_{13} d\varepsilon_{13}^{\sigma} + 2\sigma_{12} d\varepsilon_{12}^{\sigma}$$

is written compactly in indicial form as

$$d\mathcal{U} = \sigma_{ij}\left(\varepsilon_{pq}^{\sigma}\right) d\varepsilon_{ij}^{\sigma}$$

STRAIN-ENERGY DENSITY FOR THERMAL LOADING
CONTINUED

- The strain-energy density $\mathcal{U}$ is obtained by integrating $d\mathcal{U}$ over the deformation associated with a **thermomechanical loading process** that starts at a stress- and strain-free state and ends at a particular stress and strain state; that is,

$$\mathcal{U} = \int_0^{\varepsilon^{\sigma}_{pq}} \sigma_{ij}\left(\varepsilon^{\sigma}_{pq}\right)\, d\varepsilon^{\sigma}_{ij} = \mathcal{U}\left(\varepsilon^{\sigma}_{pq}\right)$$

- Because no **mechanical work** of the internal forces within a body is lost during a **conservative**, elastic, thermomechanical loading-unloading process, it follows that $\mathcal{U} = \oint \sigma_{ij}\left(\varepsilon^{\sigma}_{pq}\right)\, d\varepsilon^{\sigma}_{ij} = 0$, which implies

$$d\mathcal{U} = \frac{\partial \mathcal{U}}{\partial \varepsilon^{\sigma}_{ij}}\, d\varepsilon^{\sigma}_{ij} \quad \text{and} \quad \frac{\partial^2 \mathcal{U}}{\partial \varepsilon^{\sigma}_{ij}\, \partial \varepsilon^{\sigma}_{kl}} = \frac{\partial^2 \mathcal{U}}{\partial \varepsilon^{\sigma}_{kl}\, \partial \varepsilon^{\sigma}_{ij}}$$

STRAIN-ENERGY DENSITY FOR THERMAL LOADING
CONCLUDED

- Equating $d\mathcal{U} = \sigma_{ij}(\varepsilon^{\sigma}_{pq})\, d\varepsilon^{\sigma}_{ij}$ and $d\mathcal{U} = \dfrac{\partial \mathcal{U}}{\partial \varepsilon^{\sigma}_{ij}}\, d\varepsilon^{\sigma}_{ij}$ gives $\dfrac{\partial \mathcal{U}}{\partial \varepsilon^{\sigma}_{ij}} = \sigma_{ij}(\varepsilon^{\sigma}_{pq})$

- Then, $\dfrac{\partial^2 \mathcal{U}}{\partial \varepsilon^{\sigma}_{ij}\,\partial \varepsilon^{\sigma}_{kl}} = \dfrac{\partial^2 \mathcal{U}}{\partial \varepsilon^{\sigma}_{kl}\,\partial \varepsilon^{\sigma}_{ij}}$ and $\dfrac{\partial \mathcal{U}}{\partial \varepsilon^{\sigma}_{ij}} = \sigma_{ij}(\varepsilon^{\sigma}_{pq})$ give $\dfrac{\partial \sigma_{ij}}{\partial \varepsilon^{\sigma}_{kl}} = \dfrac{\partial \sigma_{kl}}{\partial \varepsilon^{\sigma}_{ij}}$

- $\sigma_{ij} = C_{ijrs}\varepsilon^{\sigma}_{rs}$ gives $\dfrac{\partial \sigma_{ij}}{\partial \varepsilon^{\sigma}_{kl}} = \dfrac{\partial}{\partial \varepsilon^{\sigma}_{kl}}\left[C_{ijrs}\varepsilon^{\sigma}_{rs}\right] = C_{ijrs}\dfrac{\partial \varepsilon^{\sigma}_{rs}}{\partial \varepsilon^{\sigma}_{kl}} = C_{ijrs}\delta_{rk}\delta_{sl} = C_{ijkl}$

- $\sigma_{ij} = C_{ijpq}\varepsilon^{\sigma}_{pq}$ gives $\dfrac{\partial \sigma_{kl}}{\partial \varepsilon^{\sigma}_{ij}} = \dfrac{\partial}{\partial \varepsilon^{\sigma}_{ij}}\left[C_{klpq}\varepsilon^{\sigma}_{pq}\right] = C_{klpq}\dfrac{\partial \varepsilon^{\sigma}_{pq}}{\partial \varepsilon^{\sigma}_{ij}} = C_{klpq}\delta_{pi}\delta_{qj} = C_{klij}$

- Thus, $\dfrac{\partial \sigma_{ij}}{\partial \varepsilon^{\sigma}_{kl}} = \dfrac{\partial \sigma_{kl}}{\partial \varepsilon^{\sigma}_{ij}}$ yields $C_{ijkl} = C_{klij}$

COMPLEMENTARY STRAIN-ENERGY DENSITY FOR THERMAL LOADING

- The symmetry condition $S_{ijkl} = S_{klij}$ is obtained by examining the **complementary strain-energy density function** $\mathcal{U}^*$

- An expression for $\mathcal{U}^*$ is obtained by first requiring that a *one-to-one relationship* exists between the stresses and strains, and by expressing

$$\sigma_{ij}\varepsilon_{ij} = \sigma_{ij}\left(\varepsilon_{ij}^{\sigma} + \alpha_{ij}\left(T - T_{ref}\right)\right) \quad \text{or} \quad \sigma_{ij}\varepsilon_{ij} = \sigma_{ij}\varepsilon_{ij}^{\sigma} + \sigma_{ij}\alpha_{ij}\left(T - T_{ref}\right)$$

- Next, the product rule of differentiation is used to get

$$d\left(\sigma_{ij}\varepsilon_{ij}\right) = \sigma_{ij}d\varepsilon_{ij}^{\sigma} + \varepsilon_{ij}^{\sigma}d\sigma_{ij} + d\left[\sigma_{ij}\alpha_{ij}\left(T - T_{ref}\right)\right]$$

  - In the part $\sigma_{ij}d\varepsilon_{ij}^{\sigma}$, the stress-induced, mechanical strains are taken as the independent variables

  - In the part $\varepsilon_{ij}^{\sigma}d\sigma_{ij}$, the stresses are taken as the independent variables

COMPLEMENTARY STRAIN-ENERGY DENSITY FOR THERMAL LOADING - CONTINUED

- **Next, the expression** $d(\sigma_{ij}\varepsilon_{ij}) = \sigma_{ij} d\varepsilon_{ij}^{\sigma} + \varepsilon_{ij}^{\sigma} d\sigma_{ij} + d\left[\sigma_{ij}\alpha_{ij}(T - T_{ref})\right]$ **is integrated from the initial stress- and strain- free state to the current stress and strain state; i.e.,**

$$\int_0^{\varepsilon_{pq}} d(\sigma_{ij}\varepsilon_{ij}) = \int_0^{\varepsilon_{pq}^{\sigma}} \sigma_{ij}(\varepsilon_{pq}^{\sigma})\, d\varepsilon_{ij}^{\sigma} + \int_0^{\sigma_{pq}} \varepsilon_{ij}^{\sigma}(\sigma_{pq})\, d\sigma_{ij} + \int_0^{\varepsilon_{pq}} d\left[\sigma_{ij}\alpha_{ij}(T - T_{ref})\right]$$

- **In the term** $\int_0^{\varepsilon_{pq}} d(\sigma_{ij}\varepsilon_{ij})$ **it is presumed that the stresses are known as functions of the strains**
  - **This term can also be expressed as** $\int_0^{\sigma_{pq}} d(\sigma_{ij}\varepsilon_{ij})$ **, where it is presumed that the strains are known as functions of the stresses**
  - **Both terms yield** $\sigma_{ij}\varepsilon_{ij}$ **, the current values of the stresses and strains**

COMPLEMENTARY STRAIN-ENERGY DENSITY FOR THERMAL LOADING - CONCLUDED

- Using the previous expression and the definition of the strain-energy density function $\mathcal{U}$ gives $\sigma_{ij}\varepsilon_{ij} = \mathcal{U}\left(\varepsilon^{\sigma}_{pq}\right) + \int_0^{\sigma_{pq}} \varepsilon^{\sigma}_{ij}\left(\sigma_{pq}\right) d\sigma_{ij} + \sigma_{ij}\alpha_{ij}\left(T - T_{ref}\right)$

- The complementary strain-energy density function $\mathcal{U}^*$ is defined as

$\mathcal{U}^* = \int_0^{\sigma_{pq}} \varepsilon^{\sigma}_{ij}\left(\sigma_{pq}\right) d\sigma_{ij} + \sigma_{ij}\alpha_{ij}\left(T - T_{ref}\right)$ such that $\sigma_{ij}\varepsilon_{ij} = \mathcal{U}\left(\varepsilon^{\sigma}_{pq}\right) + \mathcal{U}^*\left(\sigma_{pq}, T\right)$

  - Note that $d\mathcal{U}^* = \varepsilon^{\sigma}_{ij} d\sigma_{ij} + d\left[\sigma_{ij}\alpha_{ij}\left(T - T_{ref}\right)\right]$

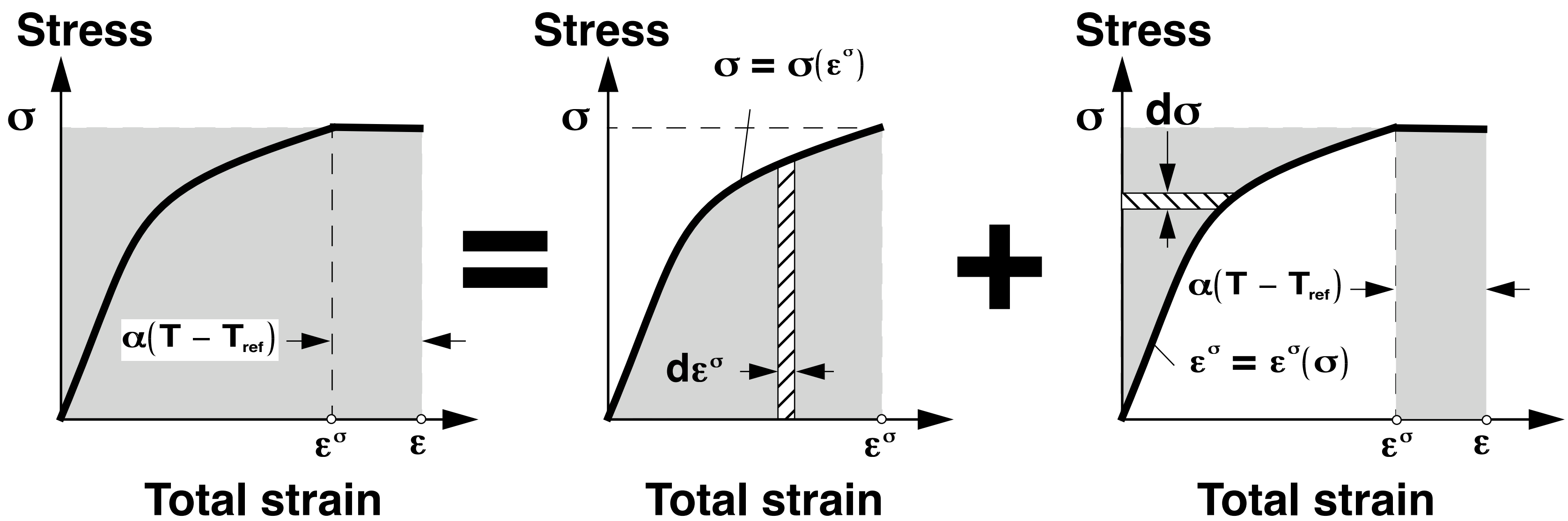
  - Legendre's transformation takes the form $\mathcal{U}^*\left(\sigma_{pq}, T\right) = \sigma_{ij}\varepsilon_{ij} - \mathcal{U}\left(\varepsilon^{\sigma}_{pq}\right)$

- The complementary relationship of the strain-energy density function and the complementary strain-energy density function for the thermoelastic case are illustrated on the next two charts for a *one-dimensional* stress and strain state

ILLUSTRATION OF THERMOELASTIC ENERGY DENSITIES

ONE-DIMENSIONAL CASE

$$\sigma\varepsilon = \mathcal{U}(\varepsilon^{\sigma}) + \mathcal{U}^{*}(\sigma,T)$$



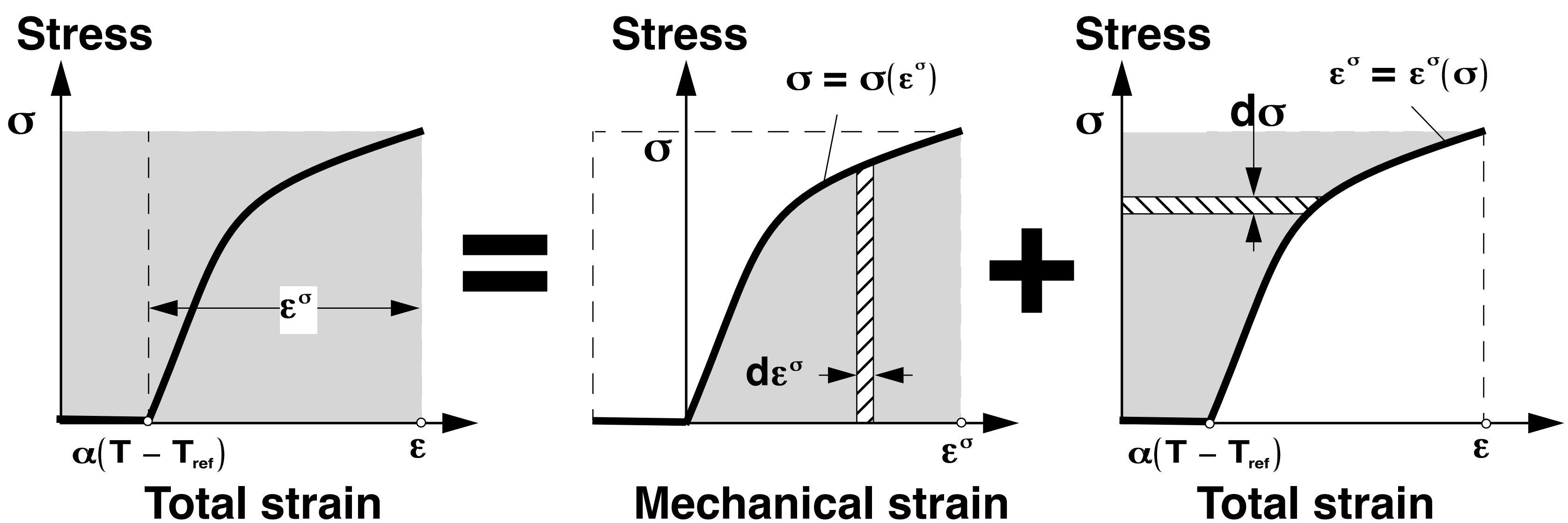
Area $= \sigma\varepsilon$ **Area** $= \int_0^{\varepsilon^{\sigma}} \sigma(\varepsilon^{\sigma})\, d\varepsilon^{\sigma} = \mathcal{U}(\varepsilon^{\sigma})$ **Area** $= \int_0^{\sigma} \varepsilon^{\sigma}(\sigma)d\sigma + \sigma\alpha(T - T_{ref}) = \mathcal{U}^{*}(\sigma,T)$

Loading process = mechanical loading followed by thermal loading

ILLUSTRATION OF THERMOELASTIC ENERGY DENSITIES

ONE-DIMENSIONAL CASE - CONTINUED

$$\sigma\varepsilon = \mathcal{U}(\varepsilon^{\sigma}) + \mathcal{U}^{*}(\sigma,T)$$



Area $= \sigma\varepsilon$ Area $= \int_0^{\varepsilon^{\sigma}} \sigma(\varepsilon^{\sigma})\, d\varepsilon^{\sigma} = \mathcal{U}(\varepsilon^{\sigma})$ Area $= \int_0^{\sigma} \varepsilon^{\sigma}(\sigma)d\sigma + \sigma\alpha(T - T_{ref}) = \mathcal{U}^{*}(\sigma,T)$

Loading process = thermal loading followed by mechanical loading

ILLUSTRATION OF THERMOELASTIC ENERGY DENSITIES

ONE-DIMENSIONAL CASE - CONTINUED

- The previous figures illustrate that path independence of the elastic loading-unloading process implies path independence of the ***complementary strain-energy density function***

  - If the material were inelastic, the quantity of complementary strain-energy density function would depend on the loading process

- Thus, $\mathcal{U}^*(\sigma_{pq}, T) = \oint d\mathcal{U}^* = 0$ and $d\mathcal{U}^* = \frac{\partial \mathcal{U}^*}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial \mathcal{U}^*}{\partial T} dT$

- $d\mathcal{U}^* = \varepsilon_{ij}^{\sigma} d\sigma_{ij} + d\left[\sigma_{ij}\alpha_{ij}\left(T - T_{ref}\right)\right]$ gives $d\mathcal{U}^* = \left(\varepsilon_{ij}^{\sigma} + \alpha_{ij}\left(T - T_{ref}\right)\right)d\sigma_{ij} + \sigma_{ij}\alpha_{ij}dT$

  which reduces to $d\mathcal{U}^* = \varepsilon_{ij} d\sigma_{ij} + \sigma_{ij}\alpha_{ij} dT$

ILLUSTRATION OF THERMOELASTIC ENERGY DENSITIES

ONE-DIMENSIONAL CASE - CONCLUDED

- Equating $d\mathcal{U}^* = \dfrac{\partial \mathcal{U}^*}{\partial \sigma_{ij}} d\sigma_{ij} + \dfrac{\partial \mathcal{U}^*}{\partial T} dT$ with $d\mathcal{U}^* = \varepsilon_{ij} d\sigma_{ij} + \sigma_{ij}\alpha_{ij} dT$ gives

$$\frac{\partial \mathcal{U}^*}{\partial \sigma_{ij}} = \varepsilon_{ij} = \varepsilon_{ij}^{\sigma} + \alpha_{ij}\left(T - T_{ref}\right) \quad \text{and} \quad \frac{\partial \mathcal{U}^*}{\partial T} = \sigma_{ij}\alpha_{ij}$$

- Now, $\mathcal{U}^*(\sigma_{pq}, T) = \oint d\mathcal{U}^* = 0$ implies, and is implied by, the conditions

$$\frac{\partial^2 \mathcal{U}^*}{\partial \sigma_{ij}\, \partial \sigma_{kl}} = \frac{\partial^2 \mathcal{U}^*}{\partial \sigma_{kl}\, \partial \sigma_{ij}} \quad \text{and} \quad \frac{\partial^2 \mathcal{U}^*}{\partial \sigma_{ij}\, \partial T} = \frac{\partial^2 \mathcal{U}^*}{\partial T\, \partial \sigma_{ij}}$$

- $\dfrac{\partial^2 \mathcal{U}^*}{\partial \sigma_{ij}\, \partial \sigma_{kl}} = \dfrac{\partial^2 \mathcal{U}^*}{\partial \sigma_{kl}\, \partial \sigma_{ij}}$ and $\dfrac{\partial \mathcal{U}^*}{\partial \sigma_{ij}} = \varepsilon_{ij}$ give $\dfrac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \dfrac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}}$

PROOF THAT $S_{ijkl} = S_{klij}$ FOR THERMOELASTIC SOLIDS

- First, $\dfrac{\partial \mathcal{U}^*}{\partial \sigma_{ij}} = \varepsilon_{ij}^{\sigma} + \alpha_{ij}\left(T - T_{ref}\right)$ and $\dfrac{\partial \mathcal{U}^*}{\partial T} = \sigma_{ij}\alpha_{ij}$ satisfy $\dfrac{\partial^2 \mathcal{U}^*}{\partial \sigma_{ij}\, \partial T} = \dfrac{\partial^2 \mathcal{U}^*}{\partial T\, \partial \sigma_{ij}}$ identically

- $\varepsilon_{ij} = S_{ijrs}\sigma_{rs} + \alpha_{ij}\left(T - T_{ref}\right)$ gives

$$\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \frac{\partial}{\partial \sigma_{kl}}\left[S_{ijrs}\sigma_{rs} + \alpha_{ij}\left(T - T_{ref}\right)\right] = S_{ijrs}\frac{\partial \sigma_{rs}}{\partial \sigma_{kl}} = S_{ijrs}\delta_{rk}\delta_{sl} = S_{ijkl}$$

- $\varepsilon_{ij} = S_{ijpq}\sigma_{pq} + \alpha_{ij}\left(T - T_{ref}\right)$ gives

$$\frac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}}\left[S_{klpq}\sigma_{pq} + \alpha_{ij}\left(T - T_{ref}\right)\right] = S_{klpq}\frac{\partial \sigma_{pq}}{\partial \sigma_{ij}} = S_{klpq}\delta_{pi}\delta_{qj} = S_{klij}$$

- Thus, $\dfrac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} = \dfrac{\partial \varepsilon_{kl}}{\partial \sigma_{ij}}$ yields $S_{ijkl} = S_{klij}$

CLAPEYRON'S FORMULA FOR THERMOELASTIC SOLIDS

- **Clapeyron's formula** $\mathcal{U} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}^{\sigma}$ remains the same because the strain-energy density is based on stress-induced, mechanical work

- Clapeyron's formula is expressed in terms of the total strains by using $\varepsilon_{ij}^{\sigma} = \varepsilon_{ij} - \alpha_{ij}\left(T - T_{ref}\right)$ to get

$$\mathcal{U} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} - \frac{1}{2}\sigma_{ij}\alpha_{ij}\left(T - T_{ref}\right)$$

STRAIN-ENERGY DENSITY EXPRESSIONS

- By definition, $\mathcal{U} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}^{\sigma}$ for a homogeneous, anisotropic, linear-thermoelastic solid

- Using $\varepsilon_{ij}^{\sigma} = S_{ijrs}\sigma_{rs}$ gives $\mathcal{U} = \frac{1}{2}S_{ijrs}\sigma_{ij}\sigma_{rs}$

    - For **isotropic materials**, $\mathcal{U} = \frac{1}{2E}\left[(1+\nu)\sigma_{ij}\sigma_{ij} - \nu(\sigma_{kk})^2\right]$

    - In expanded form,

$$\mathcal{U} = \frac{1}{2E}\left[(\sigma_{11})^2 + (\sigma_{22})^2 + (\sigma_{33})^2\right] - \frac{\nu}{E}(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33}) + \frac{1+\nu}{E}\left[(\sigma_{12})^2 + (\sigma_{23})^2 + (\sigma_{13})^2\right]$$

STRAIN-ENERGY DENSITY EXPRESSIONS
CONCLUDED

- Using $\sigma_{ij} = C_{ijkl}\varepsilon^{\sigma}_{kl}$ gives $\mathcal{U} = \frac{1}{2}C_{ijkl}\varepsilon^{\sigma}_{ij}\varepsilon^{\sigma}_{kl}$

  - Further, using $\varepsilon^{\sigma}_{ij} = \varepsilon_{ij} - \alpha_{ij}\Theta$, $\Theta = T - T_{ref}$ and $C_{klij} = C_{ijkl}$ gives

$$\mathcal{U} = \frac{1}{2}C_{ijkl}\left(\varepsilon_{ij}\varepsilon_{kl} - 2\varepsilon_{ij}\alpha_{kl}\Theta + \alpha_{ij}\alpha_{kl}\Theta^2\right)$$

  - For **isotropic materials**,

$$\mathcal{U} = \frac{E}{2(1+\nu)}\varepsilon_{ij}\varepsilon_{ij} + \frac{E}{2(1-2\nu)}\varepsilon_{kk}\left[\frac{\nu}{(1+\nu)}\varepsilon_{kk} - \alpha\Theta\right] + \frac{3E}{2(1-2\nu)}\alpha^2\Theta^2$$

or

$$\begin{aligned}\mathcal{U} = {} & \frac{\nu E}{2(1+\nu)(1-2\nu)}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)^2 \\ & + \frac{E}{2(1+\nu)}\left[(\varepsilon_{11})^2 + (\varepsilon_{22})^2 + (\varepsilon_{33})^2 + 2(\varepsilon_{12})^2 + 2(\varepsilon_{23})^2 + 2(\varepsilon_{13})^2\right] \\ & - \frac{\alpha E}{(1-2\nu)}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)\Theta + \frac{3\alpha^2 E}{2(1-2\nu)}\Theta^2\end{aligned}$$

ABRIDGED NOTATION
AND
ELASTIC CONSTANTS

ABRIDGED NOTATION FOR CONSTITUTIVE EQUATIONS

- The ***abridged notation*** presented subsequently is attributed to ***Woldemar Voigt*** **(1850-1919)**, and was developed for expressing the constitutive equations in the simpler, more intuitive matrix notation

- The components of the stress, strain, thermal expansion, and thermal moduli tensors are written as *column vectors*

    - The order of the elements is obtained from **cyclic permutations** of the numbers 1, 2, and 3

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \rightarrow \begin{Bmatrix} \sigma_{1} \\ \sigma_{2} \\ \sigma_{3} \\ \sigma_{4} \\ \sigma_{5} \\ \sigma_{6} \end{Bmatrix} \qquad \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} \rightarrow \begin{Bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ \varepsilon_{4} \\ \varepsilon_{5} \\ \varepsilon_{6} \end{Bmatrix} \qquad \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix} \rightarrow \begin{Bmatrix} \beta_{1} \\ \beta_{2} \\ \beta_{3} \\ \beta_{4} \\ \beta_{5} \\ \beta_{6} \end{Bmatrix} \qquad \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} \rightarrow \begin{Bmatrix} \alpha_{1} \\ \alpha_{2} \\ \alpha_{3} \\ \alpha_{4} \\ \alpha_{5} \\ \alpha_{6} \end{Bmatrix}$$

- The term "tensor," as it is used today, was introduced by Voigt in 1899

ABRIDGED NOTATION FOR CONSTITUTIVE EQUATIONS
CONTINUED

- The components of the **compliance and stiffness tensors** are expressed as

$$\begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 2S_{1123} & 2S_{1113} & 2S_{1112} \\ S_{2211} & S_{2222} & S_{2233} & 2S_{2223} & 2S_{2213} & 2S_{2212} \\ S_{3311} & S_{3322} & S_{3333} & 2S_{3323} & 2S_{3313} & 2S_{3312} \\ 2S_{2311} & 2S_{2322} & 2S_{2333} & 4S_{2323} & 4S_{2313} & 4S_{2312} \\ 2S_{1311} & 2S_{1322} & 2S_{1333} & 4S_{1323} & 4S_{1313} & 4S_{1312} \\ 2S_{1211} & 2S_{1222} & 2S_{1233} & 4S_{1223} & 4S_{1213} & 4S_{1212} \end{bmatrix} \rightarrow \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix}$$

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3313} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2313} & C_{2312} \\ C_{1311} & C_{1322} & C_{1333} & C_{1323} & C_{1313} & C_{1312} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \rightarrow \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}$$

- Materials that can be characterized by the matrices given above when they are symmetric are said to possess ***complete Voigt symmetry***

ABRIDGED NOTATION FOR CONSTITUTIVE EQUATIONS
CONTINUED

- The **constitutive equations** are often expressed in a **nontensorial, indicial form** given by

$$\varepsilon_i = S_{ij}\sigma_j + \alpha_i\left(T - T_{ref}\right) \qquad \sigma_i = C_{ij}\varepsilon_j + \beta_i\left(T - T_{ref}\right)$$

- Similarly, the constitutive equations are often expressed in **matrix form** given by

$$\{\varepsilon\} = [S]\{\sigma\} + \{\alpha\}\left(T - T_{ref}\right)$$

and

$$\{\sigma\} = [C]\left\{\{\varepsilon\} - \{\alpha\}\left(T - T_{ref}\right)\right\}$$

or

$$\{\sigma\} = [C]\{\varepsilon\} + \{\beta\}\left(T - T_{ref}\right)$$

ABRIDGED NOTATION FOR CONSTITUTIVE EQUATIONS
CONTINUED

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} + \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} (T - T_{ref})$$

$$\{\varepsilon\} = [S]\{\sigma\} + \{\alpha\}(T - T_{ref})$$

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} + \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{Bmatrix} (T - T_{ref})$$

$$\{\sigma\} = [C]\{\varepsilon\} + \{\beta\}(T - T_{ref})$$

ABRIDGED NOTATION FOR CONSTITUTIVE EQUATIONS
CONTINUED

- Often, the following **mixed-abridged notation** is used

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix} (T - T_{ref})$$

ABRIDGED NOTATION FOR CONSTITUTIVE EQUATIONS
CONCLUDED

- The **thermal moduli** are given by

$$\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 2\alpha_4 \\ 2\alpha_5 \\ 2\alpha_6 \end{Bmatrix}$$

or

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix}$$

CLAPEYRON'S FORMULA IN ABRIDGED NOTATION

- **Clapeyron's formula** for the strain-energy density of a **linear-thermoelastic solid** was given previously by $\mathcal{U} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}^{\sigma}$ or

  $\mathcal{U} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} - \frac{1}{2}\sigma_{ij}\alpha_{ij}\left(T - T_{ref}\right)$ where $\varepsilon_{ij}^{\sigma} = \varepsilon_{ij} - \alpha_{ij}\left(T - T_{ref}\right)$

- A convenient matrix form of **Clapeyron's formula** is obtained by using the following notation

$$\left\{\sigma\right\}^{T} = \left\{\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{23}\ \sigma_{13}\ \sigma_{12}\right\} \qquad \left\{\varepsilon^{\sigma}\right\}^{T} = \left\{\varepsilon_{11}^{\sigma}\ \ \varepsilon_{22}^{\sigma}\ \ \varepsilon_{33}^{\sigma}\ \ 2\varepsilon_{23}^{\sigma}\ 2\varepsilon_{13}^{\sigma}\ 2\varepsilon_{12}^{\sigma}\right\}$$

$$\left\{\varepsilon\right\}^{T} = \left\{\varepsilon_{11}\ \ \varepsilon_{22}\ \ \varepsilon_{33}\ \ 2\varepsilon_{23}\ 2\varepsilon_{13}\ 2\varepsilon_{12}\right\} \qquad \left\{\alpha\right\}^{T} = \left\{\alpha_{11}\ \ \alpha_{22}\ \ \alpha_{33}\ \ 2\alpha_{23}\ 2\alpha_{13}\ 2\alpha_{12}\right\}$$

  where the superscript T denotes matrix or vector transposition

  - First, by inspection, it follows that $\left\{\varepsilon^{\sigma}\right\} = \left\{\varepsilon\right\} - \left\{\alpha\right\}\Theta$ where $\Theta = T - T_{ref}$

CLAPEYRON'S FORMULA IN ABRIDGED NOTATION
CONTINUED

- Next, noting that $\sigma_{ij}\varepsilon^{\sigma}_{ij} = \sigma_{11}\varepsilon^{\sigma}_{11} + \sigma_{22}\varepsilon^{\sigma}_{22} + \sigma_{33}\varepsilon^{\sigma}_{33} + \sigma_{23}2\varepsilon^{\sigma}_{23} + \sigma_{13}2\varepsilon^{\sigma}_{13} + \sigma_{12}2\varepsilon^{\sigma}_{12}$ it follows that $\sigma_{ij}\varepsilon^{\sigma}_{ij} = \{\sigma\}^T\{\varepsilon^{\sigma}\} = \{\varepsilon^{\sigma}\}^T\{\sigma\}$ in matrix notation

- Therefore, **Clapeyron's formula** for the strain-energy density of a **linear-thermoelastic solid** is given in matrix from by

$$\mathcal{U} = \frac{1}{2}\{\sigma\}^T\{\varepsilon^{\sigma}\} \quad \text{or} \quad \mathcal{U} = \frac{1}{2}\{\sigma\}^T\{\varepsilon\} - \frac{1}{2}\{\sigma\}^T\{\alpha\}\Theta$$

- An alternate form of **Clapeyron's formula** is obtained as follows

- First, define $\{\sigma^{\Theta}\} = [C]\{\alpha\}\Theta$ such that $\{\sigma\} = [C]\{\varepsilon^{\sigma}\}$ becomes $\{\sigma\} = [C]\{\varepsilon\} - \{\sigma^{\Theta}\}$

CLAPEYRON'S FORMULA IN ABRIDGED NOTATION
CONTINUED

- Then, $\{\sigma^\Theta\} = [C]\{\alpha\}\Theta$ gives $\{\alpha\}\Theta = [C]^{-1}\{\sigma^\Theta\}$

- Next, $\{\varepsilon^\sigma\} = \{\varepsilon\} - \{\alpha\}\Theta$ becomes $\{\varepsilon^\sigma\} = \{\varepsilon\} - [C]^{-1}\{\sigma^\Theta\}$

- Also, $\{\sigma\}^T\{\varepsilon^\sigma\} = \{\sigma\}^T\{\varepsilon\} - \{\sigma\}^T[C]^{-1}\{\sigma^\Theta\}$

- Transposition of $\{\sigma\} = [C]\{\varepsilon\} - \{\sigma^\Theta\}$ gives $\{\sigma\}^T = \{\varepsilon\}^T[C] - \{\sigma^\Theta\}^T$, where symmetry of [C] has been used

- Using the last expression with $\{\sigma\}^T\{\varepsilon^\sigma\} = \{\sigma\}^T\{\varepsilon\} - \{\sigma\}^T[C]^{-1}\{\sigma^\Theta\}$ gives

  $\{\sigma\}^T\{\varepsilon^\sigma\} = \{\sigma\}^T\{\varepsilon\} - \{\varepsilon\}^T[C][C]^{-1}\{\sigma^\Theta\} + \{\sigma^\Theta\}^T[C]^{-1}\{\sigma^\Theta\}$ which simplifies

  to $\{\sigma\}^T\{\varepsilon^\sigma\} = \left(\{\sigma\}^T - \{\sigma^\Theta\}\right)\{\varepsilon\} + \{\sigma^\Theta\}^T[C]^{-1}\{\sigma^\Theta\}$

CLAPEYRON'S FORMULA IN ABRIDGED NOTATION
CONCLUDED

- Next, using $\{\sigma^{\Theta}\} = [C]\{\alpha\}\Theta$ gives $\{\sigma^{\Theta}\}^{T}[C]^{-1}\{\sigma^{\Theta}\} = \{\alpha\}^{T}[C]\{\alpha\}\Theta^{2}$

- Using the last equation with $\{\sigma\}^{T}\{\varepsilon^{\sigma}\} = \left(\{\sigma\}^{T} - \{\sigma^{\Theta}\}\right)\{\varepsilon\} + \{\sigma^{\Theta}\}^{T}[C]^{-1}\{\sigma^{\Theta}\}$

  gives $\{\sigma\}^{T}\{\varepsilon^{\sigma}\} = \left(\{\sigma\}^{T} - \{\sigma^{\Theta}\}\right)\{\varepsilon\} + \{\alpha\}^{T}[C]\{\alpha\}\Theta^{2}$

- Therefore, **Clapeyron's formula** $\mathcal{U} = \frac{1}{2}\{\sigma\}^{T}\{\varepsilon^{\sigma}\}$ becomes

$$\mathcal{U} = \frac{1}{2}\left(\{\sigma\}^{T} - \{\sigma^{\Theta}\}^{T}\right)\{\varepsilon\} + \frac{1}{2}\{\alpha\}^{T}[C]\{\alpha\}\Theta^{2}$$

PHYSICAL MEANING OF THE ELASTIC CONSTANTS

- The **shaded terms** shown below correspond to independent interaction between pure extensional stresses and strains

$$\begin{bmatrix} \boxed{S_{11}} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & \boxed{S_{22}} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & \boxed{S_{33}} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \qquad \begin{bmatrix} \boxed{C_{11}} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & \boxed{C_{22}} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & \boxed{C_{33}} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

- The **shaded terms** shown below correspond to independent interaction between pure shearing stresses and strains

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & \boxed{S_{44}} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & \boxed{S_{55}} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & \boxed{S_{66}} \end{bmatrix} \qquad \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & \boxed{C_{44}} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & \boxed{C_{55}} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & \boxed{C_{66}} \end{bmatrix}$$

PHYSICAL MEANING OF THE ELASTIC CONSTANTS
CONTINUED

- **The shaded terms shown below correspond to coupling of interactions between pure extensional stresses and strains**

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \qquad \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

- **The shaded terms shown below correspond to coupling of interactions between pure shearing stresses and strains**

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \qquad \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

PHYSICAL MEANING OF THE ELASTIC CONSTANTS
CONCLUDED

- The shaded terms shown below correspond to coupling or interactions between extensional and shearing behavior

$$
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\
S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\
S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\
S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66}
\end{bmatrix}
\qquad
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\
C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\
C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}
\end{bmatrix}
$$

TRANSFORMATION EQUATIONS

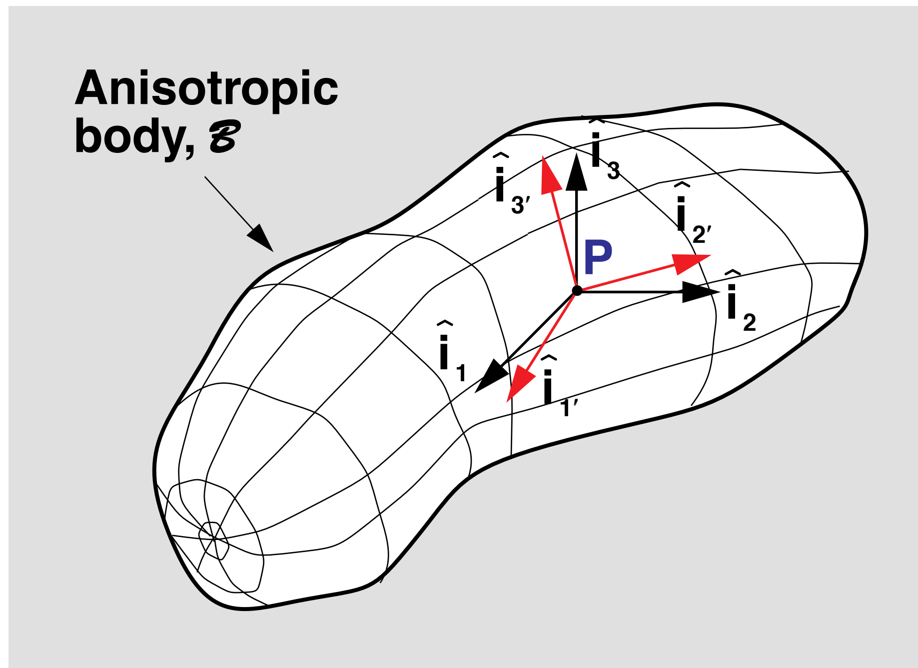
TRANSFORMATION OF [C] AND [S]

- Consider a general orthogonal transformation between the **Cartesian coordinates** $\left(x_1, x_2, x_3\right)$ and $\left(x_{1'}, x_{2'}, x_{3'}\right)$ at a fixed point **P** of a body

- The **orthonormal bases** for the two coordinate systems, for an arbitrary point **P** of a body, are indicated on the figure below

- Because only a fixed point **P** is considered, coordinate translations are excluded

- Although two **right-handed** coordinate systems are shown in the figure, there is no such restriction on the following development of the transformation equations



TRANSFORMATION OF [C] AND [S] - CONTINUED

- **The general relationship between the two orthonormal bases is given by the following matrix representations**

  - $$\begin{Bmatrix} \hat{\mathbf{i}}_{1'} \\ \hat{\mathbf{i}}_{2'} \\ \hat{\mathbf{i}}_{3'} \end{Bmatrix} = \begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_{1} \\ \hat{\mathbf{i}}_{2} \\ \hat{\mathbf{i}}_{3} \end{Bmatrix}$$ **or, in abridged form,** $\{\hat{\mathbf{i}}'\} = [a]\{\hat{\mathbf{i}}\}$

  - $$\begin{Bmatrix} \hat{\mathbf{i}}_{1} \\ \hat{\mathbf{i}}_{2} \\ \hat{\mathbf{i}}_{3} \end{Bmatrix} = \begin{bmatrix} a_{1'1} & a_{2'1} & a_{3'1} \\ a_{1'2} & a_{2'2} & a_{3'2} \\ a_{1'3} & a_{2'3} & a_{3'3} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{i}}_{1'} \\ \hat{\mathbf{i}}_{2'} \\ \hat{\mathbf{i}}_{3'} \end{Bmatrix}$$ **or, in abridged form,** $\{\hat{\mathbf{i}}\} = [a]^{-1}\{\hat{\mathbf{i}}'\}$

- **Examining these two matrices indicates that** $[a]^{-1} = [a]^{T}$

  - **Transformations of this type are known as orthogonal transformations and preserve the lengths of, and the angles between, vectors**

  - $a_{p'q} \equiv (\hat{\mathbf{i}}_{p'} \bullet \hat{\mathbf{i}}_{q}) = (\hat{\mathbf{i}}_{q} \bullet \hat{\mathbf{i}}_{p'}) = a_{qp'}$ **and** $(a_{p'q} = a_{qp'}) \neq (a_{pq'} = a_{q'p})$

TRANSFORMATION OF [C] AND [S] - CONTINUED

- The matrix equation $\left\{\hat{\mathbf{i}}'\right\} = [\mathbf{a}]\left\{\hat{\mathbf{i}}\right\}$ is expressed in indicial notation by $\hat{\mathbf{i}}_{k'} = \mathbf{a}_{k'q}\hat{\mathbf{i}}_q$

- Likewise, $\left\{\hat{\mathbf{i}}\right\} = [\mathbf{a}]^{-1}\left\{\hat{\mathbf{i}}'\right\}$ is expressed in indicial notation by $\hat{\mathbf{i}}_p = \mathbf{a}_{r'p}\hat{\mathbf{i}}_{r'}$

- Important relationships between the direction cosines $\mathbf{a}_{k'q}$ are obtained by enforcing the two **orthonormality conditions** $\hat{\mathbf{i}}_{k'} \bullet \hat{\mathbf{i}}_{p'} = \delta_{kp}$ and $\hat{\mathbf{i}}_m \bullet \hat{\mathbf{i}}_n = \delta_{mn}$

  - These conditions yield the relationships

$$\mathbf{a}_{k'q}\mathbf{a}_{p'q} = \delta_{kp} \quad \text{and} \quad \mathbf{a}_{q'k}\mathbf{a}_{q'p} = \delta_{kp}$$

  - Each indicial equation possesses six independent relations

TRANSFORMATION OF [C] AND [S] - CONTINUED

- **The total of** **twelve independent relations** **are given in tabular form below:**

k	p	$a_{k'q}a_{p'q} = \delta_{kp}$
1	1	$(a_{1'1})^2 + (a_{1'2})^2 + (a_{1'3})^2 = 1$
2	1	$a_{2'1}a_{1'1} + a_{2'2}a_{1'2} + a_{2'3}a_{1'3} = 0$
3	1	$a_{3'1}a_{1'1} + a_{3'2}a_{1'2} + a_{3'3}a_{1'3} = 0$
2	2	$(a_{2'1})^2 + (a_{2'2})^2 + (a_{2'3})^2 = 1$
3	2	$a_{3'1}a_{2'1} + a_{3'2}a_{2'2} + a_{3'3}a_{2'3} = 0$
3	3	$(a_{3'1})^2 + (a_{3'2})^2 + (a_{3'3})^2 = 1$

k	p	$a_{q'k}a_{q'p} = \delta_{kp}$
1	1	$(a_{1'1})^2 + (a_{2'1})^2 + (a_{3'1})^2 = 1$
2	1	$a_{1'2}a_{1'1} + a_{2'2}a_{2'1} + a_{3'2}a_{3'1} = 0$
3	1	$a_{1'3}a_{1'1} + a_{2'3}a_{2'1} + a_{3'3}a_{3'1} = 0$
2	2	$(a_{1'2})^2 + (a_{2'2})^2 + (a_{3'2})^2 = 1$
3	2	$a_{1'3}a_{1'2} + a_{2'3}a_{2'2} + a_{3'3}a_{3'2} = 0$
3	3	$(a_{1'3})^2 + (a_{2'3})^2 + (a_{3'3})^2 = 1$

TRANSFORMATION OF [C] AND [S] - CONTINUED

- **By using the abridged notation, matrix forms of the stress and strain transformation equations can be obtained that are given by**

$$\{\sigma'\} = [T_\sigma]\{\sigma\} \quad \text{and} \quad \{\varepsilon'\} = [T_\varepsilon]\{\varepsilon\}$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}$$

$$\{\sigma'\} = \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{Bmatrix} \sigma_{1'} \\ \sigma_{2'} \\ \sigma_{3'} \\ \sigma_{4'} \\ \sigma_{5'} \\ \sigma_{6'} \end{Bmatrix} \qquad \{\varepsilon'\} = \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{1'} \\ \varepsilon_{2'} \\ \varepsilon_{3'} \\ \varepsilon_{4'} \\ \varepsilon_{5'} \\ \varepsilon_{6'} \end{Bmatrix} \quad \text{and}$$

TRANSFORMATION OF [C] AND [S] - CONTINUED

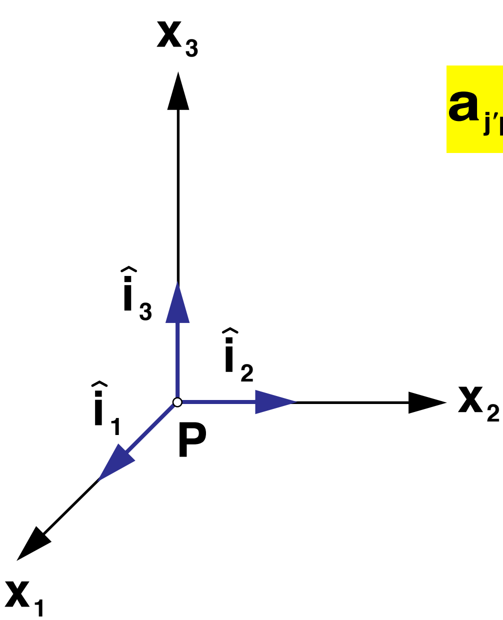
$$
\left[T_\sigma\right] = \left[\begin{array}{ccc|c|c|c}
(a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & 2a_{1'2}a_{1'3} & 2a_{1'1}a_{1'3} & 2a_{1'1}a_{1'2} \\
(a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & 2a_{2'2}a_{2'3} & 2a_{2'1}a_{2'3} & 2a_{2'1}a_{2'2} \\
(a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & 2a_{3'2}a_{3'3} & 2a_{3'1}a_{3'3} & 2a_{3'1}a_{3'2} \\
a_{2'1}a_{3'1} & a_{2'2}a_{3'2} & a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\
a_{1'1}a_{3'1} & a_{1'2}a_{3'2} & a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\
a_{1'1}a_{2'1} & a_{1'2}a_{2'2} & a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1})
\end{array}\right]
$$

$$
\left[T_\varepsilon\right] = \left[\begin{array}{ccc|c|c|c}
(a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & a_{1'2}a_{1'3} & a_{1'1}a_{1'3} & a_{1'1}a_{1'2} \\
(a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & a_{2'2}a_{2'3} & a_{2'1}a_{2'3} & a_{2'1}a_{2'2} \\
(a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & a_{3'2}a_{3'3} & a_{3'1}a_{3'3} & a_{3'1}a_{3'2} \\
2a_{2'1}a_{3'1} & 2a_{2'2}a_{3'2} & 2a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\
2a_{1'1}a_{3'1} & 2a_{1'2}a_{3'2} & 2a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\
2a_{1'1}a_{2'1} & 2a_{1'2}a_{2'2} & 2a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1})
\end{array}\right]
$$

$$
a_{j'p} \equiv \left(\hat{i}_{j'} \bullet \hat{i}_p\right)
$$

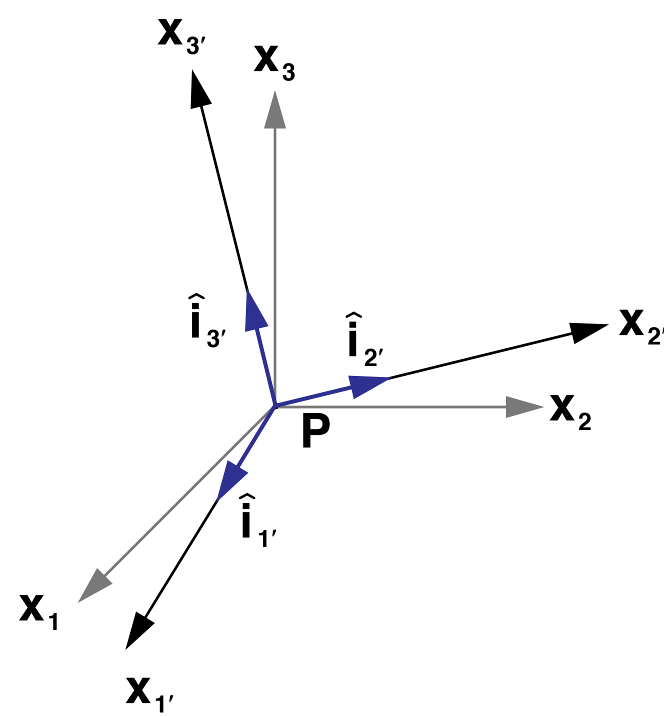
- Inspection of the matrices shown above indicates that when the **off-diagonal terms vanish**, which happens for certain transformations, the two matrices are identical

TRANSFORMATION OF [C] AND [S] - CONTINUED



Initial coordinate frame

$$a_{j'p} \equiv \left(\hat{i}_{j'} \bullet \hat{i}_p\right)$$



Initial and new coordinate frame

TRANSFORMATION OF [C] AND [S] - CONTINUED

- Recall that the **thermoelastic constitutive equations** are expressed in symbolic form by

$$\{\varepsilon\} = [S]\{\sigma\} + \{\alpha\}\Theta \quad \text{and} \quad \{\sigma\} = [C]\{\varepsilon\} + \{\beta\}\Theta$$

where $\Theta = T - T_{ref}$

- In the $\{x_{1'}, x_{2'}, x_{3'}\}$ coordinate frame the thermoelastic constitutive equations are expressed in symbolic form by

$$\{\varepsilon'\} = [S']\{\sigma'\} + \{\alpha'\}\Theta \quad \text{and} \quad \{\sigma'\} = [C']\{\varepsilon'\} + \{\beta'\}\Theta$$

- By using the matrix form of the stress and strain transformation equations,

$$\{\sigma\} = [C]\{\varepsilon\} + \{\beta\}\Theta \quad \text{becomes} \quad [T_\sigma]^{-1}\{\sigma'\} = [C][T_\varepsilon]^{-1}\{\varepsilon'\} + \{\beta\}\Theta$$

TRANSFORMATION OF [C] AND [S] - CONTINUED

- Premultiplying by $\left[T_\sigma\right]$ gives

$$\{\sigma'\} = [T_\sigma][C][T_\varepsilon]^{-1}\{\varepsilon'\} + [T_\sigma]\{\beta\}\Theta$$

- Comparing this equation with $\{\sigma'\} = \left[C'\right]\{\varepsilon'\} + \{\beta'\}\Theta$ it follows that

$$[C'] = [T_\sigma][C][T_\varepsilon]^{-1} \quad \text{and} \quad \{\beta'\} = [T_\sigma]\{\beta\}$$

- Thus, $[C] = [T_\sigma]^{-1}[C'][T_\varepsilon]$

TRANSFORMATION OF [C] AND [S] - CONTINUED

- Next, by using the matrix form of the stress and strain transformation equations, $\{\varepsilon\} = [S]\{\sigma\} + \{\alpha\}\Theta$ becomes

$$[T_\varepsilon]^{-1}\{\varepsilon'\} = [S][T_\sigma]^{-1}\{\sigma'\} + \{\alpha\}\Theta$$

- Premultiplying by $[T_\varepsilon]$ gives

$$\{\varepsilon'\} = [T_\varepsilon][S][T_\sigma]^{-1}\{\sigma'\} + [T_\varepsilon]\{\alpha\}\Theta$$

- Comparing this equation with $\{\varepsilon'\} = [S']\{\sigma'\} + \{\alpha'\}\Theta$ it follows that

$$[S'] = [T_\varepsilon][S][T_\sigma]^{-1} \quad \text{and} \quad \{\alpha'\} = [T_\varepsilon]\{\alpha\}$$

- Thus, $[S'] = [T_\varepsilon][S][T_\sigma]^{-1}$ gives $[S] = [T_\varepsilon]^{-1}[S'][T_\sigma]$

TRANSFORMATION OF [C] AND [S]
CONCLUDED

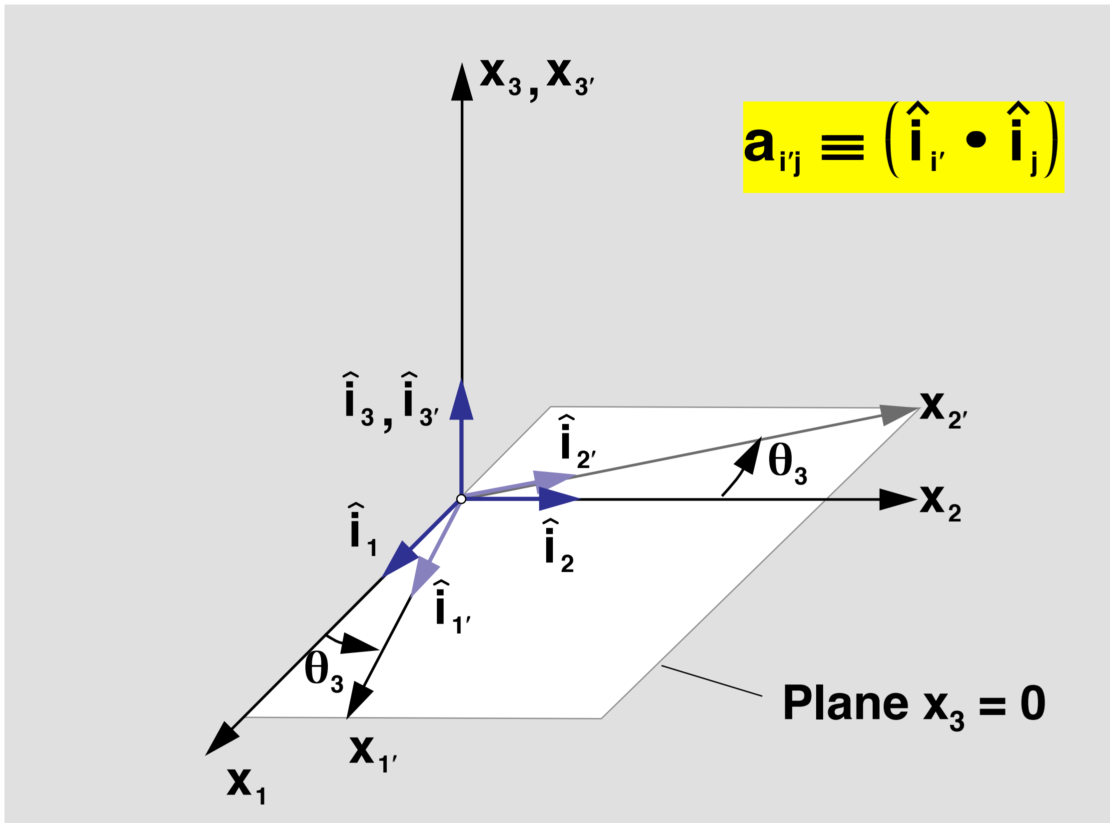
- **In summary:**

$$[S'] = [T_\varepsilon][S][T_\sigma]^{-1} \qquad [S] = [T_\varepsilon]^{-1}[S'][T_\sigma]$$

$$[C'] = [T_\sigma][C][T_\varepsilon]^{-1} \qquad [C] = [T_\sigma]^{-1}[C'][T_\varepsilon]$$

$$\{\alpha'\} = [T_\varepsilon]\{\alpha\} \qquad \{\beta'\} = [T_\sigma]\{\beta\}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS



- The term **"dextral"** refers to a **right-handed rotation**

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- **The matrix form of the stress-tensor transformation law is given by**

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -2\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

and is expressed symbolically by

$$\{\sigma'\} = [T_\sigma(\theta_3)]\{\sigma\}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- Similarly, the matrix form of the inverse transformation law is given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & -2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & 2\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & \sin\theta_3 & 0 \\ 0 & 0 & 0 & -\sin\theta_3 & \cos\theta_3 & 0 \\ \sin\theta_3\cos\theta_3 & -\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix}$$

and is expressed symbolically by

$$\{\sigma\} = \left[T_\sigma(\theta_3)\right]^{-1}\{\sigma'\}$$

where

$$\left[T_\sigma(\theta_3)\right]^{-1} = \left[T_\sigma(-\theta_3)\right]$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- **By using the second-order, symmetric tensor transformation equations, the transformation law for the vector of engineering strains is given by**

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

which is expressed symbolically by

$$\{\varepsilon'\} = [T_\varepsilon(\theta_3)]\{\varepsilon\}$$

- **Note that** $$[T_\varepsilon(\theta_3)]^T = [T_\sigma(\theta_3)]^{-1} = [T_\sigma(-\theta_3)]$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- **Similarly, the matrix form of the inverse transformation law is given by**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & -\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & \sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & \sin\theta_3 & 0 \\ 0 & 0 & 0 & -\sin\theta_3 & \cos\theta_3 & 0 \\ 2\sin\theta_3\cos\theta_3 & -2\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix}$$

and is expressed symbolically by

$$\{\varepsilon\} = \left[\mathbf{T}_\varepsilon(\theta_3)\right]^{-1}\{\varepsilon'\}$$

where

$$\left[\mathbf{T}_\varepsilon(\theta_3)\right]^{-1} = \left[\mathbf{T}_\varepsilon(-\theta_3)\right] = \left[\mathbf{T}_\sigma(\theta_3)\right]^{T}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- The general expression for the transformation of the stiffness coefficients and thermal moduli are

$$[C'] = [T_\sigma][C][T_\varepsilon]^{-1} \quad \text{and} \quad \{\beta'\} = [T_\sigma]\{\beta\}$$

- Noting that $[T_\varepsilon]^{-1} = [T_\sigma]^T$ for a dextral rotation about the x_3 axis gives

$$[C'] = [T_\sigma][C][T_\sigma]^T$$

- Similarly, the general expression for the inverse transformation of the stiffness coefficients is $[C] = [T_\sigma]^{-1}[C'][T_\varepsilon]$

- Noting that $[T_\sigma]^{-1} = [T_\varepsilon]^T$ for a dextral rotation about the x_3 axis gives

$$[C] = [T_\varepsilon]^T[C'][T_\varepsilon]$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- **The general expression for the transformation of the compliance coefficients and thermal expansion coefficients are**

$$[S'] = [T_\varepsilon][S][T_\sigma]^{-1} \quad \text{and} \quad \{\alpha'\} = [T_\varepsilon]\{\alpha\}$$

- **Noting that** $[T_\sigma]^{-1} = [T_\varepsilon]^T$ **for a dextral rotation about the x_3 axis gives**

$$[S'] = [T_\varepsilon][S][T_\varepsilon]^T$$

- **Similarly, the general expression for the inverse transformation of the compliance coefficients is** $[S] = [T_\varepsilon]^{-1}[S'][T_\sigma]$

- **Noting that** $[T_\varepsilon]^{-1} = [T_\sigma]^T$ **for a dextral rotation about the x_3 axis gives**

$$[S] = [T_\sigma]^T[S'][T_\sigma]$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - SUMMARY

$$[S'] = [T_\varepsilon][S][T_\varepsilon]^T \qquad [S] = [T_\sigma]^T[S'][T_\sigma]$$

$$[C'] = [T_\sigma][C][T_\sigma]^T \qquad [C] = [T_\varepsilon]^T[C'][T_\varepsilon]$$

$$[T_\sigma] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -2\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

$$[T_\varepsilon] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - SUMMARY

$$\{\alpha'\} = [T_\varepsilon]\{\alpha\} \qquad \{\alpha\} = [T_\sigma]^T\{\alpha'\}$$

$$\{\beta'\} = [T_\sigma]\{\beta\} \qquad \{\beta\} = [T_\varepsilon]^T\{\beta'\}$$

$$[T_\sigma] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -2\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

$$[T_\varepsilon] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- **Let $m = \cos\theta_3$ and $n = \sin\theta_3$ for convenience**

- **Performing the calculations given by $[C'] = [T_\sigma][C][T_\sigma]^T$ yields**

$$C_{1'1'} = m^4 C_{11} + 2m^2 n^2 (C_{12} + 2C_{66}) + 4mn(m^2 C_{16} + n^2 C_{26}) + n^4 C_{22}$$

$$C_{1'2'} = m^2 n^2 (C_{11} + C_{22} - 4C_{66}) - 2mn(m^2 - n^2)(C_{16} - C_{26}) + (m^4 + n^4) C_{12}$$

$$C_{1'3'} = m^2 C_{13} + n^2 C_{23} + 2mnC_{36}$$

$$C_{1'4'} = m^3 C_{14} + m^2 n (2C_{46} - C_{15}) - mn^2 (2C_{56} - C_{24}) - n^3 C_{25}$$

$$C_{1'5'} = m^3 C_{15} + m^2 n (2C_{56} + C_{14}) + mn^2 (2C_{46} + C_{25}) + n^3 C_{24}$$

$$C_{1'6'} = m^2 (m^2 - 3n^2) C_{16} - m^3 n (C_{11} - C_{12} - 2C_{66}) + mn^3 (C_{22} - C_{12} - 2C_{66}) - n^2 (n^2 - 3m^2) C_{26}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

$$C_{2'2'} = m^4C_{22} + 2m^2n^2\left(C_{12} + 2C_{66}\right) - 4mn\left(m^2C_{26} + n^2C_{16}\right) + n^4C_{11}$$

$$C_{2'3'} = m^2C_{23} + n^2C_{13} - 2mnC_{36}$$

$$C_{2'4'} = m^3C_{24} - m^2n\left(2C_{46} + C_{25}\right) + mn^2\left(2C_{56} + C_{14}\right) - n^3C_{15}$$

$$C_{2'5'} = m^3C_{25} - m^2n\left(2C_{56} - C_{24}\right) - mn^2\left(2C_{46} - C_{15}\right) + n^3C_{14}$$

$$\begin{aligned} C_{2'6'} = m^2\left(m^2 - 3n^2\right)C_{26} + m^3n\left(C_{22} - C_{12} - 2C_{66}\right) \\ - mn^3\left(C_{11} - C_{12} - 2C_{66}\right) - n^2\left(n^2 - 3m^2\right)C_{16} \end{aligned}$$

$$C_{3'3'} = C_{33}$$

$$C_{3'4'} = mC_{34} - nC_{35}$$

$$C_{3'5'} = mC_{35} + nC_{34}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

$$C_{3'6'} = (m^2 - n^2)C_{36} + mn(C_{23} - C_{13})$$

$$C_{4'4'} = m^2C_{44} + n^2C_{55} - 2mnC_{45}$$

$$C_{4'5'} = (m^2 - n^2)C_{45} + mn(C_{44} - C_{55})$$

$$C_{4'6'} = m^3C_{46} - m^2n(C_{56} + C_{14} - C_{24}) - mn^2(C_{46} - C_{15} + C_{25}) + n^3C_{56}$$

$$C_{5'5'} = m^2C_{55} + n^2C_{44} + 2mnC_{45}$$

$$C_{5'6'} = m^3C_{56} + m^2n(C_{46} + C_{25} - C_{15}) - mn^2(C_{56} + C_{14} - C_{24}) - n^3C_{46}$$

$$C_{6'6'} = m^2n^2(C_{11} + C_{22} - 2C_{12}) - 2mn(m^2 - n^2)(C_{16} - C_{26}) + (m^2 - n^2)^2C_{66}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- **Again, let** $m = \cos\theta_3$ **and** $n = \sin\theta_3$

- **Performing the calculations given by** $[C] = [T_\varepsilon]^T[C'][T_\varepsilon]$ **yields**

$$C_{11} = m^4C_{1'1'} + 2m^2n^2(C_{1'2'} + 2C_{6'6'}) - 4mn(m^2C_{1'6'} + n^2C_{2'6'}) + n^4C_{2'2'}$$

$$C_{12} = m^2n^2(C_{1'1'} + C_{2'2'} - 4C_{6'6'}) + 2mn(m^2 - n^2)(C_{1'6'} - C_{2'6'}) + (m^4 + n^4)C_{1'2'}$$

$$C_{13} = m^2C_{1'3'} + n^2C_{2'3'} - 2mnC_{3'6'}$$

$$C_{14} = m^3C_{1'4'} - m^2n(2C_{4'6'} - C_{1'5'}) - mn^2(2C_{5'6'} - C_{2'4'}) + n^3C_{2'5'}$$

$$C_{15} = m^3C_{1'5'} - m^2n(2C_{5'6'} + C_{1'4'}) + mn^2(2C_{4'6'} + C_{2'5'}) - n^3C_{2'4'}$$

$$\begin{aligned} C_{16} = m^2(m^2 - 3n^2)C_{1'6'} &+ m^3n(C_{1'1'} - C_{1'2'} - 2C_{6'6'}) \\ &- mn^3(C_{2'2'} - C_{1'2'} - 2C_{6'6'}) - n^2(n^2 - 3m^2)C_{2'6'} \end{aligned}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

$$C_{22} = m^4 C_{2'2'} + 2m^2n^2\left(C_{1'2'} + 2C_{6'6'}\right) + 4mn\left(m^2 C_{2'6'} + n^2 C_{1'6'}\right) + n^4 C_{1'1'}$$

$$C_{23} = m^2 C_{2'3'} + n^2 C_{1'3'} + 2mnC_{3'6'}$$

$$C_{24} = m^3 C_{2'4'} + m^2 n\left(2C_{4'6'} + C_{2'5'}\right) + mn^2\left(2C_{5'6'} + C_{1'4'}\right) + n^3 C_{1'5'}$$

$$C_{25} = m^3 C_{2'5'} + m^2 n\left(2C_{5'6'} - C_{2'4'}\right) - mn^2\left(2C_{4'6'} - C_{1'5'}\right) - n^3 C_{1'4'}$$

$$\begin{aligned} C_{26} = m^2\left(m^2 - 3n^2\right)C_{2'6'} &- m^3 n\left(C_{2'2'} - C_{1'2'} - 2C_{6'6'}\right) \\ &+ mn^3\left(C_{1'1'} - C_{1'2'} - 2C_{6'6'}\right) - n^2\left(n^2 - 3m^2\right)C_{1'6'} \end{aligned}$$

$$C_{33} = C_{3'3'} \qquad C_{34} = mC_{3'4'} + nC_{3'5'} \qquad C_{35} = mC_{3'5'} - nC_{3'4'}$$

$$C_{36} = \left(m^2 - n^2\right)C_{3'6'} - mn\left(C_{2'3'} - C_{1'3'}\right)$$

$$C_{44} = m^2 C_{4'4'} + n^2 C_{5'5'} + 2mnC_{4'5'}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

$$C_{45} = (m^2 - n^2)C_{4'5'} - mn(C_{4'4'} - C_{5'5'})$$

$$C_{46} = m^3C_{4'6'} + m^2n(C_{5'6'} + C_{1'4'} - C_{2'4'}) - mn^2(C_{4'6'} - C_{1'5'} + C_{2'5'}) - n^3C_{5'6'}$$

$$C_{55} = m^2C_{5'5'} + n^2C_{4'4'} - 2mnC_{4'5'}$$

$$C_{56} = m^3C_{5'6'} - m^2n(C_{4'6'} + C_{2'5'} - C_{1'5'}) - mn^2(C_{5'6'} + C_{1'4'} - C_{2'4'}) + n^3C_{4'6'}$$

$$C_{66} = m^2n^2(C_{1'1'} + C_{2'2'} - 2C_{1'2'}) + 2mn(m^2 - n^2)(C_{1'6'} - C_{2'6'}) + (m^2 - n^2)^2C_{6'6'}$$

- Note that $[C']$ and $[C]$ can be expressed as

$$[C'] = [T_\sigma(\theta_3)][C][T_\varepsilon(-\theta_3)] \quad \text{and} \quad [C] = [T_\sigma(-\theta_3)][C'][T_\varepsilon(\theta_3)]$$

  - Thus, one set of transformed stiffness expressions can be obtained from the other by simply interchanging the primed and unprimed indices and replacing n with -n

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- Let $m = \cos\theta_3$ and $n = \sin\theta_3$
- Performing the calculations given by $[S'] = [T_\varepsilon][S][T_\varepsilon]^T$ yields

$$S_{1'1'} = m^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + 2mn(m^2 S_{16} + n^2 S_{26}) + n^4 S_{22}$$

$$S_{1'2'} = m^2 n^2 (S_{11} + S_{22} - S_{66}) - mn(m^2 - n^2)(S_{16} - S_{26}) + (m^4 + n^4) S_{12}$$

$$S_{1'3'} = m^2 S_{13} + n^2 S_{23} + mnS_{36}$$

$$S_{1'4'} = m^3 S_{14} + m^2 n(S_{46} - S_{15}) - mn^2 (S_{56} - S_{24}) - n^3 S_{25}$$

$$S_{1'5'} = m^3 S_{15} + m^2 n(S_{56} + S_{14}) + mn^2 (S_{46} + S_{25}) + n^3 S_{24}$$

$$S_{1'6'} = m^2 (m^2 - 3n^2) S_{16} - m^3 n(2S_{11} - 2S_{12} - S_{66}) + mn^3 (2S_{22} - 2S_{12} - S_{66}) - n^2 (n^2 - 3m^2) S_{26}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

$$S_{2'2'} = m^4 S_{22} + m^2 n^2 \left(2S_{12} + S_{66}\right) - 2mn\left(m^2 S_{26} + n^2 S_{16}\right) + n^4 S_{11}$$

$$S_{2'3'} = m^2 S_{23} + n^2 S_{13} - mnS_{36}$$

$$S_{2'4'} = m^3 S_{24} - m^2 n\left(S_{46} + S_{25}\right) + mn^2\left(S_{56} + S_{14}\right) - n^3 S_{15}$$

$$S_{2'5'} = m^3 S_{25} - m^2 n\left(S_{56} - S_{24}\right) - mn^2\left(S_{46} - S_{15}\right) + n^3 S_{14}$$

$$S_{2'6'} = m^2\left(m^2 - 3n^2\right)S_{26} + m^3 n\left(2S_{22} - 2S_{12} - S_{66}\right) - mn^3\left(2S_{11} - 2S_{12} - S_{66}\right) - n^2\left(n^2 - 3m^2\right)S_{16}$$

$$S_{3'3'} = S_{33}$$

$$S_{3'4'} = mS_{34} - nS_{35}$$

$$S_{3'5'} = mS_{35} + nS_{34}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

$$S_{3'6'} = \left(m^2 - n^2\right)S_{36} + 2mn\left(S_{23} - S_{13}\right)$$

$$S_{4'4'} = m^2S_{44} + n^2S_{55} - 2mnS_{45}$$

$$S_{4'5'} = \left(m^2 - n^2\right)S_{45} + mn\left(S_{44} - S_{55}\right)$$

$$S_{4'6'} = m^3S_{46} - m^2n\left(S_{56} + 2S_{14} - 2S_{24}\right) - mn^2\left(S_{46} - 2S_{15} + 2S_{25}\right) + n^3S_{56}$$

$$S_{5'5'} = m^2S_{55} + n^2S_{44} + 2mnS_{45}$$

$$S_{5'6'} = m^3S_{56} + m^2n\left(S_{46} + 2S_{25} - 2S_{15}\right) - mn^2\left(S_{56} + 2S_{14} - 2S_{24}\right) - n^3S_{46}$$

$$S_{6'6'} = 4m^2n^2\left(S_{11} + S_{22} - 2S_{12}\right) - 4mn\left(m^2 - n^2\right)\left(S_{16} - S_{26}\right) + \left(m^2 - n^2\right)^2S_{66}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- Let $m = \cos\theta_3$ and $n = \sin\theta_3$

- Performing the calculations given by $[S] = [T_\sigma]^T[S'][T_\sigma]$ yields

$$S_{11} = m^4 S_{1'1'} + m^2 n^2 (2S_{1'2'} + S_{6'6'}) - 2mn(m^2 S_{1'6'} + n^2 S_{2'6'}) + n^4 S_{2'2'}$$

$$S_{12} = m^2 n^2 (S_{1'1'} + S_{2'2'} - S_{6'6'}) + mn(m^2 - n^2)(S_{1'6'} - S_{2'6'}) + (m^4 + n^4) S_{1'2'}$$

$$S_{13} = m^2 S_{1'3'} + n^2 S_{2'3'} - mn S_{3'6'}$$

$$S_{14} = m^3 S_{1'4'} - m^2 n (S_{4'6'} - S_{1'5'}) - mn^2 (S_{5'6'} - S_{2'4'}) + n^3 S_{2'5'}$$

$$S_{15} = m^3 S_{1'5'} - m^2 n (S_{5'6'} + S_{1'4'}) + mn^2 (S_{4'6'} + S_{2'5'}) - n^3 S_{2'4'}$$

$$S_{16} = m^2 (m^2 - 3n^2) S_{1'6'} + m^3 n (2S_{1'1'} - 2S_{1'2'} - S_{6'6'}) - mn^3 (2S_{2'2'} - 2S_{1'2'} - S_{6'6'}) - n^2 (n^2 - 3m^2) S_{2'6'}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

$$S_{22} = m^4 S_{2'2'} + m^2 n^2 \left(2S_{1'2'} + S_{6'6'}\right) + 2mn\left(m^2 S_{2'6'} + n^2 S_{1'6'}\right) + n^4 S_{1'1'}$$

$$S_{23} = m^2 S_{2'3'} + n^2 S_{1'3'} + mnS_{3'6'}$$

$$S_{24} = m^3 S_{2'4'} + m^2 n\left(S_{4'6'} + S_{2'5'}\right) + mn^2\left(S_{5'6'} + S_{1'4'}\right) + n^3 S_{1'5'}$$

$$S_{25} = m^3 S_{2'5'} + m^2 n\left(S_{5'6'} - S_{2'4'}\right) - mn^2\left(S_{4'6'} - S_{1'5'}\right) - n^3 S_{1'4'}$$

$$S_{26} = m^2\left(m^2 - 3n^2\right)S_{2'6'} - m^3 n\left(2S_{2'2'} - 2S_{1'2'} - S_{6'6'}\right) + mn^3\left(2S_{1'1'} - 2S_{1'2'} - S_{6'6'}\right) - n^2\left(n^2 - 3m^2\right)S_{1'6'}$$

$$S_{33} = S_{3'3'} \qquad S_{34} = mS_{3'4'} + nS_{3'5'} \qquad S_{35} = mS_{3'5'} - nS_{3'4'}$$

$$S_{36} = \left(m^2 - n^2\right)S_{3'6'} - 2mn\left(S_{2'3'} - S_{1'3'}\right)$$

$$S_{44} = m^2 S_{4'4'} + n^2 S_{5'5'} + 2mnS_{4'5'}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

$$S_{45} = \left(m^2 - n^2\right)S_{4'5'} - mn\left(S_{4'4'} - S_{5'5'}\right)$$

$$S_{46} = m^3 S_{4'6'} + m^2 n\left(S_{5'6'} + 2S_{1'4'} - 2S_{2'4'}\right) - mn^2\left(S_{4'6'} - 2S_{1'5'} + 2S_{2'5'}\right) - n^3 S_{5'6'}$$

$$S_{55} = m^2 S_{5'5'} + n^2 S_{4'4'} - 2mnS_{4'5'}$$

$$S_{56} = m^3 S_{5'6'} - m^2 n\left(S_{4'6'} + 2S_{2'5'} - 2S_{1'5'}\right) - mn^2\left(S_{5'6'} + 2S_{1'4'} - 2S_{2'4'}\right) + n^3 S_{4'6'}$$

$$S_{66} = 4m^2 n^2\left(S_{1'1'} + S_{2'2'} - 2S_{1'2'}\right) + 4mn\left(m^2 - n^2\right)\left(S_{1'6'} - S_{2'6'}\right) + \left(m^2 - n^2\right)^2 S_{6'6'}$$

- Note that $[S']$ and $[S]$ can be expressed as

$$[S'] = [T_\varepsilon(\theta_3)][S][T_\sigma(-\theta_3)] \quad \text{and} \quad [S] = [T_\varepsilon(-\theta_3)][S'][T_\sigma(\theta_3)]$$

  - Thus, one set of transformed compliance expressions can be obtained from the other by simply interchanging the primed and unprimed indices and replacing n with -n

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- Let $m = \cos\theta_3$ and $n = \sin\theta_3$
- Performing the calculations given by $\{\alpha'\} = [T_\varepsilon]\{\alpha\}$ yields

$$\alpha_{1'1'} = m^2\alpha_{11} + 2mn\alpha_{12} + n^2\alpha_{22}$$

$$\alpha_{2'2'} = m^2\alpha_{22} - 2mn\alpha_{12} + n^2\alpha_{11}$$

$$\alpha_{3'3'} = \alpha_{33}$$

$$\alpha_{2'3'} = m\alpha_{23} - n\alpha_{13}$$

$$\alpha_{1'3'} = m\alpha_{13} + n\alpha_{23}$$

$$\alpha_{1'2'} = (m^2 - n^2)\alpha_{12} + mn(\alpha_{22} - \alpha_{11})$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- Let $m = \cos\theta_3$ and $n = \sin\theta_3$

- Performing the calculations given by $\{\alpha\} = [T_\sigma]^T\{\alpha'\}$ yields

$$\alpha_{11} = m^2\alpha_{1'1'} - 2mn\alpha_{1'2'} + n^2\alpha_{2'2'}$$

$$\alpha_{22} = m^2\alpha_{2'2'} + 2mn\alpha_{1'2'} + n^2\alpha_{1'1'}$$

$$\alpha_{33} = \alpha_{3'3'}$$

$$\alpha_{23} = m\alpha_{2'3'} + n\alpha_{1'3'}$$

$$\alpha_{13} = m\alpha_{1'3'} - n\alpha_{2'3'}$$

$$\alpha_{12} = \left(m^2 - n^2\right)\alpha_{1'2'} - mn\left(\alpha_{2'2'} - \alpha_{1'1'}\right)$$

Note that

$$[T_\sigma(\theta_3)]^T = [T_\varepsilon(-\theta_3)]$$

and hence

$$\{\alpha\} = [T_\varepsilon(-\theta_3)]\{\alpha'\}$$

So, the expressions given here for α_{ij} can be obtained from the those previously given for $\alpha_{i'j'}$ by switching the primed and unprimed indices and replacing n with -n

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONTINUED

- Let $m = \cos\theta_3$ and $n = \sin\theta_3$

- Performing the calculations given by $\{\beta'\} = [T_\sigma]\{\beta\}$ yields

$$\beta_{1'1'} = m^2\beta_{11} + 2mn\beta_{12} + n^2\beta_{22}$$

$$\beta_{2'2'} = m^2\beta_{22} - 2mn\beta_{12} + n^2\beta_{11}$$

$$\beta_{3'3'} = \beta_{33}$$

$$\beta_{2'3'} = m\beta_{23} - n\beta_{13}$$

$$\beta_{1'3'} = m\beta_{13} + n\beta_{23}$$

$$\beta_{1'2'} = (m^2 - n^2)\beta_{12} + mn(\beta_{22} - \beta_{11})$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_3 AXIS - CONCLUDED

- **Let** $m = \cos\theta_3$ **and** $n = \sin\theta_3$
- **Performing the calculations given by** $\{\beta\} = [T_\varepsilon]^T\{\beta'\}$ **yields**

$$\beta_{11} = m^2\beta_{1'1'} - 2mn\beta_{1'2'} + n^2\beta_{2'2'}$$

$$\beta_{22} = m^2\beta_{2'2'} + 2mn\beta_{1'2'} + n^2\beta_{1'1'}$$

$$\beta_{33} = \beta_{3'3'}$$

$$\beta_{23} = m\beta_{2'3'} + n\beta_{1'3'}$$

$$\beta_{13} = m\beta_{1'3'} - n\beta_{2'3'}$$

$$\beta_{12} = \left(m^2 - n^2\right)\beta_{1'2'} - mn\left(\beta_{2'2'} - \beta_{1'1'}\right)$$

Note that

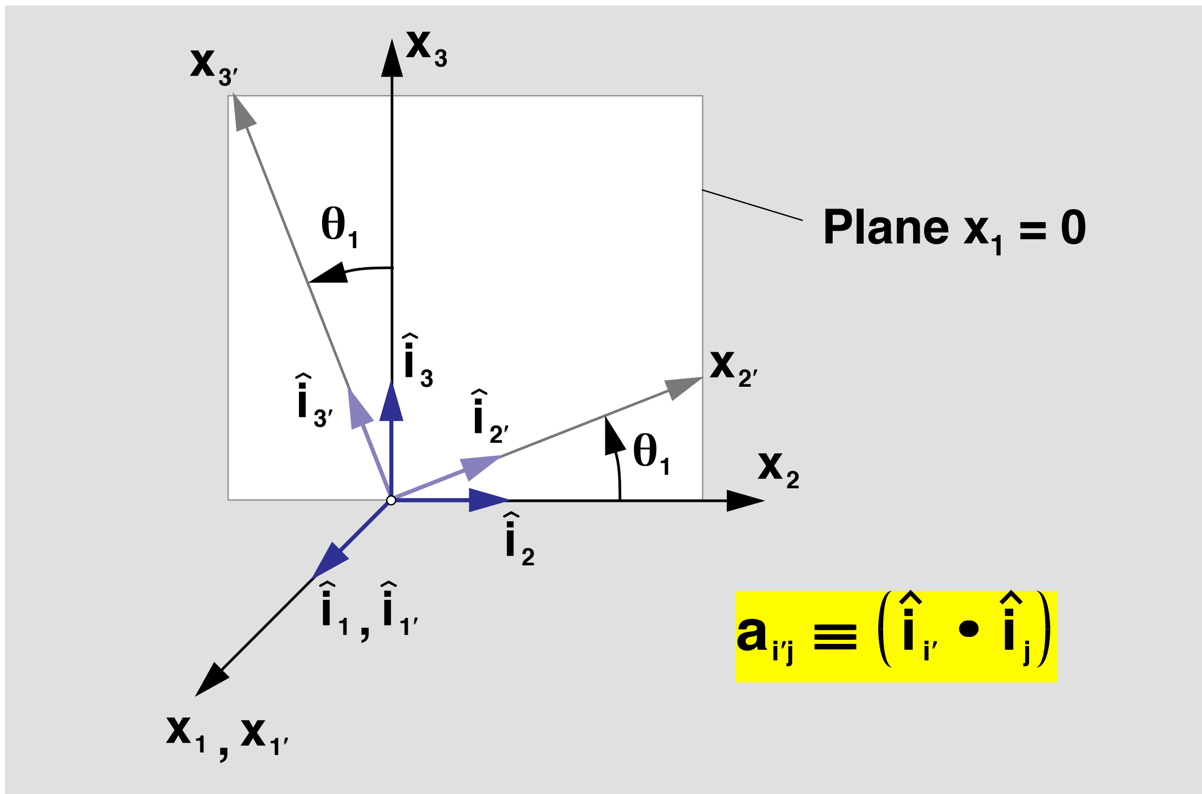
$$[T_\varepsilon(\theta_3)]^T = [T_\sigma(-\theta_3)]$$

and hence

$$\{\beta\} = [T_\sigma(-\theta_3)]\{\beta'\}$$

So, the expressions given here for β_{ij} can be obtained from the those previously given for $\beta_{i'j'}$ by switching the primed and unprimed indices and replacing n with -n

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS



- The term "**dextral**" refers to a right-handed rotation

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- The matrix form of the stress-tensor transformation law is given by

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & 2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -\sin\theta_1\cos\theta_1 & \sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

and is expressed symbolically by

$$\{\sigma'\} = [T_\sigma(\theta_1)]\{\sigma\}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- Similarly, the matrix form of the inverse transformation law is given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & -2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & 2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin\theta_1\cos\theta_1 & -\sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix}$$

and is expressed symbolically by

$$\{\boldsymbol{\sigma}\} = [\mathbf{T}_\sigma(\theta_1)]^{-1}\{\boldsymbol{\sigma}'\}$$

where
$$[\mathbf{T}_\sigma(\theta_1)]^{-1} = [\mathbf{T}_\sigma(-\theta_1)]$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- **By using the second-order, symmetric tensor transformation equations, the transformation law for the vector of engineering strains is given by**

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & \sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -2\sin\theta_1\cos\theta_1 & 2\sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

which is expressed symbolically by

$$\{\varepsilon'\} = [\mathbf{T}_\varepsilon(\theta_1)]\{\varepsilon\}$$

- **Note that** $$[\mathbf{T}_\varepsilon(\theta_1)]^T = [\mathbf{T}_\sigma(\theta_1)]^{-1} = [\mathbf{T}_\sigma(-\theta_1)]$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- **Similarly, the matrix form of the inverse transformation law is given by**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & -\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & \sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & 2\sin\theta_1\cos\theta_1 & -2\sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix}$$

and is expressed symbolically by

$$\{\boldsymbol{\varepsilon}\} = \left[\mathbf{T}_\varepsilon(\theta_1)\right]^{-1}\{\boldsymbol{\varepsilon}'\}$$

where

$$\left[\mathbf{T}_\varepsilon(\theta_1)\right]^{-1} = \left[\mathbf{T}_\varepsilon(-\theta_1)\right] = \left[\mathbf{T}_\sigma(\theta_1)\right]^{T}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- **The general expression for the transformation of the stiffness coefficients and thermal moduli are**

$$[C'] = [T_\sigma][C][T_\varepsilon]^{-1} \quad \text{and} \quad \{\beta'\} = [T_\sigma]\{\beta\}$$

- **Noting that $[T_\varepsilon]^{-1} = [T_\sigma]^T$ for a dextral rotation about the x_1 axis gives**

$$[C'] = [T_\sigma][C][T_\sigma]^T$$

- **Similarly, the general expression for the inverse transformation of the stiffness coefficients is** $[C] = [T_\sigma]^{-1}[C'][T_\varepsilon]$

- **Noting that $[T_\sigma]^{-1} = [T_\varepsilon]^T$ for a dextral rotation about the x_1 axis gives**

$$[C] = [T_\varepsilon]^T[C'][T_\varepsilon]$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- **The general expression for the transformation of the compliance coefficients and thermal expansion coefficients are**

$$[S'] = [T_\varepsilon][S][T_\sigma]^{-1} \quad \text{and} \quad \{\alpha'\} = [T_\varepsilon]\{\alpha\}$$

- **Noting that** $[T_\sigma]^{-1} = [T_\varepsilon]^T$ **for a dextral rotation about the x_1 axis gives**

$$[S'] = [T_\varepsilon][S][T_\varepsilon]^T$$

- **Similarly, the general expression for the inverse transformation of the compliance coefficients is** $[S] = [T_\varepsilon]^{-1}[S'][T_\sigma]$

- **Noting that** $[T_\varepsilon]^{-1} = [T_\sigma]^T$ **for a dextral rotation about the x_1 axis gives**

$$[S] = [T_\sigma]^T[S'][T_\sigma]$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - SUMMARY

$$[S'] = [T_\varepsilon][S][T_\varepsilon]^T \qquad [S] = [T_\sigma]^T[S'][T_\sigma]$$

$$[C'] = [T_\sigma][C][T_\sigma]^T \qquad [C] = [T_\varepsilon]^T[C'][T_\varepsilon]$$

$$[T_\sigma] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & 2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -\sin\theta_1\cos\theta_1 & \sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$[T_\varepsilon] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & \sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -2\sin\theta_1\cos\theta_1 & 2\sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - SUMMARY

$$\{\alpha'\} = [T_\varepsilon]\{\alpha\} \qquad \{\alpha\} = [T_\sigma]^T\{\alpha'\}$$

$$\{\beta'\} = [T_\sigma]\{\beta\} \qquad \{\beta\} = [T_\varepsilon]^T\{\beta'\}$$

$$[T_\sigma] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & 2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -\sin\theta_1\cos\theta_1 & \sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$[T_\varepsilon] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & \sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -2\sin\theta_1\cos\theta_1 & 2\sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- Let $m = \cos\theta_1$ and $n = \sin\theta_1$

- Performing the calculations given by $[C'] = [T_\sigma][C][T_\sigma]^T$ yields

$$C_{1'1'} = C_{11}$$

$$C_{1'2'} = m^2C_{12} + n^2C_{13} + 2mnC_{14}$$

$$C_{1'3'} = m^2C_{13} + n^2C_{12} - 2mnC_{14}$$

$$C_{1'4'} = \left(m^2 - n^2\right)C_{14} + mn\left(C_{13} - C_{12}\right)$$

$$C_{1'5'} = mC_{15} - nC_{16}$$

$$C_{1'6'} = mC_{16} + nC_{15}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

$$C_{2'2'} = m^4C_{22} + 2m^2n^2(C_{23} + 2C_{44}) + 4mn(m^2C_{24} + n^2C_{34}) + n^4C_{33}$$

$$C_{2'3'} = m^2n^2(C_{22} + C_{33} - 4C_{44}) - 2mn(m^2 - n^2)(C_{24} - C_{34}) + (m^4 + n^4)C_{23}$$

$$C_{2'4'} = m^2(m^2 - 3n^2)C_{24} - m^3n(C_{22} - C_{23} - 2C_{44}) + mn^3(C_{33} - C_{23} - 2C_{44}) - n^2(n^2 - 3m^2)C_{34}$$

$$C_{2'5'} = m^3C_{25} + m^2n(2C_{45} - C_{26}) - mn^2(2C_{46} - C_{35}) - n^3C_{36}$$

$$C_{2'6'} = m^3C_{26} + m^2n(2C_{46} + C_{25}) + mn^2(2C_{45} + C_{36}) + n^3C_{35}$$

$$C_{3'3'} = m^4C_{33} + 2m^2n^2(C_{23} + 2C_{44}) - 4mn(m^2C_{34} + n^2C_{24}) + n^4C_{22}$$

$$C_{3'4'} = m^2(m^2 - 3n^2)C_{34} + m^3n(C_{33} - C_{23} - 2C_{44}) - mn^3(C_{22} - C_{23} - 2C_{44}) - n^2(n^2 - 3m^2)C_{24}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

$$C_{3'5'} = m^3C_{35} - m^2n(2C_{45} + C_{36}) + mn^2(2C_{46} + C_{25}) - n^3C_{26}$$

$$C_{3'6'} = m^3C_{36} - m^2n(2C_{46} - C_{35}) - mn^2(2C_{45} - C_{26}) + n^3C_{25}$$

$$C_{4'4'} = m^2n^2(C_{22} + C_{33} - 2C_{23}) - 2mn(m^2 - n^2)(C_{24} - C_{34}) + (m^2 - n^2)^2C_{44}$$

$$C_{4'5'} = m^3C_{45} - m^2n(C_{46} + C_{25} - C_{35}) - mn^2(C_{45} - C_{26} + C_{36}) + n^3C_{46}$$

$$C_{4'6'} = m^3C_{46} + m^2n(C_{45} + C_{36} - C_{26}) - mn^2(C_{46} + C_{25} - C_{35}) - n^3C_{45}$$

$$C_{5'5'} = m^2C_{55} + n^2C_{66} - 2mnC_{56}$$

$$C_{5'6'} = (m^2 - n^2)C_{56} + mn(C_{55} - C_{66})$$

$$C_{6'6'} = m^2C_{66} + n^2C_{55} + 2mnC_{56}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- Let $m = \cos\theta_1$ and $n = \sin\theta_1$
- Performing the calculations given by $[C] = [T_\varepsilon]^T[C'][T_\varepsilon]$ yields

$$C_{11} = C_{1'1'}$$

$$C_{12} = m^2C_{1'2'} + n^2C_{1'3'} - 2mnC_{1'4'}$$

$$C_{13} = m^2C_{1'3'} + n^2C_{1'2'} + 2mnC_{1'4'}$$

$$C_{14} = (m^2 - n^2)C_{1'4'} - mn(C_{1'3'} - C_{1'2'})$$

$$C_{15} = mC_{1'5'} + nC_{1'6'}$$

$$C_{16} = mC_{1'6'} - nC_{1'5'}$$

$$C_{22} = m^4C_{2'2'} + 2m^2n^2(C_{2'3'} + 2C_{4'4'}) - 4mn(m^2C_{2'4'} + n^2C_{3'4'}) + n^4C_{3'3'}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

$$C_{23} = m^2n^2\left(C_{2'2'} + C_{3'3'} - 4C_{4'4'}\right) + 2mn\left(m^2 - n^2\right)\left(C_{2'4'} - C_{3'4'}\right) + \left(m^4 + n^4\right)C_{2'3'}$$

$$C_{24} = m^2\left(m^2 - 3n^2\right)C_{2'4'} + m^3n\left(C_{2'2'} - C_{2'3'} - 2C_{4'4'}\right) - mn^3\left(C_{3'3'} - C_{2'3'} - 2C_{4'4'}\right) - n^2\left(n^2 - 3m^2\right)C_{3'4'}$$

$$C_{25} = m^3C_{2'5'} - m^2n\left(2C_{4'5'} - C_{2'6'}\right) - mn^2\left(2C_{4'6'} - C_{3'5'}\right) + n^3C_{3'6'}$$

$$C_{26} = m^3C_{2'6'} - m^2n\left(2C_{4'6'} + C_{2'5'}\right) + mn^2\left(2C_{4'5'} + C_{3'6'}\right) - n^3C_{3'5'}$$

$$C_{33} = m^4C_{3'3'} + 2m^2n^2\left(C_{2'3'} + 2C_{4'4'}\right) + 4mn\left(m^2C_{3'4'} + n^2C_{2'4'}\right) + n^4C_{2'2'}$$

$$C_{34} = m^2\left(m^2 - 3n^2\right)C_{3'4'} - m^3n\left(C_{3'3'} - C_{2'3'} - 2C_{4'4'}\right) + mn^3\left(C_{2'2'} - C_{2'3'} - 2C_{4'4'}\right) - n^2\left(n^2 - 3m^2\right)C_{2'4'}$$

$$C_{35} = m^3C_{3'5'} + m^2n\left(2C_{4'5'} + C_{3'6'}\right) + mn^2\left(2C_{4'6'} + C_{2'5'}\right) + n^3C_{2'6'}$$

$$C_{36} = m^3C_{3'6'} + m^2n\left(2C_{4'6'} - C_{3'5'}\right) - mn^2\left(2C_{4'5'} - C_{2'6'}\right) - n^3C_{2'5'}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

$$C_{44} = m^2n^2(C_{2'2'} + C_{3'3'} - 2C_{2'3'}) + 2mn(m^2 - n^2)(C_{2'4'} - C_{3'4'}) + (m^2 - n^2)^2 C_{4'4'}$$

$$C_{45} = m^3C_{4'5'} + m^2n(C_{4'6'} + C_{2'5'} - C_{3'5'}) - mn^2(C_{4'5'} - C_{2'6'} + C_{3'6'}) - n^3C_{4'6'}$$

$$C_{46} = m^3C_{4'6'} - m^2n(C_{4'5'} + C_{3'6'} - C_{2'6'}) - mn^2(C_{4'6'} + C_{2'5'} - C_{3'5'}) + n^3C_{4'5'}$$

$$C_{55} = m^2C_{5'5'} + n^2C_{6'6'} + 2mnC_{5'6'}$$

$$C_{56} = (m^2 - n^2)C_{5'6'} - mn(C_{5'5'} - C_{6'6'})$$

$$C_{66} = m^2C_{6'6'} + n^2C_{5'5'} - 2mnC_{5'6'}$$

- Note that $[C']$ and $[C]$ can be expressed as

$$[C'] = [T_\sigma(\theta_1)][C][T_\varepsilon(-\theta_1)] \quad \text{and} \quad [C] = [T_\sigma(-\theta_1)][C'][T_\varepsilon(\theta_1)]$$

  - Thus, one set of transformed stiffness expressions can be obtained from the other by simply interchanging the primed and unprimed indices and replacing n with -n

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- Let $m = \cos\theta_1$ and $n = \sin\theta_1$
- Performing the calculations given by $[S'] = [T_\varepsilon][S][T_\varepsilon]^T$ yields

$$S_{1'1'} = S_{11}$$

$$S_{1'2'} = m^2S_{12} + n^2S_{13} + mnS_{14}$$

$$S_{1'3'} = m^2S_{13} + n^2S_{12} - mnS_{14}$$

$$S_{1'4'} = \left(m^2 - n^2\right)S_{14} + 2mn\left(S_{13} - S_{12}\right)$$

$$S_{1'5'} = mS_{15} - nS_{16}$$

$$S_{1'6'} = mS_{16} + nS_{15}$$

$$S_{2'2'} = m^4S_{22} + m^2n^2\left(2S_{23} + S_{44}\right) + 2mn\left(m^2S_{24} + n^2S_{34}\right) + n^4S_{33}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

$$S_{2'3'} = m^2n^2(S_{22} + S_{33} - S_{44}) - mn(m^2 - n^2)(S_{24} - S_{34}) + (m^4 + n^4)S_{23}$$

$$S_{2'4'} = m^2(m^2 - 3n^2)S_{24} - m^3n(2S_{22} - 2S_{23} - S_{44}) + mn^3(2S_{33} - 2S_{23} - S_{44}) - n^2(n^2 - 3m^2)S_{34}$$

$$S_{2'5'} = m^3S_{25} + m^2n(S_{45} - S_{26}) - mn^2(S_{46} - S_{35}) - n^3S_{36}$$

$$S_{2'6'} = m^3S_{26} + m^2n(S_{46} + S_{25}) + mn^2(S_{45} + S_{36}) + n^3S_{35}$$

$$S_{3'3'} = m^4S_{33} + m^2n^2(2S_{23} + S_{44}) - 2mn(m^2S_{34} + n^2S_{24}) + n^4S_{22}$$

$$S_{3'4'} = m^2(m^2 - 3n^2)S_{34} + m^3n(2S_{33} - 2S_{23} - S_{44}) - mn^3(2S_{22} - 2S_{23} - S_{44}) - n^2(n^2 - 3m^2)S_{24}$$

$$S_{3'5'} = m^3S_{35} - m^2n(S_{45} + S_{36}) + mn^2(S_{46} + S_{25}) - n^3S_{26}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

$$S_{3'6'} = m^3 S_{36} - m^2 n(S_{46} - S_{35}) - mn^2(S_{45} - S_{26}) + n^3 S_{25}$$

$$S_{4'4'} = 4m^2 n^2 (S_{22} + S_{33} - 2S_{23}) - 4mn(m^2 - n^2)(S_{24} - S_{34}) + (m^2 - n^2)^2 S_{44}$$

$$S_{4'5'} = m^3 S_{45} - m^2 n(S_{46} + 2S_{25} - 2S_{35}) - mn^2(S_{45} - 2S_{26} + 2S_{36}) + n^3 S_{46}$$

$$S_{4'6'} = m^3 S_{46} + m^2 n(S_{45} + 2S_{36} - 2S_{26}) - mn^2(S_{46} + 2S_{25} - 2S_{35}) - n^3 S_{45}$$

$$S_{5'5'} = m^2 S_{55} + n^2 S_{66} - 2mnS_{56}$$

$$S_{5'6'} = (m^2 - n^2)S_{56} + mn(S_{55} - S_{66})$$

$$S_{6'6'} = m^2 S_{66} + n^2 S_{55} + 2mnS_{56}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- **Let** $m = \cos\theta_1$ **and** $n = \sin\theta_1$
- **Performing the calculations given by** $[S] = [T_\sigma]^T [S'][T_\sigma]$ **yields**

$$S_{11} = S_{1'1'}$$

$$S_{12} = m^2 S_{1'2'} + n^2 S_{1'3'} - mnS_{1'4'}$$

$$S_{13} = m^2 S_{1'3'} + n^2 S_{1'2'} + mnS_{1'4'}$$

$$S_{14} = \left(m^2 - n^2\right) S_{1'4'} - 2mn\left(S_{1'3'} - S_{1'2'}\right)$$

$$S_{15} = mS_{1'5'} + nS_{1'6'}$$

$$S_{16} = mS_{1'6'} - nS_{1'5'}$$

$$S_{22} = m^4 S_{2'2'} + m^2 n^2 \left(2S_{2'3'} + S_{4'4'}\right) - 2mn\left(m^2 S_{2'4'} + n^2 S_{3'4'}\right) + n^4 S_{3'3'}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

$$S_{23} = m^2n^2(S_{2'2'} + S_{3'3'} - S_{4'4'}) + mn(m^2 - n^2)(S_{2'4'} - S_{3'4'}) + (m^4 + n^4)S_{2'3'}$$

$$S_{24} = m^2(m^2 - 3n^2)S_{2'4'} + m^3n(2S_{2'2'} - 2S_{2'3'} - S_{4'4'}) - mn^3(2S_{3'3'} - 2S_{2'3'} - S_{4'4'}) - n^2(n^2 - 3m^2)S_{3'4'}$$

$$S_{25} = m^3S_{2'5'} - m^2n(S_{4'5'} - S_{2'6'}) - mn^2(S_{4'6'} - S_{3'5'}) + n^3S_{3'6'}$$

$$S_{26} = m^3S_{2'6'} - m^2n(S_{4'6'} + S_{2'5'}) + mn^2(S_{4'5'} + S_{3'6'}) - n^3S_{3'5'}$$

$$S_{33} = m^4S_{3'3'} + m^2n^2(2S_{2'3'} + S_{4'4'}) + 2mn(m^2S_{3'4'} + n^2S_{2'4'}) + n^4S_{2'2'}$$

$$S_{34} = m^2(m^2 - 3n^2)S_{3'4'} - m^3n(2S_{3'3'} - 2S_{2'3'} - S_{4'4'}) + mn^3(2S_{2'2'} - 2S_{2'3'} - S_{4'4'}) - n^2(n^2 - 3m^2)S_{2'4'}$$

$$S_{35} = m^3S_{3'5'} + m^2n(S_{4'5'} + S_{3'6'}) + mn^2(S_{4'6'} + S_{2'5'}) + n^3S_{2'6'}$$

$$S_{36} = m^3S_{3'6'} + m^2n(S_{4'6'} - S_{3'5'}) - mn^2(S_{4'5'} - S_{2'6'}) - n^3S_{2'5'}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

$$S_{44} = 4m^2n^2(S_{2'2'} + S_{3'3'} - 2S_{2'3'}) + 4mn(m^2 - n^2)(S_{2'4'} - S_{3'4'}) + (m^2 - n^2)^2 S_{4'4'}$$

$$S_{45} = m^3 S_{4'5'} + m^2 n(S_{4'6'} + 2S_{2'5'} - 2S_{3'5'}) - mn^2(S_{4'5'} - 2S_{2'6'} + 2S_{3'6'}) - n^3 S_{4'6'}$$

$$S_{46} = m^3 S_{4'6'} - m^2 n(S_{4'5'} + 2S_{3'6'} - 2S_{2'6'}) - mn^2(S_{4'6'} + 2S_{2'5'} - 2S_{3'5'}) + n^3 S_{4'5'}$$

$$S_{55} = m^2 S_{5'5'} + n^2 S_{6'6'} + 2mnS_{5'6'}$$

$$S_{56} = (m^2 - n^2)S_{5'6'} - mn(S_{5'5'} - S_{6'6'})$$

$$S_{66} = m^2 S_{6'6'} + n^2 S_{5'5'} - 2mnS_{5'6'}$$

- Note that $[S']$ and $[S]$ can be expressed as

$$[S'] = [T_\varepsilon(\theta_1)][S][T_\sigma(-\theta_1)] \quad \text{and} \quad [S] = [T_\varepsilon(-\theta_1)][S'][T_\sigma(\theta_1)]$$

  - Thus, one set of transformed compliance expressions can be obtained from the other by simply interchanging the primed and unprimed indices and replacing n with -n

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- **Let** $m = \cos\theta_1$ **and** $n = \sin\theta_1$

- **Performing the calculations given by** $\{\alpha'\} = [T_\varepsilon]\{\alpha\}$ **yields**

$$\alpha_{1'1'} = \alpha_{11}$$

$$\alpha_{2'2'} = m^2\alpha_{22} + 2mn\alpha_{23} + n^2\alpha_{33}$$

$$\alpha_{3'3'} = m^2\alpha_{33} - 2mn\alpha_{23} + n^2\alpha_{22}$$

$$\alpha_{2'3'} = \left(m^2 - n^2\right)\alpha_{23} + mn\left(\alpha_{33} - \alpha_{22}\right)$$

$$\alpha_{1'3'} = m\alpha_{13} - n\alpha_{12}$$

$$\alpha_{1'2'} = m\alpha_{12} + n\alpha_{13}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- Let $m = \cos\theta_1$ and $n = \sin\theta_1$
- Performing the calculations given by $\{\alpha\} = [T_\sigma]^T\{\alpha'\}$ yields

$$\alpha_{11} = \alpha_{1'1'}$$

$$\alpha_{22} = m^2\alpha_{2'2'} - 2mn\alpha_{2'3'} + n^2\alpha_{3'3'}$$

$$\alpha_{33} = m^2\alpha_{3'3'} + 2mn\alpha_{2'3'} + n^2\alpha_{2'2'}$$

$$\alpha_{23} = \left(m^2 - n^2\right)\alpha_{2'3'} - mn\left(\alpha_{3'3'} - \alpha_{2'2'}\right)$$

$$\alpha_{13} = m\alpha_{1'3'} + n\alpha_{1'2'}$$

$$\alpha_{12} = m\alpha_{1'2'} - n\alpha_{1'3'}$$

Note that

$$[T_\sigma(\theta_1)]^T = [T_\varepsilon(-\theta_1)]$$

and hence

$$\{\alpha\} = [T_\varepsilon(-\theta_1)]\{\alpha'\}$$

So, the expressions given here for α_{ij} can be obtained from the those previously given for $\alpha_{i'j'}$ by switching the primed and unprimed indices and replacing n with -n

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- Let $m = \cos\theta_1$ and $n = \sin\theta_1$
- Performing the calculations given by $\{\beta'\} = [T_\sigma]\{\beta\}$ yields

$$\beta_{1'1'} = \beta_{11}$$

$$\beta_{2'2'} = m^2\beta_{22} + 2mn\beta_{23} + n^2\beta_{33}$$

$$\beta_{3'3'} = m^2\beta_{33} - 2mn\beta_{23} + n^2\beta_{22}$$

$$\beta_{2'3'} = (m^2 - n^2)\beta_{23} + mn(\beta_{33} - \beta_{22})$$

$$\beta_{1'3'} = m\beta_{13} - n\beta_{12}$$

$$\beta_{1'2'} = m\beta_{12} + n\beta_{13}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- **Let** $m = \cos\theta_1$ **and** $n = \sin\theta_1$

- **Performing the calculations given by** $\{\beta\} = [T_\varepsilon]^T\{\beta'\}$ **yields**

$$\beta_{11} = \beta_{1'1'}$$

$$\beta_{22} = m^2\beta_{2'2'} - 2mn\beta_{2'3'} + n^2\beta_{3'3'}$$

$$\beta_{33} = m^2\beta_{3'3'} + 2mn\beta_{2'3'} + n^2\beta_{2'2'}$$

$$\beta_{23} = \left(m^2 - n^2\right)\beta_{2'3'} - mn\left(\beta_{3'3'} - \beta_{2'2'}\right)$$

$$\beta_{13} = m\beta_{1'3'} + n\beta_{1'2'}$$

$$\beta_{12} = m\beta_{1'2'} - n\beta_{1'3'}$$

Note that

$$[T_\varepsilon(\theta_1)]^T = [T_\sigma(-\theta_1)]$$

and hence

$$\{\beta\} = [T_\sigma(-\theta_1)]\{\beta'\}$$

So, the expressions given here for β_{ij} can be obtained from the those previously given for $\beta_{i'j'}$ by switching the primed and unprimed indices and replacing n with -n

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- **The algebra involved in computing**

$$[C'] = [T_\sigma][C][T_\sigma]^T \qquad [S'] = [T_\varepsilon][S][T_\varepsilon]^T \qquad \{\alpha'\} = [T_\varepsilon]\{\alpha\}$$

$$[C] = [T_\varepsilon]^T[C'][T_\varepsilon] \qquad [S] = [T_\sigma]^T[S'][T_\sigma] \qquad \{\alpha\} = [T_\sigma]^T\{\alpha'\}$$

$$\{\beta'\} = [T_\sigma]\{\beta\} \qquad \{\beta\} = [T_\varepsilon]^T\{\beta'\}$$

  is definitely tedious when done by hand

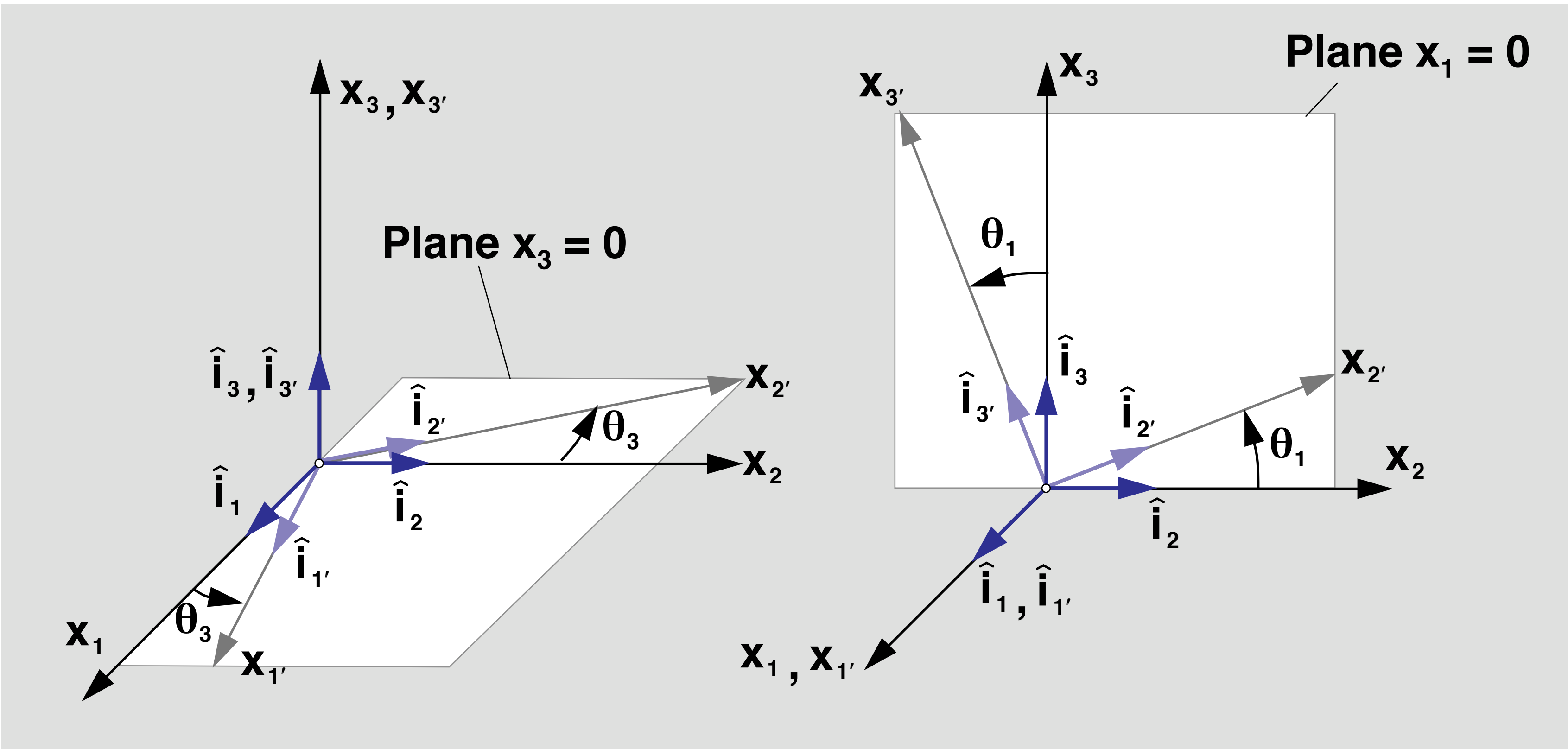
- **When the expressions for these matrix operations are known for either dextral rotations about the x_3 axis or about the x_2 axis, a simpler and much less tedious method for obtaining the transformed stiffnesses, compliances, thermal moduli, and thermal-expansion coefficients for dextral rotations about the x_1 axis is available**

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- **Consider the case in which the transformation expressions are known for dextral rotations about the x_3 axis and one wishes to find similar expressions for dextral rotations about the x_1 axis**

- **The desired tranformation equations are found by simply determining the renumbering of the indices that brings the following figure shown for dextral rotations about the x_3 axis into congruence with the following figure shown for dextral rotations about the x_1 axis**

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED



TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- Inspection of the figures indicates the following transformation of the indices: $1 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 1$
- Next, it must be realized that the exchanging of indices must be used with the indices of tensors to determine the indices used with the abridged notation (matrix)
  - The following index pairs relate the tensor indices to the matrix indices

tensor notation	11	22	33	23, 32	31, 13	12, 21
matrix notation	1	2	3	4	5	6

- Using this information along with $1 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 1$ gives the relations: $4 \rightarrow 5$, $5 \rightarrow 6$, and $6 \rightarrow 4$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONTINUED

- Likewise, the transformation of index pairs that appear in the abridged notation are given by

$11 \to 22$					
$12 \to 23$	$22 \to 33$				
$13 \to 12$	$23 \to 13$	$33 \to 11$			
$14 \to 25$	$24 \to 35$	$34 \to 15$	$44 \to 55$		
$15 \to 26$	$25 \to 36$	$35 \to 16$	$45 \to 56$	$55 \to 66$	
$16 \to 24$	$26 \to 34$	$36 \to 14$	$46 \to 45$	$56 \to 46$	$66 \to 44$

- Consider $C_{1'1'} = m^4 C_{11} + 2m^2n^2\left(C_{12} + 2C_{66}\right) + 4mn\left(m^2C_{16} + n^2C_{26}\right) + n^4C_{22}$, where $m = \cos\theta_3$ and $n = \sin\theta_3$

- The transformation of indices gives $m = \cos\theta_1$, $n = \sin\theta_1$, and

$$C_{2'2'} = m^4 C_{22} + 2m^2n^2\left(C_{23} + 2C_{44}\right) + 4mn\left(m^2C_{24} + n^2C_{34}\right) + n^4C_{33}$$

TRANSFORMATIONS FOR DEXTRAL ROTATIONS ABOUT THE x_1 AXIS - CONCLUDED

- **Applying the index transformation to the transformed stiffnesses, compliances, thermal moduli, and thermal-expansion coefficients for dextral rotations about the x_3 axis yields exactly the same expressions given herein previously for dextral rotations about the x_1 axis**

MATERIAL SYMMETRIES

MATERIAL SYMMETRIES

- The next logical step in the development of linear thermoelastic constitutive equations is the search for analytical conditions for which dilatation and distortion uncouple
  - For example, experience with common metals indicates that there are classes of materials for which dilatation and distortion uncouple
- Also, from a practical viewpoint, there is a need to find ways to minimize the number of laboratory experiments needed to fully characterize a given material
- Together, these considerations suggest a need for a systematic way to reduce the number of independent elastic constants and the number of independent thermal-expansion coefficients
- Previously, it was shown herein that there exists 21 independent elastic constants for an elastic anisotropic material - a finding that is substantiated by experimental evidence

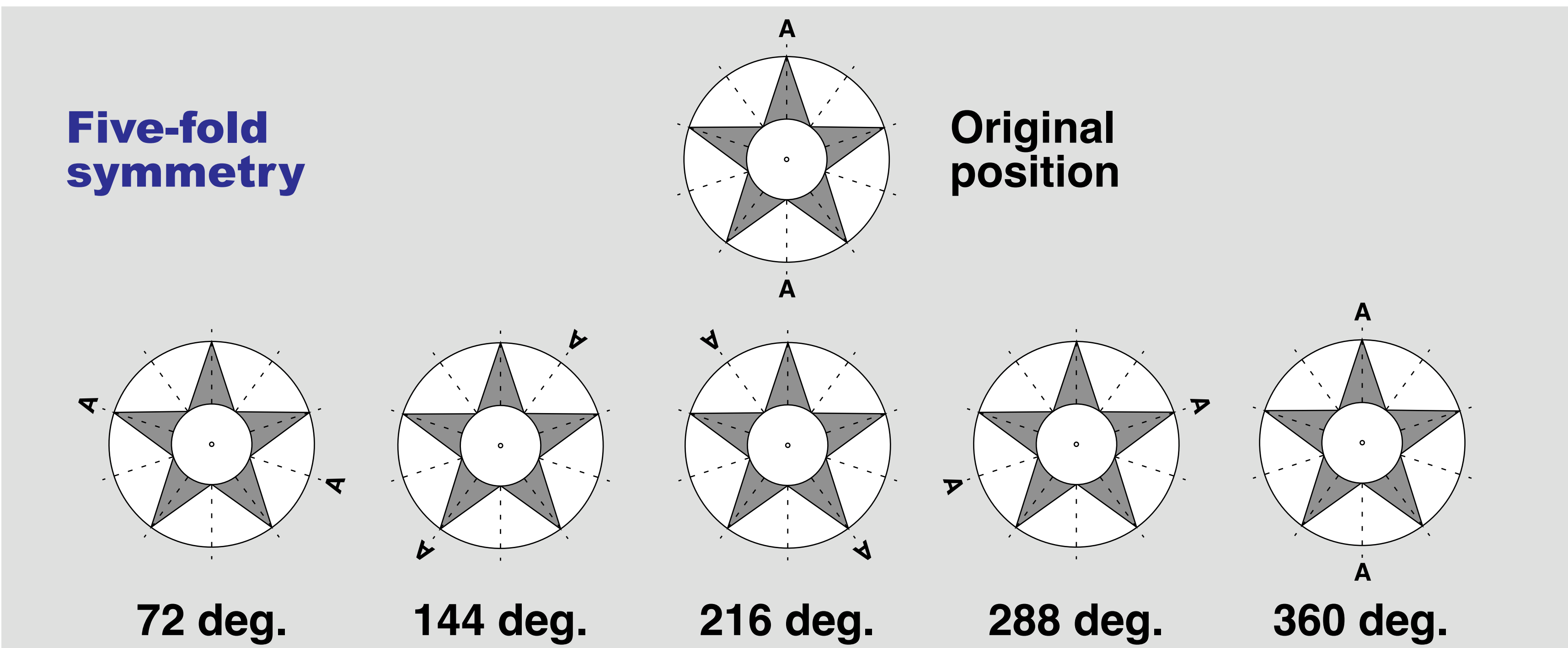
MATERIAL SYMMETRIES - CONCLUDED

- However, the number of independent constants needed to fully characterize an anisotropic material was the subject of a lengthy controversy
  - In the early to mid 19th century, **A. L. Cauchy** (1789-1857) and **S. D. Poisson** (1781-1840) formulated specialized mathematical models of the molecular interaction in solids, and argued that the number of independent constants could not exceed 15
- Investigations have indicated that when a solid exhibits a geometry symmetry, the elastic properties are identical in certain directions
- However, experience has shown that geometric symmetry is not equivalent to elastic symmetry; that is, it is possible to have elastic symmetry in directions that do not exhibit geometric symmetry
- Just as concepts of symmetry are used to reduce complexity of geometric objects, they are also used to reduce the complexity of material properties

MATHEMATICAL CHARACTERIZATION OF SYMMETRY

- The commonplace notion of symmetry is usually concerned with **geometric objects**

- For example, a two-dimensional geometric object may possess a shape that can be rotated about a central point by a finite angle with no appearent change in shape

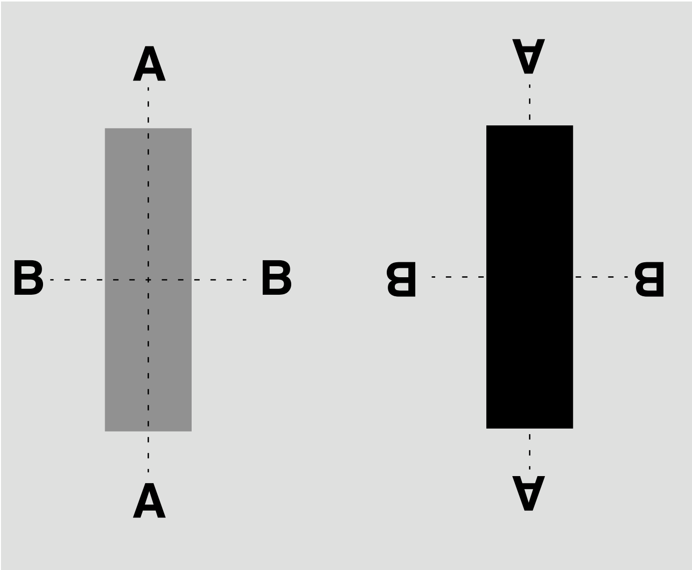


MATHEMATICAL CHARACTERIZATION OF SYMMETRY
CONTINUED

- **For the object shown in the previous figure, it can be rotated incrementally by 72 degrees into the identical shape**

  - $\dfrac{360 \text{ deg}}{72 \text{ deg}} = 5$ **and the shape is said to possess five-fold symmetry**

- **A situation of particular interest herein is the case when a geometric object possesses only two-fold symmetry**



- **For this object, the line B-B is described as a line of reflective symmetry (or mirror symmetry)**

- **For each of the geometric objects, undergoing the given rotations, a transformation occurs in which the objects appears unchanged; it is said to remain invariant under the transformation**

MATHEMATICAL CHARACTERIZATION OF SYMMETRY
CONTINUED

- For any geometric object in three-dimensional Euclidean space, the object can be represented by a set of points, whose position in space can be determined by a **coordinate frame** and a **coordinate domain** for the set of points

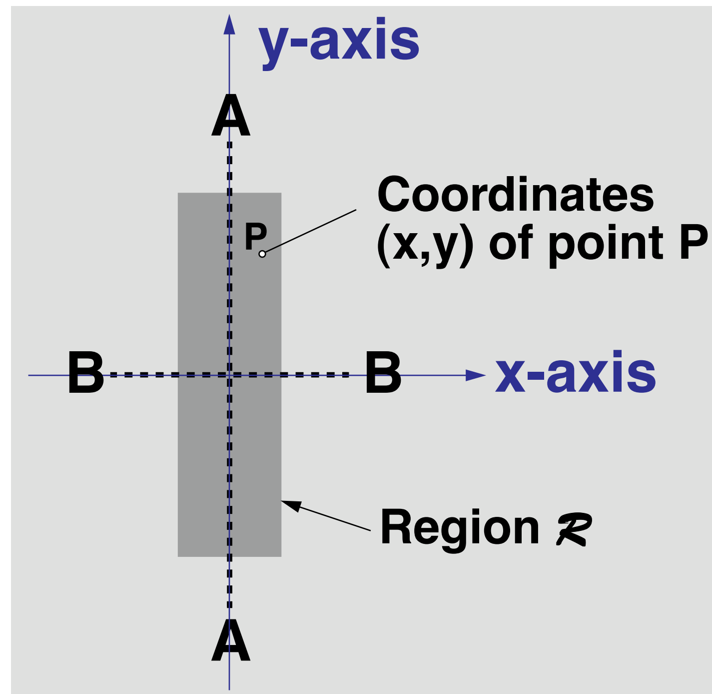
- For the figure, a generic point P of the region $\mathcal{R}$ has the coordinates (x,y), with respect to the coordinate frame shown

- The coordinate domain is given by

  $-\frac{1}{2}w \le x \le \frac{1}{2}w$ and $-\frac{1}{2}h \le y \le \frac{1}{2}h$,

  where w and h are the width and height of the rectangle, respectively

- For other geometries, curvilinear coordinates may be more suitable



MATHEMATICAL CHARACTERIZATION OF SYMMETRY
CONTINUED

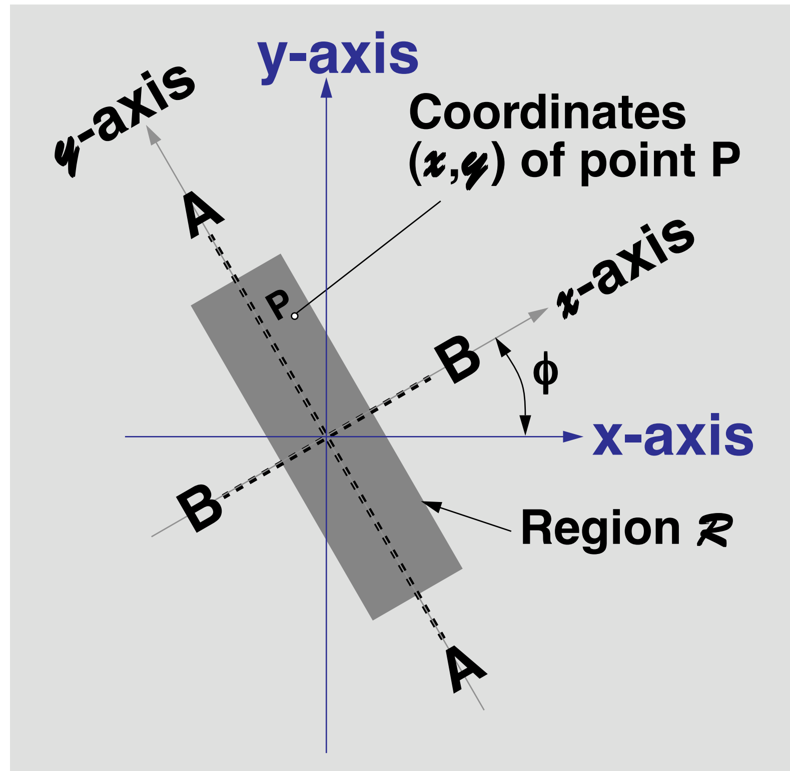
- For the purpose of investigating symmetry, it is convenient to place the origin at the central point of the rectangle

- Now consider a second set of coordinates ($\mathcal{x}$,$\mathcal{y}$), for which

  $-\frac{1}{2}w \le \mathcal{x} \le \frac{1}{2}w$ and $-\frac{1}{2}h \le \mathcal{y} \le \frac{1}{2}h$

  - This domain also describes the same rectangle with respect to a rotated coordinate frame, as shown in the figure

- For values of the angle ϕ equal to 180 and 360 deg, the rectangle is brought into coincidence with the initial configuration



MATHEMATICAL CHARACTERIZATION OF SYMMETRY
CONTINUED

- **For ϕ = 180 deg and ϕ = 360 deg, the geometric shape is invariant, and the transformation of coordinates given symbolically by $\mathcal{x} = \mathcal{x}(x,y)$ and $\mathcal{y} = \mathcal{y}(x,y)$ is called a symmetry transformation for the rectangle**

- **Obviously, this process of characterizing symmetry is easily extended to three dimensions**

- **Moreover, the functional characterization of symmetry in geometric shapes can be extended intuitively to symmetry in functions**
  - **A transformation of coordinates that leaves the *structural form* of the rule that defines a given function unchanged (invariant) is defined as a symmetry transformation for that function**

- **The use of algebraic structure and sets of transformations for quantifying symmetry in (real and abstract) objects is part of a branch of mathematics known as group theory**

MATHEMATICAL CHARACTERIZATION OF SYMMETRY
CONCLUDED

- Consider the function $F(x, y) = x^2 + y^2$ and the transformation of coordinates given by $x = -\mathscr{x}$ and $y = -\mathscr{y}$

- Applying the **transformation of coordinates** gives

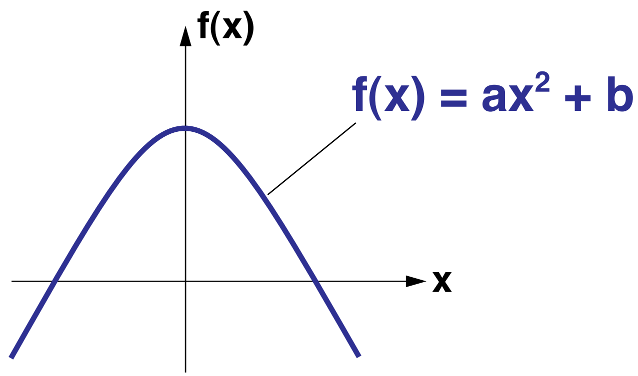
$$F(x, y) \rightarrow (-\mathscr{x})^2 + (-\mathscr{y})^2 = \mathscr{x}^2 + \mathscr{y}^2 = \mathscr{F}(\mathscr{x}, \mathscr{y}) = F(-x, -y)$$

  - The structural rules given by $F(x, y) = x^2 + y^2$ and $\mathscr{F}(\mathscr{x}, \mathscr{y}) = \mathscr{x}^2 + \mathscr{y}^2$ are identical; thus, $x = -\mathscr{x}$ and $y = -\mathscr{y}$ **define a symmetry transformation** for the function

  - The more common, and more succinct, way of describing the symmetry is given by writing $F(x, y) = F(-x, -y)$

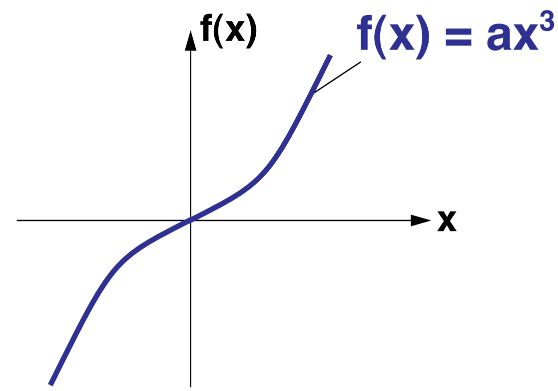
SOME TYPES OF SYMMETRY IN TWO DIMENSIONS

GRAPHS OF FUNCTIONS



Line of reflective symmetry

$f(-x) = f(x)$

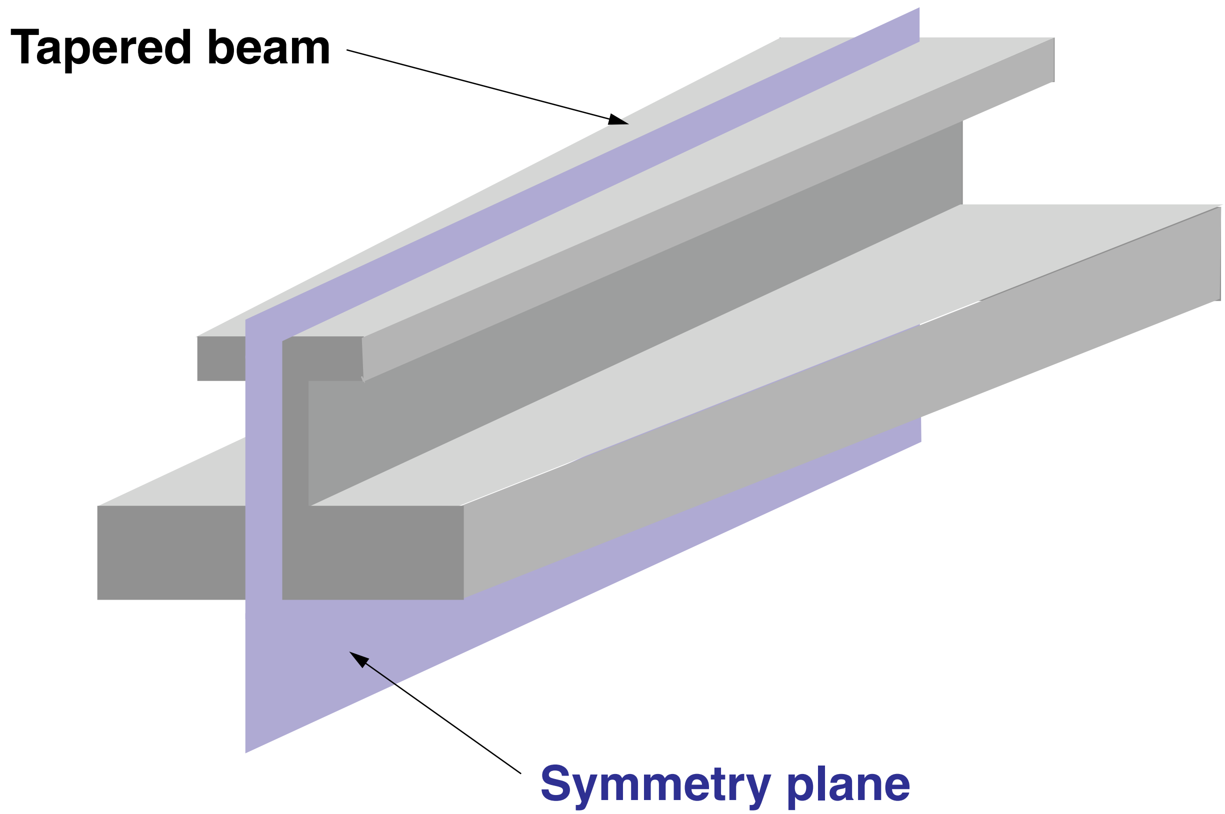


Line of reflective antisymmetry

$f(-x) = -f(x)$

SOME TYPES OF SYMMETRY IN THREE DIMENSIONS

PLANE OF REFLECTIVE SYMMETRY

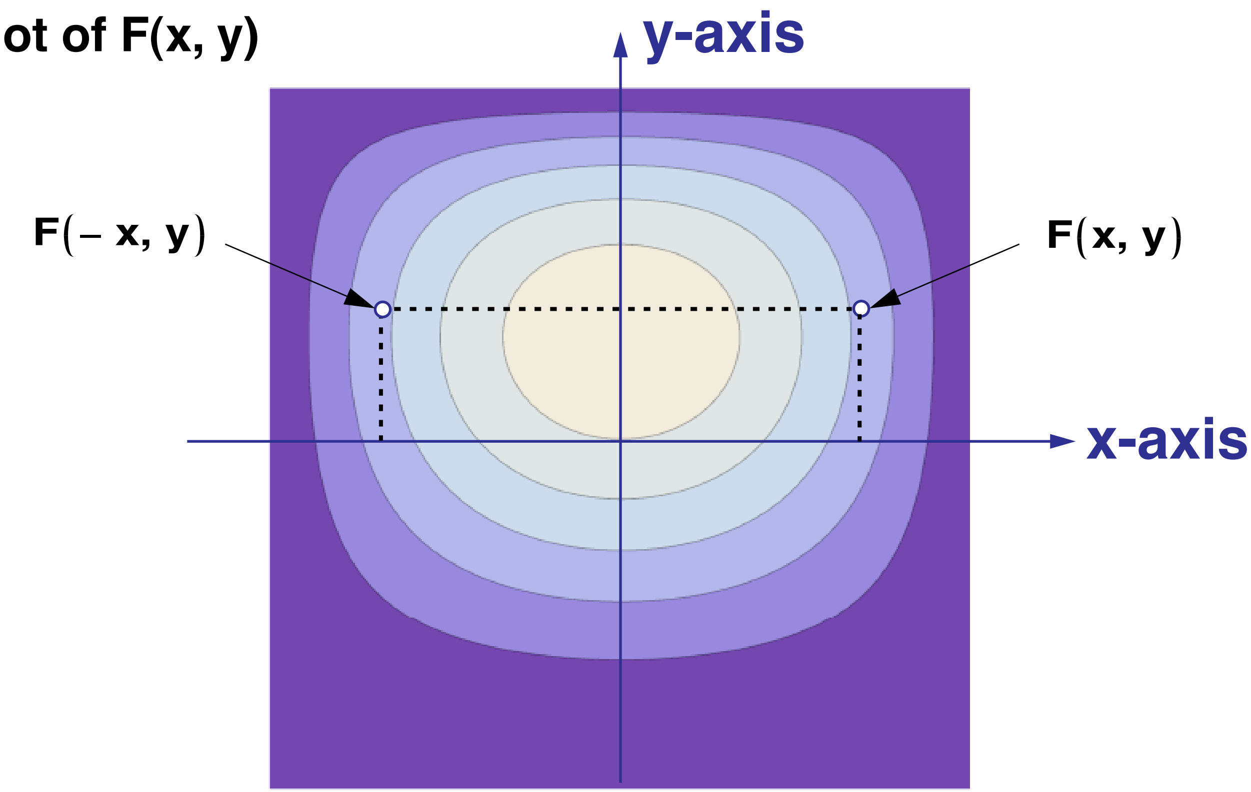


SOME TYPES OF SYMMETRY IN THREE DIMENSIONS

PLANE OF REFLECTIVE SYMMETRY

$$F(-x, y) = F(x, y)$$

Contour plot of F(x, y)



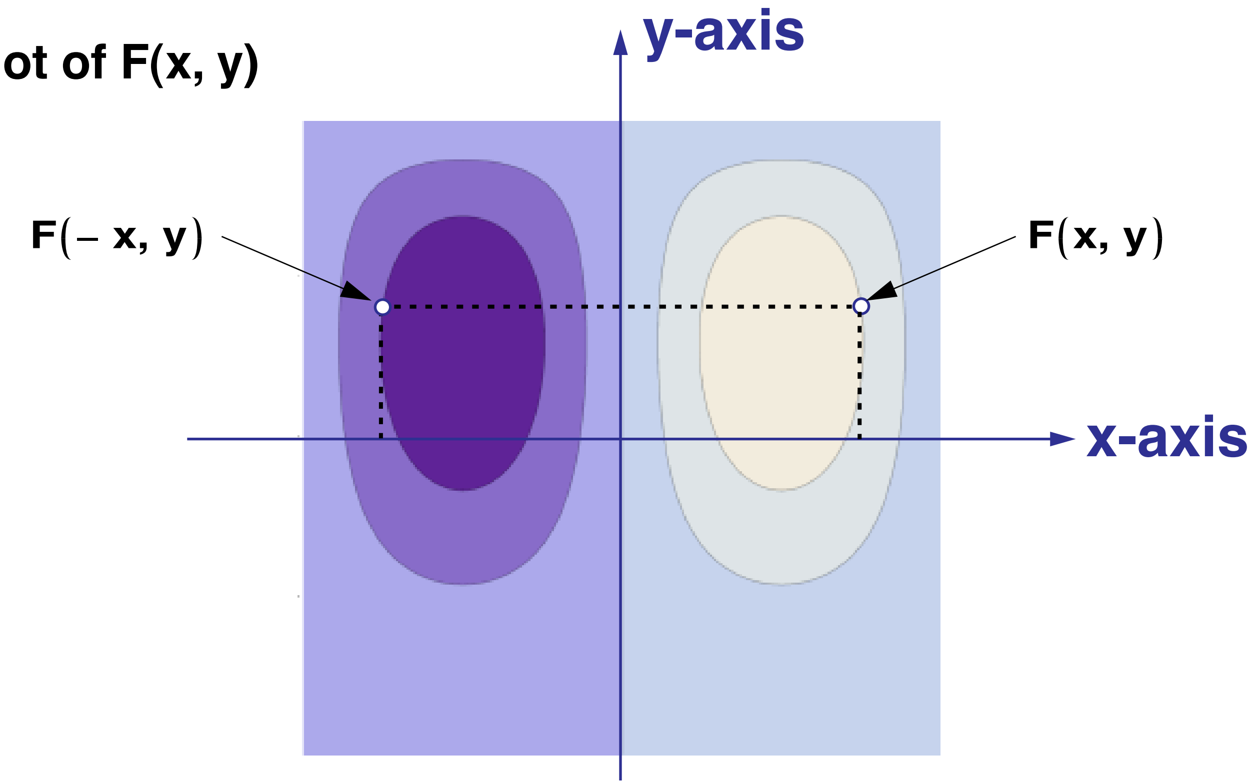
$$F(x, y) = y\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi y}{2}\right) \qquad -1 \le x \le 1 \qquad 0 \le y \le 2$$

SOME TYPES OF SYMMETRY IN THREE DIMENSIONS

PLANE OF REFLECTIVE ANTISYMMETRY

$$F(-x, y) = -F(x, y)$$

Contour plot of F(x, y)



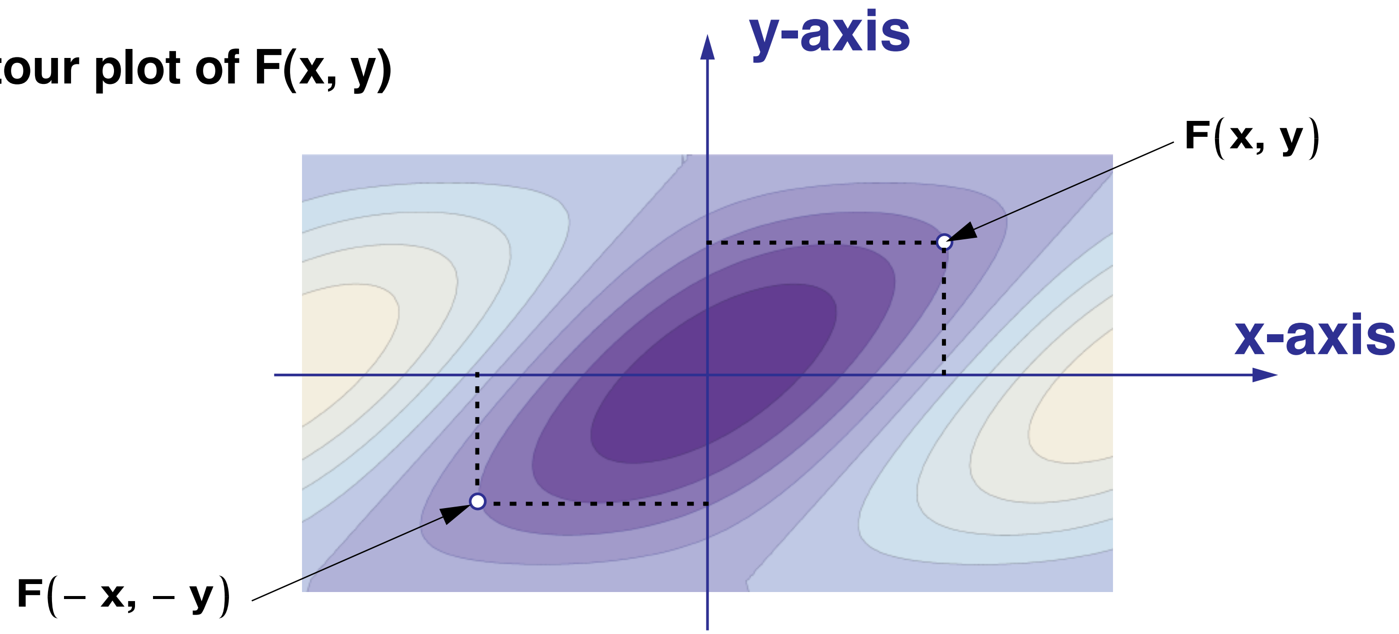
$$F(x, y) = y \sin(\pi x) \sin\left(\frac{\pi y}{2}\right)$$ $-1 \le x \le 1$ $0 \le y \le 2$

SOME TYPES OF SYMMETRY IN THREE DIMENSIONS

CENTRAL POINT OF INVERSION SYMMETRY (POLAR SYMMETRY)

$$F(-x, -y) = F(x, y)$$

Contour plot of F(x, y)



$$F(x, y) = \sin\left[\pi\left(x - \frac{9y}{10}\right)\right]\sin(\pi y)$$

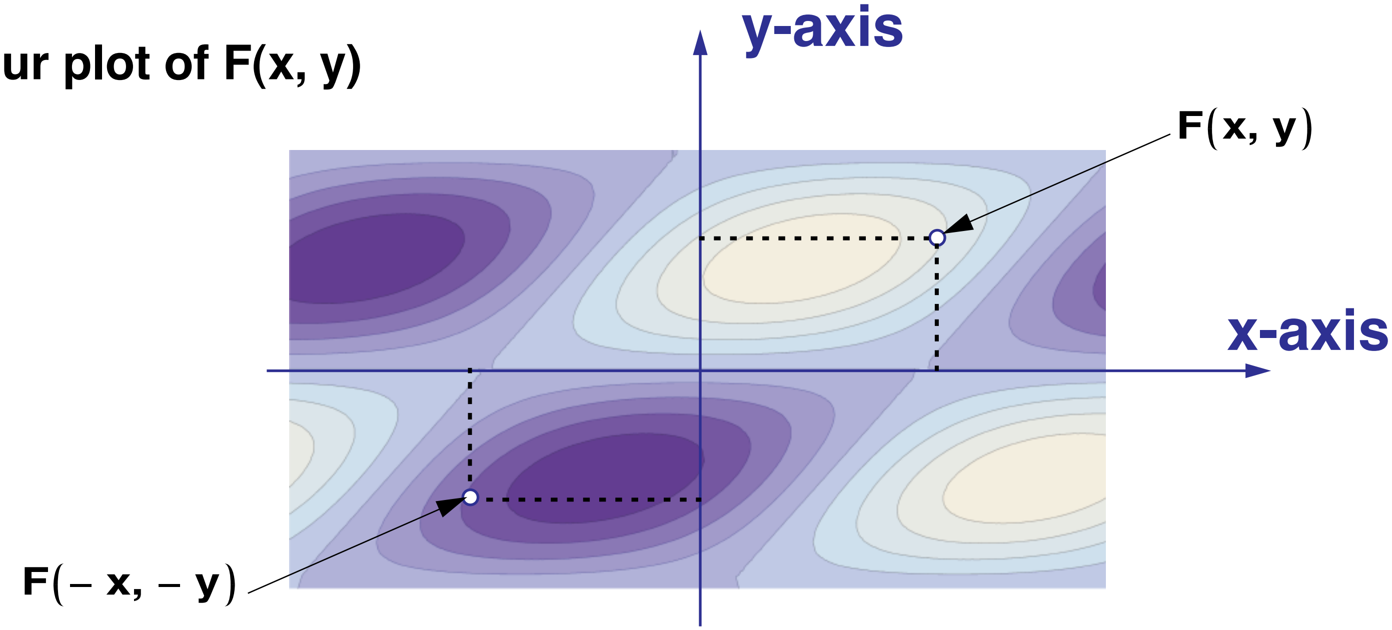
$-1 \leq x \leq 1$ $\quad$ $0 \leq y \leq 1$

SOME TYPES OF SYMMETRY IN THREE DIMENSIONS

CENTRAL POINT OF INVERSION ANTISYMMETRY

$$F(-x, -y) = -F(x, y)$$

Contour plot of F(x, y)



$$F(x, y) = \sin\left[\pi\left(x - \frac{9y}{10}\right)\right]\sin(2\pi y)$$

$-1 \leq x \leq 1 \qquad 0 \leq y \leq 1$

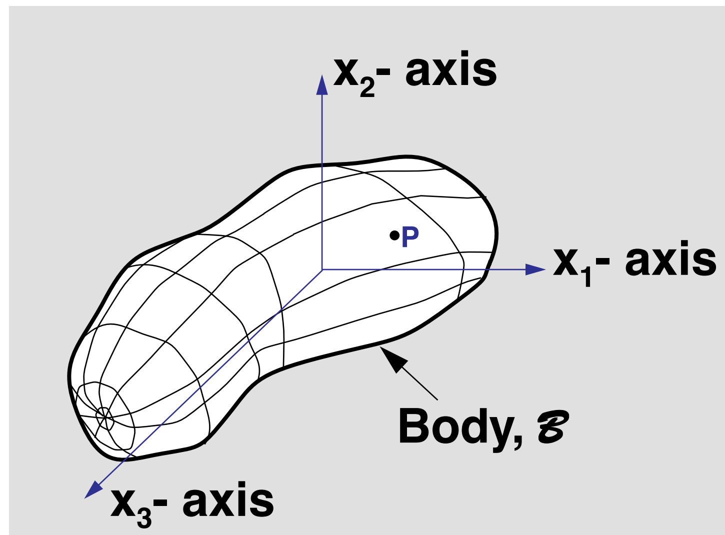
CRITERIA FOR MATERIAL SYMMETRY

- To define the conditions on the stiffness or compliance coefficients for a given type of symmetry to exist, one must first realize that the stresses, strains, and stiffness or compliance coefficients are functions of position within a given material body

  - Let the coordinates (x_1, x_2, x_3) and the corresponding coordinate frame be a coordinate system for a material body and its properties

  - The point P of the material body shown in the figure has coordinates (x_1, x_2, x_3)

  - The stresses, strains, stiffness and compliance matrices, thermal moduli, and thermal-expansion coefficients for this coordinate system are σ_{ij}, ε_{ij}, [C], [S], β_{ij}, and α_{ij}, respectively



CRITERIA FOR MATERIAL SYMMETRY - CONTINUED

- **Recall that the abridged forms of the thermoelastic constitutive equations for the material in the (x_1, x_2, x_3) coordinates are given by**

$$\{\sigma\} = [C]\{\varepsilon\} + \{\beta\}\,(T - T_{ref}) \quad \text{or} \quad \{\varepsilon\} = [S]\{\sigma\} + \{\alpha\}\,(T - T_{ref})$$

- **Now, consider a general orthogonal transformation between the rectangular Cartesian coordinates (x_1, x_2, x_3) and $(x_{1'}, x_{2'}, x_{3'})$, that define a generic point P of the material body**

    - **There is no need to place the restriction that $(x_{1'}, x_{2'}, x_{3'})$ be the coordinates of a right-handed (dextral) coordinate system**

    - **The stresses, strains, stiffness and compliance matrices, thermal moduli, and thermal-expansion coefficients for this coordinate system are $\sigma_{i'j'}$, $\varepsilon_{i'j'}$, $[C']$, $[S']$, $\beta_{i'j'}$, and $\alpha_{i'j'}$, respectively**

CRITERIA FOR MATERIAL SYMMETRY - CONTINUED

- The abridged forms of the thermoelastic constitutive equations for the material in the $(x_{1'}, x_{2'}, x_{3'})$ coordinate system are given by

$$\{\sigma'\} = [C']\{\varepsilon'\} + \{\beta'\}(T - T_{ref}) \quad \text{or} \quad \{\varepsilon'\} = [S']\{\sigma'\} + \{\alpha'\}(T - T_{ref})$$

- Moreover, it was shown previously that, for the given arbitrary transformation (rotation) of coordinates,

$$[C'] = [T_\sigma][C][T_\varepsilon]^{-1} \qquad \{\beta'\} = [T_\sigma]\{\beta\}$$

$$[S'] = [T_\varepsilon][S][T_\sigma]^{-1} \qquad \{\alpha'\} = [T_\varepsilon]\{\alpha\}$$

- When the mathematical description of the material properties are identical for two different coordinate systems (reference frames), a certain type of symmetry exists, whose character depends on the type of transformation between the two coordinates systems

CRITERIA FOR MATERIAL SYMMETRY - CONCLUDED

- Now, for $x_k = x_k(x_{1'}, x_{2'}, x_{3'})$ to define a symmetry transformation, such that a predetermined state of *symmetry* exist at a point **P** of the body, the structural form (rule) of the constitutive equations must remain invariant; specifically:

    - The matrix $[C']$ must be **invariant** under the transformation given by $[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$; that is, $[C] = [T_\sigma][C][T_\varepsilon]^{-1}$ must hold

    - Similarly, $[S] = [T_\varepsilon][S][T_\sigma]^{-1}$, $\{\alpha\} = [T_\varepsilon]\{\alpha\}$, and $\{\beta\} = [T_\sigma]\{\beta\}$ must hold

- Collectively, these invariance conditions are the criteria for a state of *material symmetry* to exist, and are sufficient conditions because $x_k = x_k(x_{1'}, x_{2'}, x_{3'})$ is presumed to be a **symmetry transformation**

CLASSES OF MATERIAL SYMMETRY

- Presently, there exists **eight distinct classes** of elastic-material symmetry
    - Many of these classes were discovered while studying the composition of various crystals
    - The classes are distinguished by the number of, and orientation of, **planes of elastic symmetry**
- A **plane of elastic symmetry**, at a point of an elastic material body, is defined as a plane for which the material exhibits reflective symmetry
- A **plane of isotropy**, at a point of an elastic material body, is defined as a plane for which there exists an infinite number of perpendicular planes of elastic symmetry (also called axisymmetry)

CLASSES OF MATERIAL SYMMETRY
CONTINUED

- The **eight distinct classes** of elastic-material symmetry are given by:
  - **Triclinic** materials - no inherent symmetry (fully anisotropic)
  - **Monoclinic** materials - one plane of symmetry
  - **Orthotropic** materials - three perpendicular planes of symmetry
  - **Trigonal** materials - three aligned planes of symmetry that are spaced 60 degrees apart
  - **Tetragonal** materials - four aligned planes of symmetry that are spaced 45 degrees apart and that are all perpendicular to one additional symmetry plane
  - **Transversely isotropic** materials - one plane of isotropy that is perpendicular to two other mutually perpendicular symmetry planes

CLASSES OF MATERIAL SYMMETRY
CONTINUED

- **Cubic** materials - three mutually perpendicular planes of symmetry and six additional symmetry planes, in which two of the six are aligned with one of the perpendicular planes and intersect it at 45 degrees

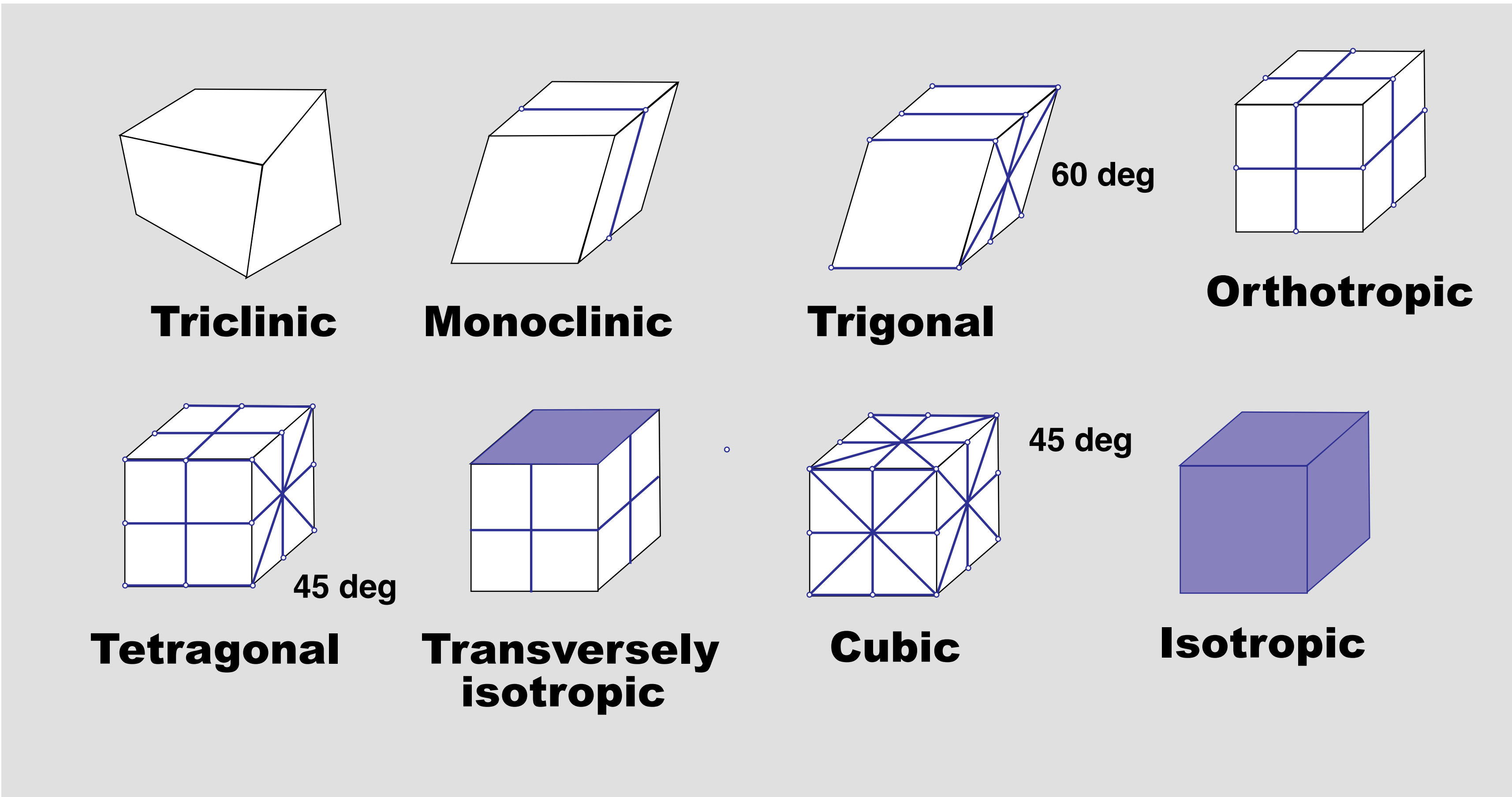
- Completely **isotropic** materials - an infinite number of planes of isotropy exist

- There are four classes of elastic materials that are of great practical importance in engineering

  - These classes of materials are monoclinic, orthotropic, transversely isotropic, and isotropic materials

CLASSES OF MATERIAL SYMMETRY

PICTORIAL REPRESENTATIONS



- The blue lines represent the edge of a symmetry plane

MONOCLINIC MATERIALS

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$

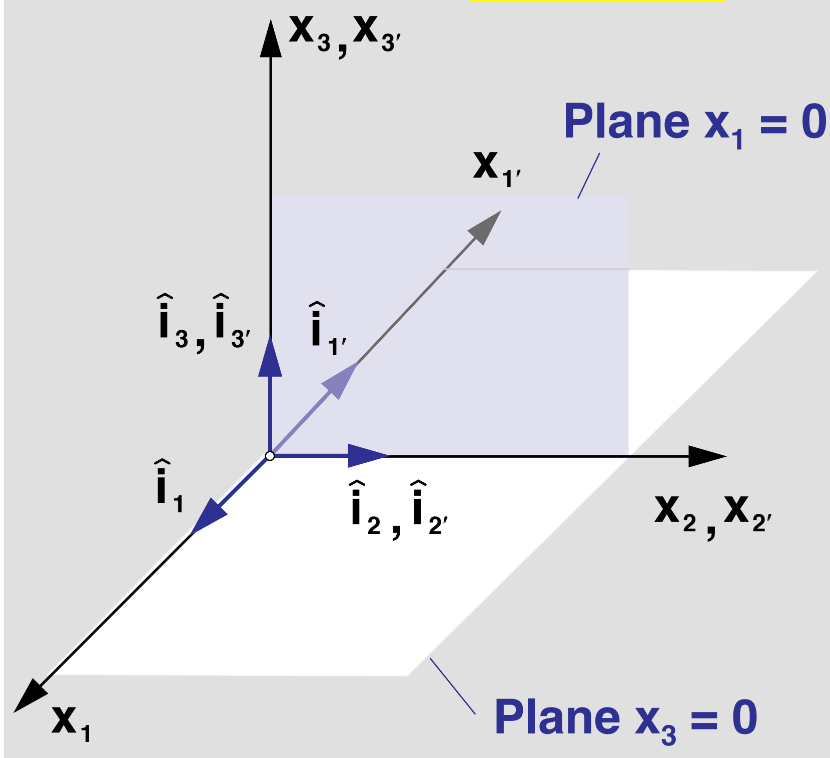
- First, consider the case in which the material exhibits elastic symmetry about the plane $x_1 = 0$

- The coordinate transformation for this symmetry is shown in the figure and is given by $x_{1'} = -x_1$, $x_{2'} = x_2$, and $x_{3'} = x_3$

- The corresponding matrix of **direction cosines** is given by

$$\begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_{i'j} \equiv \left(\hat{i}_{i'} \bullet \hat{i}_j\right)$$



MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- For this special case, the general transformation matrix

$$\left[T_\sigma\right] = \begin{bmatrix} (a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & 2a_{1'2}a_{1'3} & 2a_{1'1}a_{1'3} & 2a_{1'1}a_{1'2} \\ (a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & 2a_{2'2}a_{2'3} & 2a_{2'1}a_{2'3} & 2a_{2'1}a_{2'2} \\ (a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & 2a_{3'2}a_{3'3} & 2a_{3'1}a_{3'3} & 2a_{3'1}a_{3'2} \\ a_{2'1}a_{3'1} & a_{2'2}a_{3'2} & a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\ a_{1'1}a_{3'1} & a_{1'2}a_{3'2} & a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\ a_{1'1}a_{2'1} & a_{1'2}a_{2'2} & a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1}) \end{bmatrix}$$

reduces to

the diagonal matrix

$$\left[T_\sigma\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

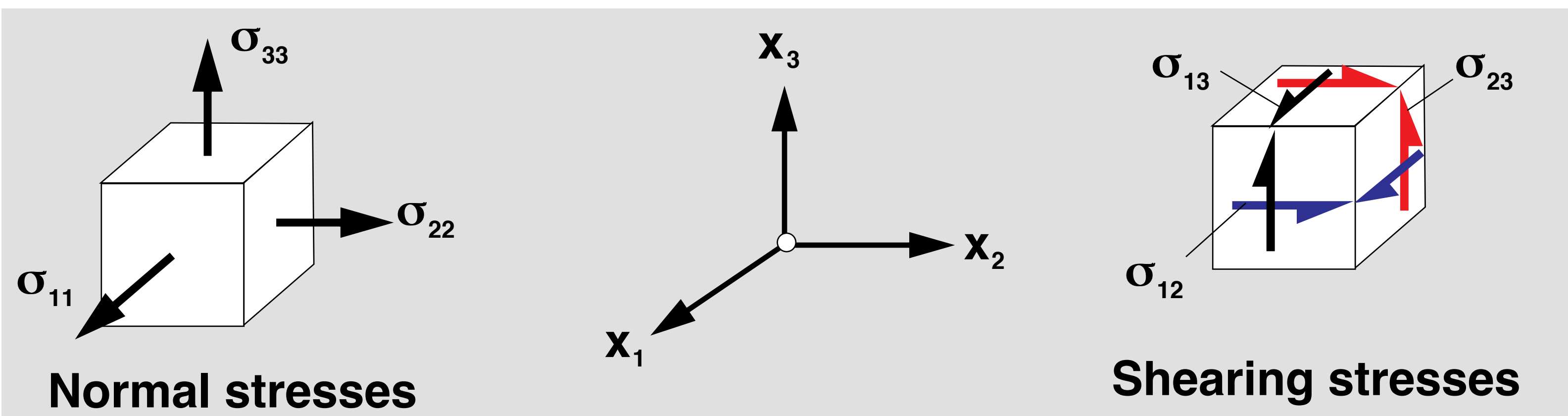
MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- Moreover, because $\left[T_\sigma\right]$ is a diagonal matrix, it follows that $\left[T_\varepsilon\right] = \left[T_\sigma\right]$

- It is worthwhile to mention that $\left[T_\sigma\right]$ can be deduced directly, and quickly, by a direction comparison of the positive-valued stresses that act on a differential volume element

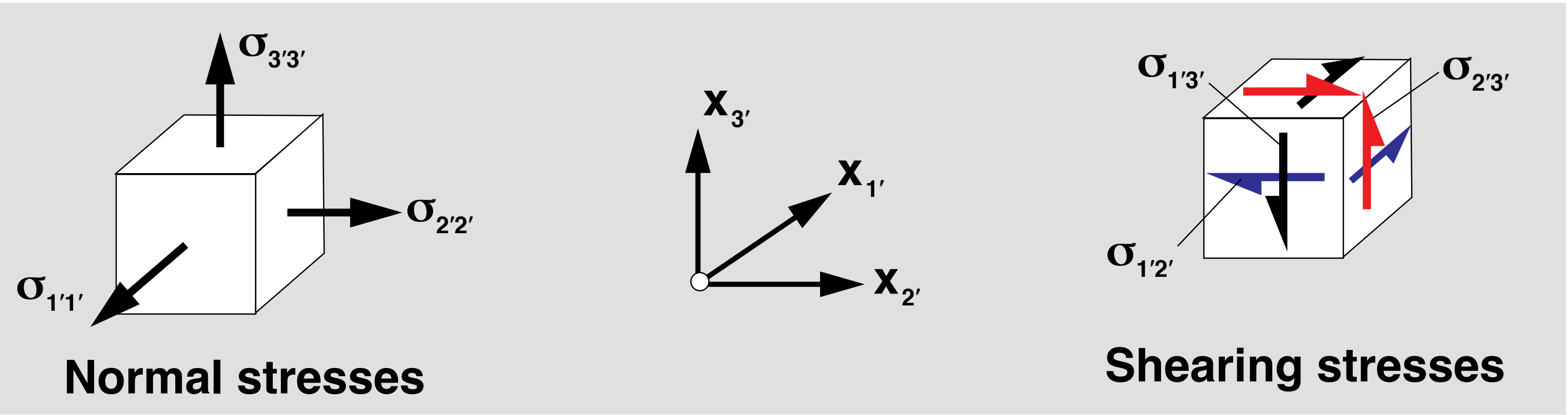
- First, sketch the positive-valued stresses that act on a differential volume element when described by the (x_1, x_2, x_3) coordinates



MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- **Then, sketch the positive-valued stresses that act on a differential volume element when described by the $(x_{1'}, x_{2'}, x_{3'})$ coordinates**



- **Direct comparison of the stresses yields the relationships**

$$\sigma_{1'1'} = \sigma_{11} \qquad \sigma_{2'2'} = \sigma_{22} \qquad \sigma_{3'3'} = \sigma_{33}$$

$$\sigma_{2'3'} = \sigma_{23} \qquad \sigma_{1'3'} = -\sigma_{13} \qquad \sigma_{1'2'} = -\sigma_{12}$$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- Expressing the relationships in matrix form gives

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

- Thus, $\{\sigma'\} = [T_\sigma]\{\sigma\}$ gives $[T_\sigma] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$ directly

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- Now, for a state of *reflective symmetry* about the plane $x_1 = 0$ to exist at a point **P** of the body, it was shown herein that the matrix $[C']$ must be **invariant** under the transformation given by $[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$
  - That is, $[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$ must become $[C] = [T_\sigma][C][T_\varepsilon]^{-1}$
- A more convenient form of this invariance condition is obtained for this particular transformation as follows
- First, postmultiplying the last expression by $[T_\varepsilon]$ gives $[C][T_\varepsilon] = [T_\sigma][C]$ as the (sufficient) condition for symmetry
- Next, noting that $[T_\varepsilon] = [T_\sigma]$ for this particular symmetry transformation, it follows that $[C][T_\varepsilon] = [T_\sigma][C]$ becomes $[C][T_\sigma] = [T_\sigma][C]$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- Also, because $\left[T_\sigma\right] = \left[T_\sigma\right]^T$ for this particular symmetry transformation and $\left[C\right] = \left[C\right]^T$, it follows that $\left(\left[C\right]\left[T_\sigma\right]\right)^T = \left[T_\sigma\right]^T\left[C\right]^T = \left[T_\sigma\right]\left[C\right]$

- Thus, the sufficient condition for symmetry, $\left[C\right]\left[T_\sigma\right] = \left[T_\sigma\right]\left[C\right]$, becomes $\left[C\right]\left[T_\sigma\right] = \left(\left[C\right]\left[T_\sigma\right]\right)^T$; that is, $\left[C\right]\left[T_\sigma\right]$ must be a **symmetric matrix**

- Computing $\left[C\right]\left[T_\sigma\right]$ gives

$$[C][T_\sigma] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{15} & -C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & -C_{25} & -C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & -C_{35} & -C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & -C_{45} & -C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & -C_{55} & -C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & -C_{56} & -C_{66} \end{bmatrix}$$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- **Enforcing $[C][T_\sigma] = ([C][T_\sigma])^T$ yields the following requirements on the stiffnesses in order to exhibit a state of elastic *reflective symmetry* about the plane $x_1 = 0$:**

$C_{15} = 0$, $C_{16} = 0$, $C_{25} = 0$, $C_{26} = 0$, $C_{35} = 0$, $C_{36} = 0$, $C_{45} = 0$, and $C_{46} = 0$

- **Thus, the stiffness matrix for a monoclinic material, which exhibits elastic *reflective symmetry* about the plane $x_1 = 0$, has the form**

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix}$$

which has **13 independent stiffnesses**

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- Similarly, for a state of *elastic reflective symmetry* about the plane $x_1 = 0$ to exist at a point **P** of the body, the matrix $[S']$ must be **invariant** under the transformation given by $[S'] = [T_\varepsilon][S][T_\sigma]^{-1}$

    - That is, $[S'] = [T_\varepsilon][S][T_\sigma]^{-1}$ must become $[S] = [T_\varepsilon][S][T_\sigma]^{-1}$

- Postmultiplying the last expressing by $[T_\sigma]$ gives $[S][T_\sigma] = [T_\varepsilon][S]$ as the (sufficient) condition for symmetry

- Next, noting that $[T_\varepsilon] = [T_\sigma]$ for this particular transformation, it follows that $[S][T_\sigma] = [T_\varepsilon][S]$ becomes $[S][T_\sigma] = [T_\sigma][S]$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- Like before, because $[T_\sigma] = [T_\sigma]^T$ for this particular transformation and $[S] = [S]^T$, it follows that $\left([S][T_\sigma]\right)^T = [T_\sigma]^T[S]^T = [T_\sigma][S]$

- Thus, the sufficient condition for symmetry, $[S][T_\sigma] = [T_\sigma][S]$, becomes $[S][T_\sigma] = \left([S][T_\sigma]\right)^T$; that is, $[S][T_\sigma]$ must be a **symmetric matrix**

- Like for the stiffness matrix, computing $[S][T_\sigma]$ and enforcing $[S][T_\sigma] = \left([S][T_\sigma]\right)^T$ yields the following requirements on the compliances in order to exhibit a state of *elastic reflective symmetry* about the plane $x_1 = 0$:

$S_{15} = 0$, $S_{16} = 0$, $S_{25} = 0$, $S_{26} = 0$, $S_{35} = 0$, $S_{36} = 0$, $S_{45} = 0$, and $S_{46} = 0$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- **Thus, the compliance matrix for a monoclinic material that a state of *elastic reflective symmetry* about the plane $x_1 = 0$ has the form**

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ S_{12} & S_{22} & S_{23} & S_{24} & 0 & 0 \\ S_{13} & S_{23} & S_{33} & S_{34} & 0 & 0 \\ S_{14} & S_{24} & S_{34} & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & S_{56} \\ 0 & 0 & 0 & 0 & S_{56} & S_{66} \end{bmatrix}$$

which has 13 independent compliances

- **Comparison of the compliance and stiffness matrices for this case indicates that they have the same form**

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- The requirements on the coefficients of thermal expansion and the thermal moduli for a state of *reflective symmetry* about the plane $x_1 = 0$ to exist are simpler than those for the stiffnesses and compliances

- The requirements on the coefficients of thermal expansion are given by the requirement that the vector $\{\alpha'\}$ must be **invariant** under the transformation given by $\{\alpha'\} = [T_\varepsilon]\{\alpha\}$

  - That is, $\{\alpha'\} = [T_\varepsilon]\{\alpha\}$ must become $\{\alpha\} = [T_\varepsilon]\{\alpha\}$

- Similarly, $\{\beta'\}$ must be **invariant** under the transformation given by $\{\beta'\} = [T_\sigma]\{\beta\}$; that is, $\{\beta'\} = [T_\sigma]\{\beta\}$ must become $\{\beta\} = [T_\sigma]\{\beta\}$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- Computing $\{\alpha\} = [T_\varepsilon]\{\alpha\}$ gives

$$\begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix}$$

which reduces to

$$\begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} = \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ -2\alpha_{13} \\ -2\alpha_{12} \end{Bmatrix}$$

- Thus, enforcing $\{\alpha\} = [T_\varepsilon]\{\alpha\}$ requires $\alpha_{12} = \alpha_{13} = 0$ in order for a state of *reflective symmetry* about the plane $x_1 = 0$ to exist

- Similarly, enforcing $\{\beta\} = [T_\sigma]\{\beta\}$ requires $\beta_{12} = \beta_{13} = 0$ in order for a state of *reflective symmetry* about the plane $x_1 = 0$ to exist

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONTINUED)

- Applying all the simplifications, the **linear thermoelastic constitutive equations** become

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ S_{12} & S_{22} & S_{23} & S_{24} & 0 & 0 \\ S_{13} & S_{23} & S_{33} & S_{34} & 0 & 0 \\ S_{14} & S_{24} & S_{34} & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & S_{56} \\ 0 & 0 & 0 & 0 & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_1 = 0$ (CONCLUDED)

- Finally, the nonzero thermal moduli are given in terms of the coefficients of thermal expansion by

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \end{Bmatrix}$$

- The constitutive equations show that the normal stresses, or a temperature change, produce shearing deformations only in the plane $x_1 = 0$

  - Extension and shearing are totally uncoupled in the planes $x_2 = 0$ and $x_3 = 0$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_2 = 0$

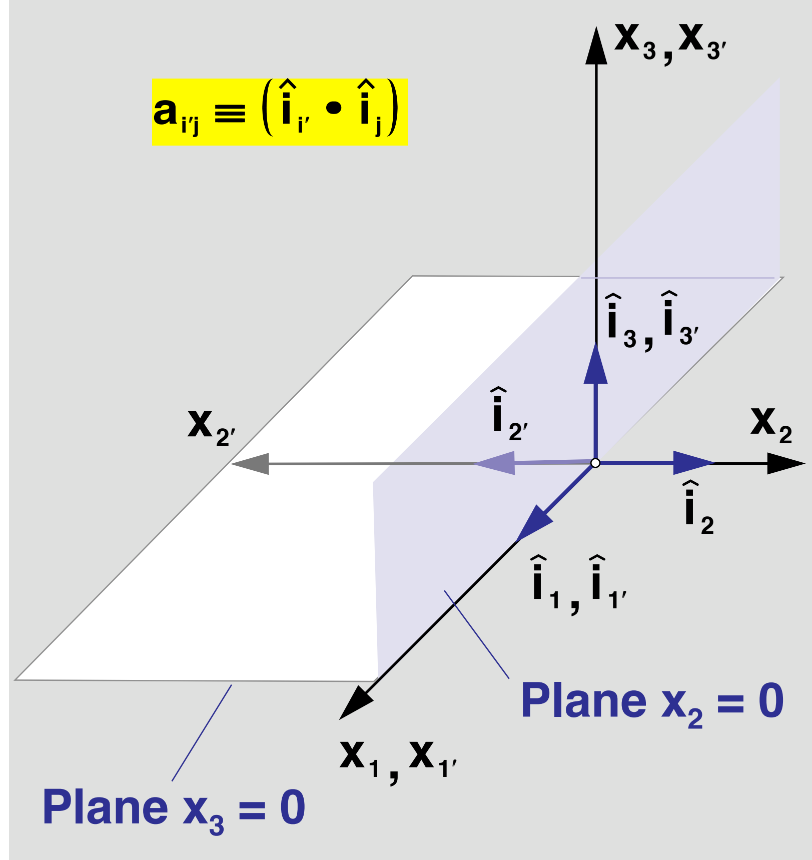
- Next, consider the case in which the material exhibits elastic symmetry about the plane $x_2 = 0$

- The coordinate transformation for this symmetry is shown in the figure and is given by $x_{1'} = x_1$, $x_{2'} = -x_2$, and $x_{3'} = x_3$

- The corresponding matrix of **direction cosines** is given by

$$\begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_{i'j} \equiv \left(\hat{i}_{i'} \bullet \hat{i}_j\right)$$



MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_2 = 0$ (CONTINUED)

- For this special case, the general transformation matrix

$$[T_\sigma] = \begin{bmatrix} (a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & 2a_{1'2}a_{1'3} & 2a_{1'1}a_{1'3} & 2a_{1'1}a_{1'2} \\ (a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & 2a_{2'2}a_{2'3} & 2a_{2'1}a_{2'3} & 2a_{2'1}a_{2'2} \\ (a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & 2a_{3'2}a_{3'3} & 2a_{3'1}a_{3'3} & 2a_{3'1}a_{3'2} \\ a_{2'1}a_{3'1} & a_{2'2}a_{3'2} & a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\ a_{1'1}a_{3'1} & a_{1'2}a_{3'2} & a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\ a_{1'1}a_{2'1} & a_{1'2}a_{2'2} & a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1}) \end{bmatrix}$$

reduces to

the diagonal matrix

$$[T_\sigma] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_2 = 0$ (CONTINUED)

- Also, because $\left[T_\sigma\right]$ is a diagonal matrix, it follows that $\left[T_\varepsilon\right] = \left[T_\sigma\right]$

- Like before, the sufficient conditions for a state of *elastic reflective symmetry* about the plane $x_2 = 0$ to exist are found from

$$[C][T_\sigma] = \left([C][T_\sigma]\right)^T \qquad [S][T_\sigma] = \left([S][T_\sigma]\right)^T \qquad \{\alpha\} = [T_\varepsilon]\{\alpha\} \qquad \{\beta\} = [T_\sigma]\{\beta\}$$

- Computing $[C][T_\sigma]$ gives

$$[C][T_\sigma] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & -C_{14} & C_{15} & -C_{16} \\ C_{12} & C_{22} & C_{23} & -C_{24} & C_{25} & -C_{26} \\ C_{13} & C_{23} & C_{33} & -C_{34} & C_{35} & -C_{36} \\ C_{14} & C_{24} & C_{34} & -C_{44} & C_{45} & -C_{46} \\ C_{15} & C_{25} & C_{35} & -C_{45} & C_{55} & -C_{56} \\ C_{16} & C_{26} & C_{36} & -C_{46} & C_{56} & -C_{66} \end{bmatrix}$$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_2 = 0$ (CONTINUED)

- **Enforcing $[C][T_\sigma] = ([C][T_\sigma])^T$ yields the following requirements on the stiffnesses:**

  $C_{14} = 0$, $C_{16} = 0$, $C_{24} = 0$, $C_{26} = 0$, $C_{34} = 0$, $C_{36} = 0$, $C_{45} = 0$, **and** $C_{56} = 0$

- **Thus, the stiffness matrix for a monoclinic material that a state of *elastic reflective symmetry* about the plane $x_2 = 0$ has the form**

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{46} \\ C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{46} & 0 & C_{66} \end{bmatrix}$$

which also has 13 independent stiffnesses

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_2 = 0$ (CONTINUED)

- **Likewise, the compliance matrix for a monoclinic material that a state of *elastic reflective symmetry* about the plane $x_2 = 0$ has the form**

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & S_{15} & 0 \\ S_{12} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ S_{13} & S_{23} & S_{33} & 0 & S_{35} & 0 \\ 0 & 0 & 0 & S_{44} & 0 & S_{46} \\ S_{15} & S_{25} & S_{35} & 0 & S_{55} & 0 \\ 0 & 0 & 0 & S_{46} & 0 & S_{66} \end{bmatrix}$$

which also has 13 independent compliances

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_2 = 0$ (CONTINUED)

- Next, computing $\{\alpha\} = [T_\varepsilon]\{\alpha\}$ gives

$$\begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix}$$

which reduces to

$$\begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} = \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ -2\alpha_{23} \\ 2\alpha_{13} \\ -2\alpha_{12} \end{Bmatrix}$$

- Thus, enforcing $\{\alpha\} = [T_\varepsilon]\{\alpha\}$ requires $\alpha_{12} = \alpha_{23} = 0$ in order for a state of *elastic reflective symmetry* about the plane $x_2 = 0$ to exist

- Similarly, enforcing $\{\beta\} = [T_\sigma]\{\beta\}$ requires $\beta_{12} = \beta_{23} = 0$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_2 = 0$ (CONTINUED)

- Applying all the simplifications, the ***linear thermoelastic constitutive equations*** become

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & S_{15} & 0 \\ S_{12} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ S_{13} & S_{23} & S_{33} & 0 & S_{35} & 0 \\ 0 & 0 & 0 & S_{44} & 0 & S_{46} \\ S_{15} & S_{25} & S_{35} & 0 & S_{55} & 0 \\ 0 & 0 & 0 & S_{46} & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 2\alpha_{13} \\ 0 \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{46} \\ C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{46} & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ \beta_{13} \\ 0 \end{Bmatrix} (T - T_{ref})$$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_2 = 0$ (CONCLUDED)

- The nonzero thermal moduli are given in terms of the coefficients of thermal expansion by

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{13} \end{Bmatrix} = - \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{15} \\ C_{12} & C_{22} & C_{23} & C_{25} \\ C_{13} & C_{23} & C_{33} & C_{35} \\ C_{15} & C_{25} & C_{35} & C_{55} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{13} \end{Bmatrix}$$

- The constitutive equations show that the normal stresses, or a temperature change, produce shearing deformations only in the plane $x_2 = 0$

  - Extension and shearing are totally uncoupled in the planes $x_1 = 0$ and $x_3 = 0$

MONOCLINIC MATERIALS

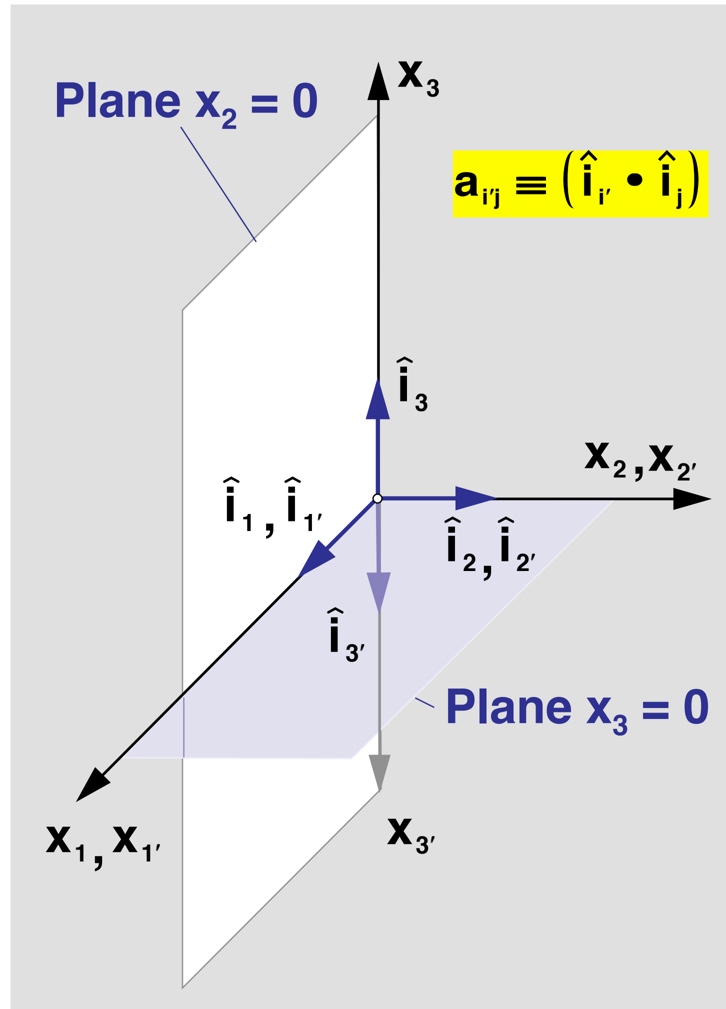
REFLECTIVE SYMMETRY ABOUT THE PLANE $x_3 = 0$

- Now, consider the case in which the material exhibits symmetry about the plane $x_3 = 0$

- The coordinate transformation for this symmetry is shown in the figure and is given by $x_{1'} = x_1$, $x_{2'} = x_2$, and $x_{3'} = -x_3$

- The corresponding matrix of **direction cosines** is given by

$$\begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_3 = 0$ (CONTINUED)

- For this special case, the general transformation matrix

$$
[T_\sigma] = \begin{bmatrix}
(a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & 2a_{1'2}a_{1'3} & 2a_{1'1}a_{1'3} & 2a_{1'1}a_{1'2} \\
(a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & 2a_{2'2}a_{2'3} & 2a_{2'1}a_{2'3} & 2a_{2'1}a_{2'2} \\
(a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & 2a_{3'2}a_{3'3} & 2a_{3'1}a_{3'3} & 2a_{3'1}a_{3'2} \\
a_{2'1}a_{3'1} & a_{2'2}a_{3'2} & a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\
a_{1'1}a_{3'1} & a_{1'2}a_{3'2} & a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\
a_{1'1}a_{2'1} & a_{1'2}a_{2'2} & a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1})
\end{bmatrix}
$$

reduces to

the diagonal matrix

$$
[T_\sigma] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_3 = 0$ (CONTINUED)

- **Also, because $[T_\sigma]$ is a diagonal matrix, it follows that $[T_\varepsilon] = [T_\sigma]$**

- **Like before, the sufficient conditions for a state of *elastic reflective symmetry* about the plane $x_3 = 0$ to exist are found from**

$$[C][T_\sigma] = ([C][T_\sigma])^T \qquad [S][T_\sigma] = ([S][T_\sigma])^T \qquad \{\alpha\} = [T_\varepsilon]\{\alpha\} \qquad \{\beta\} = [T_\sigma]\{\beta\}$$

- **Computing $[C][T_\sigma]$ gives**

$$[C][T_\sigma] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & -C_{14} & -C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & -C_{24} & -C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & -C_{34} & -C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & -C_{44} & -C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & -C_{45} & -C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & -C_{46} & -C_{56} & C_{66} \end{bmatrix}$$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_3 = 0$ (CONTINUED)

- Enforcing $[C][T_\sigma] = ([C][T_\sigma])^T$ yields the following requirements on the stiffnesses:

$C_{14} = 0$, $C_{15} = 0$, $C_{24} = 0$, $C_{25} = 0$, $C_{34} = 0$, $C_{35} = 0$, $C_{46} = 0$, and $C_{56} = 0$

- Thus, the stiffness matrix for a **monoclinic material** that a state of *elastic reflective symmetry* about the plane $x_3 = 0$ has the form

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

which also has **13 independent stiffnesses**

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_3 = 0$ (CONTINUED)

- **Likewise, the compliance matrix for a monoclinic material that a state of *elastic reflective symmetry* about the plane $x_3 = 0$ has the form**

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix}$$

which also has 13 independent compliances

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_3 = 0$ (CONTINUED)

- Next, computing $\{\alpha\} = [T_\varepsilon]\{\alpha\}$ gives

$$\begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix}$$

which reduces to

$$\begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} = \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ -2\alpha_{23} \\ -2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix}$$

- Thus, enforcing $\{\alpha\} = [T_\varepsilon]\{\alpha\}$ requires $\alpha_{13} = \alpha_{23} = 0$ in order for a state of *elastic reflective symmetry* about the plane $x_3 = 0$ to exist

- Similarly, enforcing $\{\beta\} = [T_\sigma]\{\beta\}$ requires $\beta_{13} = \beta_{23} = 0$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_3 = 0$ (CONTINUED)

- Applying all the simplifications, the ***linear thermoelastic constitutive equations*** become

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right)$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ \beta_{12} \end{Bmatrix} \left(T - T_{ref}\right)$$

MONOCLINIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANE $x_3 = 0$ (CONCLUDED)

- The nonzero thermal moduli are given in terms of the coefficients of thermal expansion by

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{12} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{36} \\ C_{16} & C_{26} & C_{36} & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{12} \end{Bmatrix}$$

- The constitutive equations show that the normal stresses, or a temperature change, produce shearing deformations only in the plane $x_3 = 0$

    - Extension and shearing are totally uncoupled in the planes $x_1 = 0$ and $x_2 = 0$

ORTHOTROPIC MATERIALS

ORTHOTROPIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$ AND $x_2 = 0$

- Now, consider the case in which a monoclinic material, which exhibits symmetry about the plane $x_1 = 0$, also exhibits symmetry about the plane $x_2 = 0$, which is perpendicular to the plane $x_1 = 0$

- For this *monoclinic material*, it was shown previously that the material properties are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

and

ORTHOTROPIC MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$ AND $x_2 = 0$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ S_{12} & S_{22} & S_{23} & S_{24} & 0 & 0 \\ S_{13} & S_{23} & S_{33} & S_{34} & 0 & 0 \\ S_{14} & S_{24} & S_{34} & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & S_{56} \\ 0 & 0 & 0 & 0 & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

- **To determine the effects of the second symmetry plane, $x_2 = 0$, the coordinate transformation given by $x_{1'} = x_1$, $x_{2'} = -x_2$, and $x_{3'} = x_3$ is applied to the material properties of the monoclinic material**
  - **This process is the same as applying the transformation for symmetry about the plane $x_2 = 0$ in succession to applying the transformation for symmetry about the plane $x_1 = 0$ to the initial anisotropic-material properties**

ORTHOTROPIC MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$ AND $x_2 = 0$

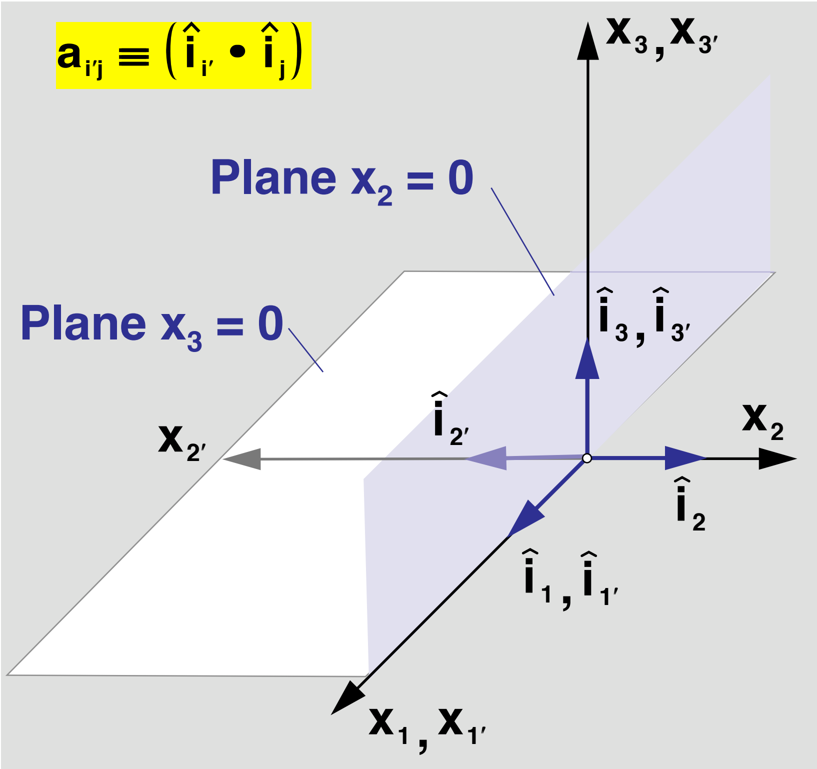
- The effect of the second transformation, for symmetry about the plane $x_2 = 0$, is obtained directly from the results given previously for a monoclinic material that exhibits symmetry about the plane $x_2 = 0$

  - That is, the second coordinate transformation was given by

    $x_{1'} = x_1$, $x_{2'} = -x_2$, and $x_{3'} = x_3$

  - And, the corresponding matrix of **direction cosines** were shown to be given by

$$\begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



ORTHOTROPIC MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$ AND $x_2 = 0$

- For this case, $$[T_\varepsilon] = [T_\sigma] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

- It was also shown that, for symmetry about the plane $x_2 = 0$,

$C_{14} = 0$, $C_{16} = 0$, $C_{24} = 0$, $C_{26} = 0$, $C_{34} = 0$, $C_{36} = 0$, $C_{45} = 0$, $C_{56} = 0$,

$S_{14} = 0$, $S_{16} = 0$, $S_{24} = 0$, $S_{26} = 0$, $S_{34} = 0$, $S_{36} = 0$, $S_{45} = 0$, $S_{56} = 0$,

$\alpha_{12} = 0$, $\alpha_{23} = 0$, $\beta_{12} = 0$, and $\beta_{23} = 0$

ORTHOTROPIC MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$ AND $x_2 = 0$

- The net effects of applying the two symmetry transformations successively is obtained by applying the conditions given on the previous page to the constitutive equations for the monoclinic material that exhibits symmetry about the plane $x_1 = 0$

- This process yields

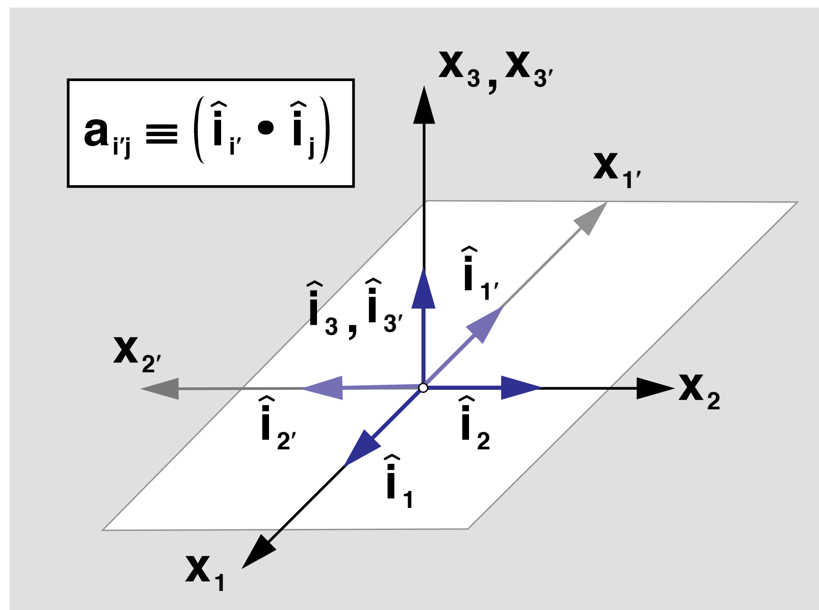
$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

ORTHOTROPIC MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$ AND $x_2 = 0$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

- It is worth pointing out at this point in the development, that the single coordinate transformation given by $x_{1'} = -x_1$, $x_{2'} = -x_2$, and $x_{3'} = x_3$ does not produce the same result as the two successive symmetry transformations



ORTHOTROPIC MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$ AND $x_2 = 0$

- In particular, the corresponding matrix of **direction cosines** is given by

$$\begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The corresponding stress and strain transformation matrices are

$$[T_\sigma] = [T_\varepsilon] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

ORTHOTROPIC MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$ AND $x_2 = 0$

- Next, the condition $[C] = [T_\sigma][C][T_\varepsilon]^{-1}$ for a state of symmetry to exist at a point **P** of the body is applied

  - Using that $[T_\varepsilon] = [T_\sigma]$ for this particular transformation, it was shown that the condition $[C] = [T_\sigma][C][T_\varepsilon]^{-1}$ simplifies to the condition that $[C][T_\sigma]$ must be a **symmetric matrix**

- Computing $[C][T_\sigma]$ gives

$$[C][T_\sigma] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & -C_{14} & -C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & -C_{24} & -C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & -C_{34} & -C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & -C_{44} & -C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & -C_{45} & -C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & -C_{46} & -C_{56} & C_{66} \end{bmatrix}$$

ORTHOTROPIC MATERIALS - CONCLUDED

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$ AND $x_2 = 0$

- Enforcing $[C][T_\sigma] = ([C][T_\sigma])^T$ yields the following requirements on the stiffnesses:

  $C_{14} = 0$, $C_{15} = 0$, $C_{24} = 0$, $C_{25} = 0$, $C_{34} = 0$, $C_{35} = 0$, $C_{46} = 0$, and $C_{56} = 0$

- Inspection of the conditions and comparison with the previous result for the two successive transformations shows that the results are different

ORTHOTROPIC MATERIALS

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$, $x_2 = 0$, AND $x_3 = 0$

- **The next step in the analysis of an orthotropic material is to consider the effects of yet a third, successive transformation**
  - **That is, a transformation for which the material exhibits symmetry about the perpendicular plane $x_3 = 0$, in addition to symmetry about the perpendicular planes $x_1 = 0$ and $x_2 = 0$**

- **The effect of this third symmetry transformation is obtained directly by applying the conditions obtained for a monoclinic that exhibits symmetry about the plane $x_3 = 0$ to the constitutive equations that were obtained previously for the two successive symmetry transformations**
  - **The conditions for symmetry about the plane $x_3 = 0$ are $C_{14} = 0$, $C_{15} = 0$, $C_{24} = 0$, $C_{25} = 0$, $C_{34} = 0$, $C_{35} = 0$, $C_{46} = 0$, $C_{56} = 0$, $S_{14} = 0$, $S_{15} = 0$, $S_{24} = 0$, $S_{25} = 0$, $S_{34} = 0$, $S_{35} = 0$, $S_{46} = 0$, $S_{56} = 0$, $\alpha_{13} = 0$, $\alpha_{23} = 0$, $\beta_{13} = 0$, and $\beta_{23} = 0$**

ORTHOTROPIC MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT THE PLANES $x_1 = 0$, $x_2 = 0$, AND $x_3 = 0$

- **Examination of these conditions indicates that the third successive transformation yields no new conditions on the constitutive equations that are not obtained from the first two successive transformations**

  - **Therefore, two perpendiular planes of material symmetry imply the existence of a third mutually perpendicular plane**

- **An orthotropic material (that is, an *orthogonally anisotropic* material) is defined as a material that has three mutually perpendicular planes of elastic symmetry**

- **An orthotropic material has 9 independent stiffnesses, 9 independent compliances, 3 independent coefficients of thermal expansion, and 3 independent thermal moduli**

CONSTITUTIVE EQUATIONS

- The constitutive equations for a linear, thermoelastic, orthotropic material are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right) \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

CONSTITUTIVE EQUATIONS - CONCLUDED

- The **thermal moduli** are given in terms of the coefficients of thermal expansion by

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \end{Bmatrix} = - \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \end{Bmatrix}$$

- The constitutive equations show that extension and shearing are totally uncoupled in the planes $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$

- When using orthotropic materials with various directional orientations in a structure, the structural coordinate system must be distinguished from the coordinate systems of the orthotropic materials

  - The structural coordinate system is typically picked to facilitate the geometric representation of the structure

- For convenience, the coordinate system of an orthotropic material with the previously derived constitutive equations is defined as the **principal material coordinate system** and the material is referred to as a **specially orthotropic material**

SPECIALLY ORTHOTROPIC MATERIALS

- **Any material that is fully characterized by the following constitutive equations is defined as a specially orthotropic material**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

- **The corresponding coordinate system used to describe this material is defined as the principal material coordinate system**

GENERALLY ORTHOTROPIC MATERIALS

- For an arbitrary coordinate transformation, from a principal $\{x_1, x_2, x_3\}$ coordinate frame to a $\{x_{1'}, x_{2'}, x_{3'}\}$ coordinate frame, the transformation matrices $[T_\varepsilon]$ and $[T_\sigma]$ are fully populated

- Thus, when the elastic stiffness coefficients of a specially orthotropic solid are transformed from the $\{x_1, x_2, x_3\}$ coordinate frame to the $\{x_{1'}, x_{2'}, x_{3'}\}$ coordinate frame, the matrices of transformed elastic constants are also fully populated

    - To an observer, without prior knowledge of the material, the solid **appears to be anisotropic**

- When a coordinate frame exists for a solid in which it is specially orthotropic, the material is referred to ***generally orthotropic***, to distinguish it from an anisotropic material

GENERALLY ORTHOTROPIC MATERIALS

CONTINUED

- For a **dextral rotation** about the x_3 - axis, with $m = \cos\theta_3$ and $n = \sin\theta_3$, the transformed elastic constants are given by

$$C_{1'1'} = m^4 C_{11} + 2m^2 n^2 (C_{12} + 2C_{66}) + n^4 C_{22}$$

$$C_{1'2'} = m^2 n^2 (C_{11} + C_{22} - 4C_{66}) + (m^4 + n^4) C_{12}$$

$$C_{1'3'} = m^2 C_{13} + n^2 C_{23} \qquad C_{1'4'} = 0 \qquad C_{1'5'} = 0$$

$$C_{1'6'} = mn[m^2 - n^2](C_{12} + 2C_{66}) + mn(n^2 C_{22} - m^2 C_{11})$$

$$C_{2'2'} = m^4 C_{22} + 2m^2 n^2 (C_{12} + 2C_{66}) + n^4 C_{11}$$

$$C_{2'3'} = m^2 C_{23} + n^2 C_{13} \qquad C_{2'4'} = 0 \qquad C_{2'5'} = 0$$

$$C_{2'6'} = mn[n^2 - m^2](C_{12} + 2C_{66}) + mn(m^2 C_{22} - n^2 C_{11})$$

GENERALLY ORTHOTROPIC MATERIALS

CONTINUED

$$C_{3'3'} = C_{33} \qquad C_{3'4'} = 0 \qquad C_{3'5'} = 0 \qquad C_{3'6'} = mn\left(C_{23} - C_{13}\right)$$

$$C_{4'4'} = m^2 C_{44} + n^2 C_{55} \qquad C_{4'5'} = mn\left(C_{44} - C_{55}\right) \qquad C_{4'6'} = 0 \qquad C_{5'6'} = 0$$

$$C_{5'5'} = m^2 C_{55} + n^2 C_{44} \qquad C_{6'6'} = m^2 n^2\left(C_{11} + C_{22} - 2C_{12}\right) + \left(m^2 - n^2\right)^2 C_{66}$$

- The population of the matrix of transformed elastic stiffnesses is given by

$$\begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & 0 & 0 & C_{1'6'} \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & 0 & 0 & C_{2'6'} \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & 0 & 0 & C_{3'6'} \\ 0 & 0 & 0 & C_{4'4'} & C_{4'5'} & 0 \\ 0 & 0 & 0 & C_{4'5'} & C_{5'5'} & 0 \\ C_{1'6'} & C_{2'6'} & C_{3'6'} & 0 & 0 & C_{6'6'} \end{bmatrix}$$

- For the $\left\{x_{1'}, x_{2'}, x_{3'}\right\}$ coordinate frame, the specially orthotropic material appears to have the properties of a **monoclinic material**

GENERALLY ORTHOTROPIC MATERIALS

CONTINUED

- **Similarly,**

$$S_{1'1'} = m^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + n^4 S_{22}$$

$$S_{1'2'} = m^2 n^2 (S_{11} + S_{22} - S_{66}) + (m^4 + n^4) S_{12}$$

$$S_{1'3'} = m^2 S_{13} + n^2 S_{23} \qquad S_{1'4'} = 0 \qquad S_{1'5'} = 0$$

$$S_{1'6'} = mn[m^2 - n^2](2S_{12} + S_{66}) + 2mn(n^2 S_{22} - m^2 S_{11})$$

$$S_{2'2'} = m^4 S_{22} + m^2 n^2 (2S_{12} + S_{66}) + n^4 S_{11} \qquad S_{2'3'} = m^2 S_{23} + n^2 S_{13}$$

$$S_{2'4'} = 0 \qquad S_{2'5'} = 0 \qquad S_{3'3'} = S_{33} \qquad S_{3'4'} = 0 \qquad S_{3'5'} = 0$$

$$S_{2'6'} = mn[n^2 - m^2](2S_{12} + S_{66}) + 2mn(m^2 S_{22} - n^2 S_{11})$$

$$S_{3'6'} = 2mn(S_{23} - S_{13})$$

GENERALLY ORTHOTROPIC MATERIALS

CONTINUED

$$S_{4'4'} = m^2 S_{44} + n^2 S_{55} \qquad S_{4'5'} = mn\left(S_{44} - S_{55}\right) \qquad S_{4'6'} = 0$$

$$S_{5'5'} = m^2 S_{55} + n^2 S_{44} \qquad S_{5'6'} = 0$$

$$S_{6'6'} = 4m^2 n^2 \left(S_{11} + S_{22} - 2S_{12}\right) + \left(m^2 - n^2\right)^2 S_{66}$$

- The population of the matrix of transformed elastic compliances is given by

$$\begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & 0 & 0 & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & 0 & 0 & S_{2'6'} \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & 0 & 0 & S_{3'6'} \\ 0 & 0 & 0 & S_{4'4'} & S_{4'5'} & 0 \\ 0 & 0 & 0 & S_{4'5'} & S_{5'5'} & 0 \\ S_{1'6'} & S_{2'6'} & S_{3'6'} & 0 & 0 & S_{6'6'} \end{bmatrix}$$

- For the $\left\{x_{1'}, x_{2'}, x_{3'}\right\}$ coordinate frame, the specially orthotropic material, again, appears to have the properties of a *monoclinic material*

GENERALLY ORTHOTROPIC MATERIALS

CONTINUED

- For a **dextral rotation** about the x_3 - axis, the transformed thermal-expansion coefficients are given by

$$\alpha_{1'1'} = m^2\alpha_{11} + n^2\alpha_{22} \qquad \alpha_{2'2'} = m^2\alpha_{22} + n^2\alpha_{11} \qquad \alpha_{3'3'} = \alpha_{33}$$

$$\alpha_{2'3'} = 0 \qquad \alpha_{1'3'} = 0 \qquad \alpha_{1'2'} = mn\left(\alpha_{22} - \alpha_{11}\right)$$

- Similarly, the transformed thermal moduli are given by

$$\beta_{1'1'} = m^2\beta_{11} + n^2\beta_{22} \qquad \beta_{2'2'} = m^2\beta_{22} + n^2\beta_{11} \qquad \beta_{3'3'} = \beta_{33}$$

$$\beta_{2'3'} = 0 \qquad \beta_{1'3'} = 0 \qquad \beta_{1'2'} = mn\left(\beta_{22} - \beta_{11}\right)$$

GENERALLY ORTHOTROPIC MATERIALS
CONCLUDED

- Thus, for a **dextral rotation** about the x_3 - axis, the transformed constitutive equations for a specially orthotropic material are given by

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & 0 & 0 & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & 0 & 0 & S_{2'6'} \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & 0 & 0 & S_{3'6'} \\ 0 & 0 & 0 & S_{4'4'} & S_{4'5'} & 0 \\ 0 & 0 & 0 & S_{4'5'} & S_{5'5'} & 0 \\ S_{1'6'} & S_{2'6'} & S_{3'6'} & 0 & 0 & S_{6'6'} \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 0 \\ 0 \\ 2\alpha_{1'2'} \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & 0 & 0 & C_{1'6'} \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & 0 & 0 & C_{2'6'} \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & 0 & 0 & C_{3'6'} \\ 0 & 0 & 0 & C_{4'4'} & C_{4'5'} & 0 \\ 0 & 0 & 0 & C_{4'5'} & C_{5'5'} & 0 \\ C_{1'6'} & C_{2'6'} & C_{3'6'} & 0 & 0 & C_{6'6'} \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \beta_{1'1'} \\ \beta_{2'2'} \\ \beta_{3'3'} \\ 0 \\ 0 \\ \beta_{1'2'} \end{Bmatrix} (T - T_{ref})$$

TRIGONAL MATERIALS

TRIGONAL MATERIALS

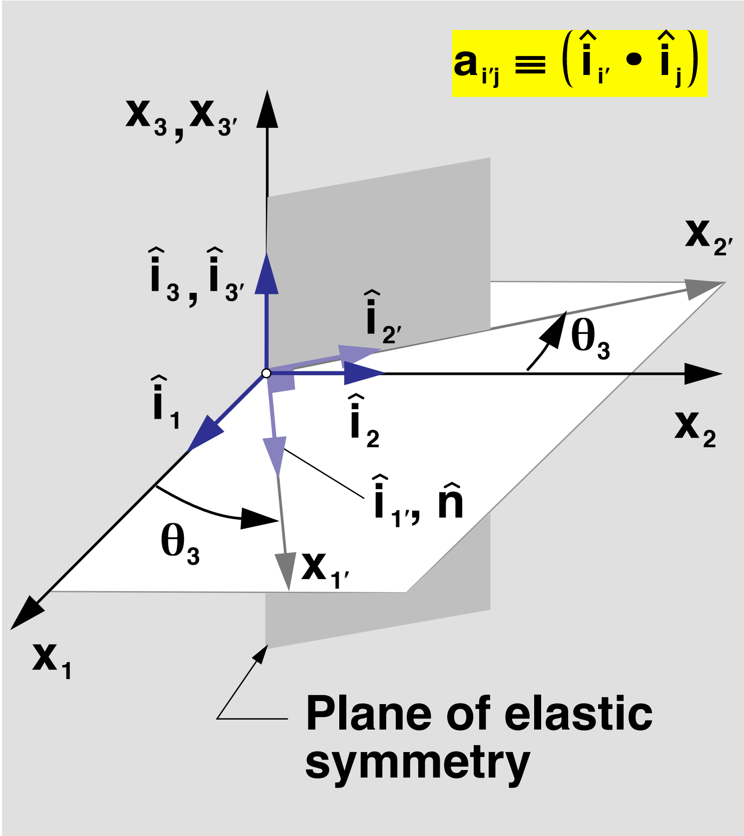
REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- To determine the conditions on the constitutive equations for **trigonal materials**, it is necessary to consider a plane of elastic symmetry that is oriented arbitrarily, with respect to two of the coordinate axes

- In particular, consider a **plane of elastic symmetry** whose normal $\hat{n}$ lies in the plane $x_3 = 0$ and makes an angle θ_3 with the x_1 axis, as shown in the figure

- The angle θ_3 is defined to be in the range $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$, because $\theta_3 = -\frac{\pi}{2}$ and $\theta_3 = \frac{\pi}{2}$ define the same plane

- In addition, let the $(x_{1'}, x_{2'}, x_{3'})$ be the coordinates used to define the material symmetry



TRIGONAL MATERIALS - CONTINUED

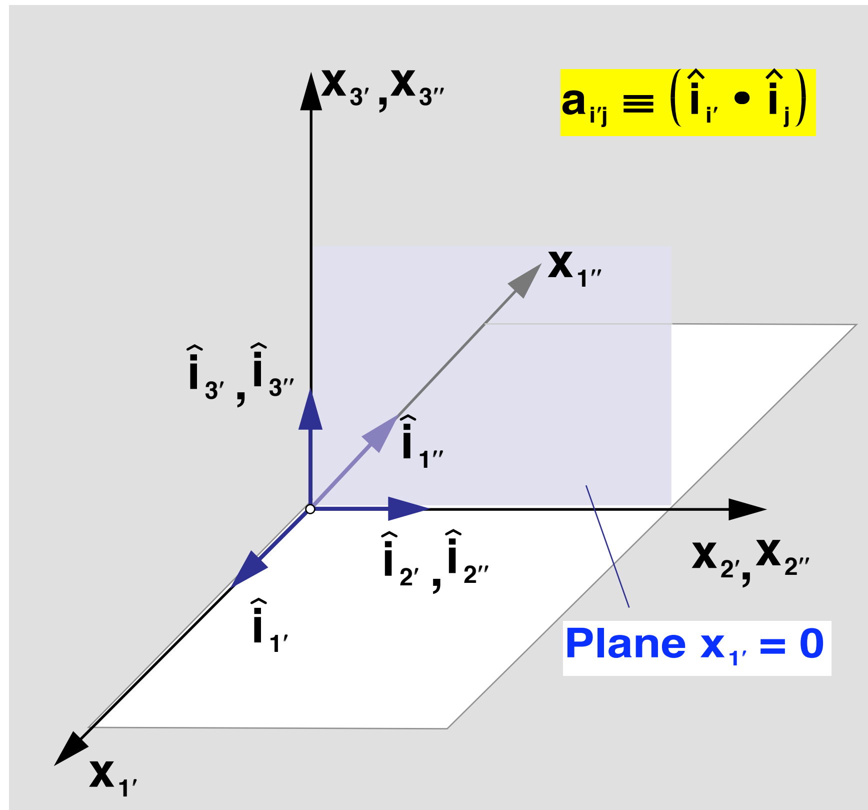
REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- Specifically, for a plane of elastic symmetry given by $x_{1'} = 0$, the **symmetry transformation** is shown in the figure and is given by $x_{1''} = -x_{1'}$, $x_{2''} = x_{2'}$, and $x_{3''} = x_{3'}$

- The corresponding matrix of **direction cosines** is given by

$$\begin{bmatrix} a_{1''1'} & a_{1''2'} & a_{1''3'} \\ a_{2''1'} & a_{2''2'} & a_{2''3'} \\ a_{3''1'} & a_{3''2'} & a_{3''3'} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For the $(x_{1'}, x_{2'}, x_{3'})$ coordinate system, the general constitutive equations are expressed as follows



TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & S_{1'4'} & S_{1'5'} & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & S_{2'4'} & S_{2'5'} & S_{2'6'} \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & S_{3'4'} & S_{3'5'} & S_{3'6'} \\ S_{1'4'} & S_{2'4'} & S_{3'4'} & S_{4'4'} & S_{4'5'} & S_{4'6'} \\ S_{1'5'} & S_{2'5'} & S_{3'5'} & S_{4'5'} & S_{5'5'} & S_{5'6'} \\ S_{1'6'} & S_{2'6'} & S_{3'6'} & S_{4'6'} & S_{5'6'} & S_{6'6'} \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 2\alpha_{2'3'} \\ 2\alpha_{1'3'} \\ 2\alpha_{1'2'} \end{Bmatrix} \left(T - T_{ref}\right)$$

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & C_{1'4'} & C_{1'5'} & C_{1'6'} \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & C_{2'4'} & C_{2'5'} & C_{2'6'} \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & C_{3'4'} & C_{3'5'} & C_{3'6'} \\ C_{1'4'} & C_{2'4'} & C_{3'4'} & C_{4'4'} & C_{4'5'} & C_{4'6'} \\ C_{1'5'} & C_{2'5'} & C_{3'5'} & C_{4'5'} & C_{5'5'} & C_{5'6'} \\ C_{1'6'} & C_{2'6'} & C_{3'6'} & C_{4'6'} & C_{5'6'} & C_{6'6'} \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \beta_{1'1'} \\ \beta_{2'2'} \\ \beta_{3'3'} \\ 2\beta_{2'3'} \\ 2\beta_{1'3'} \\ 2\beta_{1'2'} \end{Bmatrix} \left(T - T_{ref}\right)$$

or by

$\{\varepsilon'\} = [S']\{\sigma'\} + \{\alpha'\}\Theta$ and $\{\sigma'\} = [C']\{\varepsilon'\} + \{\beta'\}\Theta$,

where $\Theta = T - T_{ref}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- For the $(x_{1''}, x_{2''}, x_{3''})$ coordinate system, the constitutive equations are identical in form and are obtained by replacing the index pair $i'j'$ with $i''j''$

  - In the abridged notation,

$$\{\varepsilon''\} = [S'']\{\sigma''\} + \{\alpha''\}\Theta \quad \text{and}$$

$$\{\sigma''\} = [C'']\{\varepsilon''\} + \{\beta''\}\Theta, \quad \text{where} \quad \Theta = T - T_{ref}$$

- For the transformation of coordinates defined by the symmetry transformation (a reflection about the plane $x_{1'} = 0$),

$$\{\sigma''\} = [T_{\sigma}^{r_{1'}}]\{\sigma'\} \quad \text{and} \quad \{\varepsilon''\} = [T_{\varepsilon}^{r_{1'}}]\{\varepsilon'\} \quad \text{where}$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

$$\left[T_{\sigma}^{r_{1'}}\right] = \left[T_{\varepsilon}^{r_{1'}}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

- **Likewise,**

$$[S''] = \left[T_{\varepsilon}^{r_{1'}}\right][S']\left[T_{\sigma}^{r_{1'}}\right]^{-1} \qquad [S'] = \left[T_{\varepsilon}^{r_{1'}}\right]^{-1}[S'']\left[T_{\sigma}^{r_{1'}}\right]$$

$$[C''] = \left[T_{\sigma}^{r_{1'}}\right][C']\left[T_{\varepsilon}^{r_{1'}}\right]^{-1} \qquad [C'] = \left[T_{\sigma}^{r_{1'}}\right]^{-1}[C'']\left[T_{\varepsilon}^{r_{1'}}\right]$$

$$\{\alpha''\} = \left[T_{\varepsilon}^{r_{1'}}\right]\{\alpha'\} \qquad \{\beta''\} = \left[T_{\sigma}^{r_{1'}}\right]\{\beta'\}$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The **conditions for invariance** under the symmetry transformation are given by

$$[S'] = \left[T_{\varepsilon}^{r_{1'}}\right][S']\left[T_{\sigma}^{r_{1'}}\right]^{-1} \qquad [C'] = \left[T_{\sigma}^{r_{1'}}\right][C']\left[T_{\varepsilon}^{r_{1'}}\right]^{-1}$$

$$\{\alpha'\} = \left[T_{\varepsilon}^{r_{1'}}\right]\{\alpha'\} \qquad \{\beta'\} = \left[T_{\sigma}^{r_{1'}}\right]\{\beta'\}$$

- Rather than calculating the outcome of the invariance conditions, the outcome can be found by direct comparison with the results given previously for a material that is **monoclinic** with respect to the plane $x_1 = 0$

  - Direct comparison reveals that the material is **monoclinic** with respect to the plane $x_{1'} = 0$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- That is, direct comparison yields

$$
\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & S_{1'4'} & 0 & 0 \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & S_{2'4'} & 0 & 0 \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & S_{3'4'} & 0 & 0 \\ S_{1'4'} & S_{2'4'} & S_{3'4'} & S_{4'4'} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{5'5'} & S_{5'6'} \\ 0 & 0 & 0 & 0 & S_{5'6'} & S_{6'6'} \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 2\alpha_{2'3'} \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}
$$

$$
\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & C_{1'4'} & 0 & 0 \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & C_{2'4'} & 0 & 0 \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & C_{3'4'} & 0 & 0 \\ C_{1'4'} & C_{2'4'} & C_{3'4'} & C_{4'4'} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{5'5'} & C_{5'6'} \\ 0 & 0 & 0 & 0 & C_{5'6'} & C_{6'6'} \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \beta_{1'1'} \\ \beta_{2'2'} \\ \beta_{3'3'} \\ \beta_{2'3'} \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})
$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- Next, the compliances, stiffnesses, thermal-expansion coefficients, and thermal moduli, expressed in $(x_{1'}, x_{2'}, x_{3'})$ coordinates, are referred back to the original (x_1, x_2, x_3) coordinates

- The transformation corresponds to the dextral rotation about the x_3 axis, shown in the figure, and is given by

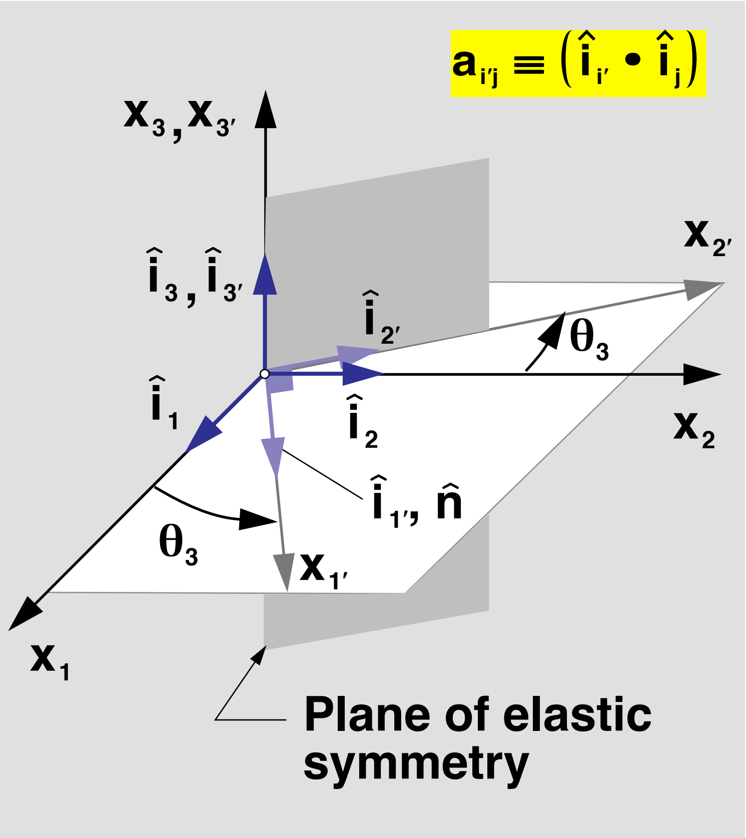
  $x_{1'} = x_1\cos\theta_3 + x_2\sin\theta_3$,

  $x_{2'} = -x_1\sin\theta_3 + x_2\cos\theta_3$, and $x_{3'} = x_3$,

  with $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$

- The corresponding matrix of **direction cosines** is given by

$$\begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The corresponding stress and strain transformation matrices were shown previously to be given by

$$\left[T_\sigma(\theta_3)\right] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -2\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$ and

$$\left[T_\varepsilon(\theta_3)\right] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The transformation laws for the compliances, stiffnesses, thermal-expansion coefficients, and thermal moduli have been given as

$$[C'] = [T_\sigma][C][T_\varepsilon]^{-1} \qquad [S'] = [T_\varepsilon][S][T_\sigma]^{-1}$$

$$\{\beta'\} = [T_\sigma]\{\beta\} \qquad \{\alpha'\} = [T_\varepsilon]\{\alpha\}$$

- These laws transform the previously obtained invariance conditions on $[C']$, $[S']$, $\{\alpha'\}$, and $\{\beta'\}$ into invariance conditions on $[C]$, $[S]$, $\{\alpha\}$, and $\{\beta\}$

- Specific expressions for these transformation laws (dextral rotation about the x_3 axis) were given previously for a fully anisotropic material (triclinic)

  - Note that the matrices $[C]$ and $[S]$ are fully populated

  - Also, the vectors $\{\alpha\}$ and $\{\beta\}$ are fully populated

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- Because of the invariance conditions, the matrices $[C']$ and $[S']$ have the form

$$\begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & C_{1'4'} & 0 & 0 \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & C_{2'4'} & 0 & 0 \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & C_{3'4'} & 0 & 0 \\ C_{1'4'} & C_{2'4'} & C_{3'4'} & C_{4'4'} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{5'5'} & C_{5'6'} \\ 0 & 0 & 0 & 0 & C_{5'6'} & C_{6'6'} \end{bmatrix} \text{ and } \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & S_{1'4'} & 0 & 0 \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & S_{2'4'} & 0 & 0 \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & S_{3'4'} & 0 & 0 \\ S_{1'4'} & S_{2'4'} & S_{3'4'} & S_{4'4'} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{5'5'} & S_{5'6'} \\ 0 & 0 & 0 & 0 & S_{5'6'} & S_{6'6'} \end{bmatrix}$$

- Also, the vectors $\{\alpha'\}$ and $\{\beta'\}$ have the form $\begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 2\alpha_{2'3'} \\ 0 \\ 0 \end{Bmatrix}$ and $\begin{Bmatrix} \beta_{1'1'} \\ \beta_{2'2'} \\ \beta_{3'3'} \\ \beta_{2'3'} \\ 0 \\ 0 \end{Bmatrix}$

- The corresponding forms of $[C]$, $[S]$, $\{\alpha\}$, and $\{\beta\}$ are obtained from the transformation laws once a value for the angle θ_3 is specified

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- For example, specifying $\theta_3 = 0$ yields

$$\begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & C_{1'4'} & 0 & 0 \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & C_{2'4'} & 0 & 0 \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & C_{3'4'} & 0 & 0 \\ C_{1'4'} & C_{2'4'} & C_{3'4'} & C_{4'4'} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{5'5'} & C_{5'6'} \\ 0 & 0 & 0 & 0 & C_{5'6'} & C_{6'6'} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \qquad \begin{Bmatrix} \beta_{1'1'} \\ \beta_{2'2'} \\ \beta_{3'3'} \\ \beta_{2'3'} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix}$$

$$\begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & S_{1'4'} & 0 & 0 \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & S_{2'4'} & 0 & 0 \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & S_{3'4'} & 0 & 0 \\ S_{1'4'} & S_{2'4'} & S_{3'4'} & S_{4'4'} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{5'5'} & S_{5'6'} \\ 0 & 0 & 0 & 0 & S_{5'6'} & S_{6'6'} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \qquad \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 2\alpha_{2'3'} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix}$$

- Enforcing the invariance conditions on $[C']$ gives

$C_{15} = 0$, $C_{16} = 0$, $C_{25} = 0$, $C_{26} = 0$, $C_{35} = 0$, $C_{36} = 0$, $C_{45} = 0$, and $C_{46} = 0$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- Likewise, enforcing the invariance conditions on $[S']$ gives

  $S_{15} = 0$, $S_{16} = 0$, $S_{25} = 0$, $S_{26} = 0$, $S_{35} = 0$, $S_{36} = 0$, $S_{45} = 0$, and $S_{46} = 0$

- Enforcing the invariance conditions on $\{\alpha'\}$ and $\{\beta'\}$ gives

  $$\alpha_{12} = \alpha_{13} = 0 \quad \text{and} \quad \beta_{12} = \beta_{13} = 0$$

- These conditions are identical to the conditions previously obtained for a material that is **monoclinic** with respect to the plane $x_1 = 0$

- Similarly, specifying $\theta_3 = \frac{\pi}{2}$ and enforcing the invariance conditions on $[C']$, $[S']$, $\{\alpha'\}$, and $\{\beta'\}$ yields the conditions previously obtained for a material that is **monoclinic** with respect to the plane $x_2 = 0$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- Now consider an arbitrary value for the angle θ_3 such that $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$

- The previous example for $\theta_3 = 0$ shows that the invariance conditions on $[C]$, $[S]$, $\{\alpha\}$, and $\{\beta\}$ were obtained by the terms of $[C']$, $[S']$, $\{\alpha'\}$, and $\{\beta'\}$ that were zero valued

- Using the transformation equations for a dextral rotation about the x_3 axis that were given previously for a (triclinic) fully anisotropic, elastic material gives the following results for the invariance conditions on $[C']$

$$C_{1'5'} = 0: \quad m^3C_{15} + m^2n\left(2C_{56} + C_{14}\right) + mn^2\left(2C_{46} + C_{25}\right) + n^3C_{24} = 0$$

$$\begin{aligned} C_{1'6'} = 0: \quad & m^2\left(m^2 - 3n^2\right)C_{16} - m^3n\left(C_{11} - C_{12} - 2C_{66}\right) \\ & + mn^3\left(C_{22} - C_{12} - 2C_{66}\right) - n^2\left(n^2 - 3m^2\right)C_{26} = 0 \end{aligned}$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

$$C_{2'5'} = 0: \quad m^3C_{25} - m^2n(2C_{56} - C_{24}) - mn^2(2C_{46} - C_{15}) + n^3C_{14} = 0$$

$$C_{2'6'} = 0: \quad m^2(m^2 - 3n^2)C_{26} + m^3n(C_{22} - C_{12} - 2C_{66}) - mn^3(C_{11} - C_{12} - 2C_{66}) - n^2(n^2 - 3m^2)C_{16} = 0$$

$$C_{3'5'} = 0: \quad mC_{35} + nC_{34} = 0$$

$$C_{3'6'} = 0: \quad (m^2 - n^2)C_{36} + mn(C_{23} - C_{13}) = 0$$

$$C_{4'5'} = 0: \quad (m^2 - n^2)C_{45} + mn(C_{44} - C_{55}) = 0$$

$$C_{4'6'} = 0: \quad m^3C_{46} - m^2n(C_{56} + C_{14} - C_{24}) - mn^2(C_{46} - C_{15} + C_{25}) + n^3C_{56} = 0$$

with $m = \cos\theta_3$ and $n = \sin\theta_3$

- These conditions give 8 equations and 20 unknowns

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The invariance conditions on $[S']$ are given by

$$S_{1'5'} = 0: \quad m^3 S_{15} + m^2 n(S_{56} + S_{14}) + mn^2(S_{46} + S_{25}) + n^3 S_{24} = 0$$

$$S_{1'6'} = 0: \quad m^2(m^2 - 3n^2)S_{16} - m^3 n(2S_{11} - 2S_{12} - S_{66}) + mn^3(2S_{22} - 2S_{12} - S_{66}) - n^2(n^2 - 3m^2)S_{26} = 0$$

$$S_{2'5'} = 0: \quad m^3 S_{25} - m^2 n(S_{56} - S_{24}) - mn^2(S_{46} - S_{15}) + n^3 S_{14} = 0$$

$$S_{2'6'} = 0: \quad m^2(m^2 - 3n^2)S_{26} + m^3 n(2S_{22} - 2S_{12} - S_{66}) - mn^3(2S_{11} - 2S_{12} - S_{66}) - n^2(n^2 - 3m^2)S_{16} = 0$$

$$S_{3'5'} = 0: \quad mS_{35} + nS_{34} = 0 \qquad S_{3'6'} = 0: \quad (m^2 - n^2)S_{36} + 2mn(S_{23} - S_{13}) = 0$$

$$S_{4'5'} = 0: \quad (m^2 - n^2)S_{45} + mn(S_{44} - S_{55}) = 0$$

$$S_{4'6'} = 0: \quad m^3 S_{46} - m^2 n(S_{56} + 2S_{14} - 2S_{24}) - mn^2(S_{46} - 2S_{15} + 2S_{25}) + n^3 S_{56} = 0$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The invariance conditions on $\{\alpha'\}$ and $\{\beta'\}$ yield

$$\alpha_{1'3'} = 0: \quad m\alpha_{13} + n\alpha_{23} = 0 \qquad \alpha_{1'2'} = 0: \quad \left(m^2 - n^2\right)\alpha_{12} + mn\left(\alpha_{22} - \alpha_{11}\right) = 0$$

$$\beta_{1'3'} = 0: \quad m\beta_{13} + n\beta_{23} = 0 \qquad \beta_{1'2'} = 0: \quad \left(m^2 - n^2\right)\beta_{12} + mn\left(\beta_{22} - \beta_{11}\right) = 0$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- **In determining the restrictions on the compliances, stiffnesses, thermal-expansion coefficients, and thermal moduli for materials that possess more than one plane of elastic symmetry, all of which contain the x_3 axis, it is convenient to select the first plane to be given by $\theta_3 = 0$**

  - **Thus, the material is monoclinic with respect to the plane $x_1 = 0$ and, as a result, the following conditions hold**

$$C_{15} = 0 \quad C_{16} = 0 \quad C_{25} = 0 \quad C_{26} = 0 \quad C_{35} = 0 \quad C_{36} = 0 \quad C_{45} = 0 \quad C_{46} = 0$$

$$S_{15} = 0 \quad S_{16} = 0 \quad S_{25} = 0 \quad S_{26} = 0 \quad S_{35} = 0 \quad S_{36} = 0 \quad S_{45} = 0 \quad S_{46} = 0$$

$$\alpha_{12} = 0 \quad \alpha_{13} = 0 \quad \beta_{12} = 0 \quad \beta_{13} = 0$$

  - **These relations, and the fact that $n = \sin\theta_3 \neq 0$ for nonzero values of $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$, are used to simplify the previously given invariance conditions into the following three uncoupled groups**

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- **Group 1**

$$C_{1'5'} = 0: \quad m^2(2C_{56} + C_{14}) + n^2C_{24} = 0$$

$$m = \cos\theta_3 \qquad n = \sin\theta_3$$

$$C_{2'5'} = 0: \quad -m^2(2C_{56} - C_{24}) + n^2C_{14} = 0$$

$$C_{4'6'} = 0: \quad n^2C_{56} - m^2(C_{56} + C_{14} - C_{24}) = 0$$

- **Group 2**

$$C_{3'5'} = 0: \quad C_{34} = 0$$

$$C_{3'6'} = 0: \quad m(C_{23} - C_{13}) = 0$$

$$C_{4'5'} = 0: \quad m(C_{44} - C_{55}) = 0$$

- **Group 3**

$$C_{1'6'} = 0: \quad mn^2(C_{22} - C_{12} - 2C_{66}) - m^3(C_{11} - C_{12} - 2C_{66}) = 0$$

$$C_{2'6'} = 0: \quad m^3(C_{22} - C_{12} - 2C_{66}) - mn^2(C_{11} - C_{12} - 2C_{66}) = 0$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The first group can be written as

$$C_{1'5'} = 0: \quad -C_{24} = m^2(2C_{56} + C_{14} - C_{24}) \qquad m = \cos\theta_3 \qquad n = \sin\theta_3$$

$$C_{2'5'} = 0: \quad C_{14} = m^2(2C_{56} + C_{14} - C_{24})$$

$$C_{4'6'} = 0: \quad C_{56} = m^2(2C_{56} + C_{14} - C_{24})$$

- Because the right-hand side of the equations are identical, it follows that the left-hand sides are equal; that is, $C_{56} = -C_{24} = C_{14} = \Gamma$

  - Each equation can be expressed as $\Gamma = 4\Gamma\cos^2\theta_3$, which is satisfied by $\Gamma = 0$ and by $\cos^2\theta_3 = \frac{1}{4}$

  - $\Gamma = 0$ implies $C_{56} = -C_{24} = C_{14} = 0$ and $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$

  - $\cos^2\theta_3 = \frac{1}{4}$ implies $\theta_3 = \pm\frac{\pi}{3}$, $C_{56} = C_{14}$, and $C_{24} = -C_{14}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The third group of equations is simplified by first adding the two equations together and then by subtracting the two equations

  - Adding gives $C_{1'6'} + C_{2'6'} = 0: \quad (C_{22} - C_{11})\cos\theta_3 = 0$,

    which is satisfied for all $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$ if $C_{22} = C_{11}$, or for all $C_{22} \neq C_{11}$ if $\theta_3 = \frac{\pi}{2}$

  - Subtracting gives

    $$C_{1'6'} - C_{2'6'} = 0: \quad (2C_{12} + 4C_{66} - C_{11} - C_{22})\cos\theta_3\cos 2\theta_3 = 0 ,$$

    which is satisfied for all $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$ if $2C_{12} + 4C_{66} - C_{11} - C_{22} = 0$, or for all $2C_{12} + 4C_{66} - C_{11} - C_{22} \neq 0$ if $\theta_3 = \pm\frac{\pi}{4}$ or $\frac{\pi}{2}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- Group 4

$$S_{1'5'} = 0: \quad m^2(S_{56} + S_{14}) + n^2 S_{24} = 0 \qquad m = \cos\theta_3 \qquad n = \sin\theta_3$$

$$S_{2'5'} = 0: \quad -m^2(S_{56} - S_{24}) + n^2 S_{14} = 0$$

$$S_{4'6'} = 0: \quad -m^2(S_{56} + 2S_{14} - 2S_{24}) + n^2 S_{56} = 0$$

- Group 5

$$S_{3'5'} = 0: \quad S_{34} = 0$$

$$S_{3'6'} = 0: \quad m(S_{23} - S_{13}) = 0$$

$$S_{4'5'} = 0: \quad m(S_{44} - S_{55}) = 0$$

- Group 6

$$S_{1'6'} = 0: \quad mn^2(2S_{22} - 2S_{12} - S_{66}) - m^3(2S_{11} - 2S_{12} - S_{66}) = 0$$

$$S_{2'6'} = 0: \quad m^3(2S_{22} - 2S_{12} - S_{66}) - mn^2(2S_{11} - 2S_{12} - S_{66}) = 0$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- Like for the stiffnesses, the fourth group can also be written as

$$S_{1'5'} = 0: \quad -S_{24} = m^2(S_{56} + S_{14} - S_{24}) \qquad m = \cos\theta_3 \qquad n = \sin\theta_3$$

$$S_{2'5'} = 0: \quad S_{14} = m^2(S_{56} + S_{14} - S_{24})$$

$$S_{4'6'} = 0: \quad \tfrac{1}{2}S_{56} = m^2(S_{56} + S_{14} - S_{24})$$

- Because the right-hand side of the equations are identical, it follows that the left-hand sides are equal; that is, $\tfrac{1}{2}S_{56} = -S_{24} = S_{14} = \Delta$
  - Each equation can be expressed as $\Delta = 4\Delta\cos^2\theta_3$, which is satisfied by $\Delta = 0$ and by $\cos^2\theta_3 = \tfrac{1}{4}$
  - $\Delta = 0$ implies $S_{56} = S_{24} = S_{14} = 0$ and $-\tfrac{\pi}{2} < \theta_3 \leq \tfrac{\pi}{2}$
  - $\cos^2\theta_3 = \tfrac{1}{4}$ implies $\theta_3 = \pm\tfrac{\pi}{3}$, $S_{56} = 2S_{14}$, and $S_{24} = -S_{14}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The sixth group of equations is also simplified by first adding the two equations together and then by subtracting the two equations

  - Adding gives $S_{1'6'} + S_{2'6'} = 0: \quad (S_{22} - S_{11})\cos\theta_3 = 0$,

    which is satisfied for all $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$ if $S_{22} = S_{11}$, or for all $S_{22} \neq S_{11}$ if $\theta_3 = \frac{\pi}{2}$

  - Subtracting gives

    $$S_{1'6'} - S_{2'6'} = 0: \quad (S_{11} + S_{22} - 2S_{12} - S_{66})\cos\theta_3\cos 2\theta_3 = 0,$$

    which is satisfied for all $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$ if $2S_{12} + S_{66} - S_{11} - S_{22} = 0$,

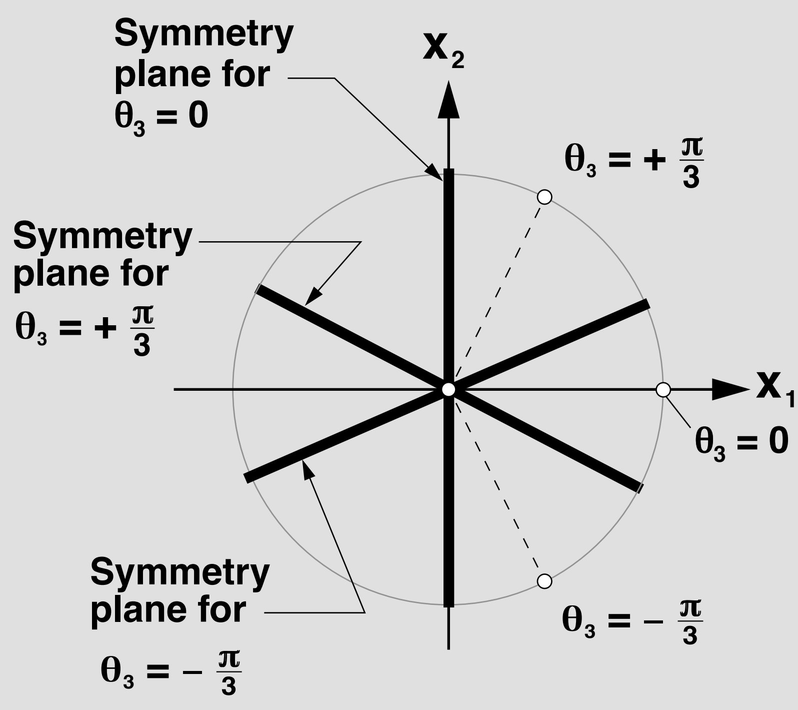
    or for all $2S_{12} + S_{66} - S_{11} - S_{22} \neq 0$ if $\theta_3 = \pm\frac{\pi}{4}$ or $\frac{\pi}{2}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The symmetry properties for **trigonal materials** arise from the solution for the invariance conditions that are given by $\cos^2\theta_3 = \frac{1}{4}$

- For this solution, the *planes of elastic symmetry* are all parallel to the x_3 axis and are given by $\theta_3 = 0$ and $\pm\frac{\pi}{3}$



Plan view of symmetry planes

- For $\theta_3 = \pm\frac{\pi}{3}$, $m = \cos\left(\pm\frac{\pi}{3}\right) = \frac{1}{2} \neq 0$ and $n = \sin\left(\pm\frac{\pi}{3}\right) = \pm\frac{\sqrt{3}}{2} \neq 0$

- The stiffness equations in group 2 yield the invariance conditions $C_{34} = 0$, $C_{23} = C_{13}$, and $C_{55} = C_{44}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- Likewise, the stiffness equations in group 1 yield the invariance conditions $C_{56} = C_{14}$ and $C_{24} = -C_{14}$

- Furthermore, the stiffness equations in group 3 yield the invariance conditions $C_{22} = C_{11}$ and $C_{66} = \frac{1}{2}(C_{11} - C_{12})$

- The compliance equations in group 5 yield the invariance conditions $S_{34} = 0$, $S_{23} = S_{13}$, and $S_{55} = S_{44}$

- The compliance equations in group 4 yield the invariance conditions $S_{56} = 2S_{14}$ and $S_{24} = -S_{14}$

- The compliance equations in group 6 yield the invariance conditions $S_{22} = S_{11}$ and $S_{66} = 2(S_{11} - S_{12})$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

- The invariance conditions $\alpha_{1'3'} = 0: \quad m\alpha_{13} + n\alpha_{23} = 0$ and $\alpha_{1'2'} = 0: \quad (m^2 - n^2)\alpha_{12} + mn(\alpha_{22} - \alpha_{11}) = 0$ yield

$$\alpha_{23} = \alpha_{13} = \alpha_{12} = 0 \quad \text{and} \quad \alpha_{22} = \alpha_{11}$$

- The invariance conditions $\beta_{1'3'} = 0: \quad m\beta_{13} + n\beta_{23} = 0$ and $\beta_{1'2'} = 0: \quad (m^2 - n^2)\beta_{12} + mn(\beta_{22} - \beta_{11}) = 0$ yield

$$\beta_{23} = \beta_{13} = \beta_{12} = 0 \quad \text{and} \quad \beta_{22} = \beta_{11}$$

- Together, the invariance conditions yield the following constitutive equations for a trigonal material

TRIGONAL MATERIALS - CONCLUDED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_3 AXIS

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & C_{14} \\ 0 & 0 & 0 & 0 & C_{14} & \dfrac{C_{11}-C_{12}}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ S_{12} & S_{11} & S_{13} & -S_{14} & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ S_{14} & -S_{14} & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 2S_{14} \\ 0 & 0 & 0 & 0 & 2S_{14} & 2(S_{11}-S_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{11} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

- Therefore, a **trigonal material** has six independent elastic constants and two independent thermal-expansion or thermal-compliance parameters

TRIGONAL MATERIALS

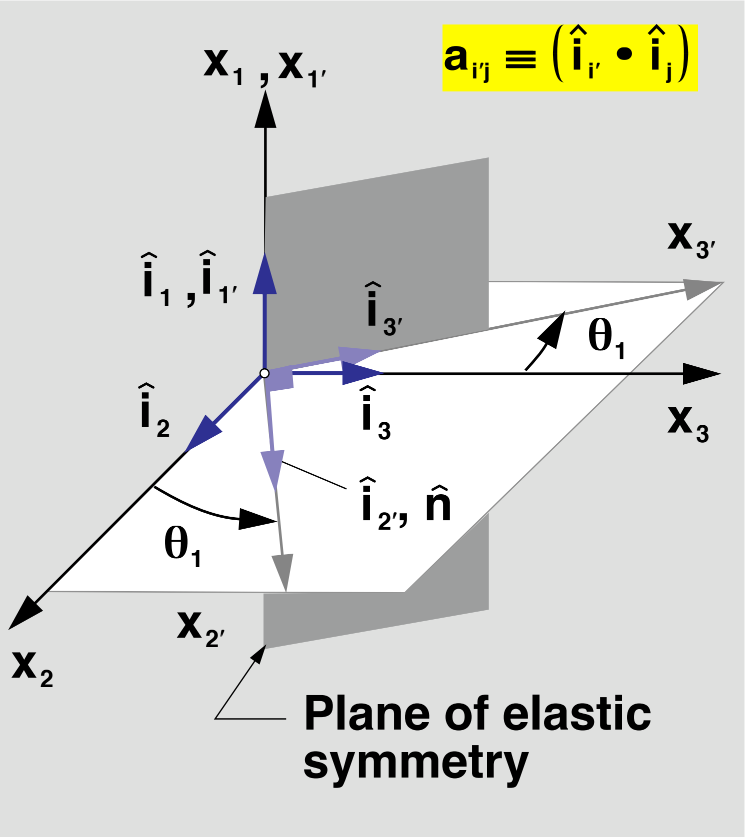
REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- To determine the conditions on the constitutive equations for **trigonal materials**, it is necessary to consider a plane of elastic symmetry that is oriented arbitrarily, with respect to two of the coordinate axes

- In particular, consider a **plane of elastic symmetry** whose normal $\hat{n}$ lies in the plane $x_1 = 0$ and makes an angle θ_1 with the x_2 axis, as shown in the figure

- The angle θ_1 is defined to be in the range $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$, because $\theta_1 = -\frac{\pi}{2}$ and $\theta_1 = \frac{\pi}{2}$ define the same plane

- In addition, let the $(x_{1'}, x_{2'}, x_{3'})$ be the coordinates used to define the material symmetry



TRIGONAL MATERIALS - CONTINUED

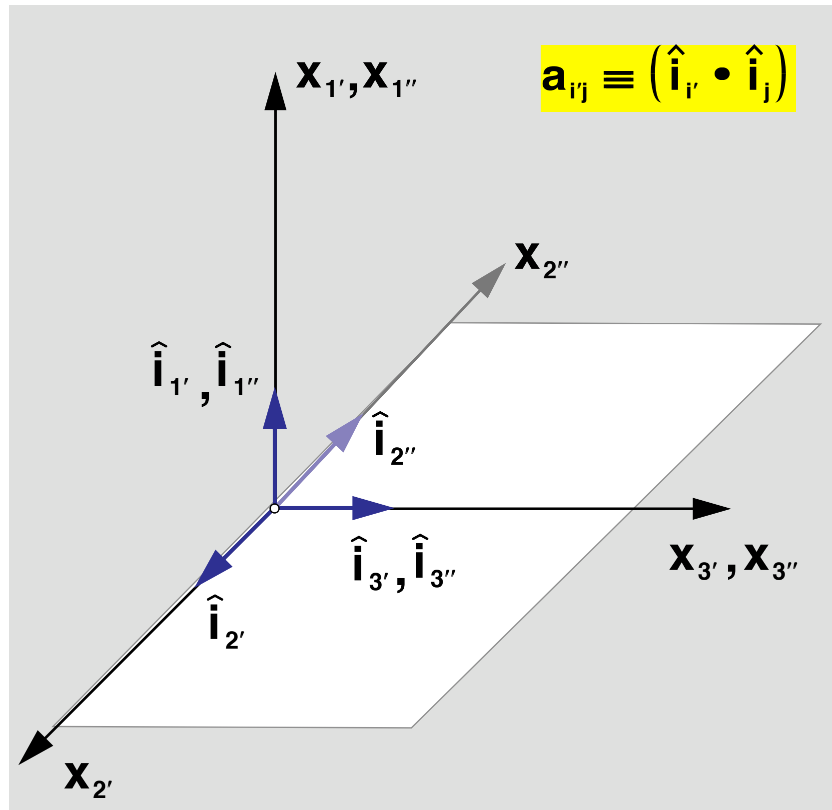
REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- Specifically, for a plane of elastic symmetry given by $x_{2'} = 0$, the **symmetry transformation** is shown in the figure and is given by $x_{1''} = x_{1'}$, $x_{2''} = -x_{2'}$, and $x_{3''} = x_{3'}$

- The corresponding matrix of **direction cosines** is given by

$$\begin{bmatrix} a_{1''1'} & a_{1''2'} & a_{1''3'} \\ a_{2''1'} & a_{2''2'} & a_{2''3'} \\ a_{3''1'} & a_{3''2'} & a_{3''3'} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- For the $(x_{1'}, x_{2'}, x_{3'})$ coordinate system, the constitutive equations are expressed as follows



TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & S_{1'4'} & S_{1'5'} & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & S_{2'4'} & S_{2'5'} & S_{2'6'} \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & S_{3'4'} & S_{3'5'} & S_{3'6'} \\ S_{1'4'} & S_{2'4'} & S_{3'4'} & S_{4'4'} & S_{4'5'} & S_{4'6'} \\ S_{1'5'} & S_{2'5'} & S_{3'5'} & S_{4'5'} & S_{5'5'} & S_{5'6'} \\ S_{1'6'} & S_{2'6'} & S_{3'6'} & S_{4'6'} & S_{5'6'} & S_{6'6'} \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 2\alpha_{2'3'} \\ 2\alpha_{1'3'} \\ 2\alpha_{1'2'} \end{Bmatrix} \left(T - T_{ref}\right)$$

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & C_{1'4'} & C_{1'5'} & C_{1'6'} \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & C_{2'4'} & C_{2'5'} & C_{2'6'} \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & C_{3'4'} & C_{3'5'} & C_{3'6'} \\ C_{1'4'} & C_{2'4'} & C_{3'4'} & C_{4'4'} & C_{4'5'} & C_{4'6'} \\ C_{1'5'} & C_{2'5'} & C_{3'5'} & C_{4'5'} & C_{5'5'} & C_{5'6'} \\ C_{1'6'} & C_{2'6'} & C_{3'6'} & C_{4'6'} & C_{5'6'} & C_{6'6'} \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \beta_{1'1'} \\ \beta_{2'2'} \\ \beta_{3'3'} \\ 2\beta_{2'3'} \\ 2\beta_{1'3'} \\ 2\beta_{1'2'} \end{Bmatrix} \left(T - T_{ref}\right)$$

or by

$\{\varepsilon'\} = [S']\{\sigma'\} + \{\alpha'\}\Theta$ and $\{\sigma'\} = [C']\{\varepsilon'\} + \{\beta'\}\Theta$,

where $\Theta = T - T_{ref}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- For the $(x_{1''}, x_{2''}, x_{3''})$ coordinate system, the constitutive equations are identical in form and are obtained by replacing the index pair $i'j'$ with $i''j''$

  - In the abridged notation,

$$\{\varepsilon''\} = [S'']\{\sigma''\} + \{\alpha''\}\Theta \quad \text{and}$$

$$\{\sigma''\} = [C'']\{\varepsilon''\} + \{\beta''\}\Theta, \quad \text{where} \quad \Theta = T - T_{ref}$$

- For the transformation of coordinates defined by the symmetry transformation (a reflection about the plane $x_{2'} = 0$),

$$\{\sigma''\} = [T_{\sigma}^{r_{2'}}]\{\sigma'\} \quad \text{and} \quad \{\varepsilon''\} = [T_{\varepsilon}^{r_{2'}}]\{\varepsilon'\} \quad \text{where}$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

$$\left[T_\sigma^{r_{2'}}\right] = \left[T_\varepsilon^{r_{2'}}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

- **Likewise,**

$$[S''] = \left[T_\varepsilon^{r_{2'}}\right][S']\left[T_\sigma^{r_{2'}}\right]^{-1} \qquad [S'] = \left[T_\varepsilon^{r_{2'}}\right]^{-1}[S'']\left[T_\sigma^{r_{2'}}\right]$$

$$[C''] = \left[T_\sigma^{r_{2'}}\right][C']\left[T_\varepsilon^{r_{2'}}\right]^{-1} \qquad [C'] = \left[T_\sigma^{r_{2'}}\right]^{-1}[C'']\left[T_\varepsilon^{r_{2'}}\right]$$

$$\{\alpha''\} = \left[T_\varepsilon^{r_{2'}}\right]\{\alpha'\} \qquad \{\beta''\} = \left[T_\sigma^{r_{2'}}\right]\{\beta'\}$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The **conditions for invariance** under the symmetry transformation are given by

$$[S'] = \left[T_{\varepsilon}^{r_{2'}}\right][S']\left[T_{\sigma}^{r_{2'}}\right]^{-1} \qquad [C'] = \left[T_{\sigma}^{r_{2'}}\right][C']\left[T_{\varepsilon}^{r_{2'}}\right]^{-1}$$

$$\{\alpha'\} = \left[T_{\varepsilon}^{r_{2'}}\right]\{\alpha'\} \qquad \{\beta'\} = \left[T_{\sigma}^{r_{2'}}\right]\{\beta'\}$$

- Rather than calculating the outcome of the invariance conditions, the outcome can be found by direct comparison with the results given previously for a material that is **monoclinic** with respect to the plane $x_2 = 0$

  - Direct comparison reveals that the material is **monoclinic** with respect to the plane $x_{2'} = 0$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- That is, direct comparison yields

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & 0 & S_{1'5'} & 0 \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & 0 & S_{2'5'} & 0 \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & 0 & S_{3'5'} & 0 \\ 0 & 0 & 0 & S_{4'4'} & 0 & S_{4'6'} \\ S_{1'5'} & S_{2'5'} & S_{3'5'} & 0 & S_{5'5'} & 0 \\ 0 & 0 & 0 & S_{4'6'} & 0 & S_{6'6'} \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 0 \\ 2\alpha_{1'3'} \\ 0 \end{Bmatrix} \left(T - T_{ref} \right) \quad \text{and}$$

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & 0 & C_{1'5'} & 0 \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & 0 & C_{2'5'} & 0 \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & 0 & C_{3'5'} & 0 \\ 0 & 0 & 0 & C_{4'4'} & 0 & C_{4'6'} \\ C_{1'5'} & C_{2'5'} & C_{3'5'} & 0 & C_{5'5'} & 0 \\ 0 & 0 & 0 & C_{4'6'} & 0 & C_{6'6'} \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \beta_{1'1'} \\ \beta_{2'2'} \\ \beta_{3'3'} \\ 0 \\ \beta_{1'3'} \\ 0 \end{Bmatrix} \left(T - T_{ref} \right)$$

TRIGONAL MATERIALS - CONTINUED

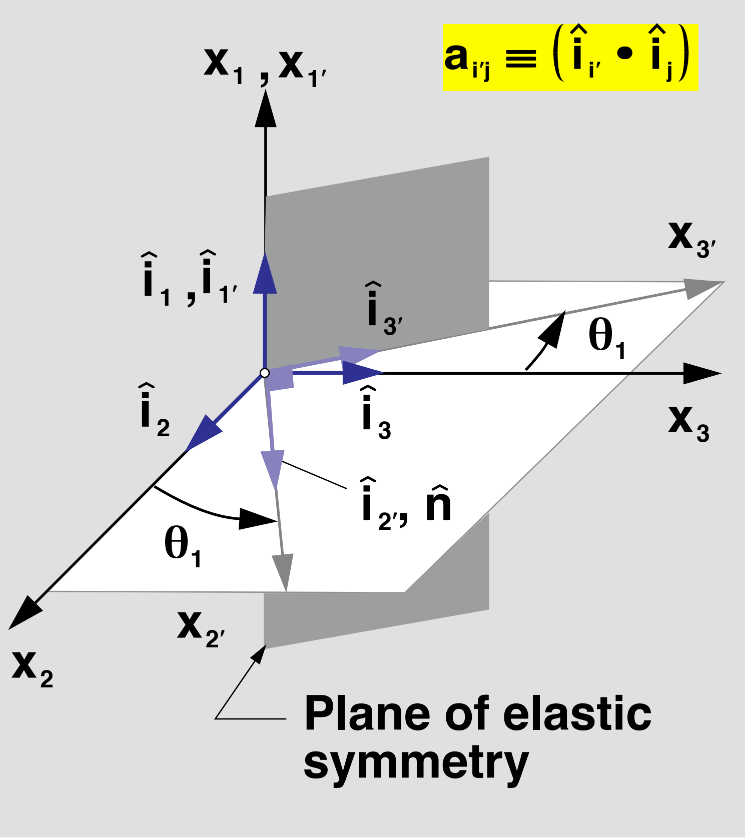
REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- Next, the compliances, stiffnesses, thermal-expansion coefficients, and thermal moduli, expressed in $(x_{1'}, x_{2'}, x_{3'})$ coordinates, are expressed in the original (x_1, x_2, x_3) coordinates

- The transformation corresponds to the dextral rotation about the x_1 axis, shown in the figure, and is given by $x_{1'} = x_1$, $x_{2'} = x_2\cos\theta_1 + x_3\sin\theta_1$, and $x_{3'} = -x_2\sin\theta_1 + x_3\cos\theta_1$, with $-\frac{\pi}{2} < \theta_1 \le \frac{\pi}{2}$

- The corresponding matrix of direction cosines is given by

$$\begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$



TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The corresponding stress and strain transformation matrices were shown previously to be given by

$$[T_\sigma(\theta_1)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & 2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -\sin\theta_1\cos\theta_1 & \sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

and

$$[T_\varepsilon(\theta_1)] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & \sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -2\sin\theta_1\cos\theta_1 & 2\sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The transformation laws for the compliances, stiffnesses, thermal-expansion coefficients, and thermal moduli have been given as

$$[C'] = [T_\sigma][C][T_\varepsilon]^{-1} \qquad [S'] = [T_\varepsilon][S][T_\sigma]^{-1}$$

$$\{\beta'\} = [T_\sigma]\{\beta\} \qquad \{\alpha'\} = [T_\varepsilon]\{\alpha\}$$

- These laws transform the invariance conditions on $[C']$, $[S']$, $\{\alpha'\}$, and $\{\beta'\}$ into invariance conditions on $[C]$, $[S]$, $\{\alpha\}$, and $\{\beta\}$

- Specific expressions for these transformation laws (dextral rotation about the x_1 axis) were given previously for a fully anisotropic material (triclinic)

    - Note that the matrices $[C]$ and $[S]$, and the vectors $\{\alpha\}$ and $\{\beta\}$, are fully populated

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- Because of the invariance conditions, the matrices $[C']$ and $[S']$ have the form $\begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & 0 & C_{1'5'} & 0 \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & 0 & C_{2'5'} & 0 \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & 0 & C_{3'5'} & 0 \\ 0 & 0 & 0 & C_{4'4'} & 0 & C_{4'6'} \\ C_{1'5'} & C_{2'5'} & C_{3'5'} & 0 & C_{5'5'} & 0 \\ 0 & 0 & 0 & C_{4'6'} & 0 & C_{6'6'} \end{bmatrix}$ and $\begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & 0 & S_{1'5'} & 0 \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & 0 & S_{2'5'} & 0 \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & 0 & S_{3'5'} & 0 \\ 0 & 0 & 0 & S_{4'4'} & 0 & S_{4'6'} \\ S_{1'5'} & S_{2'5'} & S_{3'5'} & 0 & S_{5'5'} & 0 \\ 0 & 0 & 0 & S_{4'6'} & 0 & S_{6'6'} \end{bmatrix}$

- Also, the vectors $\{\alpha'\}$ and $\{\beta'\}$ have the form $\begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 0 \\ 2\alpha_{1'3'} \\ 0 \end{Bmatrix}$ and $\begin{Bmatrix} \beta_{1'1'} \\ \beta_{2'2'} \\ \beta_{3'3'} \\ 0 \\ \beta_{1'3'} \\ 0 \end{Bmatrix}$

- The corresponding forms of $[C]$, $[S]$, $\{\alpha\}$, and $\{\beta\}$ are obtained from the transformation laws once a value for the angle θ_1 is specified

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- For example, specifying $\theta_1 = 0$ yields

$$\begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'3'} & 0 & C_{1'5'} & 0 \\ C_{1'2'} & C_{2'2'} & C_{2'3'} & 0 & C_{2'5'} & 0 \\ C_{1'3'} & C_{2'3'} & C_{3'3'} & 0 & C_{3'5'} & 0 \\ 0 & 0 & 0 & C_{4'4'} & 0 & C_{4'6'} \\ C_{1'5'} & C_{2'5'} & C_{3'5'} & 0 & C_{5'5'} & 0 \\ 0 & 0 & 0 & C_{4'6'} & 0 & C_{6'6'} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \qquad \begin{Bmatrix} \beta_{1'1'} \\ \beta_{2'2'} \\ \beta_{3'3'} \\ 0 \\ \beta_{1'3'} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix}$$

$$\begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & 0 & S_{1'5'} & 0 \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & 0 & S_{2'5'} & 0 \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & 0 & S_{3'5'} & 0 \\ 0 & 0 & 0 & S_{4'4'} & 0 & S_{4'6'} \\ S_{1'5'} & S_{2'5'} & S_{3'5'} & 0 & S_{5'5'} & 0 \\ 0 & 0 & 0 & S_{4'6'} & 0 & S_{6'6'} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \qquad \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 0 \\ 2\alpha_{1'3'} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix}$$

- Enforcing the invariance conditions on $[C']$ gives

$C_{14} = 0$, $C_{16} = 0$, $C_{24} = 0$, $C_{26} = 0$, $C_{34} = 0$, $C_{36} = 0$, $C_{45} = 0$, and $C_{56} = 0$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- Enforcing the invariance conditions on $[S']$ gives

  $S_{14} = 0$, $S_{16} = 0$, $S_{24} = 0$, $S_{26} = 0$, $S_{34} = 0$, $S_{36} = 0$, $S_{45} = 0$, and $S_{56} = 0$

- Enforcing the invariance conditions on $\{\alpha'\}$ and $\{\beta'\}$ gives

$$\alpha_{12} = \alpha_{23} = 0 \quad \text{and} \quad \beta_{12} = \beta_{23} = 0$$

- The conditions are identical to the conditions previously obtained for a material that is **monoclinic** with respect to the plane $x_2 = 0$

- Similarly, specifying $\theta_1 = \frac{\pi}{2}$ and enforcing the invariance conditions on $[C']$, $[S']$, $\{\alpha'\}$, and $\{\beta'\}$ yields the conditions previously obtained for a material that is **monoclinic** with respect to the plane $x_3 = 0$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- **Now consider an arbitrary value for the angle θ_1 such that $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$**

- **The previous example for $\theta_1 = 0$ showed that the invariance conditions on $[C]$, $[S]$, $\{\alpha\}$, and $\{\beta\}$ were obtained by the terms of $[C']$, $[S']$, $\{\alpha'\}$, and $\{\beta'\}$ that were zero valued**

- **Using the transformation equations for a dextral rotation about the x_1 axis that were given previously for a (triclinic) fully anisotropic, elastic material gives the following results for the invariance conditions on $[C']$**

$$C_{1'4'} = 0: \quad \left(m^2 - n^2\right)C_{14} + mn\left(C_{13} - C_{12}\right) = 0$$

$$C_{1'6'} = 0: \quad mC_{16} + nC_{15} = 0$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

$$C_{2'4'} = 0: \quad m^2(m^2 - 3n^2)C_{24} - m^3n(C_{22} - C_{23} - 2C_{44}) + mn^3(C_{33} - C_{23} - 2C_{44}) - n^2(n^2 - 3m^2)C_{34} = 0$$

$$C_{2'6'} = 0: \quad m^3C_{26} + m^2n(2C_{46} + C_{25}) + mn^2(2C_{45} + C_{36}) + n^3C_{35} = 0$$

$$C_{3'4'} = 0: \quad m^2(m^2 - 3n^2)C_{34} + m^3n(C_{33} - C_{23} - 2C_{44}) - mn^3(C_{22} - C_{23} - 2C_{44}) - n^2(n^2 - 3m^2)C_{24} = 0$$

$$C_{3'6'} = 0: \quad m^3C_{36} - m^2n(2C_{46} - C_{35}) - mn^2(2C_{45} - C_{26}) + n^3C_{25} = 0$$

$$C_{4'5'} = 0: \quad m^3C_{45} - m^2n(C_{46} + C_{25} - C_{35}) - mn^2(C_{45} - C_{26} + C_{36}) + n^3C_{46} = 0$$

$$C_{5'6'} = 0: \quad (m^2 - n^2)C_{56} + mn(C_{55} - C_{66}) = 0$$

with $m = \cos\theta_1$ **and** $n = \sin\theta_1$

- **These conditions give 8 equations and 20 unknowns**

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The invariance conditions on $[S']$ are given by

$$S_{1'4'} = 0: \quad (m^2 - n^2)S_{14} + 2mn(S_{13} - S_{12}) = 0$$

$$S_{1'6'} = 0: \quad mS_{16} + nS_{15} = 0$$

$$S_{2'4'} = 0: \quad m^2(m^2 - 3n^2)S_{24} - m^3n(2S_{22} - 2S_{23} - S_{44}) + mn^3(2S_{33} - 2S_{23} - S_{44}) - n^2(n^2 - 3m^2)S_{34} = 0$$

$$S_{2'6'} = 0: \quad m^3S_{26} + m^2n(S_{46} + S_{25}) + mn^2(S_{45} + S_{36}) + n^3S_{35} = 0$$

$$S_{3'4'} = 0: \quad m^2(m^2 - 3n^2)S_{34} + m^3n(2S_{33} - 2S_{23} - S_{44}) - mn^3(2S_{22} - 2S_{23} - S_{44}) - n^2(n^2 - 3m^2)S_{24} = 0$$

$$S_{3'6'} = 0: \quad m^3S_{36} - m^2n(S_{46} - S_{35}) - mn^2(S_{45} - S_{26}) + n^3S_{25} = 0$$

$$S_{4'5'} = 0: \quad m^3S_{45} - m^2n(S_{46} + 2S_{25} - 2S_{35}) - mn^2(S_{45} - 2S_{26} + 2S_{36}) + n^3S_{46} = 0$$

$$S_{5'6'} = 0: \quad (m^2 - n^2)S_{56} + mn(S_{55} - S_{66}) = 0$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The invariance conditions on $\{\alpha'\}$ and $\{\beta'\}$ yield

$$\alpha_{1'2'} = 0: \quad m\alpha_{12} + n\alpha_{13} = 0 \qquad \alpha_{2'3'} = 0: \quad (m^2 - n^2)\alpha_{23} + mn(\alpha_{33} - \alpha_{22}) = 0$$

$$\beta_{1'2'} = 0: \quad m\beta_{12} + n\beta_{13} = 0 \qquad \beta_{2'3'} = 0: \quad (m^2 - n^2)\beta_{23} + mn(\beta_{33} - \beta_{22}) = 0$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- In determining the restrictions on the compliances, stiffnesses, thermal-expansion coefficients, and thermal moduli for materials that possess more than one plane of elastic symmetry, all of which contain the x_1 axis, it is convenient to select the first plane to be given by $\theta_1 = 0$

  - Thus, the material is **monoclinic** with respect to the plane $x_2 = 0$ and, as a result, the following conditions hold

    $C_{14} = 0$ $C_{16} = 0$ $C_{24} = 0$ $C_{26} = 0$ $C_{34} = 0$ $C_{36} = 0$ $C_{45} = 0$ $C_{56} = 0$

    $S_{14} = 0$ $S_{16} = 0$ $S_{24} = 0$ $S_{26} = 0$ $S_{34} = 0$ $S_{36} = 0$ $S_{45} = 0$ $S_{56} = 0$

    $\alpha_{12} = 0$ $\alpha_{23} = 0$ $\beta_{12} = 0$ $\beta_{23} = 0$

  - These relations, and the fact that $n = \sin\theta_1 \neq 0$ for nonzero values of $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$, are used to simplify the previously given invariance conditions into the following three uncoupled groups

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- **Group 1**

$$C_{2'6'} = 0: \quad m^2(2C_{46} + C_{25}) + n^2 C_{35} = 0$$

$$m = \cos\theta_1 \qquad n = \sin\theta_1$$

$$C_{3'6'} = 0: \quad -m^2(2C_{46} - C_{35}) + n^2 C_{25} = 0$$

$$C_{4'5'} = 0: \quad -m^2(C_{46} + C_{25} - C_{35}) + n^2 C_{46} = 0$$

- **Group 2**

$$C_{1'4'} = 0: \quad m(C_{13} - C_{12}) = 0$$

$$C_{1'6'} = 0: \quad C_{15} = 0$$

$$C_{5'6'} = 0: \quad m(C_{55} - C_{66}) = 0$$

- **Group 3**

$$C_{2'4'} = 0: \quad -m^3(C_{22} - C_{23} - 2C_{44}) + mn^2(C_{33} - C_{23} - 2C_{44}) = 0$$

$$C_{3'4'} = 0: \quad m^3(C_{33} - C_{23} - 2C_{44}) - mn^2(C_{22} - C_{23} - 2C_{44}) = 0$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The first group can be written as

$$C_{2'6'} = 0: \quad -C_{35} = m^2\left(2C_{46} + C_{25} - C_{35}\right) \qquad m = \cos\theta_1 \qquad n = \sin\theta_1$$

$$C_{3'6'} = 0: \quad C_{25} = m^2\left(2C_{46} + C_{25} - C_{35}\right)$$

$$C_{4'5'} = 0: \quad C_{46} = m^2\left(2C_{46} + C_{25} - C_{35}\right)$$

- Because the right-hand side of the equations are identical, it follows that the left-hand sides are equal; that is, $C_{46} = -C_{35} = C_{25} = \Gamma$

  - Each equation can be expressed as $\Gamma = 4\Gamma\cos^2\theta_1$, which is satisfied by $\Gamma = 0$ and by $\cos^2\theta_1 = \frac{1}{4}$

  - $\Gamma = 0$ implies $C_{46} = -C_{35} = C_{25} = 0$ and $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$

  - $\cos^2\theta_1 = \frac{1}{4}$ implies $\theta_1 = \pm\frac{\pi}{3}$, $C_{46} = C_{25}$, and $C_{35} = -C_{25}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The third group of equations is simplified by first adding the two equations together and then by subtracting the two equations

  - Adding gives $C_{2'4'} + C_{3'4'} = 0: \quad (C_{33} - C_{22})\cos\theta_1 = 0$,

    which is satisfied for all $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$ if $C_{33} = C_{22}$, or for all $C_{33} \neq C_{22}$ if $\theta_1 = \frac{\pi}{2}$

  - Subtracting gives

    $$C_{2'4'} - C_{3'4'} = 0: \quad (2C_{23} + 4C_{44} - C_{22} - C_{33})\cos\theta_1\cos 2\theta_1 = 0,$$

    which is satisfied for all $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$ if $2C_{23} + 4C_{44} - C_{22} - C_{33} = 0$, or for all $2C_{23} + 4C_{44} - C_{22} - C_{33} \neq 0$ if $\theta_1 = \pm\frac{\pi}{4}$ or $\frac{\pi}{2}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- **Group 4**

$$S_{2'6'} = 0: \quad m^2(S_{46} + S_{25}) + n^2 S_{35} = 0$$

$$m = \cos\theta_1 \qquad n = \sin\theta_1$$

$$S_{3'6'} = 0: \quad -m^2(S_{46} - S_{35}) + n^2 S_{25} = 0$$

$$S_{4'5'} = 0: \quad -m^2(S_{46} + 2S_{25} - 2S_{35}) + n^2 S_{46} = 0$$

- **Group 5**

$$S_{1'4'} = 0: \quad m(S_{13} - S_{12}) = 0$$

$$S_{1'6'} = 0: \quad S_{15} = 0$$

$$S_{5'6'} = 0: \quad m(S_{55} - S_{66}) = 0$$

- **Group 6**

$$S_{2'4'} = 0: \quad -m^3(2S_{22} - 2S_{23} - S_{44}) + mn^2(2S_{33} - 2S_{23} - S_{44}) = 0$$

$$S_{3'4'} = 0: \quad m^3(2S_{33} - 2S_{23} - S_{44}) - mn^2(2S_{22} - 2S_{23} - S_{44}) = 0$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- Like for the stiffnesses, the fourth group can also be written as

$$S_{2'6'} = 0: \quad -S_{35} = m^2(S_{46} + S_{25} - S_{35}) \qquad m = \cos\theta_1 \qquad n = \sin\theta_1$$

$$S_{3'6'} = 0: \quad S_{25} = m^2(S_{46} + S_{25} - S_{35})$$

$$S_{4'5'} = 0: \quad \tfrac{1}{2}S_{46} = m^2(S_{46} + S_{25} - S_{35})$$

- Because the right-hand side of the equations are identical, it follows that the left-hand sides are equal; that is, $\frac{1}{2}S_{46} = -S_{35} = S_{25} = \Delta$
    - Each equation can be expressed as $\Delta = 4\Delta\cos^2\theta_3$, which is satisfied by $\Delta = 0$ and by $\cos^2\theta_1 = \frac{1}{4}$
    - $\Delta = 0$ implies $S_{46} = S_{35} = S_{25} = 0$ and $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$
    - $\cos^2\theta_1 = \frac{1}{4}$ implies $\theta_1 = \pm\frac{\pi}{3}$, $S_{46} = 2S_{25}$, and $S_{35} = -S_{25}$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The sixth group of equations is also simplified by first adding the two equations together and then by subtracting the two equations

  - Adding gives $S_{2'4'} + S_{3'4'} = 0\text{:} \quad (S_{33} - S_{22})\cos\theta_1 = 0$,

    which is satisfied for all $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$ if $S_{33} = S_{22}$, or for all $S_{33} \neq S_{22}$ if $\theta_1 = \frac{\pi}{2}$

  - Subtracting gives

    $$S_{2'4'} - S_{3'4'} = 0\text{:} \quad (2S_{23} + S_{44} - S_{22} - S_{33})\cos\theta_1\cos 2\theta_1 = 0,$$

    which is satisfied for all $-\frac{\pi}{2} < \theta_1 \leq \frac{\pi}{2}$ if $2S_{23} + S_{44} - S_{22} - S_{33} = 0$, or for all $2S_{23} + S_{44} - S_{22} - S_{33} \neq 0$ if $\theta_1 = \pm\frac{\pi}{4}$ or $\frac{\pi}{2}$

TRIGONAL MATERIALS - CONTINUED

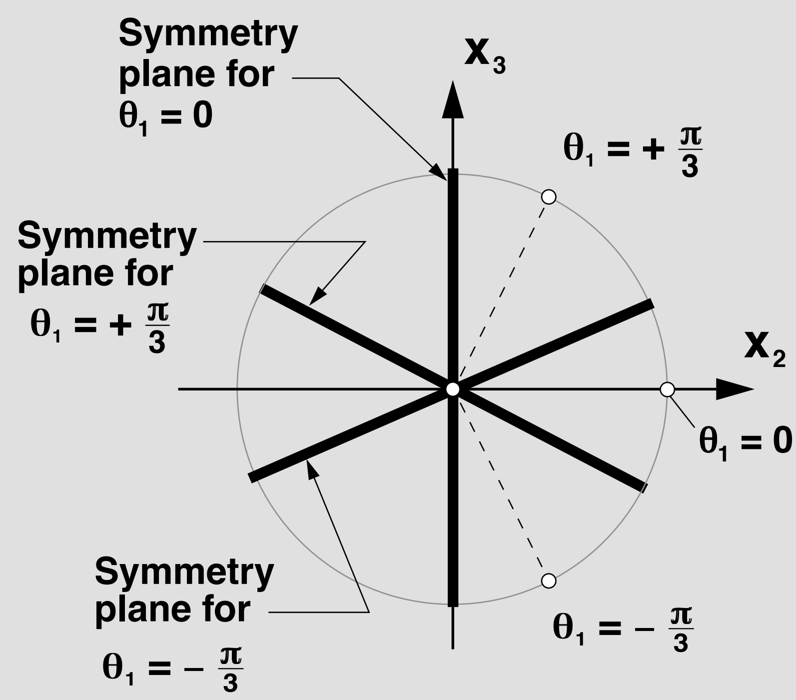
REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The symmetry properties for **trigonal materials** arise from the solution for the invariance conditions that is given by $\cos^2\theta_1 = \frac{1}{4}$

- For this solution, the *planes of elastic symmetry* are all parallel to the x_1 axis and are given by $\theta_1 = 0$ and $\pm\frac{\pi}{3}$

- For $\theta_1 = \pm\frac{\pi}{3}$, $m = \cos\left(\pm\frac{\pi}{3}\right) = \frac{1}{2} \neq 0$ and $n = \sin\left(\pm\frac{\pi}{3}\right) = \pm\frac{\sqrt{3}}{2} \neq 0$

- The stiffness equations in group 2 yield the invariance conditions $C_{15} = 0$, $C_{13} = C_{12}$, and $C_{66} = C_{55}$



Plan view of symmetry planes

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- **Likewise, the stiffness equations in group 1 yield the invariance conditions $C_{46} = C_{25}$ and $C_{35} = -C_{25}$**

- **Furthermore, the stiffness equations in group 3 yield the invariance conditions $C_{33} = C_{22}$ and $C_{44} = \frac{1}{2}(C_{22} - C_{23})$**

- **The compliance equations in group 5 yield the invariance conditions $S_{15} = 0$, $S_{13} = S_{12}$, and $S_{66} = S_{55}$**

- **The compliance equations in group 4 yield the invariance conditions $S_{46} = 2S_{25}$ and $S_{35} = -S_{25}$**

- **The compliance equations in group 6 yield the invariance conditions $S_{33} = S_{22}$ and $S_{44} = 2(S_{22} - S_{23})$**

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- The invariance conditions $\alpha_{1'2'} = 0: \quad m\alpha_{12} + n\alpha_{13} = 0$ and $\alpha_{2'3'} = 0: \quad (m^2 - n^2)\alpha_{23} + mn(\alpha_{33} - \alpha_{22}) = 0$ yield

$$\alpha_{23} = \alpha_{13} = \alpha_{12} = 0 \quad \text{and} \quad \alpha_{33} = \alpha_{22}$$

- The invariance conditions $\beta_{1'2'} = 0: \quad m\beta_{12} + n\beta_{13} = 0$ and $\beta_{2'3'} = 0: \quad (m^2 - n^2)\beta_{23} + mn(\beta_{33} - \beta_{22}) = 0$ yield

$$\beta_{23} = \beta_{13} = \beta_{12} = 0 \quad \text{and} \quad \beta_{33} = \beta_{22}$$

- Together, the invariance conditions yield the following constitutive equations for a trigonal material

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{12} & C_{23} & C_{22} & 0 & -C_{25} & 0 \\ 0 & 0 & 0 & \dfrac{C_{22}-C_{23}}{2} & 0 & C_{25} \\ 0 & C_{25} & -C_{25} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{25} & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ S_{12} & S_{23} & S_{22} & 0 & -S_{25} & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 2S_{25} \\ 0 & S_{25} & -S_{25} & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 2S_{25} & 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

- Again, the **trigonal material** has six independent elastic constants and two independent thermal-expansion or thermal-compliance parameters

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

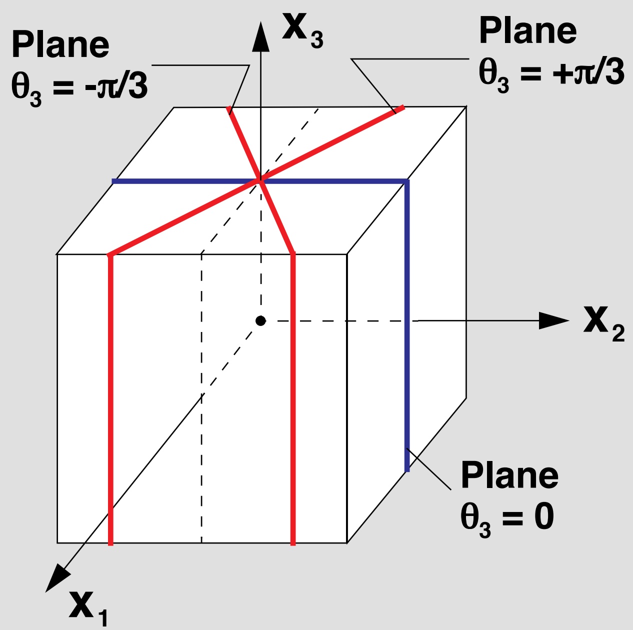
- The previous derivation of the constitutive equations for a trigonal material that has reflective symmetry planes that contain the x_1 axis is quite tedious

- These constitutive equations can be derived in alternate manner by using the corresponding equations given first for a trigonal material that has reflective symmetry planes that contain the x_3 axis, along with a juxtaposition of indices

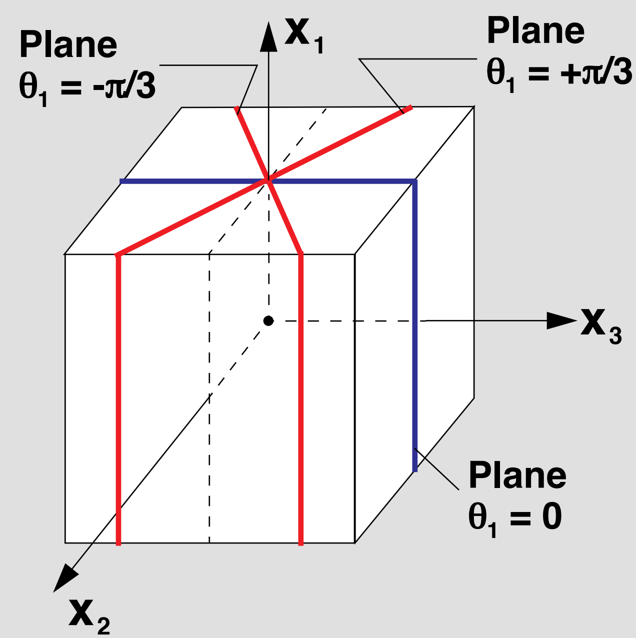
- That is, the desired constitutive equations are found by simply determining the renumbering of the indices that brings the figure shown below for symmetry planes that contain the x_3 axis into congruence with the adjacent figure shown below for symmetry planes that contain the x_1 axis

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS



Symmetry planes that contain the x_3 axis



Symmetry planes that contain the x_1 axis

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- **Inspection of the figures indicates the following transformation of the indices: $1 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 1$**

- **Next, it must be realized that the exchanging of indices must be used with the indices of tensors to determine the indices used with the abridged notation (matrix)**

  - **The following index pairs relate the tensor indices to the matrix indices**

tensor notation	11	22	33	23, 32	31, 13	12, 21
matrix notation	1	2	3	4	5	6

- **Using this information along with $1 \rightarrow 2$, $2 \rightarrow 3$, and $3 \rightarrow 1$ gives the relations: $4 \rightarrow 5$, $5 \rightarrow 6$, and $6 \rightarrow 4$**

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- Likewise, the transformation of index pairs that appear in the abridged notation are given by

$11 \rightarrow 22$					
$12 \rightarrow 23$	$22 \rightarrow 33$				
$13 \rightarrow 12$	$23 \rightarrow 13$	$33 \rightarrow 11$			
$14 \rightarrow 25$	$24 \rightarrow 35$	$34 \rightarrow 15$	$44 \rightarrow 55$		
$15 \rightarrow 26$	$25 \rightarrow 36$	$35 \rightarrow 16$	$45 \rightarrow 56$	$55 \rightarrow 66$	
$16 \rightarrow 24$	$26 \rightarrow 34$	$36 \rightarrow 14$	$46 \rightarrow 45$	$56 \rightarrow 46$	$66 \rightarrow 44$

- Consider the following constitutive equations for a trigonal material that has reflective symmetry planes that contain the x_3 axis

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & C_{14} \\ 0 & 0 & 0 & 0 & C_{14} & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

TRIGONAL MATERIALS - CONCLUDED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_1 AXIS

- Applying the index transformation to these constitutive equations gives

$$\begin{Bmatrix} \sigma_{22} \\ \sigma_{33} \\ \sigma_{11} \\ \sigma_{13} \\ \sigma_{12} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} C_{22} & C_{23} & C_{12} & C_{25} & 0 & 0 \\ C_{23} & C_{22} & C_{12} & -C_{25} & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ C_{25} & -C_{25} & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{25} \\ 0 & 0 & 0 & 0 & C_{25} & \frac{1}{2}(C_{22}-C_{23}) \end{bmatrix} \begin{Bmatrix} \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{11} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \end{Bmatrix} + \begin{Bmatrix} \beta_{22} \\ \beta_{22} \\ \beta_{11} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

- Reordering these equations into the standard form yields

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{12} & C_{23} & C_{22} & 0 & -C_{25} & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{22}-C_{23}) & 0 & C_{25} \\ 0 & C_{25} & -C_{25} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{25} & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

which are identical to the corresponding equations previously given

TRIGONAL MATERIALS

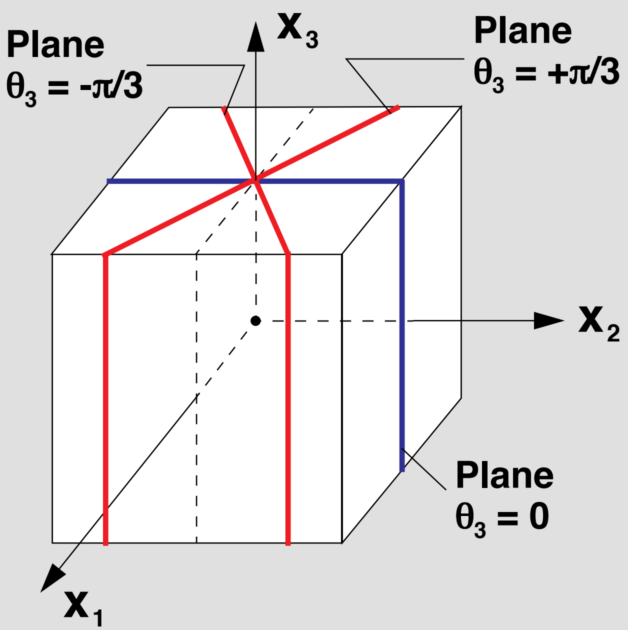
REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_2 AXIS

- These constitutive equations are derived by using the corresponding equations given first for a trigonal material that has reflective symmetry planes that contain the x_3 axis, along with a juxtaposition of indices

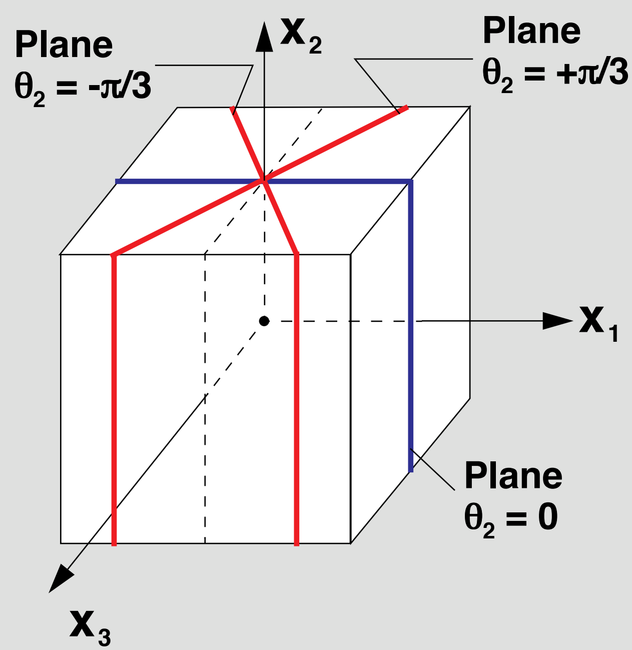
- That is, the desired constitutive equations are found by simply determining the renumbering of the indices that brings the figure shown below for symmetry planes that contain the x_3 axis into congruence with the adjacent figure shown below for symmetry planes that contain the x_2 axis

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_2 AXIS



Symmetry planes that contain the x_3 axis



Symmetry planes that contain the x_2 axis

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_2 AXIS

- **Inspection of the figures indicates the following transformation of the indices: $1 \rightarrow 3$, $2 \rightarrow 1$, and $3 \rightarrow 2$**

- **Next, it must be realized that the exchanging of indices must be used with the indices of tensors to determine the indices used with the abridged notation (matrix)**

  - **The following index pairs relate the tensor indices to the matrix indices**

tensor notation	11	22	33	23, 32	31, 13	12, 21
matrix notation	1	2	3	4	5	6

- **Using this information along with $1 \rightarrow 3$, $2 \rightarrow 1$, and $3 \rightarrow 2$ gives the relations: $4 \rightarrow 6$, $5 \rightarrow 4$, and $6 \rightarrow 5$**

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_2 AXIS

- **Likewise, the transformation of index pairs that appear in the abridged notation are given by**

$11 \rightarrow 33$					
$12 \rightarrow 13$	$22 \rightarrow 11$				
$13 \rightarrow 23$	$23 \rightarrow 12$	$33 \rightarrow 22$			
$14 \rightarrow 36$	$24 \rightarrow 16$	$34 \rightarrow 26$	$44 \rightarrow 66$		
$15 \rightarrow 34$	$25 \rightarrow 14$	$35 \rightarrow 24$	$45 \rightarrow 46$	$55 \rightarrow 44$	
$16 \rightarrow 35$	$26 \rightarrow 15$	$36 \rightarrow 25$	$46 \rightarrow 56$	$56 \rightarrow 45$	$66 \rightarrow 55$

- **Consider the following constitutive equations for a trigonal material that has reflective symmetry planes that contain the x_3 axis**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & C_{14} \\ 0 & 0 & 0 & 0 & C_{14} & \frac{1}{2}\left(C_{11} - C_{12}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

TRIGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_2 AXIS

- Applying the index transformation to these constitutive equations gives

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{33} & C_{13} & C_{23} & C_{36} & 0 & 0 \\ C_{13} & C_{33} & C_{23} & -C_{36} & 0 & 0 \\ C_{23} & C_{23} & C_{22} & 0 & 0 & 0 \\ C_{36} & -C_{36} & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & C_{36} \\ 0 & 0 & 0 & 0 & C_{36} & \frac{1}{2}\left(C_{33} - C_{13}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_{33} \\ \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} + \begin{Bmatrix} \beta_{33} \\ \beta_{33} \\ \beta_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

- Reordering these equations into the standard form yields

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{33} & C_{23} & C_{13} & 0 & 0 & -C_{36} \\ C_{23} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{66} & C_{36} & 0 \\ 0 & 0 & 0 & C_{36} & \frac{1}{2}\left(C_{33} - C_{13}\right) & 0 \\ -C_{36} & 0 & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{33} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

TRIGONAL MATERIALS - CONCLUDED

REFLECTIVE SYMMETRY ABOUT PLANES THAT CONTAIN THE x_2 AXIS

- Applying the same procedure to the inverse equations yields

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{33} & S_{23} & S_{13} & 0 & 0 & -S_{36} \\ S_{23} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{66} & 2S_{36} & 0 \\ 0 & 0 & 0 & 2S_{36} & 2(S_{33} - S_{13}) & 0 \\ -S_{36} & 0 & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ \alpha_{33} \\ \alpha_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

SUMMARY OF TRIGONAL MATERIALS

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0\\ C_{12} & C_{23} & C_{22} & 0 & -C_{25} & 0\\ 0 & 0 & 0 & \frac{1}{2}(C_{22}-C_{23}) & 0 & C_{25}\\ 0 & C_{25} & -C_{25} & 0 & C_{55} & 0\\ 0 & 0 & 0 & C_{25} & 0 & C_{55}\end{bmatrix} \begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{Bmatrix} + \begin{Bmatrix}\beta_{11}\\ \beta_{22}\\ \beta_{22}\\ 0\\ 0\\ 0\end{Bmatrix}(T - T_{ref}) \qquad x_1 \text{ axis}$$

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix} = \begin{bmatrix} C_{33} & C_{23} & C_{13} & 0 & 0 & -C_{36}\\ C_{23} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36}\\ 0 & 0 & 0 & C_{66} & C_{36} & 0\\ 0 & 0 & 0 & C_{36} & \frac{1}{2}(C_{33}-C_{13}) & 0\\ -C_{36} & 0 & C_{36} & 0 & 0 & C_{66}\end{bmatrix} \begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{Bmatrix} + \begin{Bmatrix}\beta_{33}\\ \beta_{22}\\ \beta_{33}\\ 0\\ 0\\ 0\end{Bmatrix}(T - T_{ref}) \qquad x_2 \text{ axis}$$

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0\\ C_{12} & C_{11} & C_{13} & -C_{14} & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ C_{14} & -C_{14} & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & C_{14}\\ 0 & 0 & 0 & 0 & C_{14} & \frac{1}{2}(C_{11}-C_{12})\end{bmatrix} \begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{Bmatrix} + \begin{Bmatrix}\beta_{11}\\ \beta_{11}\\ \beta_{33}\\ 0\\ 0\\ 0\end{Bmatrix}(T - T_{ref}) \qquad x_3 \text{ axis}$$

SUMMARY OF TRIGONAL MATERIALS - CONCLUDED

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & S_{25} & 0 \\ S_{12} & S_{23} & S_{22} & 0 & -S_{25} & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 2S_{25} \\ 0 & S_{25} & -S_{25} & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 2S_{25} & 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \qquad x_1 \text{ axis}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{33} & S_{23} & S_{13} & 0 & 0 & -S_{36} \\ S_{23} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{66} & 2S_{36} & 0 \\ 0 & 0 & 0 & 2S_{36} & 2(S_{33}-S_{13}) & 0 \\ -S_{36} & 0 & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ \alpha_{33} \\ \alpha_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \qquad x_2 \text{ axis}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ S_{12} & S_{11} & S_{13} & -S_{14} & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ S_{14} & -S_{14} & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 2S_{14} \\ 0 & 0 & 0 & 0 & 2S_{14} & 2(S_{11}-S_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{11} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \qquad x_3 \text{ axis}$$

TETRAGONAL MATERIALS

TETRAGONAL MATERIALS

- **Previously, it was shown that for a single plane of elastic symmetry containing the x_3 axis, special invariance conditions arise for values of $\theta_3 = \pm\frac{\pi}{4}$, in addition to $\theta_3 = 0$**

- **Likewise, it was shown that for a single plane of elastic symmetry containing the x_1 axis, special invariance conditions arise for values of $\theta_1 = \pm\frac{\pi}{4}$, in addition to $\theta_1 = 0$**

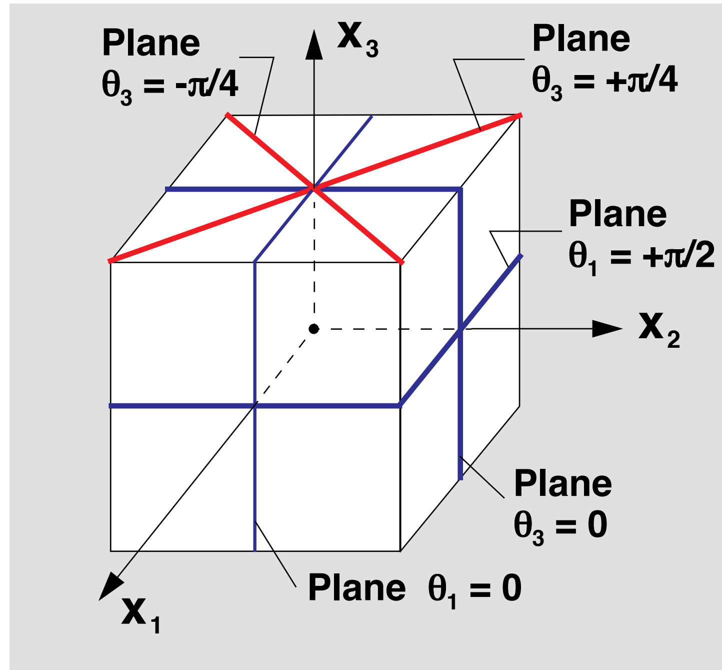
- **Furthermore, for a single plane of elastic symmetry containing the x_2 axis, special invariance conditions also arise for values of $\theta_2 = \pm\frac{\pi}{4}$, in addition to $\theta_2 = 0$**

- **Three particular cases of interest arise from these invariance conditions in which there exists five planes of elastic symmetry, four of which are perpendicular to the fifth plane**

TETRAGONAL MATERIALS - CONTINUED

- Specifically, consider a material with three planes of elastic symmetry given by $\theta_3 = 0$ and $\pm\frac{\pi}{4}$

- By taking $\theta_1 = +\frac{\pi}{2}$ as an additional plane of elastic symmetry, the fact that the planes $\theta_3 = 0$ and $\theta_1 = +\frac{\pi}{2}$ are perpendicular implies the existence of a fifth symmetry plane given by $\theta_1 = 0$
    - This fact was shown previously for specially orthotropic materials; that is, two perpendicular planes of elastic symmetry imply a third perpendicular plane of elastic symmetry

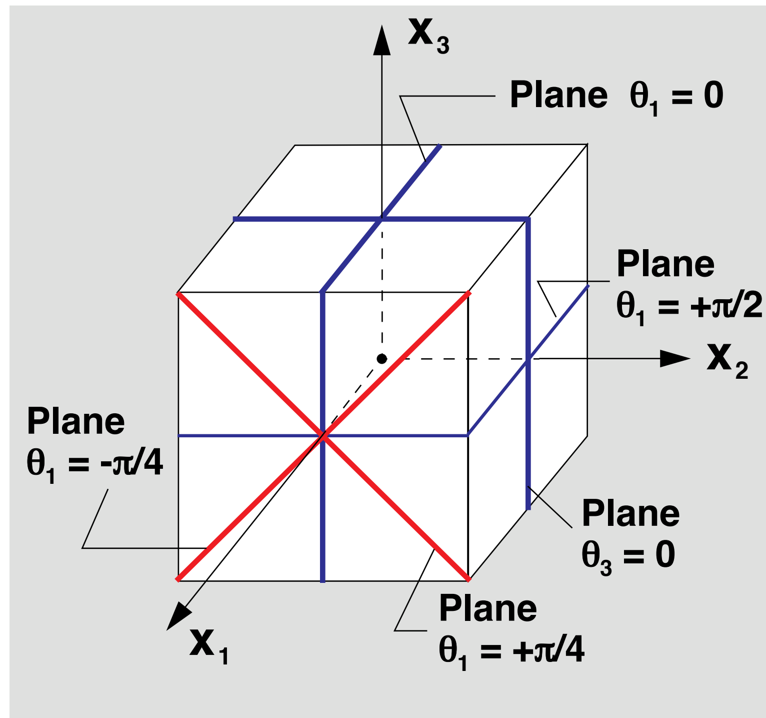


- Thus, there exists **five planes of elastic symmetry**, four of which are perpendicular to plane $\theta_1 = +\frac{\pi}{2}$

TETRAGONAL MATERIALS - CONTINUED

- **Next, consider a material with three planes of elastic symmetry given by** $\theta_1 = 0$ **and** $\pm\frac{\pi}{4}$

- **By taking** $\theta_3 = 0$ **as an additional plane of elastic symmetry, the fact that the planes** $\theta_1 = 0$ **and** $\theta_3 = 0$ **are perpendicular implies the existence of a fifth symmetry plane given by** $\theta_1 = +\frac{\pi}{2}$

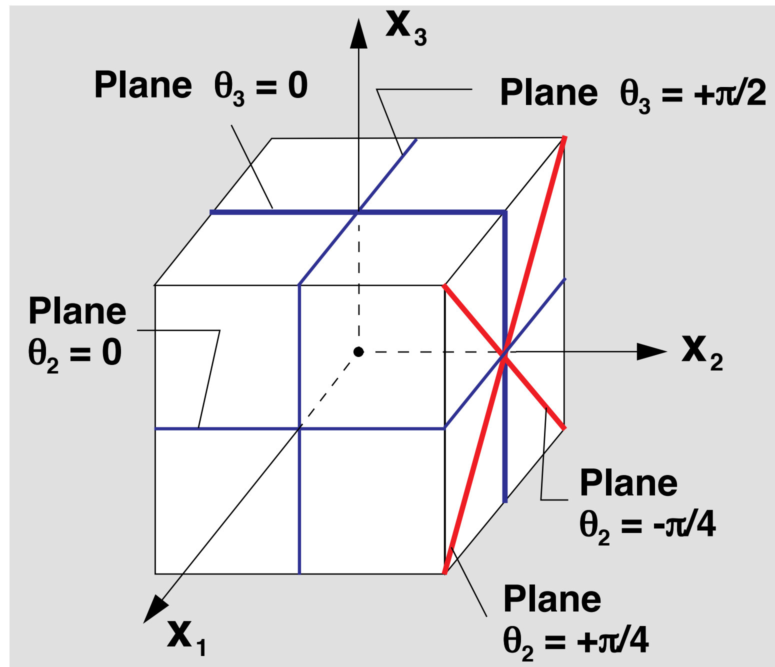


- **Thus, there exists five planes of elastic symmetry, four of which are perpendicular to plane** $\theta_3 = 0$

TETRAGONAL MATERIALS - CONTINUED

- **Finally, consider a material with three planes of elastic symmetry given by $\theta_2 = 0$ and $\pm\frac{\pi}{4}$**

- **By taking $\theta_3 = +\frac{\pi}{2}$ as an additional plane of elastic symmetry, the fact that the planes $\theta_3 = +\frac{\pi}{2}$ and $\theta_2 = 0$ are perpendicular implies the existence of a fifth symmetry plane given by $\theta_3 = 0$**



- **Thus, there exists five planes of elastic symmetry, four of which are perpendicular to plane $\theta_3 = +\frac{\pi}{2}$**

TETRAGONAL MATERIALS - CONCLUDED

- When there exists **five planes of elastic symmetry**, in which four planes are perpendicular to a fifth plane, the material is classified as a **tetragonal material**

- Three specific cases are defined as follows:

  - Planes of elastic symmetry given by $\theta_3 = 0$ and $\pm\frac{\pi}{4}$ and $\theta_1 = +\frac{\pi}{2}$

  - Planes of elastic symmetry given by $\theta_1 = 0$ and $\pm\frac{\pi}{4}$ and $\theta_3 = 0$

  - Planes of elastic symmetry given by $\theta_2 = 0$ and $\pm\frac{\pi}{4}$ and $\theta_3 = +\frac{\pi}{2}$

- For each of these three arrangements of symmetry planes, the corresponding constitutive equations can be derived directly from those for a **specially orthotropic material** by enforcing the invariance conditions for $\theta_1 = \pm\frac{\pi}{4}$, $\theta_2 = \pm\frac{\pi}{4}$, or $\theta_3 = \pm\frac{\pi}{4}$

TETRAGONAL MATERIALS

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_3 AXIS

- Consider a **tetragonal material** with planes of reflective symmetry defined by $\theta_3 = 0$ and $\pm\frac{\pi}{4}$ and by $\theta_1 = +\frac{\pi}{2}$

- For a specially orthotropic material, it was shown previously that the constitutive equations are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

TETRAGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_3 AXIS

- For this case, the additional conditions on the constitutive equations are obtained by enforcing symmetry planes given by $\theta_3 = \pm\frac{\pi}{4}$
  - For $\theta_3 = \pm\frac{\pi}{4}$, $m = \cos\left(\pm\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \neq 0$ and $n = \sin\left(\pm\frac{\pi}{4}\right) = \pm\frac{\sqrt{2}}{2} \neq 0$

- With $\theta_3 = \pm\frac{\pi}{4}$, the first group of stiffness equations previously given herein for an arbitrary plane of symmetry, defined by constant values of θ_3, requires $\Gamma = 0$
  - Recall that $\Gamma = 0$ implies $C_{56} = -C_{24} = C_{14} = 0$ for arbitrary values of the angle $-\frac{\pi}{2} < \theta_3 \leq \frac{\pi}{2}$

- Similarly, the second group of stiffness equations for an arbitrary plane of symmetry, defined by constant values of θ_3, requires $C_{34} = 0$, $C_{23} = C_{13}$, and $C_{55} = C_{44}$

TETRAGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_3 AXIS

- **Furthermore, the stiffness equations in group 3 yield the invariance condition $C_{22} = C_{11}$**
- **Likewise, the compliance equations in group 5 yield the invariance conditions $S_{34} = 0$, $S_{23} = S_{13}$, and $S_{55} = S_{44}$**
- **The compliance equations in group 4 yield the invariance conditions $S_{56} = S_{24} = S_{14} = 0$**
- **The compliance equations in group 6 yield the invariance condition $S_{22} = S_{11}$**

TETRAGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_3 AXIS

- The invariance conditions $\alpha_{1'3'} = 0: \quad m\alpha_{13} + n\alpha_{23} = 0$ and $\alpha_{1'2'} = 0: \quad (m^2 - n^2)\alpha_{12} + mn(\alpha_{22} - \alpha_{11}) = 0$ yield

$$\alpha_{23} = \alpha_{13} = \alpha_{12} = 0 \quad \text{and} \quad \alpha_{22} = \alpha_{11}$$

- The invariance conditions $\beta_{1'3'} = 0: \quad m\beta_{13} + n\beta_{23} = 0$ and $\beta_{1'2'} = 0: \quad (m^2 - n^2)\beta_{12} + mn(\beta_{22} - \beta_{11}) = 0$ yield

$$\beta_{23} = \beta_{13} = \beta_{12} = 0 \quad \text{and} \quad \beta_{22} = \beta_{11}$$

- Together, the invariance conditions yield the following constitutive equations for a **tetragonal material**

TETRAGONAL MATERIALS - CONCLUDED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_3 AXIS

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{11} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

- **Therefore, a tetragonal material has six independent elastic constants and two independent thermal-expansion or thermal-compliance parameters**

TETRAGONAL MATERIALS

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_1 AXIS

- Next, consider a **tetragonal material** with planes of reflective symmetry defined by $\theta_1 = 0$ and $\pm\frac{\pi}{4}$ and by $\theta_3 = 0$

  - For $\theta_1 = \pm\frac{\pi}{4}$, $m = \cos\left(\pm\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \neq 0$ and $n = \sin\left(\pm\frac{\pi}{4}\right) = \pm\frac{\sqrt{2}}{2} \neq 0$

- The first group of stiffness equations previously given herein for an arbitrary plane of symmetry, defined by constant values of θ_1, yields the invariance conditions $C_{46} = C_{35} = C_{25} = 0$

- The second group of stiffness equations for an arbitrary plane of symmetry, defined by constant values of θ_1, requires $C_{15} = 0$, $C_{13} = C_{12}$, and $C_{66} = C_{55}$

- The stiffness equations in group 3 yield the invariance condition $C_{33} = C_{22}$

TETRAGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_1 AXIS

- The invariance conditions $\beta_{1'2'} = 0: \quad m\beta_{12} + n\beta_{13} = 0$ and $\beta_{2'3'} = 0: \quad \left(m^2 - n^2\right)\beta_{23} + mn\left(\beta_{33} - \beta_{22}\right) = 0$ yield

$$\beta_{23} = \beta_{13} = \beta_{12} = 0 \quad \text{and} \quad \beta_{33} = \beta_{22}$$

- The resulting constitutive equation, obtained by simplifying the constitutive equation for a specialy orthotropic material, is given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

TETRAGONAL MATERIALS - CONCLUDED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_1 AXIS

- **Applying the same process to the equations for the compliances and coefficients of thermal expansion yields**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref} \right)$$

TETRAGONAL MATERIALS

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_2 AXIS

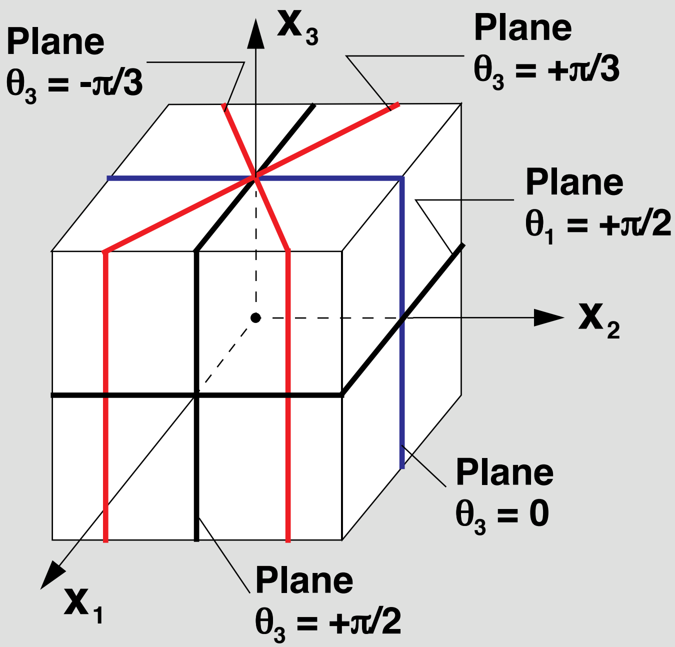
- Finally, consider a **tetragonal material** with planes of reflective symmetry defined by $\theta_2 = 0 \text{ and } \pm\frac{\pi}{4}$ and by $\theta_3 = +\frac{\pi}{2}$

- The constitutive equations are derived by using the corresponding equations given previously for a tetragonal material that has four reflective-symmetry planes that contain the x_3 axis, along with a juxtaposition of indices

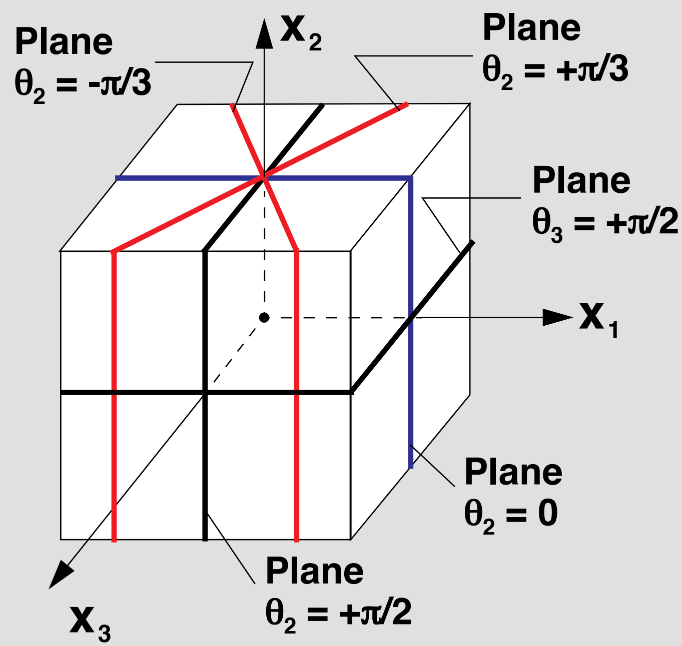
- In particular, the desired constitutive equations are found by simply determining the renumbering of the indices that brings the figure shown below with four symmetry planes that contain the x_3 axis into congruence with the adjacent figure shown below with four symmetry planes that contain the x_2 axis

TETRAGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_2 AXIS



Four symmetry planes that contain the x_3 axis



Four symmetry planes that contain the x_2 axis

TETRAGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_2 AXIS

- **Inspection of the figures indicates the following transformation of the indices: $1 \rightarrow 3$, $2 \rightarrow 1$, and $3 \rightarrow 2$**

- **Next, it must be realized that the exchanging of indices must be used with the indices of tensors to determine the indices used with the abridged notation (matrix)**

  - **The following index pairs relate the tensor indices to the matrix indices**

tensor notation	11	22	33	23, 32	31, 13	12, 21
matrix notation	1	2	3	4	5	6

- **Using this information along with $1 \rightarrow 3$, $2 \rightarrow 1$, and $3 \rightarrow 2$ gives the relations: $4 \rightarrow 6$, $5 \rightarrow 4$, and $6 \rightarrow 5$**

TETRAGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_2 AXIS

- Likewise, the transformation of index pairs that appear in the abridged notation are given by

$11 \rightarrow 33$					
$12 \rightarrow 13$	$22 \rightarrow 11$				
$13 \rightarrow 23$	$23 \rightarrow 12$	$33 \rightarrow 22$			
$14 \rightarrow 36$	$24 \rightarrow 16$	$34 \rightarrow 26$	$44 \rightarrow 66$		
$15 \rightarrow 34$	$25 \rightarrow 14$	$35 \rightarrow 24$	$45 \rightarrow 46$	$55 \rightarrow 44$	
$16 \rightarrow 35$	$26 \rightarrow 15$	$36 \rightarrow 25$	$46 \rightarrow 56$	$56 \rightarrow 45$	$66 \rightarrow 55$

- Consider the following constitutive equations for a tetragonal material that has reflective symmetry planes that contain the x_3 axis

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

TETRAGONAL MATERIALS - CONTINUED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_2 AXIS

- Applying the index transformation to these constitutive equations gives

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{33} & C_{13} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{33} & C_{23} & 0 & 0 & 0 \\ C_{23} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{33} \\ \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} + \begin{Bmatrix} \beta_{33} \\ \beta_{33} \\ \beta_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

- Reordering these equations into the standard form yields

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{33} & C_{23} & C_{13} & 0 & 0 & 0 \\ C_{23} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{33} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

TETRAGONAL MATERIALS - CONCLUDED

REFLECTIVE SYMMETRY PLANES THAT CONTAIN THE x_2 AXIS

- Applying the same procedure to the inverse equations yields

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{33} & S_{23} & S_{13} & 0 & 0 & 0 \\ S_{23} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

SUMMARY OF TETRAGONAL MATERIALS

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \qquad x_1 \text{ axis}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{33} & C_{23} & C_{13} & 0 & 0 & 0 \\ C_{23} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{33} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \qquad x_2 \text{ axis}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \qquad x_3 \text{ axis}$$

SUMMARY OF TETRAGONAL MATERIALS - CONTINUED

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

x_1 axis

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{33} & S_{23} & S_{13} & 0 & 0 & 0 \\ S_{23} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

x_2 axis

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{11} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

x_3 axis

SUMMARY OF TETRAGONAL MATERIALS
CONCLUDED

- Inspection of the equations shows that a tetragonal material is a specially orthotropic material in which the properties associated with two of the coordinate directions are identical

TRANSVERSELY ISOTROPIC MATERIALS

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$

- Now, consider the case in which a specially orthotropic material, which exhibits symmetry about the perpendicular principal-coordinate planes, also exhibits isotropy in the plane $x_3 = 0$

- For the specially orthotropic material, it was shown previously that the material properties are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

TRANSVERSELY ISOTROPIC MATERIALS

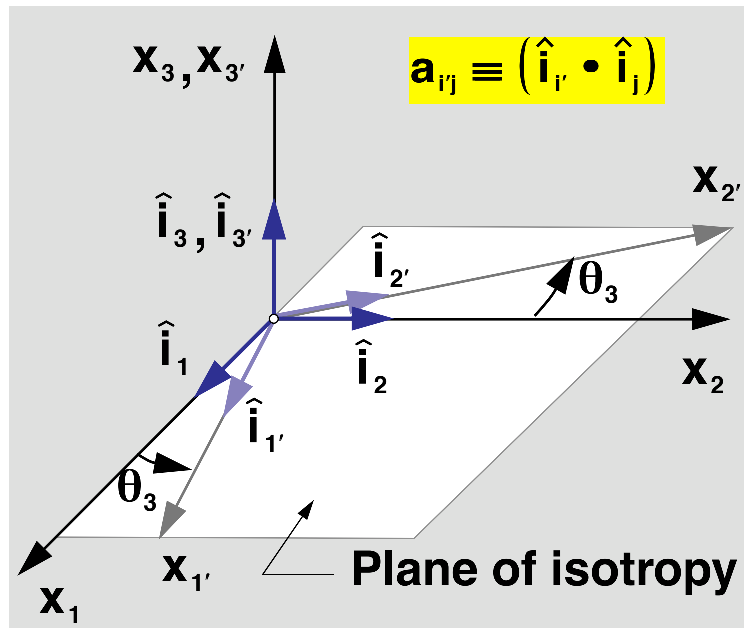
ISOTROPY PLANE $x_3 = 0$ (CONTINUED)

- **For this type of symmetry to exist, there must be an infinite number of elastic symmetry planes that are perpendicular to the plane $x_3 = 0$**
  - **Equivalently, the constitutive matrices and vectors must invariant with respect to dextral rotations about the x_3 axis**

- **The coordinate transformation for this symmetry is shown in the figure and is given by $x_{1'} = x_1\cos\theta_3 + x_2\sin\theta_3$, $x_{2'} = -x_1\sin\theta_3 + x_2\cos\theta_3$, and $x_{3'} = x_3$, with $0 \le \theta_3 < 2\pi$**

- **The corresponding matrix of direction cosines is given by**

$$\begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$ (CONTINUED)

- The corresponding stress and strain transformation matrices were shown previously to be given by

$$\left[\mathbf{T}_\sigma(\theta_3)\right] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -2\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \quad \text{and}$$

$$\left[\mathbf{T}_\varepsilon(\theta_3)\right] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$ (CONTINUED)

- Specific expressions that result from the transformation $[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$ were given previously for an anisotropic material

- For a specially orthotropic material, $[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$ yields the expressions given previously for a **generally orthotropic material**; that is,

$$C_{1'1'} = m^4 C_{11} + 2m^2 n^2 (C_{12} + 2C_{66}) + n^4 C_{22}$$

$$C_{1'2'} = m^2 n^2 (C_{11} + C_{22} - 4C_{66}) + (m^4 + n^4) C_{12}$$

$$C_{1'3'} = m^2 C_{13} + n^2 C_{23} \qquad C_{1'4'} = 0 \qquad C_{1'5'} = 0$$

$$C_{1'6'} = mn\left[m^2 - n^2\right](C_{12} + 2C_{66}) + mn\left(n^2 C_{22} - m^2 C_{11}\right)$$

$$C_{2'2'} = m^4 C_{22} + 2m^2 n^2 (C_{12} + 2C_{66}) + n^4 C_{11}$$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$ (CONTINUED)

$C_{2'3'} = m^2 C_{23} + n^2 C_{13}$ $\quad$ $C_{2'4'} = 0$ $\quad$ $C_{2'5'} = 0$

$$C_{2'6'} = mn\left[n^2 - m^2\right]\left(C_{12} + 2C_{66}\right) + mn\left(m^2 C_{22} - n^2 C_{11}\right)$$

$C_{3'3'} = C_{33}$ $\quad$ $C_{3'4'} = 0$ $\quad$ $C_{3'5'} = 0$ $\quad$ $C_{3'6'} = mn\left(C_{23} - C_{13}\right)$

$C_{4'4'} = m^2 C_{44} + n^2 C_{55}$ $\quad$ $C_{4'5'} = mn\left(C_{44} - C_{55}\right)$ $\quad$ $C_{4'6'} = 0$ $\quad$ $C_{5'6'} = 0$

$C_{5'5'} = m^2 C_{55} + n^2 C_{44}$ $\quad$ $C_{6'6'} = m^2 n^2\left(C_{11} + C_{22} - 2C_{12}\right) + \left(m^2 - n^2\right)^2 C_{66}$

with $m = \cos\theta_3$ and $n = \sin\theta_3$

- Next, the **invariance condition** $[C] = [T_\sigma][C][T_\varepsilon]^{-1}$ is enforced, which implies the conditions $C_{r's'} = C_{rs}$, where $r, s \in \{1,2,3,4,5,6\}$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$ (CONTINUED)

- Enforcing $C_{1'1'} = C_{11}$ gives $C_{11} = m^4C_{11} + 2m^2n^2\left(C_{12} + 2C_{66}\right) + n^4C_{22}$, which can only be satisfied in a general sense (arbitrary values for θ_3) if

  $C_{11} = C_{22}$ and $C_{12} + 2C_{66} = C_{11}$

- Similarly, enforcing $C_{1'3'} = C_{13}$ gives $C_{13} = m^2C_{13} + n^2C_{23}$, which yields

  $C_{23} = C_{13}$

- Likewise, enforcing $C_{4'4'} = C_{44}$ gives $C_{44} = m^2C_{44} + n^2C_{55}$, which yields

  $C_{55} = C_{44}$

- Finally, by substituting $C_{11} = C_{22}$, $C_{12} + 2C_{66} = C_{11}$, $C_{23} = C_{13}$, and $C_{55} = C_{44}$ into the remaining expressions for $C_{r's'}$ and enforcing $C_{r's'} = C_{rs}$, it is found the remaining expressions are identically satisfied

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$ (CONTINUED)

- Thus, the stiffness matrix for a **transversely isotropic material** that is *isotropic* in the plane $x_3 = 0$ has the form

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(C_{11} - C_{12}\right) \end{bmatrix}$$

which has **5 independent stiffnesses**

- Following the same procedure for the compliance coefficients by using the transformed-compliance expressions previously given for a **generally orthotropic material** and enforcing the **invariance condition** $[S] = [T_\varepsilon][S][T_\sigma]^{-1}$ or $S_{r's'} = S_{rs}$ yields similar results

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$ (CONTINUED)

- **Thus, the compliance matrix for a transversely isotropic material that is *isotropic* in the plane $x_3 = 0$ has the form**

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(S_{11} - S_{12}\right) \end{bmatrix}$$

which has 5 independent compliances

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$ (CONTINUED)

- For a specially orthotropic material, $\{\alpha'\} = [T_\varepsilon]\{\alpha\}$ yields the expressions for the transformed thermal-expansion coefficients, given previously for a **generally orthotropic material**; that is,

$$\alpha_{1'1'} = m^2\alpha_{11} + n^2\alpha_{22} \qquad \alpha_{2'2'} = m^2\alpha_{22} + n^2\alpha_{11} \qquad \alpha_{3'3'} = \alpha_{33}$$

$$\alpha_{2'3'} = 0 \qquad \alpha_{1'3'} = 0 \qquad \alpha_{1'2'} = mn(\alpha_{22} - \alpha_{11})$$

- Enforcing the **invariance condition** $\{\alpha\} = [T_\varepsilon]\{\alpha\}$ or $\alpha_{r's'} = \alpha_{rs}$ yields the requirements that $\alpha_{22} = \alpha_{11}$ and $\alpha_{12} = 0$

- Following the same procedure for the thermal moduli by using the transformed-thermal-moduli expressions previously given for a **generally orthotropic material** and enforcing the **invariance condition** $\{\beta\} = [T_\sigma]\{\beta\}$ or $\beta_{r's'} = \beta_{rs}$ yields similar results; that is, $\beta_{22} = \beta_{11}$ and $\beta_{12} = 0$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$ (CONTINUED)

- Applying all these simplifications, the linear thermoelastic constitutive equations become

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(S_{11} - S_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{11} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_3 = 0$ (CONCLUDED)

- **Likewise** $$\begin{Bmatrix} \beta_{11} \\ \beta_{33} \end{Bmatrix} = - \begin{bmatrix} (C_{11} + C_{12}) & C_{13} \\ 2C_{13} & C_{33} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{33} \end{Bmatrix}$$

- **Thus, the transversely isotropic material considered has 5 independent stiffness coefficients, 5 independent compliance coefficients, 2 independent thermal-expansion coefficients, and 2 independent thermal-moduli coefficients**

- **It is interesting to point out that hexagonal materials, defined by planes of reflective symmetry given by $\theta_3 = 0, \pm\frac{\pi}{6}$, have the same number of independent elastic constants and thermal parameters as the transversely isotropic material**

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$

- Now, consider the case in which a **specially orthotropic material**, which exhibits symmetry about the perpendicular principal-coordinate planes, also exhibits isotropy in the plane $x_1 = 0$

- For the specially orthotropic material, it was shown previously that the material properties are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

TRANSVERSELY ISOTROPIC MATERIALS

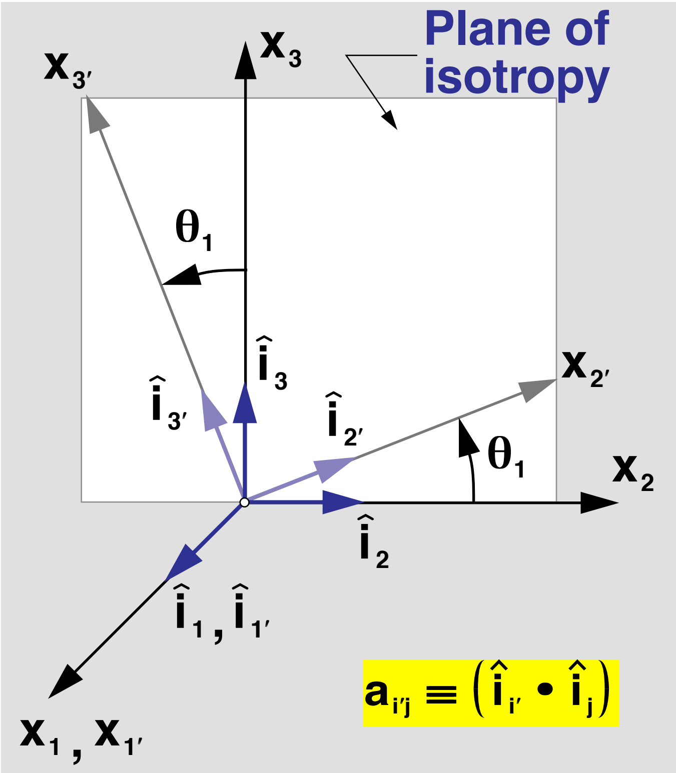
ISOTROPY PLANE $x_1 = 0$ (CONTINUED)

- For this type of symmetry to exist, there must be an infinite number of elastic symmetry planes that are perpendicular to the plane $x_1 = 0$
    - Equivalently, the constitutive matrices and vectors must invariant with respect to dextral rotations about the x_1 axis

- The coordinate transformation for this symmetry is shown in the figure and is given by $x_{1'} = x_1$, $x_{2'} = x_2\cos\theta_1 + x_3\sin\theta_1$, and $x_{3'} = -x_2\sin\theta_1 + x_3\cos\theta_1$, with $0 \le \theta_1 < 2\pi$

- The corresponding matrix of **direction cosines** is given by

$$\begin{bmatrix} a_{1'1} & a_{1'2} & a_{1'3} \\ a_{2'1} & a_{2'2} & a_{2'3} \\ a_{3'1} & a_{3'2} & a_{3'3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$



TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$ (CONTINUED)

- **The corresponding stress and strain transformation matrices were shown previously to be given by**

$$[T_\sigma] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & 2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -2\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -\sin\theta_1\cos\theta_1 & \sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

and

$$[T_\varepsilon] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos^2\theta_1 & \sin^2\theta_1 & \sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & \sin^2\theta_1 & \cos^2\theta_1 & -\sin\theta_1\cos\theta_1 & 0 & 0 \\ 0 & -2\sin\theta_1\cos\theta_1 & 2\sin\theta_1\cos\theta_1 & \cos^2\theta_1 - \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & 0 & 0 & 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$ (CONTINUED)

- **Specific expressions that result from the transformation $[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$ were given previously for an anisotropic material**

- **For a specially orthotropic material, $[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$ yields the expressions for a generally orthotropic material; that is,**

$$C_{1'1'} = C_{11} \qquad C_{1'2'} = m^2C_{12} + n^2C_{13} \qquad C_{1'3'} = m^2C_{13} + n^2C_{12}$$

$$C_{1'4'} = mn\left(C_{13} - C_{12}\right) \qquad C_{1'5'} = 0 \qquad C_{1'6'} = 0$$

$$C_{2'2'} = m^4C_{22} + 2m^2n^2\left(C_{23} + 2C_{44}\right) + n^4C_{33}$$

$$C_{2'3'} = m^2n^2\left(C_{22} + C_{33} - 4C_{44}\right) + \left(m^4 + n^4\right)C_{23}$$

$$C_{2'4'} = mn^3\left(C_{33} - C_{23} - 2C_{44}\right) - m^3n\left(C_{22} - C_{23} - 2C_{44}\right)$$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$ (CONTINUED)

$C_{2'5'} = 0$ $\quad$ $C_{2'6'} = 0$ $\quad$ $C_{3'3'} = m^4 C_{33} + 2m^2 n^2 \left(C_{23} + 2C_{44}\right) + n^4 C_{22}$

$C_{3'4'} = m^3 n\left(C_{33} - C_{23} - 2C_{44}\right) - mn^3\left(C_{22} - C_{23} - 2C_{44}\right)$

$C_{3'5'} = 0$ $\quad$ $C_{3'6'} = 0$ $\quad$ $C_{4'4'} = m^2 n^2\left(C_{22} + C_{33} - 2C_{23}\right) + \left(m^2 - n^2\right)^2 C_{44}$

$C_{4'5'} = 0$ $\quad$ $C_{4'6'} = 0$ $\quad$ $C_{5'5'} = m^2 C_{55} + n^2 C_{66}$

$C_{5'6'} = mn\left(C_{55} - C_{66}\right)$ $\quad$ $C_{6'6'} = m^2 C_{66} + n^2 C_{55}$

with $m = \cos\theta_1$ and $n = \sin\theta_1$

- Next, the **invariance condition** $[C] = [T_\sigma][C][T_\varepsilon]^{-1}$ is enforced, which implies the conditions $C_{r's'} = C_{rs}$, where $r, s \in \{1, 2, 3, 4, 5, 6\}$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$ (CONTINUED)

- Enforcing $C_{2'2'} = C_{22}$ gives $C_{22} = m^4C_{22} + 2m^2n^2\left(C_{23} + 2C_{44}\right) + n^4C_{33}$, which can only be satisfied in a general sense (arbitrary values for θ_1) if

  $C_{33} = C_{22}$ and $C_{23} + 2C_{44} = C_{22}$

- Similarly, enforcing $C_{1'3'} = C_{13}$ gives $C_{13} = m^2C_{13} + n^2C_{12}$, which yields

  $C_{12} = C_{13}$

- Likewise, enforcing $C_{5'5'} = C_{55}$ gives $C_{55} = m^2C_{55} + n^2C_{66}$, which yields

  $C_{66} = C_{55}$

- Finally, by substituting $C_{33} = C_{22}$, $C_{23} + 2C_{44} = C_{22}$, $C_{12} = C_{13}$, and $C_{66} = C_{55}$ into the remaining expressions for $C_{r's'}$ and enforcing $C_{r's'} = C_{rs}$, it is found the remaining expressions are identically satisfied

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$ (CONTINUED)

- Thus, the stiffness matrix for a transversely isotropic material that is *isotropic* in the plane $x_1 = 0$ has the form

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{22} - C_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}$$

which has **5 independent stiffnesses**

- Following the same procedure for the compliance coefficients by using the transformed-compliance expressions previously given for a **generally orthotropic material** and enforcing the **invariance condition** $[S] = [T_\varepsilon][S][T_\sigma]^{-1}$ or $S_{r's'} = S_{rs}$ yields similar results

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$ (CONTINUED)

- Thus, the compliance matrix for a transversely isotropic material that is *isotropic* in the plane $x_1 = 0$ has the form

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(S_{22} - S_{23}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix}$$

which has 5 independent compliances

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$ (CONTINUED)

- For a specially orthotropic material, $\{\alpha'\} = [T_\varepsilon]\{\alpha\}$ yields the expressions for the transformed thermal-expansion coefficients for a **generally orthotropic material**; that is,

$$\alpha_{1'1'} = \alpha_{11} \qquad \alpha_{2'2'} = m^2\alpha_{22} + n^2\alpha_{33} \qquad \alpha_{3'3'} = m^2\alpha_{33} + n^2\alpha_{22}$$

$$\alpha_{2'3'} = mn(\alpha_{33} - \alpha_{22}) \qquad \alpha_{1'3'} = 0 \qquad \alpha_{1'2'} = 0$$

- Enforcing the **invariance condition** $\{\alpha\} = [T_\varepsilon]\{\alpha\}$ or $\alpha_{r's'} = \alpha_{rs}$ yields the requirements that $\alpha_{33} = \alpha_{22}$ and $\alpha_{23} = 0$

- Following the same procedure for the thermal moduli by using the transformed-thermal-moduli expressions previously given for a **generally orthotropic material** and enforcing the **invariance condition** $\{\beta\} = [T_\sigma]\{\beta\}$ or $\beta_{r's'} = \beta_{rs}$ yields similar results; that is, $\beta_{33} = \beta_{22}$ and $\beta_{23} = 0$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$ (CONTINUED)

- **Applying all these simplifications, the linear thermoelastic constitutive equations become**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{22} - C_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

and

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(S_{22} - S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{22} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

TRANSVERSELY ISOTROPIC MATERIALS

ISOTROPY PLANE $x_1 = 0$ (CONCLUDED)

- **Likewise** $$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \end{Bmatrix} = -\begin{bmatrix} C_{11} & 2C_{12} \\ C_{12} & (C_{22} + C_{23}) \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \end{Bmatrix}$$

- **Thus, the transversely isotropic material considered has 5 independent stiffness coefficients, 5 independent compliance coefficients, 2 independent thermal-expansion coefficients, and 2 independent thermal-moduli coefficients**

- **It is interesting to point out that hexagonal materials, defined by planes of reflective symmetry given by $\theta_1 = 0, \pm\frac{\pi}{6}$, have the same number of independent elastic constants and thermal parameters as the transversely isotropic material**

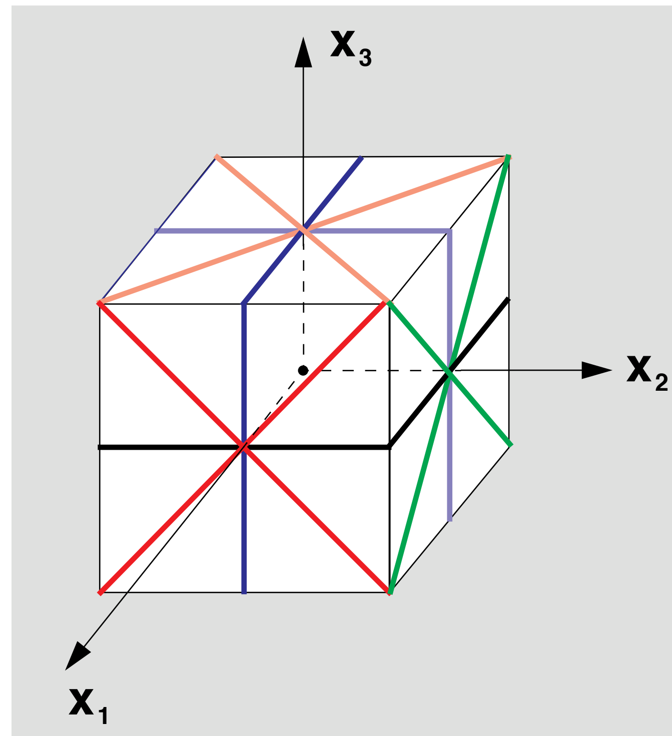
CUBIC MATERIALS

CUBIC MATERIALS

- A cubic material arises from the invariance conditions that are obtained for a single plane of elastic reflective symmetry that contains a coordinate axis

- Precisely, a cubic material is obtained by enforcing the invariance conditions for a tetragonal material, for each of the three coordinate axes

- That is, there are **nine planes** of elastic symmetry that are given by $\theta_1 = 0, \pm\frac{\pi}{4}$, $\theta_2 = 0, \pm\frac{\pi}{4}$, and $\theta_3 = 0, \pm\frac{\pi}{4}$



- Expressions for the constitutive equations are obtained directly by enforcing the three sets of invariance conditions for the previously given cases for tetragonal materials sequentially

CUBIC MATERIALS - CONCLUDED

The resulting constitutive equations are given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{11} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{44} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{11} \\ \alpha_{11} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

- **Therefore, a cubic material has three independent elastic constants and one independent thermal-expansion or thermal-compliance parameter**

COMPLETELY ISOTROPIC MATERIALS

COMPLETELY ISOTROPIC MATERIALS

- First, consider the case in which both the planes $x_1 = 0$ and $x_2 = 0$ are planes of isotropy

  - Applying the results of both of the corresponding symmetry transformations successively yields the following constitutive equations

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{11} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

  - These equations possess **two** independent elastic stiffnesses and **one** independent thermal moduli; the same as a completely **isotropic material**

COMPLETELY ISOTROPIC MATERIALS
(CONTINUED)

- **Similarly,**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(S_{11} - S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(S_{11} - S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(S_{11} - S_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{11} \\ \alpha_{11} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

$$\beta_{11} = -\left(C_{11} + 2C_{12}\right)\alpha_{11}$$

- **Now consider a general transformation of the stiffness matrix previously obtained for a material with two perpendicular planes of isotropy**

COMPLETELY ISOTROPIC MATERIALS
(CONTINUED)

- The general transformation law is given by $[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$, where

$$[T_\sigma] = \begin{bmatrix} (a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & 2a_{1'2}a_{1'3} & 2a_{1'1}a_{1'3} & 2a_{1'1}a_{1'2} \\ (a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & 2a_{2'2}a_{2'3} & 2a_{2'1}a_{2'3} & 2a_{2'1}a_{2'2} \\ (a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & 2a_{3'2}a_{3'3} & 2a_{3'1}a_{3'3} & 2a_{3'1}a_{3'2} \\ a_{2'1}a_{3'1} & a_{2'2}a_{3'2} & a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\ a_{1'1}a_{3'1} & a_{1'2}a_{3'2} & a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\ a_{1'1}a_{2'1} & a_{1'2}a_{2'2} & a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1}) \end{bmatrix}$$

and

$$[T_\varepsilon] = \begin{bmatrix} (a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & a_{1'2}a_{1'3} & a_{1'1}a_{1'3} & a_{1'1}a_{1'2} \\ (a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & a_{2'2}a_{2'3} & a_{2'1}a_{2'3} & a_{2'1}a_{2'2} \\ (a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & a_{3'2}a_{3'3} & a_{3'1}a_{3'3} & a_{3'1}a_{3'2} \\ 2a_{2'1}a_{3'1} & 2a_{2'2}a_{3'2} & 2a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\ 2a_{1'1}a_{3'1} & 2a_{1'2}a_{3'2} & 2a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\ 2a_{1'1}a_{2'1} & 2a_{1'2}a_{2'2} & 2a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1}) \end{bmatrix}$$

COMPLETELY ISOTROPIC MATERIALS

(CONTINUED)

- In addition, the direction cosines that appear in the transformation matrices satisfy the following conditions, that are given in expanded, tabular form:

k	p	$a_{k'q}a_{p'q} = \delta_{kp}$
1	1	$(a_{1'1})^2 + (a_{1'2})^2 + (a_{1'3})^2 = 1$
2	1	$a_{2'1}a_{1'1} + a_{2'2}a_{1'2} + a_{2'3}a_{1'3} = 0$
3	1	$a_{3'1}a_{1'1} + a_{3'2}a_{1'2} + a_{3'3}a_{1'3} = 0$
2	2	$(a_{2'1})^2 + (a_{2'2})^2 + (a_{2'3})^2 = 1$
3	2	$a_{3'1}a_{2'1} + a_{3'2}a_{2'2} + a_{3'3}a_{2'3} = 0$
3	3	$(a_{3'1})^2 + (a_{3'2})^2 + (a_{3'3})^2 = 1$

k	p	$a_{q'k}a_{q'p} = \delta_{kp}$
1	1	$(a_{1'1})^2 + (a_{2'1})^2 + (a_{3'1})^2 = 1$
2	1	$a_{1'2}a_{1'1} + a_{2'2}a_{2'1} + a_{3'2}a_{3'1} = 0$
3	1	$a_{1'3}a_{1'1} + a_{2'3}a_{2'1} + a_{3'3}a_{3'1} = 0$
2	2	$(a_{1'2})^2 + (a_{2'2})^2 + (a_{3'2})^2 = 1$
3	2	$a_{1'3}a_{1'2} + a_{2'3}a_{2'2} + a_{3'3}a_{3'2} = 0$
3	3	$(a_{1'3})^2 + (a_{2'3})^2 + (a_{3'3})^2 = 1$

COMPLETELY ISOTROPIC MATERIALS
(CONTINUED)

- For the general transformation to be a symmetry transformation,

  $[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$ becomes $[C] = [T_\sigma][C][T_\varepsilon]^{-1}$

  - It is convenient to rewrite this expression as $[C][T_\varepsilon] - [T_\sigma][C] = [0]$,

    where $C_{44} = \frac{1}{2}(C_{11} - C_{12})$,

$$[C][T_\varepsilon] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} (a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & a_{1'2}a_{1'3} & a_{1'1}a_{1'3} & a_{1'1}a_{1'2} \\ (a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & a_{2'2}a_{2'3} & a_{2'1}a_{2'3} & a_{2'1}a_{2'2} \\ (a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & a_{3'2}a_{3'3} & a_{3'1}a_{3'3} & a_{3'1}a_{3'2} \\ 2a_{2'1}a_{3'1} & 2a_{2'2}a_{3'2} & 2a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\ 2a_{1'1}a_{3'1} & 2a_{1'2}a_{3'2} & 2a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\ 2a_{1'1}a_{2'1} & 2a_{1'2}a_{2'2} & 2a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1}) \end{bmatrix}$$

and

$$[T_\sigma][C] = \begin{bmatrix} (a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & 2a_{1'2}a_{1'3} & 2a_{1'1}a_{1'3} & 2a_{1'1}a_{1'2} \\ (a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & 2a_{2'2}a_{2'3} & 2a_{2'1}a_{2'3} & 2a_{2'1}a_{2'2} \\ (a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & 2a_{3'2}a_{3'3} & 2a_{3'1}a_{3'3} & 2a_{3'1}a_{3'2} \\ a_{2'1}a_{3'1} & a_{2'2}a_{3'2} & a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\ a_{1'1}a_{3'1} & a_{1'2}a_{3'2} & a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\ a_{1'1}a_{2'1} & a_{1'2}a_{2'2} & a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1}) \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}$$

COMPLETELY ISOTROPIC MATERIALS
(CONTINUED)

- **Performing the calculations for each element of the matrix equation** $[C][T_\varepsilon] - [T_\sigma][C] = [0]$ **gives**

$$11:\ C_{12}\left[(a_{2'1})^2 + (a_{3'1})^2 - (a_{1'2})^2 - (a_{1'3})^2\right] = 0$$

$$12:\ C_{12}\left[(a_{2'2})^2 + (a_{3'2})^2 - (a_{1'2})^2 - (a_{1'3})^2\right] = 0$$

$$13:\ C_{12}\left[(a_{2'3})^2 + (a_{3'3})^2 - (a_{1'1})^2 - (a_{1'2})^2\right] = 0$$

$$14:\ C_{12}\left[a_{1'3}a_{1'2} + a_{2'3}a_{2'2} + a_{3'3}a_{3'2}\right] = 0$$

$$15:\ C_{12}\left[a_{1'3}a_{1'1} + a_{2'3}a_{2'1} + a_{3'3}a_{3'1}\right] = 0$$

$$16:\ C_{12}\left[a_{1'2}a_{1'1} + a_{2'2}a_{2'1} + a_{3'2}a_{3'1}\right] = 0$$

$$21:\ C_{12}\left[(a_{1'1})^2 + (a_{3'1})^2 - (a_{2'2})^2 - (a_{2'3})^2\right] = 0$$

COMPLETELY ISOTROPIC MATERIALS

(CONTINUED)

$$22:\ C_{12}\left[(a_{1'2})^2 + (a_{3'2})^2 - (a_{2'1})^2 - (a_{2'3})^2\right] = 0$$

$$23:\ C_{12}\left[(a_{1'3})^2 + (a_{3'3})^2 - (a_{2'1})^2 - (a_{2'2})^2\right] = 0$$

$$24:\ C_{12}\left[a_{1'3}a_{1'2} + a_{2'3}a_{2'2} + a_{3'3}a_{3'2}\right] = 0$$

$$25:\ C_{12}\left[a_{1'3}a_{1'1} + a_{2'3}a_{2'1} + a_{3'3}a_{3'1}\right] = 0$$

$$26:\ C_{12}\left[a_{1'2}a_{1'1} + a_{2'2}a_{2'1} + a_{3'2}a_{3'1}\right] = 0$$

$$31:\ C_{12}\left[(a_{1'1})^2 + (a_{2'1})^2 - (a_{3'2})^2 - (a_{3'3})^2\right] = 0$$

$$32:\ C_{12}\left[(a_{1'2})^2 + (a_{2'2})^2 - (a_{3'1})^2 - (a_{3'3})^2\right] = 0$$

$$33:\ C_{12}\left[(a_{1'3})^2 + (a_{2'3})^2 - (a_{3'1})^2 - (a_{3'2})^2\right] = 0$$

COMPLETELY ISOTROPIC MATERIALS

(CONTINUED)

$$34:\ C_{12}\left[a_{1'3}a_{1'2} + a_{2'3}a_{2'2} + a_{3'3}a_{3'2}\right] = 0$$

$$35:\ C_{12}\left[a_{1'3}a_{1'1} + a_{2'3}a_{2'1} + a_{3'3}a_{3'1}\right] = 0$$

$$36:\ C_{12}\left[a_{1'2}a_{1'1} + a_{2'2}a_{2'1} + a_{3'2}a_{3'1}\right] = 0$$

$$41:\ C_{12}\left[a_{3'1}a_{2'1} + a_{3'2}a_{2'2} + a_{3'3}a_{2'3}\right] = 0$$

$$42:\ C_{12}\left[a_{3'1}a_{2'1} + a_{3'2}a_{2'2} + a_{3'3}a_{2'3}\right] = 0$$

$$43:\ C_{12}\left[a_{3'1}a_{2'1} + a_{3'2}a_{2'2} + a_{3'3}a_{2'3}\right] = 0$$

$$51:\ C_{12}\left[a_{3'1}a_{1'1} + a_{3'2}a_{1'2} + a_{3'3}a_{1'3}\right] = 0$$

$$52:\ C_{12}\left[a_{3'1}a_{1'1} + a_{3'2}a_{1'2} + a_{3'3}a_{1'3}\right] = 0$$

$$53:\ C_{12}\left[a_{3'1}a_{1'1} + a_{3'2}a_{1'2} + a_{3'3}a_{1'3}\right] = 0$$

COMPLETELY ISOTROPIC MATERIALS

(CONTINUED)

$$61:\ C_{12}\left[a_{2'1}a_{1'1} + a_{2'2}a_{1'2} + a_{2'3}a_{1'3}\right] = 0$$

$$62:\ C_{12}\left[a_{2'1}a_{1'1} + a_{2'2}a_{1'2} + a_{2'3}a_{1'3}\right] = 0$$

$$63:\ C_{12}\left[a_{2'1}a_{1'1} + a_{2'2}a_{1'2} + a_{2'3}a_{1'3}\right] = 0$$

- The elements of the matrix equation $[C][T_\varepsilon] - [T_\sigma][C] = [0]$ that are not listed above are satisfied identically

- Now, consider $11:\ C_{12}\left[(a_{2'1})^2 + (a_{3'1})^2 - (a_{1'2})^2 - (a_{1'3})^2\right] = 0$

- The condition $(a_{1'1})^2 + (a_{1'2})^2 + (a_{1'3})^2 = 1$ given in the previous table yields $(a_{1'2})^2 + (a_{1'3})^2 = 1 - (a_{1'1})^2$; hence,

$$11:\ C_{12}\left[(a_{1'1})^2 + (a_{2'1})^2 + (a_{3'1})^2 - 1\right] = 0$$

COMPLETELY ISOTROPIC MATERIALS
(CONTINUED)

- Next, using the condition $(a_{1'1})^2 + (a_{2'1})^2 + (a_{3'1})^2 = 1$, given in the previous table, shows that $11{:}\ C_{12}\left[(a_{1'1})^2 + (a_{2'1})^2 + (a_{3'1})^2 - 1\right] = 0$ is identically satisfied

- By following a similar procedure or by using direct substitution of the conditions $a_{k'q}a_{p'q} = \delta_{kp}$ and $a_{q'k}a_{q'p} = \delta_{kp}$ given in the previous table, it can be shown that the invariance condition $[C][T_\varepsilon] - [T_\sigma][C] = [0]$ is identically satisfied

- Therefore, *two orthogonal planes of isotropy imply that every plane is a plane of isotropy* because a symmetry transformation for any plane can be obtained from the general transformation

  - An **isotropic material** has two independent stiffnesses and is the simplest known material, with no dependence on direction

COMPLETELY ISOTROPIC MATERIALS

(CONTINUED)

- Now consider a **general transformation of the thermal moduli** previously obtained for a material with two perpendicular plane of isotropy

  - The transformation law is given by $\{\beta'\} = [T_\sigma]\{\beta\}$

- For the general transformation to be a symmetry transformation,

  $\{\beta'\} = [T_\sigma]\{\beta\}$ becomes $\{\beta\} - [T_\sigma]\{\beta\} = \{0\}$

  - The expanded form is given by

$$\begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{11} \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} (a_{1'1})^2 & (a_{1'2})^2 & (a_{1'3})^2 & 2a_{1'2}a_{1'3} & 2a_{1'1}a_{1'3} & 2a_{1'1}a_{1'2} \\ (a_{2'1})^2 & (a_{2'2})^2 & (a_{2'3})^2 & 2a_{2'2}a_{2'3} & 2a_{2'1}a_{2'3} & 2a_{2'1}a_{2'2} \\ (a_{3'1})^2 & (a_{3'2})^2 & (a_{3'3})^2 & 2a_{3'2}a_{3'3} & 2a_{3'1}a_{3'3} & 2a_{3'1}a_{3'2} \\ a_{2'1}a_{3'1} & a_{2'2}a_{3'2} & a_{2'3}a_{3'3} & (a_{2'2}a_{3'3} + a_{2'3}a_{3'2}) & (a_{2'1}a_{3'3} + a_{2'3}a_{3'1}) & (a_{2'1}a_{3'2} + a_{2'2}a_{3'1}) \\ a_{1'1}a_{3'1} & a_{1'2}a_{3'2} & a_{1'3}a_{3'3} & (a_{1'2}a_{3'3} + a_{1'3}a_{3'2}) & (a_{1'1}a_{3'3} + a_{1'3}a_{3'1}) & (a_{1'1}a_{3'2} + a_{1'2}a_{3'1}) \\ a_{1'1}a_{2'1} & a_{1'2}a_{2'2} & a_{1'3}a_{2'3} & (a_{1'2}a_{2'3} + a_{1'3}a_{2'2}) & (a_{1'1}a_{2'3} + a_{1'3}a_{2'1}) & (a_{1'1}a_{2'2} + a_{1'2}a_{2'1}) \end{bmatrix} \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{11} \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 0$$

COMPLETELY ISOTROPIC MATERIALS
(CONTINUED)

- By using direct substitution of the conditions $a_{k'q}a_{p'q} = \delta_{kp}$ and $a_{q'k}a_{q'p} = \delta_{kp}$ given in the previous table, it can be shown that the invariance condition $\{\beta\} - [T_\sigma]\{\beta\} = \{0\}$ is identically satisfied
  - Therefore, an **isotropic material** has **one** independent thermal moduli
- For **general transformations of the compliance matrix** and the **thermal-expansion coefficents** previously obtained for a material with two perpendicular plane of isotropy, the transformation laws are given by $[S'] = [T_\varepsilon][S][T_\sigma]^{-1}$ and $\{\alpha'\} = [T_\varepsilon]\{\alpha\}$, respectively
- For the general transformations to be symmetry transformations,

$[S'] = [T_\varepsilon][S][T_\sigma]^{-1}$ becomes $[T_\varepsilon][S] - [S][T_\sigma] = [0]$ and

$\{\alpha'\} = [T_\varepsilon]\{\alpha\}$ becomes $\{\alpha\} - [T_\varepsilon]\{\alpha\} = \{0\}$

COMPLETELY ISOTROPIC MATERIALS
(CONCLUDED)

- By performing the calculations for each element of the matrix equation $[T_\varepsilon][S] - [S][T_\sigma] = [0]$, and using the conditions $a_{k'q}a_{p'q} = \delta_{kp}$ and $a_{q'k}a_{q'p} = \delta_{kp}$ given in the previous table, it can be shown that the invariance condition is satisfied identically

- Similarly, the invariance condition $\{\alpha\} - [T_\varepsilon]\{\alpha\} = \{0\}$ is also satisfied identically

- Therefore, an **isotropic material** has two independent compliances and **one** coefficient of thermal expansion

CLASSES OF MATERIAL SYMMETRY

SUMMARY OF INDEPENDENT MATERIAL CONSTANTS

- The **eight distinct classes** of elastic-material symmetry are classified by the number of independent material constants as follows:
  - **Triclinic** materials - 21 elastic, 6 thermal
  - **Monoclinic** materials - 13 elastic, 4 thermal
  - **Orthotropic** materials - 9 elastic, 3 thermal
  - **Trigonal** materials - 6 elastic, 2 thermal
  - **Tetragonal** materials - 6 elastic, 2 thermal
  - **Transversely isotropic** materials - 5 elastic, 2 thermal
  - **Cubic** materials - 3 elastic, 1 thermal
  - Completely **isotropic** materials - 2 elastic, 1 thermal

ENGINEERING CONSTANTS FOR ELASTIC MATERIALS

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS

- **The compliances of a homogeneous, elastic, anisotropic solid are usually expressed in terms of *engineering constants* when practical applications are under consideration**
  - **These constants are determined from experiments**
  - ***21 independent elastic constants* imply 21 separate experiments**
- **To determine expressions for the compliances in terms of engineering constants, it is useful to examine the meaning of each term in the general, unsymmetric compliance matrix given below**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right)$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **It is important to remember, that stresses cannot be measured in a laboratory experiment; only strains**

- **The terms S_{11}, S_{22}, and S_{33} relate the *normal strain* to the corresponding *normal stress***

  - **The engineering constants used to represent these relationships are called elastic moduli or moduli of elasticity**

  - **In particular, the symbol E_j is used herein to denote the elastic modulus in the x_j - coordinate direction**

  - **In general,** $$E_j \equiv \frac{\text{normal stress } \sigma_{jj}}{\text{normal strain } \varepsilon_{jj} \text{ caused by } \sigma_{jj}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- The terms S_{44}, S_{55}, and S_{66} relate the shearing strain in each coordinate plane to the corresponding shearing stress
  - The engineering constants used to represent these relationships are called *shear moduli* or moduli of rigidity
  - In particular, the symbol G_{ij} is used herein to denote the **shear modulus** in the x_i - x_j coordinate plane
  - In general, $G_{ij} \equiv \dfrac{\text{shearing stress } \sigma_{ij}}{\text{shearing strain } 2\varepsilon_{ij} \text{ caused by } \sigma_{ij}}$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **The terms S_{12}, S_{21}, S_{13}, S_{31}, S_{23}, and S_{32} relate the <u>lateral contraction or expansion</u> to the expansion or contraction of the solid, in the direction of a given normal stress**

  - **The engineering constants used to represent these relationships are called *Poisson's ratios***

  - **In particular, the symbol ν_{ij} is used herein to denote the lateral contraction or expansion in the x_j - coordinate direction caused by a normal stress applied in the x_i - coordinate direction**

  - **In general,** $$\nu_{ij} \equiv -\frac{\text{normal strain } \varepsilon_{jj} \text{ caused by normal stress } \sigma_{ii}}{\text{normal strain } \varepsilon_{ii} \text{ caused by } \sigma_{ii}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **The terms S_{14}, S_{15}, S_{16}, S_{24}, S_{25}, S_{26}, S_{34}, S_{35}, and S_{36} relate normal strains to shearing stresses**

  - **The engineering constants used to represent these relationships are generalizations of Poisson's ratios and are called *coefficients of interaction (or mutual influence) of the first kind*, and are attributed to A. L. Rabinovich**

  - **In particular, the symbol $\eta_{k,ij}$ is used herein to relate the contraction or expansion in the x_k - coordinate direction induced by a shearing stress applied in the x_i - x_j coordinate plane**

  - **That is,** $\eta_{k,\,ij} \equiv \dfrac{\text{normal strain } \varepsilon_{kk} \text{ caused by shearing stress } \sigma_{ij}}{\text{shearing strain } 2\varepsilon_{ij} \text{ caused by shearing stress } \sigma_{ij}}$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **The terms S_{41}, S_{42}, S_{43}, S_{51}, S_{52}, S_{53}, S_{61}, S_{62}, and S_{63} relate shearing strains to normal stresses**

  - **The engineering constants used to represent these relationships are also generalizations of Poisson's ratios and are called *coefficients of interaction (or mutual influence) of the second kind*, and are also attributed to A. L. Rabinovich (circa 1946)**

  - **In particular, the symbol $\eta_{ij,k}$ is used herein to relate the shearing strain in the x_i - x_j coordinate plane induced by the action of a normal stress applied in the x_k - coordinate direction**

  - **That is,** $$\eta_{ij,k} = \frac{\text{shearing strain } 2\varepsilon_{ij} \text{ caused by normal stress } \sigma_{kk}}{\text{normal strain } \varepsilon_{kk} \text{ caused by } \sigma_{kk}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **The terms S_{45}, S_{46}, S_{54}, S_{56}, S_{64}, and S_{65} relate <u>shearing strains</u> to <u>noncorresponding shearing stresses</u>**

  - **The engineering constants used to represent these relationships are called *Chentsov's coefficients*, and are attributed to N. G. Chentsov**

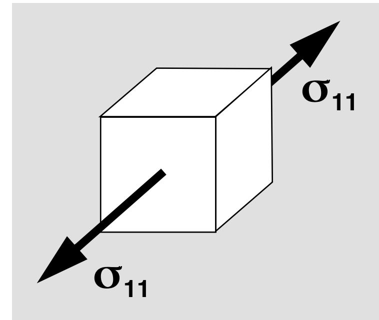
  - **In particular, the symbol $\mu_{ij,kl}$ is used herein to relate the shearing strain in the x_i - x_j coordinate plane induced by a shearing stress applied in the x_k - x_l coordinate plane**

  - **That is,** $$\mu_{ij,\,kl} \equiv \frac{\text{shearing } 2\varepsilon_{ij} \text{ caused by shearing stress } \sigma_{kl}}{\text{shearing strain } 2\varepsilon_{kl} \text{ caused by shearing stress } \sigma_{kl}}$$

  - **Note that $\mu_{ij,\,kl} = \mu_{ji,\,kl} = \mu_{ji,\,lk} = \mu_{ij,\,lk}$ because of symmetry of $2\varepsilon_{ij}$ and σ_{kl}**

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Now, consider a **parallelopiped** of homogeneous material that is subjected to only a constant value of σ_{11} and no thermal loading, like a stress state that might be exist in an experiment like a **tensile test**



  - For this case,

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

  - This equation indicates that, in general, the parallelopiped will extend in the x_1 - coordinate direction, expand or contract in the x_2 - and x_3 - coordinate directions, and shear in each face for this very simple state of uniaxial stress

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **From the definition of the elastic moduli, it follows that** $E_1 = \dfrac{\sigma_{11}}{\varepsilon_{11}}$
  - **Noting that the previous matrix equation gives** $\varepsilon_{11} = S_{11}\sigma_{11}$ **for this simple state of stress, it also follows that** $S_{11} = \dfrac{1}{E_1}$

- **Next, from the definition for the Poisson's ratios, it follows that** $\nu_{12} = -\dfrac{\varepsilon_{22}}{\varepsilon_{11}}$ **and** $\nu_{13} = -\dfrac{\varepsilon_{33}}{\varepsilon_{11}}$ **for this simple state of stress**
  - **Noting that the previous matrix equation also gives** $\varepsilon_{22} = S_{21}\sigma_{11}$ **and** $\varepsilon_{33} = S_{31}\sigma_{11}$ **for this simple state of stress, and using** $\varepsilon_{11} = S_{11}\sigma_{11}$**, it also follows that** $\nu_{12} = -\dfrac{S_{21}}{S_{11}}$ **and** $\nu_{13} = -\dfrac{S_{31}}{S_{11}}$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Using $S_{11} = \frac{1}{E_1}$ gives the results $S_{21} = -\frac{\nu_{12}}{E_1}$ and $S_{31} = -\frac{\nu_{13}}{E_1}$

- Next, from the definition for the *coefficients of interaction of the second kind*, it follows that $\eta_{23,1} = \frac{2\varepsilon_{23}}{\varepsilon_{11}}$, $\eta_{13,1} = \frac{2\varepsilon_{13}}{\varepsilon_{11}}$, and $\eta_{12,1} = \frac{2\varepsilon_{12}}{\varepsilon_{11}}$ for this simple state of stress

- Noting that the previous matrix equation also gives $2\varepsilon_{23} = S_{41}\sigma_{11}$, $2\varepsilon_{13} = S_{51}\sigma_{11}$, and $2\varepsilon_{12} = S_{61}\sigma_{11}$ for this simple state of stress, and using $\varepsilon_{11} = S_{11}\sigma_{11}$, it also follows that

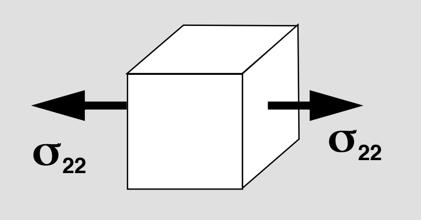
$$\eta_{23,1} = \frac{S_{41}}{S_{11}}, \quad \eta_{13,1} = \frac{S_{51}}{S_{11}}, \quad \text{and} \quad \eta_{12,1} = \frac{S_{61}}{S_{11}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Using $S_{11} = \dfrac{1}{E_1}$ gives the results

$$S_{41} = \frac{\eta_{23,\,1}}{E_1}\,, \quad S_{51} = \frac{\eta_{13,\,1}}{E_1}\,, \quad \text{and} \quad S_{61} = \frac{\eta_{12,\,1}}{E_1}$$

- Now, consider a parallelopiped of homogeneous material that is subjected to only a constant value of σ_{22} and no thermal loading



- For this case,

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ \sigma_{22} \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- This equation indicates that, in general, the parallelopiped will extend in the x_2 - coordinate direction, expand or contract in the x_1 - and x_3 - coordinate directions, and shear in each face for this very simple state of uniaxial stress

- From the definition of the elastic moduli, it follows that $E_2 = \frac{\sigma_{22}}{\varepsilon_{22}}$

  - Noting that the previous matrix equation gives $\varepsilon_{22} = S_{22}\sigma_{22}$ for this simple state of stress, it also follows that $S_{22} = \frac{1}{E_2}$

- Next, from the definition for the Poisson's ratios, it follows that

$\nu_{21} = -\frac{\varepsilon_{11}}{\varepsilon_{22}}$ and $\nu_{23} = -\frac{\varepsilon_{33}}{\varepsilon_{22}}$ for this simple state of stress

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Noting that the previous matrix equation also gives $\varepsilon_{11} = S_{12}\sigma_{22}$ and $\varepsilon_{33} = S_{32}\sigma_{22}$ for this simple state of stress, and using $\varepsilon_{22} = S_{22}\sigma_{22}$, it also follows that $\nu_{21} = -\dfrac{S_{12}}{S_{22}}$ and $\nu_{23} = -\dfrac{S_{32}}{S_{22}}$

- Using $S_{22} = \dfrac{1}{E_2}$ gives the results $S_{12} = -\dfrac{\nu_{21}}{E_2}$ and $S_{32} = -\dfrac{\nu_{23}}{E_2}$

- Next, from the definition for the *coefficients of interaction of the second kind*, it follows that $\eta_{23,2} = \dfrac{2\varepsilon_{23}}{\varepsilon_{22}}$, $\eta_{13,2} = \dfrac{2\varepsilon_{13}}{\varepsilon_{22}}$, and $\eta_{12,2} = \dfrac{2\varepsilon_{12}}{\varepsilon_{22}}$ for this simple state of stress

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Noting that the previous matrix equation also gives $2\varepsilon_{23} = S_{42}\sigma_{22}$, $2\varepsilon_{13} = S_{52}\sigma_{22}$, and $2\varepsilon_{12} = S_{62}\sigma_{22}$ for this simple state of stress, and using $\varepsilon_{22} = S_{22}\sigma_{22}$, it also follows that

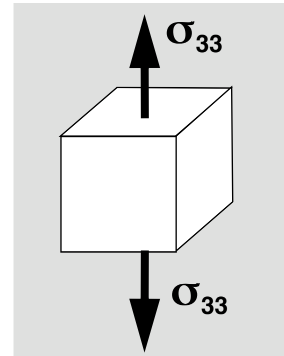
$$\eta_{23,2} = \frac{S_{42}}{S_{22}}, \quad \eta_{13,2} = \frac{S_{52}}{S_{22}}, \quad \text{and} \quad \eta_{12,2} = \frac{S_{62}}{S_{22}}$$

- Using $S_{22} = \frac{1}{E_2}$ gives the results

$$S_{42} = \frac{\eta_{23,2}}{E_2}, \quad S_{52} = \frac{\eta_{13,2}}{E_2}, \quad \text{and} \quad S_{62} = \frac{\eta_{12,2}}{E_2}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Now, consider a parallelopiped of homogeneous material that is subjected to only a constant value of σ_{33} and no thermal loading



  - For this case,

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \sigma_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

  - This equation indicates that, in general, the parallelopiped will extend in the x_3 - coordinate direction, expand or contract in the x_1 - and x_2 - coordinate directions, and shear in each face for this very simple state of uniaxial stress

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- From the definition of the elastic moduli, it follows that $E_3 = \frac{\sigma_{33}}{\varepsilon_{33}}$
  - Noting that the previous matrix equation gives $\varepsilon_{33} = S_{33}\sigma_{33}$ for this simple state of stress, it also follows that $S_{33} = \frac{1}{E_3}$
- Next, from the definition for the Poisson's ratios, it follows that $\nu_{31} = -\frac{\varepsilon_{11}}{\varepsilon_{33}}$ and $\nu_{32} = -\frac{\varepsilon_{22}}{\varepsilon_{33}}$ for this simple state of stress
  - Noting that the previous matrix equation also gives $\varepsilon_{11} = S_{13}\sigma_{33}$ and $\varepsilon_{22} = S_{23}\sigma_{33}$ for this simple state of stress, and using $\varepsilon_{33} = S_{33}\sigma_{33}$, it also follows that $\nu_{31} = -\frac{S_{13}}{S_{33}}$ and $\nu_{32} = -\frac{S_{23}}{S_{33}}$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Using $S_{33} = \frac{1}{E_3}$ gives the results $S_{13} = -\frac{\nu_{31}}{E_3}$ and $S_{23} = -\frac{\nu_{32}}{E_3}$

- Next, from the definition for the *coefficients of interaction of the second kind*, it follows that $\eta_{23,3} = \frac{2\varepsilon_{23}}{\varepsilon_{33}}$, $\eta_{13,3} = \frac{2\varepsilon_{13}}{\varepsilon_{33}}$, and $\eta_{12,3} = \frac{2\varepsilon_{12}}{\varepsilon_{33}}$ for this simple state of stress

  - Noting that the previous matrix equation also gives $2\varepsilon_{23} = S_{43}\sigma_{33}$, $2\varepsilon_{13} = S_{53}\sigma_{33}$, and $2\varepsilon_{12} = S_{63}\sigma_{33}$ for this simple state of stress, and using $\varepsilon_{33} = S_{33}\sigma_{33}$, it also follows that

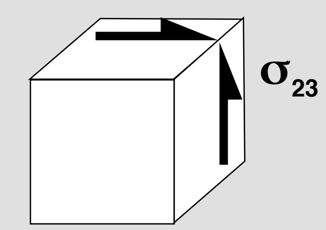
$$\eta_{23,3} = \frac{S_{43}}{S_{33}}, \quad \eta_{13,3} = \frac{S_{53}}{S_{33}}, \quad \text{and} \quad \eta_{12,3} = \frac{S_{63}}{S_{33}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Using $S_{33} = \dfrac{1}{E_3}$ gives the results

$$S_{43} = \frac{\eta_{23,3}}{E_3}, \quad S_{53} = \frac{\eta_{13,3}}{E_3}, \quad \text{and} \quad S_{63} = \frac{\eta_{12,3}}{E_3}$$

- Now, consider a parallelopiped of homogeneous material that is subjected to only a constant value of σ_{23} and no thermal loading



- For this case,

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \sigma_{23} \\ 0 \\ 0 \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **This equation also indicates that, in general, the parallelopiped will expand or contract along all three coordinate directions, and shear in each face for this very simple state of pure shearing stress**

- **From the definition of the shear moduli, it follows that** $G_{23} = \dfrac{\sigma_{23}}{2\varepsilon_{23}}$

  - **Noting that the previous matrix equation gives** $2\varepsilon_{23} = S_{44}\sigma_{23}$ **for this simple state of stress, it also follows that** $S_{44} = \dfrac{1}{G_{23}}$

- **Next, from the definition for the *coefficients of interaction of the first kind*, it follows that** $\eta_{1,23} = \dfrac{\varepsilon_{11}}{2\varepsilon_{23}}$ **,** $\eta_{2,23} = \dfrac{\varepsilon_{22}}{2\varepsilon_{23}}$ **, and** $\eta_{3,23} = \dfrac{\varepsilon_{33}}{2\varepsilon_{23}}$ **for this simple state of stress**

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Noting that the previous matrix equation also gives $\varepsilon_{11} = S_{14}\sigma_{23}$, $\varepsilon_{22} = S_{24}\sigma_{23}$, and $\varepsilon_{33} = S_{34}\sigma_{23}$ f or this simple state of stress, and using $2\varepsilon_{23} = S_{44}\sigma_{23}$, it also follows that

$$\eta_{1,23} = \frac{S_{14}}{S_{44}}, \quad \eta_{2,23} = \frac{S_{24}}{S_{44}}, \quad \text{and} \quad \eta_{3,23} = \frac{S_{34}}{S_{44}}$$

- Using $S_{44} = \dfrac{1}{G_{23}}$ gives the results

$$S_{14} = \frac{\eta_{1,23}}{G_{23}}, \quad S_{24} = \frac{\eta_{2,23}}{G_{23}}, \quad \text{and} \quad S_{34} = \frac{\eta_{3,23}}{G_{23}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Next, from the definition for the *Chentsov's coefficients*, it follows that $\mu_{13,23} = \dfrac{2\varepsilon_{13}}{2\varepsilon_{23}}$ and $\mu_{12,23} = \dfrac{2\varepsilon_{12}}{2\varepsilon_{23}}$ for this simple state of stress

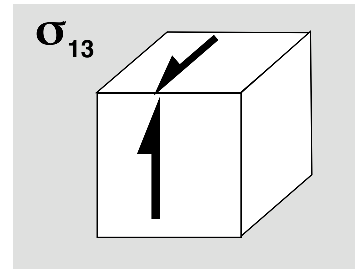
  - Noting that the previous matrix equation also gives $2\varepsilon_{13} = S_{54}\sigma_{23}$ and $2\varepsilon_{12} = S_{64}\sigma_{23}$ for this simple state of stress, and using $2\varepsilon_{23} = S_{44}\sigma_{23}$, it also follows that $\mu_{13,23} = \dfrac{S_{54}}{S_{44}}$ and $\mu_{12,23} = \dfrac{S_{64}}{S_{44}}$

  - Using $S_{44} = \dfrac{1}{G_{23}}$ gives the results

$$S_{54} = \frac{\mu_{13,23}}{G_{23}} \quad \text{and} \quad S_{64} = \frac{\mu_{12,23}}{G_{23}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **Now, consider a parallelopiped of homogeneous material that is subjected to only a constant value of σ_{13} and no thermal loading**



  - **For this case,**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \sigma_{13} \\ 0 \end{Bmatrix}$$

  - **This equation also indicates that, in general, the parallelopiped will expand or contract along all three coordinate directions, and shear in each face for this very simple state of pure shearing stress**

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- From the definition of the shear moduli, it follows that $G_{13} = \dfrac{\sigma_{13}}{2\varepsilon_{13}}$
  - Noting that the previous matrix equation gives $2\varepsilon_{13} = S_{55}\sigma_{13}$ for this simple state of stress, it also follows that $S_{55} = \dfrac{1}{G_{13}}$
- Next, from the definition for the *coefficients of interaction of the first kind*, it follows that $\eta_{1,13} = \dfrac{\varepsilon_{11}}{2\varepsilon_{13}}$, $\eta_{2,13} = \dfrac{\varepsilon_{22}}{2\varepsilon_{13}}$, and $\eta_{3,13} = \dfrac{\varepsilon_{33}}{2\varepsilon_{13}}$ for this simple state of stress

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Noting that the previous matrix equation also gives $\varepsilon_{11} = S_{15}\sigma_{13}$, $\varepsilon_{22} = S_{25}\sigma_{13}$, and $\varepsilon_{33} = S_{35}\sigma_{13}$ for this simple state of stress, and using $2\varepsilon_{13} = S_{55}\sigma_{13}$, it also follows that

$$\eta_{1,13} = \frac{S_{15}}{S_{55}}, \quad \eta_{2,13} = \frac{S_{25}}{S_{55}}, \quad \text{and} \quad \eta_{3,13} = \frac{S_{35}}{S_{55}}$$

- Using $S_{55} = \frac{1}{G_{13}}$ gives the results

$$S_{15} = \frac{\eta_{1,13}}{G_{13}}, \quad S_{25} = \frac{\eta_{2,13}}{G_{13}}, \quad \text{and} \quad S_{35} = \frac{\eta_{3,13}}{G_{13}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Next, from the definition for the *Chentsov's coefficients*, it follows that $\mu_{23,\,13} = \dfrac{2\varepsilon_{23}}{2\varepsilon_{13}}$ and $\mu_{12,\,13} = \dfrac{2\varepsilon_{12}}{2\varepsilon_{13}}$ for this simple state of stress

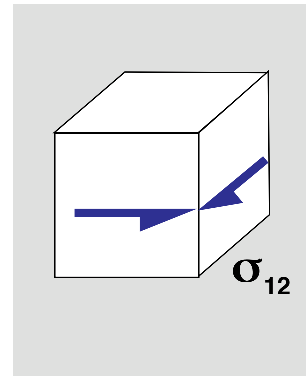
  - Noting that the previous matrix equation also gives $2\varepsilon_{23} = S_{45}\sigma_{13}$ and $2\varepsilon_{12} = S_{65}\sigma_{13}$ for this simple state of stress, and using $2\varepsilon_{13} = S_{55}\sigma_{13}$, it also follows that $\mu_{23,\,13} = \dfrac{S_{45}}{S_{55}}$ and $\mu_{12,\,13} = \dfrac{S_{65}}{S_{55}}$

  - Using $S_{55} = \dfrac{1}{G_{13}}$ gives the results

$$S_{45} = \frac{\mu_{23,\,13}}{G_{13}} \quad \text{and} \quad S_{65} = \frac{\mu_{12,\,13}}{G_{13}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **Finally, consider a parallelopiped of homogeneous material that is subjected to only a constant value of σ_{12} and no thermal loading**



  - **For this case,**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \sigma_{12} \end{Bmatrix}$$

  - **This equation also indicates that, in general, the parallelopiped will expand or contract along all three coordinate directions, and shear in each face for this very simple state of pure shearing stress**

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- **From the definition of the shear moduli, it follows that** $G_{12} = \dfrac{\sigma_{12}}{2\varepsilon_{12}}$
    - **Noting that the previous matrix equation gives** $2\varepsilon_{12} = S_{66}\sigma_{12}$ **for this simple state of stress, it also follows that** $S_{66} = \dfrac{1}{G_{12}}$

- **Next, from the definition for the *coefficients of interaction of the first kind*, it follows that** $\eta_{1,12} = \dfrac{\varepsilon_{11}}{2\varepsilon_{12}}$, $\eta_{2,12} = \dfrac{\varepsilon_{22}}{2\varepsilon_{12}}$, **and** $\eta_{3,12} = \dfrac{\varepsilon_{33}}{2\varepsilon_{12}}$ **for this simple state of stress**

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Noting that the previous matrix equation also gives $\varepsilon_{11} = S_{16}\sigma_{12}$, $\varepsilon_{22} = S_{26}\sigma_{12}$, and $\varepsilon_{33} = S_{36}\sigma_{12}$ for this simple state of stress, and using $2\varepsilon_{12} = S_{66}\sigma_{12}$, it also follows that

$$\eta_{1,12} = \frac{S_{16}}{S_{66}}, \quad \eta_{2,12} = \frac{S_{26}}{S_{66}}, \quad \text{and} \quad \eta_{3,12} = \frac{S_{36}}{S_{66}}$$

- Using $S_{66} = \dfrac{1}{G_{12}}$ gives the results

$$S_{16} = \frac{\eta_{1,12}}{G_{12}}, \quad S_{26} = \frac{\eta_{2,12}}{G_{12}}, \quad \text{and} \quad S_{36} = \frac{\eta_{3,12}}{G_{12}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Next, from the definition for the *Chentsov's coefficients*, it follows that $\mu_{23,12} = \frac{2\varepsilon_{23}}{2\varepsilon_{12}}$ and $\mu_{13,12} = \frac{2\varepsilon_{13}}{2\varepsilon_{12}}$ for this simple state of stress

  - Noting that the previous matrix equation also gives $2\varepsilon_{23} = S_{46}\sigma_{12}$ and $2\varepsilon_{13} = S_{56}\sigma_{12}$ for this simple state of stress, and using $2\varepsilon_{12} = S_{66}\sigma_{12}$, it also follows that $\mu_{23,12} = \frac{S_{46}}{S_{66}}$ and $\mu_{13,12} = \frac{S_{56}}{S_{66}}$

  - Using $S_{66} = \frac{1}{G_{12}}$ gives the results

$$S_{46} = \frac{\mu_{23,12}}{G_{12}} \quad \text{and} \quad S_{56} = \frac{\mu_{13,12}}{G_{12}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Using all of the derived expressions for S_{ij}, the constitutive equation

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref})$$

becomes

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & \frac{\eta_{1,23}}{G_{23}} & \frac{\eta_{1,13}}{G_{13}} & \frac{\eta_{1,12}}{G_{12}} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & \frac{\eta_{2,23}}{G_{23}} & \frac{\eta_{2,13}}{G_{13}} & \frac{\eta_{2,12}}{G_{12}} \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & \frac{\eta_{3,23}}{G_{23}} & \frac{\eta_{3,13}}{G_{13}} & \frac{\eta_{3,12}}{G_{12}} \\ \hline \frac{\eta_{23,1}}{E_1} & \frac{\eta_{23,2}}{E_2} & \frac{\eta_{23,3}}{E_3} & \frac{1}{G_{23}} & \frac{\mu_{23,13}}{G_{13}} & \frac{\mu_{23,12}}{G_{12}} \\ \frac{\eta_{13,1}}{E_1} & \frac{\eta_{13,2}}{E_2} & \frac{\eta_{13,3}}{E_3} & \frac{\mu_{13,23}}{G_{23}} & \frac{1}{G_{13}} & \frac{\mu_{13,12}}{G_{12}} \\ \frac{\eta_{12,1}}{E_1} & \frac{\eta_{12,2}}{E_2} & \frac{\eta_{12,3}}{E_3} & \frac{\mu_{12,23}}{G_{23}} & \frac{\mu_{12,13}}{G_{13}} & \frac{1}{G_{12}} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref})$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONTINUED

- Relationships between the engineering constants are obtained by **enforcing symmetry** of the matrix [S]; that is:

$$S_{12} = S_{21} \longrightarrow \frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1} \qquad S_{13} = S_{31} \longrightarrow \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1} \qquad S_{14} = S_{41} \longrightarrow \frac{\eta_{1,23}}{G_{23}} = \frac{\eta_{23,1}}{E_1}$$

$$S_{15} = S_{51} \longrightarrow \frac{\eta_{1,13}}{G_{13}} = \frac{\eta_{13,1}}{E_1} \qquad S_{16} = S_{61} \longrightarrow \frac{\eta_{1,12}}{G_{12}} = \frac{\eta_{12,1}}{E_1} \qquad S_{23} = S_{32} \longrightarrow \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}$$

$$S_{24} = S_{42} \longrightarrow \frac{\eta_{2,23}}{G_{23}} = \frac{\eta_{23,2}}{E_2} \qquad S_{25} = S_{52} \longrightarrow \frac{\eta_{2,13}}{G_{13}} = \frac{\eta_{13,2}}{E_2} \qquad S_{26} = S_{62} \longrightarrow \frac{\eta_{2,12}}{G_{12}} = \frac{\eta_{12,2}}{E_2}$$

$$S_{34} = S_{43} \longrightarrow \frac{\eta_{3,23}}{G_{23}} = \frac{\eta_{23,3}}{E_3} \qquad S_{35} = S_{53} \longrightarrow \frac{\eta_{3,13}}{G_{13}} = \frac{\eta_{13,3}}{E_3} \qquad S_{36} = S_{63} \longrightarrow \frac{\eta_{3,12}}{G_{12}} = \frac{\eta_{12,3}}{E_3}$$

$$S_{45} = S_{54} \longrightarrow \frac{\mu_{23,13}}{G_{13}} = \frac{\mu_{13,23}}{G_{23}} \qquad S_{46} = S_{64} \longrightarrow \frac{\mu_{23,12}}{G_{12}} = \frac{\mu_{12,23}}{G_{23}} \qquad S_{56} = S_{65} \longrightarrow \frac{\mu_{13,12}}{G_{12}} = \frac{\mu_{12,13}}{G_{13}}$$

CONSTITUTIVE EQUATIONS IN TERMS OF ENGINEERING CONSTANTS - CONCLUDED

- **Using the previous symmetry conditions, the constitutive equations are expressed as**

$$\varepsilon_{11} = \frac{1}{E_1}\left[\sigma_{11} - \nu_{12}\sigma_{22} - \nu_{13}\sigma_{33} + \eta_{23,1}\sigma_{23} + \eta_{13,1}\sigma_{13} + \eta_{12,1}\sigma_{12}\right] + \alpha_{11}\left(T - T_{ref}\right)$$

$$\varepsilon_{22} = \frac{1}{E_2}\left[-\nu_{21}\sigma_{11} + \sigma_{22} - \nu_{23}\sigma_{33} + \eta_{23,2}\sigma_{23} + \eta_{13,2}\sigma_{13} + \eta_{12,2}\sigma_{12}\right] + \alpha_{22}\left(T - T_{ref}\right)$$

$$\varepsilon_{33} = \frac{1}{E_3}\left[-\nu_{31}\sigma_{11} - \nu_{32}\sigma_{22} + \sigma_{33} + \eta_{23,3}\sigma_{23} + \eta_{13,3}\sigma_{13} + \eta_{12,3}\sigma_{12}\right] + \alpha_{33}\left(T - T_{ref}\right)$$

$$2\varepsilon_{23} = \frac{1}{G_{23}}\left[\eta_{1,23}\sigma_{11} + \eta_{2,23}\sigma_{22} + \eta_{3,23}\sigma_{33} + \sigma_{23} + \mu_{13,23}\sigma_{13} + \mu_{12,23}\sigma_{12}\right] + 2\alpha_{23}\left(T - T_{ref}\right)$$

$$2\varepsilon_{13} = \frac{1}{G_{13}}\left[\eta_{1,13}\sigma_{11} + \eta_{2,13}\sigma_{22} + \eta_{3,13}\sigma_{33} + \mu_{23,13}\sigma_{23} + \sigma_{13} + \mu_{12,13}\sigma_{12}\right] + 2\alpha_{13}\left(T - T_{ref}\right)$$

$$2\varepsilon_{12} = \frac{1}{G_{12}}\left[\eta_{1,12}\sigma_{11} + \eta_{2,12}\sigma_{22} + \eta_{3,12}\sigma_{33} + \mu_{23,12}\sigma_{23} + \mu_{13,12}\sigma_{13} + \sigma_{12}\right] + 2\alpha_{12}\left(T - T_{ref}\right)$$

ENGINEERING CONSTANTS OF A SPECIALLY ORTHOTROPIC MATERIAL

- The engineering constants for an anisotropic material have been presented previously herein, for an $\{x_1, x_2, x_3\}$ coordinate frame

  - The subset of engineering constants for a specially orthotropic material are given by

$$E_1 = \frac{1}{S_{11}} \qquad \nu_{12} = -\frac{S_{12}}{S_{11}} \qquad G_{23} = \frac{1}{S_{44}}$$

$$E_2 = \frac{1}{S_{22}} \qquad \nu_{13} = -\frac{S_{13}}{S_{11}} \qquad G_{13} = \frac{1}{S_{55}}$$

$$E_3 = \frac{1}{S_{33}} \qquad \nu_{23} = -\frac{S_{23}}{S_{22}} \qquad G_{12} = \frac{1}{S_{66}}$$

ENGINEERING CONSTANTS OF A SPECIALLY ORTHOTROPIC MATERIAL - CONTINUED

- Substituting these expressions into

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

gives

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

ENGINEERING CONSTANTS OF A SPECIALLY ORTHOTROPIC MATERIAL - CONTINUED

- **Inverting these matrix equations yields**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right) \quad \text{where}$$

$$C_{11} = \frac{E_1}{\Delta}\left(1 - \nu_{23}\,\nu_{32}\right) \qquad C_{12} = \frac{E_1}{\Delta}\left(\nu_{21} + \nu_{23}\,\nu_{31}\right) = \frac{E_2}{\Delta}\left(\nu_{12} + \nu_{13}\,\nu_{32}\right)$$

$$C_{13} = \frac{E_1}{\Delta}\left(\nu_{31} + \nu_{21}\,\nu_{32}\right) = \frac{E_3}{\Delta}\left(\nu_{13} + \nu_{12}\,\nu_{23}\right) \qquad C_{22} = \frac{E_2}{\Delta}\left(1 - \nu_{13}\,\nu_{31}\right)$$

$$C_{23} = \frac{E_2}{\Delta}\left(\nu_{32} + \nu_{12}\,\nu_{31}\right) = \frac{E_3}{\Delta}\left(\nu_{23} + \nu_{13}\,\nu_{21}\right) \qquad C_{33} = \frac{E_3}{\Delta}\left(1 - \nu_{12}\,\nu_{21}\right)$$

$$C_{44} = G_{23} \qquad C_{55} = G_{13} \qquad C_{66} = G_{12}$$

$$\Delta = 1 - \nu_{12}\,\nu_{21} - \nu_{23}\,\nu_{32} - \nu_{13}\,\nu_{31} - 2\nu_{21}\,\nu_{32}\,\nu_{13}$$

ENGINEERING CONSTANTS OF A SPECIALLY ORTHOTROPIC MATERIAL - CONCLUDED

- **And, where the thermal moduli are given by**

$$\beta_{11} = -E_1 \frac{\alpha_{11}(1 - \nu_{23}\nu_{32}) + \alpha_{22}(\nu_{21} + \nu_{23}\nu_{31}) + \alpha_{33}(\nu_{31} + \nu_{21}\nu_{32})}{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{32}\nu_{13}}$$

$$\beta_{22} = -E_2 \frac{\alpha_{11}(\nu_{12} + \nu_{13}\nu_{32}) + \alpha_{22}(1 - \nu_{13}\nu_{31}) + \alpha_{33}(\nu_{32} + \nu_{12}\nu_{31})}{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{32}\nu_{13}}$$

$$\beta_{33} = -E_3 \frac{\alpha_{11}(\nu_{13} + \nu_{12}\nu_{23}) + \alpha_{22}(\nu_{23} + \nu_{13}\nu_{21}) + \alpha_{33}(1 - \nu_{12}\nu_{21})}{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{32}\nu_{13}}$$

ENGINEERING CONSTANTS OF A TRANSVERSELY ISOTROPIC MATERIAL

- **Consider the constitutive equations for a specially orthotropic material**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \dfrac{1}{E_1} & -\dfrac{\nu_{21}}{E_2} & -\dfrac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & -\dfrac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\dfrac{\nu_{13}}{E_1} & -\dfrac{\nu_{23}}{E_2} & \dfrac{1}{E_3} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \dfrac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{G_{12}} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

ENGINEERING CONSTANTS OF A TRANSVERSELY ISOTROPIC MATERIAL - CONTINUED

- **For a transversely isotropic material with the plane of isotropy given by $x_3 = 0$, the constitutitve equations have the following forms**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11} - S_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{11} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

ENGINEERING CONSTANTS OF A TRANSVERSELY ISOTROPIC MATERIAL - CONTINUED

- **Using the conditions on the compliance coefficients required for a transversely isotropic material, the matrix equation for specially orthotropic materials, given in terms of engineering constants, can be expressed as**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu_{31}}{E'} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu_{31}}{E'} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E} & -\frac{\nu_{13}}{E} & \frac{1}{E'} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{G'} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G'} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \alpha \\ \alpha' \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

where $E_1 = E_2 \equiv E$, $\nu_{12} = \nu_{21} \equiv \nu$, $G_{12} \equiv G = \dfrac{E}{2(1+\nu)}$, $\alpha_{11} = \alpha_{22} \equiv \alpha$,

$E_3 \equiv E'$, $G_{13} = G_{23} \equiv G'$, **and** $\alpha_{33} \equiv \alpha'$

ENGINEERING CONSTANTS OF A TRANSVERSELY ISOTROPIC MATERIAL - CONTINUED

- In this matrix equation; $E, \nu, G, \text{and } \alpha$ are the Young's modulus, the Poisson's ratio, the shear modulus, and the coefficient of thermal expansion of the material in the plane of isotropy
  - Note that $\nu_{12} = \nu_{21} \equiv \nu$ characterizes contractions in the plane of isotropy that result from only tensile stresses applied in that plane
- $E', G', \text{and } \alpha'$ are the Young's modulus, the shear modulus, and the coefficient of thermal expansion of the material in the plane perpendicular to the plane of isotropy
- For only a tensile stress σ_{33} applied perpendicular to the plane of isotropy, the contractions are characterized by $\frac{\varepsilon_{11}}{\varepsilon_{33}} = \frac{\varepsilon_{22}}{\varepsilon_{33}} = -\nu_{31} \equiv -\nu'$
- Symmetry of the compliance matrix yields $\nu_{13} = \nu_{23} = \nu'\left(\frac{E}{E'}\right)$

ENGINEERING CONSTANTS OF A TRANSVERSELY ISOTROPIC MATERIAL - CONTINUED

- Finally, the constitutive equations for a transversely isotropic material, with the plane of isotropy given by $x_3 = 0$, are expressed by

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu'}{E'} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu'}{E'} & 0 & 0 & 0 \\ -\frac{\nu'}{E'} & -\frac{\nu'}{E'} & \frac{1}{E'} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{G'} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G'} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \alpha \\ \alpha' \\ 0 \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

- The five independent elastic constants are E, ν, E', ν', and G'

- The two independent thermal parameters are α and α'

ENGINEERING CONSTANTS OF A TRANSVERSELY ISOTROPIC MATERIAL - CONCLUDED

- **The inverted form of the previous matrix constitutive equation is given by**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G' & 0 & 0 \\ 0 & 0 & 0 & 0 & G' & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right) \quad \text{where}$$

$$C_{11} = \frac{E}{1+\nu}\left[\frac{1 - \frac{E}{E'}\nu'^2}{1 - \nu - 2\frac{E}{E'}\nu'^2}\right] \qquad C_{12} = \frac{E}{1+\nu}\left[\frac{\nu + \frac{E}{E'}\nu'^2}{1 - \nu - 2\frac{E}{E'}\nu'^2}\right] \qquad C_{13} = \frac{\nu' E}{1 - \nu - 2\frac{E}{E'}\nu'^2}$$

$$C_{33} = \frac{E'(1-\nu)}{1 - \nu - 2\frac{E}{E'}\nu'^2} \qquad \beta_{11} = -\left[\frac{E\alpha\left(1 + \frac{\alpha'}{\alpha}\nu'\right)}{1 - \nu - 2\frac{E}{E'}\nu'^2}\right] \qquad \beta_{33} = -\left[\frac{E\alpha\left(2\nu' + \frac{E'\alpha'}{E\alpha}(1-\nu)\right)}{1 - \nu - 2\frac{E}{E'}\nu'^2}\right]$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL

- Results presented herein indicate that, for an $\{x_1, x_2, x_3\}$ coordinate frame, the engineering constants of an anisotropic material are given by

$$E_1 = \frac{1}{S_{11}} \qquad \nu_{12} = -\frac{S_{12}}{S_{11}} \qquad G_{23} = \frac{1}{S_{44}} \qquad \eta_{23,1} = \frac{S_{14}}{S_{11}} \qquad \eta_{13,1} = \frac{S_{15}}{S_{11}} \qquad \eta_{12,1} = \frac{S_{16}}{S_{11}}$$

$$E_2 = \frac{1}{S_{22}} \qquad \nu_{13} = -\frac{S_{13}}{S_{11}} \qquad G_{13} = \frac{1}{S_{55}} \qquad \eta_{23,2} = \frac{S_{24}}{S_{22}} \qquad \eta_{13,2} = \frac{S_{25}}{S_{22}} \qquad \eta_{12,2} = \frac{S_{26}}{S_{22}}$$

$$E_3 = \frac{1}{S_{33}} \qquad \nu_{23} = -\frac{S_{23}}{S_{22}} \qquad G_{12} = \frac{1}{S_{66}} \qquad \eta_{23,3} = \frac{S_{34}}{S_{33}} \qquad \eta_{13,3} = \frac{S_{35}}{S_{33}} \qquad \eta_{12,3} = \frac{S_{36}}{S_{33}}$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

and

$$\eta_{1,23} = \frac{S_{14}}{S_{44}} \quad \eta_{2,23} = \frac{S_{24}}{S_{44}} \quad \eta_{3,23} = \frac{S_{34}}{S_{44}} \quad \mu_{13,23} = \frac{S_{45}}{S_{44}} \quad \mu_{12,23} = \frac{S_{46}}{S_{44}}$$

$$\eta_{1,13} = \frac{S_{15}}{S_{55}} \quad \eta_{2,13} = \frac{S_{25}}{S_{55}} \quad \eta_{3,13} = \frac{S_{35}}{S_{55}} \quad \mu_{23,13} = \frac{S_{45}}{S_{55}} \quad \mu_{12,13} = \frac{S_{56}}{S_{55}}$$

$$\eta_{1,12} = \frac{S_{16}}{S_{66}} \quad \eta_{2,12} = \frac{S_{26}}{S_{66}} \quad \eta_{3,12} = \frac{S_{36}}{S_{66}} \quad \mu_{23,12} = \frac{S_{46}}{S_{66}} \quad \mu_{13,12} = \frac{S_{56}}{S_{66}}$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

- **Using these expressions, the nonzero transformed elastic compliances for a specially orthotropic material transformed by a dextral rotation about the x_3 - axis $(m = \cos\theta_3 \ \text{ and } \ n = \sin\theta_3)$ become**

$$S_{1'1'} = \frac{1}{E_1}\left[m^4 + m^2n^2\left(\frac{E_2}{G_{12}}\left(\frac{E_1}{E_2}\right) - 2\nu_{12}\right) + n^4\frac{E_1}{E_2}\right]$$

$$S_{1'3'} = -\frac{\nu_{13}}{E_1}\left[m^2 + n^2\frac{\nu_{23}}{\nu_{13}}\frac{E_1}{E_2}\right]$$

$$S_{2'2'} = \frac{1}{E_2}\left[m^4 + m^2n^2\left(\frac{E_2}{G_{12}} - 2\nu_{12}\frac{E_2}{E_1}\right) + n^4\frac{E_2}{E_1}\right]$$

$$S_{2'3'} = -\frac{\nu_{23}}{E_2}\left[m^2 + n^2\frac{\nu_{13}}{\nu_{23}}\frac{E_2}{E_1}\right]$$

$$S_{1'2'} = -\frac{\nu_{12}}{E_1}\left[m^4 + n^4 - m^2n^2\frac{1}{\nu_{12}}\left(1 + \frac{E_1}{E_2} - \frac{E_2}{G_{12}}\left(\frac{E_1}{E_2}\right)\right)\right]$$

$$S_{3'3'} = \frac{1}{E_3}$$

$$S_{1'6'} = \frac{mn}{E_1}\left[\left[m^2 - n^2\right]\left(\frac{E_2}{G_{12}}\left(\frac{E_1}{E_2}\right) - 2\nu_{12}\right) + 2\left(n^2\frac{E_1}{E_2} - m^2\right)\right]$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

and

$$S_{2'6'} = \frac{mn}{E_2}\left[\left[n^2 - m^2\right]\left(\frac{E_2}{G_{12}} - 2\nu_{12}\frac{E_2}{E_1}\right) + 2\left(m^2 - n^2\frac{E_2}{E_1}\right)\right]$$

$$S_{3'6'} = \frac{2mn}{E_3}\left[\frac{E_3}{E_1}\left(\nu_{13} - \nu_{23}\frac{E_1}{E_2}\right)\right] \qquad S_{4'4'} = \frac{1}{G_{23}}\left[m^2 + n^2\frac{G_{23}}{G_{13}}\right]$$

$$S_{4'5'} = \frac{mn}{G_{23}}\left(1 - \frac{G_{23}}{G_{13}}\right) \qquad S_{5'5'} = \frac{1}{G_{13}}\left[m^2 + n^2\frac{G_{13}}{G_{23}}\right]$$

$$S_{6'6'} = \frac{1}{G_{12}}\left[4m^2n^2\frac{G_{12}}{E_2}\left(\frac{E_2}{E_1}\right)\left(1 + 2\nu_{12} + \frac{E_1}{E_2}\right) + \left(m^2 - n^2\right)^2\right]$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

- **The matrix form of the constitutive equations is given by**

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & 0 & 0 & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & 0 & 0 & S_{2'6'} \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & 0 & 0 & S_{3'6'} \\ 0 & 0 & 0 & S_{4'4'} & S_{4'5'} & 0 \\ 0 & 0 & 0 & S_{4'5'} & S_{5'5'} & 0 \\ S_{1'6'} & S_{2'6'} & S_{3'6'} & 0 & 0 & S_{6'6'} \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 0 \\ 0 \\ 2\alpha_{1'2'} \end{Bmatrix} (T - T_{ref})$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

- **Next, by inspection of the previously given engineering constants for an anisotropic material, it follows that, for an $\{x_{1'}, x_{2'}, x_{3'}\}$ coordinate frame, the corresponding engineering constants are given by**

$$E_{1'} = \frac{1}{S_{1'1'}} \quad \nu_{1'2'} = -\frac{S_{1'2'}}{S_{1'1'}} \quad G_{2'3'} = \frac{1}{S_{4'4'}} \quad \eta_{2'3',\,1'} = \frac{S_{1'4'}}{S_{1'1'}} \quad \eta_{1'3',\,1'} = \frac{S_{1'5'}}{S_{1'1'}} \quad \eta_{1'2',\,1'} = \frac{S_{1'6'}}{S_{1'1'}}$$

$$E_{2'} = \frac{1}{S_{2'2'}} \quad \nu_{1'3'} = -\frac{S_{1'3'}}{S_{1'1'}} \quad G_{1'3'} = \frac{1}{S_{5'5'}} \quad \eta_{2'3',\,2'} = \frac{S_{2'4'}}{S_{2'2'}} \quad \eta_{1'3',\,2'} = \frac{S_{2'5'}}{S_{2'2'}} \quad \eta_{1'2',\,2'} = \frac{S_{2'6'}}{S_{2'2'}}$$

$$E_{3'} = \frac{1}{S_{3'3'}} \quad \nu_{2'3'} = -\frac{S_{2'3'}}{S_{2'2'}} \quad G_{1'2'} = \frac{1}{S_{6'6'}} \quad \eta_{2'3',\,3'} = \frac{S_{3'4'}}{S_{3'3'}} \quad \eta_{1'3',\,3'} = \frac{S_{3'5'}}{S_{3'3'}} \quad \eta_{1'2',\,3'} = \frac{S_{3'6'}}{S_{3'3'}}$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

and

$$\eta_{1',2'3'} = \frac{S_{1'4'}}{S_{4'4'}} \quad \eta_{2',2'3'} = \frac{S_{2'4'}}{S_{4'4'}} \quad \eta_{3',2'3'} = \frac{S_{3'4'}}{S_{4'4'}} \quad \mu_{1'3',2'3'} = \frac{S_{4'5'}}{S_{4'4'}} \quad \mu_{1'2',2'3'} = \frac{S_{4'6'}}{S_{4'4'}}$$

$$\eta_{1',1'3'} = \frac{S_{1'5'}}{S_{5'5'}} \quad \eta_{2',1'3'} = \frac{S_{2'5'}}{S_{5'5'}} \quad \eta_{3',1'3'} = \frac{S_{3'5'}}{S_{5'5'}} \quad \mu_{2'3',1'3'} = \frac{S_{4'5'}}{S_{5'5'}} \quad \mu_{1'2',1'3'} = \frac{S_{5'6'}}{S_{5'5'}}$$

$$\eta_{1',1'2'} = \frac{S_{1'6'}}{S_{6'6'}} \quad \eta_{2',1'2'} = \frac{S_{2'6'}}{S_{6'6'}} \quad \eta_{3',1'2'} = \frac{S_{3'6'}}{S_{6'6'}} \quad \mu_{2'3',1'2'} = \frac{S_{4'6'}}{S_{6'6'}} \quad \mu_{1'3',1'2'} = \frac{S_{5'6'}}{S_{6'6'}}$$

- It is important to note that because the transformed compliance matrix has the same structure as a monoclinic material, the conditions $S_{1'4'} = S_{1'5'} = S_{2'4'} = S_{2'5'} = S_{3'4'} = S_{3'5'} = S_{4'6'} = S_{5'6'} = 0$ are valid

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL- CONTINUED

- Thus $\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'3'} & 0 & 0 & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'3'} & 0 & 0 & S_{2'6'} \\ S_{1'3'} & S_{2'3'} & S_{3'3'} & 0 & 0 & S_{3'6'} \\ 0 & 0 & 0 & S_{4'4'} & S_{4'5'} & 0 \\ 0 & 0 & 0 & S_{4'5'} & S_{5'5'} & 0 \\ S_{1'6'} & S_{2'6'} & S_{3'6'} & 0 & 0 & S_{6'6'} \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 0 \\ 0 \\ 2\alpha_{1'2'} \end{Bmatrix} (T - T_{ref})$ becomes

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} \frac{1}{E_{1'}} & -\frac{\nu_{2'1'}}{E_{2'}} & -\frac{\nu_{3'1'}}{E_{3'}} & 0 & 0 & \frac{\eta_{1',1'2'}}{G_{1'2'}} \\ -\frac{\nu_{1'2'}}{E_{1'}} & \frac{1}{E_{2'}} & -\frac{\nu_{3'2'}}{E_{3'}} & 0 & 0 & \frac{\eta_{2',1'2'}}{G_{1'2'}} \\ -\frac{\nu_{1'3'}}{E_{1'}} & -\frac{\nu_{2'3'}}{E_{2'}} & \frac{1}{E_{3'}} & 0 & 0 & \frac{\eta_{3',1'2'}}{G_{1'2'}} \\ \hline 0 & 0 & 0 & \frac{1}{G_{2'3'}} & \frac{\mu_{2'3',1'3'}}{G_{1'3'}} & 0 \\ 0 & 0 & 0 & \frac{\mu_{1'3',2'3'}}{G_{2'3'}} & \frac{1}{G_{1'3'}} & 0 \\ \frac{\eta_{1'2',1'}}{E_{1'}} & \frac{\eta_{1'2',2'}}{E_{2'}} & \frac{\eta_{1'2',3'}}{E_{3'}} & 0 & 0 & \frac{1}{G_{1'2'}} \end{array} \right] \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ \alpha_{3'3'} \\ 0 \\ 0 \\ 2\alpha_{1'2'} \end{Bmatrix} (T - T_{ref})$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

- By using the expressions on the previous few pages, the **effective elastic and shear moduli** of a generally orthotropic material are given by

$$\frac{E_{1'}}{E_1} = \left[m^4 + m^2n^2\left(\frac{E_2}{G_{12}}\left(\frac{E_1}{E_2}\right) - 2\nu_{12}\right) + n^4\frac{E_1}{E_2}\right]^{-1}$$

$$\frac{E_{2'}}{E_2} = \left[m^4 + m^2n^2\left(\frac{E_2}{G_{12}} - 2\nu_{12}\frac{E_2}{E_1}\right) + n^4\frac{E_2}{E_1}\right]^{-1} \qquad \frac{E_{3'}}{E_3} = 1$$

$$\frac{G_{2'3'}}{G_{23}} = \left[m^2 + n^2\frac{G_{23}}{G_{13}}\right]^{-1} \qquad \frac{G_{1'3'}}{G_{13}} = \left[m^2 + n^2\frac{G_{13}}{G_{23}}\right]^{-1}$$

$$\frac{G_{1'2'}}{G_{12}} = \left[4m^2n^2\frac{G_{12}}{E_2}\left(\frac{E_2}{E_1}\right)\left(1 + 2\nu_{12} + \frac{E_1}{E_2}\right) + \left(m^2 - n^2\right)^2\right]^{-1}$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

- Similarly, the **effective Poisson's ratios** of a generally orthotropic material are given by

$$\frac{\nu_{1'2'}}{\nu_{12}} = \frac{m^4 + n^4 - m^2 n^2 \frac{1}{\nu_{12}}\left(1 + \frac{E_1}{E_2} - \frac{E_2}{G_{12}}\left(\frac{E_1}{E_2}\right)\right)}{m^4 + m^2 n^2\left(\frac{E_2}{G_{12}}\left(\frac{E_1}{E_2}\right) - 2\nu_{12}\right) + n^4 \frac{E_1}{E_2}}$$

$$\frac{\nu_{1'3'}}{\nu_{13}} = \frac{m^2 + n^2 \frac{\nu_{23}}{\nu_{13}}\left(\frac{E_1}{E_2}\right)}{m^4 + m^2 n^2\left(\frac{E_2}{G_{12}}\left(\frac{E_1}{E_2}\right) - 2\nu_{12}\right) + n^4 \frac{E_1}{E_2}}$$

$$\frac{\nu_{2'3'}}{\nu_{23}} = \frac{m^2 + n^2 \frac{\nu_{13}}{\nu_{23}}\left(\frac{E_2}{E_1}\right)}{m^4 + m^2 n^2\left(\frac{E_2}{G_{12}} - 2\nu_{12}\frac{E_2}{E_1}\right) + n^4 \frac{E_2}{E_1}}$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

- Similarly, the nonzero **effective coefficients of mutual influence of the first kind** for a generally orthotropic material are given by

$$\eta_{1',1'2'} = mn\,\frac{G_{12}}{E_2}\left(\frac{E_2}{E_1}\right)\frac{\left[m^2 - n^2\right]\left(\dfrac{E_2}{G_{12}}\left(\dfrac{E_1}{E_2}\right) - 2\nu_{12}\right) + 2\left(n^2\dfrac{E_1}{E_2} - m^2\right)}{4m^2n^2\dfrac{G_{12}}{E_2}\left(\dfrac{E_2}{E_1}\right)\left(1 + 2\nu_{12} + \dfrac{E_1}{E_2}\right) + \left(m^2 - n^2\right)^2}$$

$$\eta_{2',1'2'} = mn\,\frac{G_{12}}{E_2}\,\frac{\left[n^2 - m^2\right]\left(\dfrac{E_2}{G_{12}} - 2\nu_{12}\dfrac{E_2}{E_1}\right) + 2\left(m^2 - n^2\dfrac{E_2}{E_1}\right)}{4m^2n^2\dfrac{G_{12}}{E_2}\left(\dfrac{E_2}{E_1}\right)\left(1 + 2\nu_{12} + \dfrac{E_1}{E_2}\right) + \left(m^2 - n^2\right)^2}$$

$$\eta_{3',1'2'} = 2mn\,\frac{G_{12}}{E_2}\left(\frac{E_2}{E_1}\right)\left(\frac{E_3}{E_1}\right)^2\frac{\nu_{13} - \nu_{23}\dfrac{E_1}{E_2}}{4m^2n^2\dfrac{G_{12}}{E_2}\left(\dfrac{E_2}{E_1}\right)\left(1 + 2\nu_{12} + \dfrac{E_1}{E_2}\right) + \left(m^2 - n^2\right)^2}$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONTINUED

- The nonzero **effective coefficients of mutual influence of the second kind** for a generally orthotropic material are given by

$$\eta_{1'2',1'} = mn\,\frac{\left[m^2 - n^2\right]\left(\frac{E_2}{G_{12}}\left(\frac{E_1}{E_2}\right) - 2\nu_{12}\right) + 2\left(n^2\frac{E_1}{E_2} - m^2\right)}{m^4 + m^2n^2\left(\frac{E_2}{G_{12}}\left(\frac{E_1}{E_2}\right) - 2\nu_{12}\right) + n^4\frac{E_1}{E_2}}$$

$$\eta_{1'2',2'} = mn\,\frac{\left[n^2 - m^2\right]\left(\frac{E_2}{G_{12}} - 2\nu_{12}\frac{E_2}{E_1}\right) + 2\left(m^2 - n^2\frac{E_2}{E_1}\right)}{m^4 + m^2n^2\left(\frac{E_2}{G_{12}} - 2\nu_{12}\frac{E_2}{E_1}\right) + n^4\frac{E_2}{E_1}}$$

$$\eta_{1'2',3'} = 2mn\,\frac{E_3}{E_1}\left(\nu_{13} - \nu_{23}\frac{E_1}{E_2}\right)$$

ENGINEERING CONSTANTS OF A GENERALLY ORTHOTROPIC MATERIAL - CONCLUDED

- The nonzero **effective Chentsov coefficients** for a generally orthotropic material are given by

$$\mu_{1'3',\,2'3'} = mn\,\frac{1 - \dfrac{G_{23}}{G_{13}}}{m^2 + n^2\dfrac{G_{23}}{G_{13}}} \qquad \mu_{2'3',\,1'3'} = mn\,\frac{G_{13}}{G_{23}}\,\frac{1 - \dfrac{G_{23}}{G_{13}}}{m^2 + n^2\dfrac{G_{13}}{G_{23}}}$$

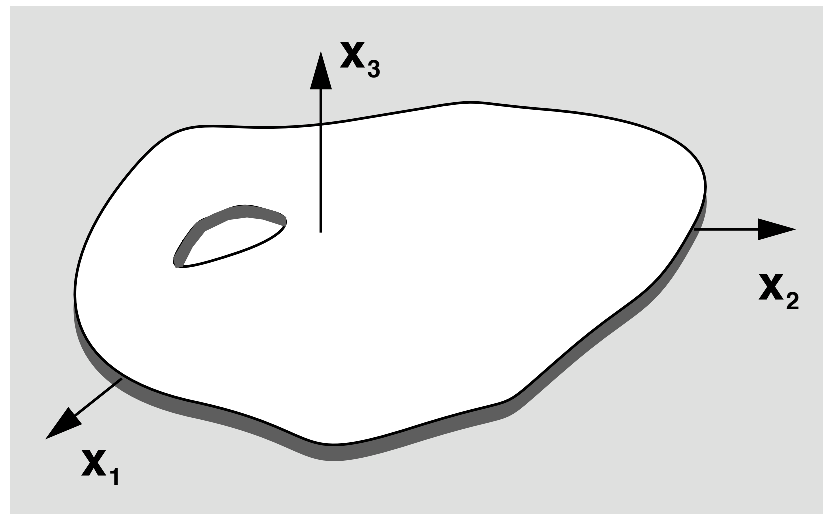
REDUCED CONSTITUTIVE EQUATIONS

CONSTITUTIVE EQUATIONS FOR PLANE STRESS

- When, analyzing solids that are ***relatively flat and thin***, simplifying assumptions are made about the stress state to facilitate analytical solution of practical problems

    - One such assumption is that the stresses in a thin, flat body, that are normal to the plane of flatness, are negligible compared to the other stresses

    - This simplification is commonly referred to as the **plane-stress assumption**



- For a ***state of plane stress*** in a homogeneous, anisotropic solid, with respect to the x_1 - x_2 plane, the stress field is approximated such that

$$\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **For this special case, the general matrix constitutive equation**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right)$$

uncouples directly into

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right) \quad \textbf{and}$$

$$\begin{Bmatrix} \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ S_{14} & S_{24} & S_{46} \\ S_{15} & S_{25} & S_{56} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix} \left(T - T_{ref}\right)$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- In terms of the engineering constants, the **plane-stress constitutive equations** for a homogeneous, ***anisotropic material*** are given by

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & \frac{\eta_{1,12}}{G_{12}} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{\eta_{2,12}}{G_{12}} \\ \frac{\eta_{12,1}}{E_1} & \frac{\eta_{12,2}}{E_2} & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{bmatrix} -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{\eta_{3,12}}{G_{12}} \\ \frac{\eta_{23,1}}{E_1} & \frac{\eta_{23,2}}{E_2} & \frac{\mu_{23,12}}{G_{12}} \\ \frac{\eta_{13,1}}{E_1} & \frac{\eta_{13,2}}{E_2} & \frac{\mu_{13,12}}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix} (T - T_{ref})$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS

CONTINUED

- However, simplification of the following general constitutive equation is not as easy

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix} \left(T - T_{ref}\right)$$

- First, the equation given above is expressed as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right) \right\}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Then, using $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$ gives**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ 0 \\ 0 \\ 0 \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\}$$

- **Rearranging the rows and columns into a convenient form gives**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \hline 0 \\ 0 \\ 0 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{16} & C_{13} & C_{14} & C_{15} \\ C_{12} & C_{22} & C_{26} & C_{23} & C_{24} & C_{25} \\ C_{16} & C_{26} & C_{66} & C_{36} & C_{46} & C_{56} \\ \hline C_{13} & C_{23} & C_{36} & C_{33} & C_{34} & C_{35} \\ C_{14} & C_{24} & C_{46} & C_{34} & C_{44} & C_{45} \\ C_{15} & C_{25} & C_{56} & C_{35} & C_{45} & C_{55} \end{array} \right] \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ \hline \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \\ \hline \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix} (T - T_{ref}) \right\}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- For convenience, let the **mechanical strains** be denoted by

$$\left\{\begin{array}{c}\varepsilon_{11}^{\sigma}\\ \varepsilon_{22}^{\sigma}\\ 2\varepsilon_{12}^{\sigma}\\ \hline \varepsilon_{33}^{\sigma}\\ 2\varepsilon_{23}^{\sigma}\\ 2\varepsilon_{13}^{\sigma}\end{array}\right\} = \left\{\begin{array}{c}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\\ \hline \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\end{array}\right\} - \left\{\begin{array}{c}\alpha_{11}\\ \alpha_{22}\\ 2\alpha_{12}\\ \hline \alpha_{33}\\ 2\alpha_{23}\\ 2\alpha_{13}\end{array}\right\}(T - T_{ref})$$

- The previous matrix constitutive equation becomes

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \hline 0\\ 0\\ 0\end{array}\right\} = \left[\begin{array}{ccc|ccc}C_{11} & C_{12} & C_{16} & C_{13} & C_{14} & C_{15}\\ C_{12} & C_{22} & C_{26} & C_{23} & C_{24} & C_{25}\\ C_{16} & C_{26} & C_{66} & C_{36} & C_{46} & C_{56}\\ \hline C_{13} & C_{23} & C_{36} & C_{33} & C_{34} & C_{35}\\ C_{14} & C_{24} & C_{46} & C_{34} & C_{44} & C_{45}\\ C_{15} & C_{25} & C_{56} & C_{35} & C_{45} & C_{55}\end{array}\right]\left\{\begin{array}{c}\varepsilon_{11}^{\sigma}\\ \varepsilon_{22}^{\sigma}\\ 2\varepsilon_{12}^{\sigma}\\ \hline \varepsilon_{33}^{\sigma}\\ 2\varepsilon_{23}^{\sigma}\\ 2\varepsilon_{13}^{\sigma}\end{array}\right\}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Next, the matrix constitutive equation**

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \hline 0\\ 0\\ 0\end{array}\right\} = \left[\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{16} & C_{13} & C_{14} & C_{15}\\ C_{12} & C_{22} & C_{26} & C_{23} & C_{24} & C_{25}\\ C_{16} & C_{26} & C_{66} & C_{36} & C_{46} & C_{56}\\ \hline C_{13} & C_{23} & C_{36} & C_{33} & C_{34} & C_{35}\\ C_{14} & C_{24} & C_{46} & C_{34} & C_{44} & C_{45}\\ C_{15} & C_{25} & C_{56} & C_{35} & C_{45} & C_{55}\end{array}\right] \left\{\begin{array}{c}\varepsilon_{11}^{\sigma}\\ \varepsilon_{22}^{\sigma}\\ 2\varepsilon_{12}^{\sigma}\\ \hline \varepsilon_{33}^{\sigma}\\ 2\varepsilon_{23}^{\sigma}\\ 2\varepsilon_{13}^{\sigma}\end{array}\right\}$$ **is separated to get**

$$\left\{\begin{array}{c}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{array}\right\} = \left[\begin{array}{ccc} C_{11} & C_{12} & C_{16}\\ C_{12} & C_{22} & C_{26}\\ C_{16} & C_{26} & C_{66}\end{array}\right] \left\{\begin{array}{c}\varepsilon_{11}^{\sigma}\\ \varepsilon_{22}^{\sigma}\\ 2\varepsilon_{12}^{\sigma}\end{array}\right\} + \left[\begin{array}{ccc} C_{13} & C_{14} & C_{15}\\ C_{23} & C_{24} & C_{25}\\ C_{36} & C_{46} & C_{56}\end{array}\right] \left\{\begin{array}{c}\varepsilon_{33}^{\sigma}\\ 2\varepsilon_{23}^{\sigma}\\ 2\varepsilon_{13}^{\sigma}\end{array}\right\}$$ **and**

$$\left\{\begin{array}{c}0\\ 0\\ 0\end{array}\right\} = \left[\begin{array}{ccc} C_{13} & C_{23} & C_{36}\\ C_{14} & C_{24} & C_{46}\\ C_{15} & C_{25} & C_{56}\end{array}\right] \left\{\begin{array}{c}\varepsilon_{11}^{\sigma}\\ \varepsilon_{22}^{\sigma}\\ 2\varepsilon_{12}^{\sigma}\end{array}\right\} + \left[\begin{array}{ccc} C_{33} & C_{34} & C_{35}\\ C_{34} & C_{44} & C_{45}\\ C_{35} & C_{45} & C_{55}\end{array}\right] \left\{\begin{array}{c}\varepsilon_{33}^{\sigma}\\ 2\varepsilon_{23}^{\sigma}\\ 2\varepsilon_{13}^{\sigma}\end{array}\right\}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Solving the previous *homogeneous equation* for $\varepsilon_{33}^{\sigma}$, $2\varepsilon_{23}^{\sigma}$, and $2\varepsilon_{13}^{\sigma}$ gives**

$$\begin{Bmatrix} \varepsilon_{33}^{\sigma} \\ 2\varepsilon_{23}^{\sigma} \\ 2\varepsilon_{13}^{\sigma} \end{Bmatrix} = -\begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \end{Bmatrix}$$

- **Back substitution of the column vector containing $\varepsilon_{33}^{\sigma}$, $2\varepsilon_{23}^{\sigma}$, and $2\varepsilon_{13}^{\sigma}$ into**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \end{Bmatrix} + \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{Bmatrix} \varepsilon_{33}^{\sigma} \\ 2\varepsilon_{23}^{\sigma} \\ 2\varepsilon_{13}^{\sigma} \end{Bmatrix}$$

yields the result

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **The matrix with the subscripted Q terms is given by**

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix}$$

- **Next, expressing the mechanical strains in terms of the total strains and the strains caused by free thermal expansion results in**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\}$$

- **This equation is manipulated further by defining**

$$\begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = - \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- Thus, the **constitutive equations for plane stress**, in terms of stiffness coefficients and thermal moduli, become

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} (T - T_{ref})$$

where

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix}$$

and

$$\begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = - \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

- The Q_{ij} and $\tilde{\beta}_{ij}$ are called the **reduced stiffness coefficients** and **reduced thermal moduli**, respectively

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- The relationship between the reduced stiffnesses and the compliances is obtained by first considering the previously derived equation

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref})$$

- Inverting gives

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix}^{-1} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Comparing**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix}^{-1} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\}$$

with

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} (T - T_{ref})$$

indicates that

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix}^{-1}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **The relationship between the reduced thermal moduli and the regular thermal moduli is obtained by first considering the equation**

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix}$$

- **This matrix equation can be separated into**

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix} \quad \textbf{and}$$

$$\begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix} = -\begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} - \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix} \begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Solving the matrix equation**

$$\begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix} = -\begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} - \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix} \begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix}$$

for the column vector containing $2\alpha_{13}$, $2\alpha_{23}$, and α_{33} gives

$$\begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix} = -\begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix} - \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Substituting**

$$\begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix} = -\begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix} - \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

into

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix}$$ **gives**

$$\begin{aligned} &\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{Bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix} = \\ &\quad - \left[\begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \right] \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \end{aligned}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Noting that**

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix},$$

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{Bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix} =$$

$$- \left[\begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \right] \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

gives

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{Bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix} = - \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Finally, noting that**

$$\begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = - \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

, the expression

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{Bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix} = - \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

gives

$$\begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{Bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- For a material that is ***monoclinic***, with $x_3 = 0$ being a plane of reflective symmetry, the plane-stress constitutive equations become

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \qquad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$

- The last matrix equation reduces to

$$\varepsilon_{33} = S_{13}\sigma_{11} + S_{23}\sigma_{22} + S_{36}\sigma_{12} + \alpha_{33}(T - T_{ref})$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **In terms of engineering constants,**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{21}}{E_2} & \dfrac{\eta_{1,12}}{G_{12}} \\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & \dfrac{\eta_{2,12}}{G_{12}} \\ \dfrac{\eta_{12,1}}{E_1} & \dfrac{\eta_{12,2}}{E_2} & \dfrac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right)$$

$$\varepsilon_{33} = -\frac{\nu_{13}}{E_1}\sigma_{11} - \frac{\nu_{23}}{E_2}\sigma_{22} + \frac{\eta_{12,3}}{E_3}\sigma_{33} + \alpha_{33}\left(T - T_{ref}\right)$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- Similarly, for a material that is ***monoclinic***, with $x_3 = 0$ being a plane of reflective symmetry, the plane-stress constitutive matrix becomes

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} - \begin{bmatrix} C_{13} & 0 & 0 \\ C_{23} & 0 & 0 \\ C_{36} & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{33} & 0 & 0 \\ 0 & C_{44} & C_{45} \\ 0 & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} \left(C_{11} - \dfrac{C_{13}C_{13}}{C_{33}}\right) & \left(C_{12} - \dfrac{C_{13}C_{23}}{C_{33}}\right) & \left(C_{16} - \dfrac{C_{13}C_{36}}{C_{33}}\right) \\ \left(C_{12} - \dfrac{C_{13}C_{23}}{C_{33}}\right) & \left(C_{22} - \dfrac{C_{23}C_{23}}{C_{33}}\right) & \left(C_{26} - \dfrac{C_{23}C_{36}}{C_{33}}\right) \\ \left(C_{16} - \dfrac{C_{13}C_{36}}{C_{33}}\right) & \left(C_{26} - \dfrac{C_{23}C_{36}}{C_{33}}\right) & \left(C_{66} - \dfrac{C_{36}C_{36}}{C_{33}}\right) \end{bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **The remaining matrix equation for the strains, given by**

$$\begin{Bmatrix} \varepsilon_{33}^{\sigma} \\ 2\varepsilon_{23}^{\sigma} \\ 2\varepsilon_{13}^{\sigma} \end{Bmatrix} = -\begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \end{Bmatrix}$$

becomes

$$\begin{Bmatrix} \varepsilon_{33}^{\sigma} \\ 2\varepsilon_{23}^{\sigma} \\ 2\varepsilon_{13}^{\sigma} \end{Bmatrix} = -\begin{bmatrix} C_{33} & 0 & 0 \\ 0 & C_{44} & C_{45} \\ 0 & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \end{Bmatrix}$$

which reduces to

$$\varepsilon_{33}^{\sigma} = -\frac{1}{C_{33}}\left[C_{13}\varepsilon_{11}^{\sigma} + C_{23}\varepsilon_{22}^{\sigma} + 2C_{36}\varepsilon_{12}^{\sigma}\right]$$ **and** $$2\varepsilon_{23}^{\sigma} = 2\varepsilon_{13}^{\sigma} = 0$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- Expressions for the reduced stiffnesses in terms of engineering constants are obtained directly by using the previously obtained results

$$\begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & \frac{\eta_{1,12}}{G_{12}} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{\eta_{2,12}}{G_{12}} \\ \frac{\eta_{12,1}}{E_1} & \frac{\eta_{12,2}}{E_2} & \frac{1}{G_{12}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix}^{-1}$$

- Inverting the matrix of compliances yields

$$Q_{11} = \frac{E_1}{\Delta}\left(1 - \eta_{12,2}\eta_{2,12}\right) \qquad Q_{12} = \frac{E_1}{\Delta}\left(\nu_{21} + \eta_{12,2}\eta_{1,12}\right) = \frac{E_2}{\Delta}\left(\nu_{12} + \eta_{12,1}\eta_{2,12}\right)$$

$$Q_{16} = -\frac{E_1}{\Delta}\left(\eta_{1,12} + \nu_{21}\eta_{2,12}\right) = -\frac{G_{12}}{\Delta}\left(\eta_{12,1} + \nu_{12}\eta_{12,2}\right) \qquad Q_{22} = \frac{E_2}{\Delta}\left(1 - \eta_{12,1}\eta_{1,12}\right)$$

$$Q_{26} = -\frac{E_2}{\Delta}\left(\eta_{2,12} + \nu_{12}\eta_{1,12}\right) = -\frac{G_{12}}{\Delta}\left(\eta_{12,2} + \nu_{21}\eta_{12,1}\right) \qquad Q_{66} = \frac{G_{12}}{\Delta}\left(1 - \nu_{12}\nu_{21}\right)$$

$$\Delta = 1 - \nu_{12}\nu_{21} - \eta_{12,1}\left(\eta_{1,12} + \nu_{21}\eta_{2,12}\right) - \eta_{12,2}\left(\eta_{2,12} + \nu_{12}\eta_{1,12}\right)$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- Similarly, the expression $\begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = -\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$ gives

$$\tilde{\beta}_{11} = -\frac{1}{\Delta}\left(E_1\alpha_{11}\left(1 - \eta_{12,2}\eta_{2,12}\right) + E_2\alpha_{22}\left(\nu_{12} + \eta_{12,1}\eta_{2,12}\right) - 2G_{12}\alpha_{12}\left(\eta_{12,1} + \nu_{12}\eta_{12,2}\right)\right)$$

$$\tilde{\beta}_{22} = -\frac{1}{\Delta}\left(E_1\alpha_{11}\left(\nu_{21} + \eta_{12,2}\eta_{1,12}\right) + E_2\alpha_{22}\left(1 - \eta_{12,1}\eta_{1,12}\right) - 2G_{12}\alpha_{12}\left(\eta_{12,2} + \nu_{21}\eta_{12,1}\right)\right)$$

$$\tilde{\beta}_{12} = \frac{1}{\Delta}\left(E_1\alpha_{11}\left(\eta_{1,12} + \nu_{21}\eta_{2,12}\right) + E_2\alpha_{22}\left(\eta_{2,12} + \nu_{12}\eta_{1,12}\right) - 2G_{12}\alpha_{12}\left(1 - \nu_{12}\nu_{21}\right)\right)$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- For a **specially orthotropic material**, the plane-stress constitutive equations become

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

$$\varepsilon_{33} = S_{13}\sigma_{11} + S_{23}\sigma_{22} + \alpha_{33}(T - T_{ref}) \quad \text{and} \quad 2\varepsilon_{23} = 2\varepsilon_{13} = 0$$

- In terms of engineering constants,

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{Bmatrix} (T - T_{ref})$$

$$\varepsilon_{33} = -\frac{\nu_{13}}{E_1}\sigma_{11} - \frac{\nu_{23}}{E_2}\sigma_{22} + \alpha_{33}(T - T_{ref}) \quad \text{and} \quad 2\varepsilon_{23} = 2\varepsilon_{13} = 0$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Similarly, for a specially orthotropic material, the plane-stress constitutive equations become**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \end{Bmatrix} + \begin{bmatrix} C_{13} & 0 & 0 \\ C_{23} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{33}^{\sigma} \\ 2\varepsilon_{23}^{\sigma} \\ 2\varepsilon_{13}^{\sigma} \end{Bmatrix} \quad \text{and}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} C_{13} & C_{23} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \end{Bmatrix} + \begin{bmatrix} C_{33} & 0 & 0 \\ 0 & C_{44} & 0 \\ 0 & 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{33}^{\sigma} \\ 2\varepsilon_{23}^{\sigma} \\ 2\varepsilon_{13}^{\sigma} \end{Bmatrix}$$

- **The last matrix equation gives**

$$\varepsilon_{33}^{\sigma} = -\frac{1}{C_{33}}\left[C_{13}\varepsilon_{11}^{\sigma} + C_{23}\varepsilon_{22}^{\sigma}\right] \quad \text{and} \quad 2\varepsilon_{23} = 2\varepsilon_{13} = 0$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS

CONTINUED

- Using the previous expressions for $\varepsilon_{33}^{\sigma}$, $2\varepsilon_{23}^{\sigma}$, and $2\varepsilon_{13}^{\sigma}$, the constitutive equations for plane stress, in terms of stiffness coefficients, become

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{Bmatrix} (T - T_{ref}) \right\}$$

where

$$\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} = \begin{bmatrix} \left(C_{11} - \dfrac{C_{13}C_{13}}{C_{33}}\right) & \left(C_{12} - \dfrac{C_{13}C_{23}}{C_{33}}\right) & 0 \\ \left(C_{12} - \dfrac{C_{13}C_{23}}{C_{33}}\right) & \left(C_{22} - \dfrac{C_{23}C_{23}}{C_{33}}\right) & 0 \\ 0 & 0 & C_{66} \end{bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **Next, noting that**

$$\begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = -\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{Bmatrix}$$ **for a specially orthotropic solid**

gives

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

- **Now, note that the general expression for the regular thermal moduli simplifies to**

$$\begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ \beta_{23} \\ \beta_{13} \\ \beta_{12} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONTINUED

- **In addition,**

$$\begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{12} \end{Bmatrix} - \begin{bmatrix} C_{13} & C_{14} & C_{15} \\ C_{23} & C_{24} & C_{25} \\ C_{36} & C_{46} & C_{56} \end{bmatrix} \begin{bmatrix} C_{33} & C_{34} & C_{35} \\ C_{34} & C_{44} & C_{45} \\ C_{35} & C_{45} & C_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_{33} \\ \beta_{23} \\ \beta_{13} \end{Bmatrix}$$

simplifies to

$$\begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ 0 \end{Bmatrix} - \begin{bmatrix} C_{13} & 0 & 0 \\ C_{23} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{33} & 0 & 0 \\ 0 & C_{44} & 0 \\ 0 & 0 & C_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \beta_{33} \\ 0 \\ 0 \end{Bmatrix}$$

which yields

$$\tilde{\beta}_{11} = \beta_{11} - \frac{C_{13}}{C_{33}} \beta_{33} \qquad \tilde{\beta}_{22} = \beta_{22} - \frac{C_{23}}{C_{33}} \beta_{33} \qquad \tilde{\beta}_{12} = 0$$

CONSTITUTIVE EQUATIONS FOR PLANE STRESS
CONCLUDED

- In terms of the engineering constants, the reduced stiffness coefficients and are given by

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \qquad Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}}$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \qquad Q_{66} = G_{12}$$

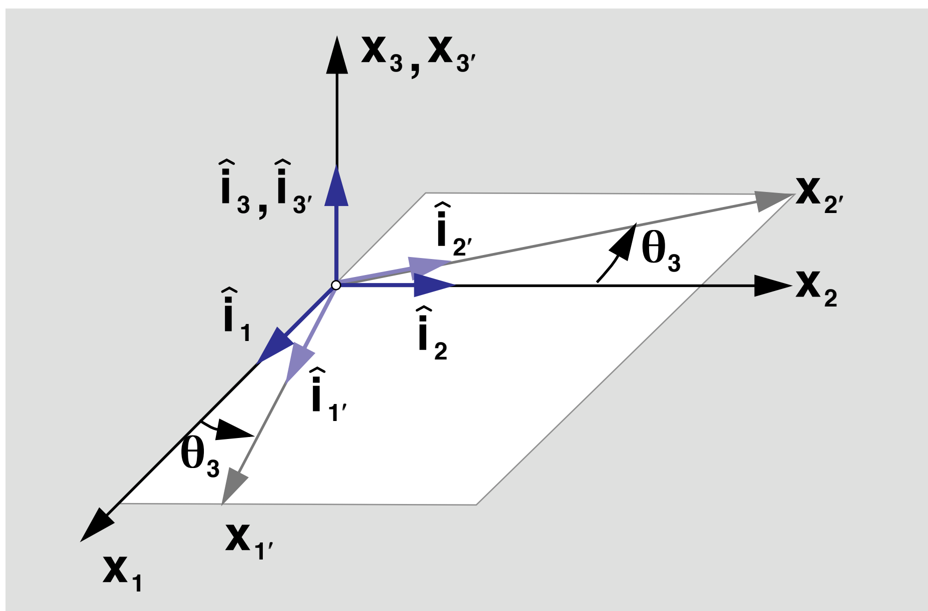
- Similarly, the nonzero reduced thermal moduli are given by

$$\tilde{\beta}_{11} = -E_1 \frac{\alpha_{11} + \alpha_{22}\,\nu_{21}}{1 - \nu_{12}\nu_{21}} \qquad \tilde{\beta}_{22} = -E_2 \frac{\alpha_{22} + \alpha_{11}\,\nu_{12}}{1 - \nu_{12}\nu_{21}}$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRESS

- Often, the relationship between the planar stresses and strains that are defined relative to two different coordinate systems is needed

- Consider the dextral (right-handed) rotation of coordinate frames shown in the figure



STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRESS - CONTINUED

- **Previously, the matrix form of the stress-transformation law for this specific transformation was given as**

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -2\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

- **Substituting $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$ into this equation yields**

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRESS - CONTINUED

- Thus, the **stress-transformation law** for a state of plane stress and a dextral rotation about the x_3 axis is given by

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} = \left[\begin{array}{c:c:c} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \hdashline \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ \hdashline -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

- This law is expressed symbolically by $\{\Sigma'\} = [T_\sigma(\theta_3)]\{\Sigma\}$

where $\{\Sigma'\} \equiv \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix}$, $\{\Sigma\} \equiv \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$, and

$$[T_\sigma(\theta_3)] \equiv \left[\begin{array}{c:c:c} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \hdashline \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ \hdashline -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{array}\right]$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRESS - CONTINUED

- **Similarly, the matrix form of the inverse stress-tensor transformation law was also given previously as**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & -2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & 2\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & \sin\theta_3 & 0 \\ 0 & 0 & 0 & -\sin\theta_3 & \cos\theta_3 & 0 \\ \sin\theta_3\cos\theta_3 & -\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix}$$

- **Following a similar process gives** $\{\Sigma\} = [T_\sigma(\theta_3)]^{-1}\{\Sigma'\}$ **where**

$$[T_\sigma(\theta_3)]^{-1} = [T_\sigma(-\theta_3)] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin\theta_3\cos\theta_3 & -\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRESS - CONTINUED

- Previously, the matrix form of the strain-transformation law for this specific transformation was given as

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

- This matrix equation can be partitioned into the following parts

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -\sin\theta_3\cos\theta_3 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix},$$

$$\begin{Bmatrix} 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \end{Bmatrix} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{bmatrix} \begin{Bmatrix} 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} \quad \text{and} \quad \varepsilon_{3'3'} = \varepsilon_{33}$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRESS - CONTINUED

- The first law is expressed symbolically by $\{\mathbf{E}'\} = [\mathbf{T}_\varepsilon(\theta_3)]\{\mathbf{E}\}$

where $\{\mathbf{E}'\} \equiv \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix}$, $\{\mathbf{E}\} \equiv \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}$, and

$$[\mathbf{T}_\varepsilon(\theta_3)] \equiv \left[\begin{array}{c|c|c} \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\cos\theta_3 \\ \hline \sin^2\theta_3 & \cos^2\theta_3 & -\sin\theta_3\cos\theta_3 \\ \hline -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{array}\right]$$

- A useful relationship that is easily verified is given by

$$[\mathbf{T}_\varepsilon(\theta_3)]^{\mathrm{T}} = [\mathbf{T}_\sigma(\theta_3)]^{-1} = [\mathbf{T}_\sigma(-\theta_3)]$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRESS - CONCLUDED

- **Similarly, the matrix form of the inverse strain-transformation law was also given previously as**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & -\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & \sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & \sin\theta_3 & 0 \\ 0 & 0 & 0 & -\sin\theta_3 & \cos\theta_3 & 0 \\ 2\sin\theta_3\cos\theta_3 & -2\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix}$$

- **From this matrix equation, it follows that** $\{\mathbf{E}\} = [\mathbf{T}_\varepsilon(\theta_3)]^{-1}\{\mathbf{E}'\}$

where $$[\mathbf{T}_\varepsilon(\theta_3)]^{-1} = [\mathbf{T}_\varepsilon(-\theta_3)] = [\mathbf{T}_\sigma(\theta_3)]^{\mathrm{T}}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS

- The two matrix equations that resulted from the reduction for a state of plane stress are

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \quad \text{and}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} (T - T_{ref}) , \quad \text{where}$$

$$\begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = -\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix}^{-1}$$

- Also, $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$ and generally $\varepsilon_{33} \neq \varepsilon_{13} \neq \varepsilon_{23} \neq 0$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- In terms of another set of coordinates $\left(x_{1'}, x_{2'}, x_{3'}\right)$ that correspond to a dextral rotation about the x_3 axis, the constitutive equations must also have the forms given as

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'6'} \\ S_{1'6'} & S_{2'6'} & S_{6'6'} \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ 2\alpha_{1'2'} \end{Bmatrix} \left(T - T_{ref}\right) \quad \text{and}$$

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} Q_{1'1'} & Q_{1'2'} & Q_{1'6'} \\ Q_{1'2'} & Q_{2'2'} & Q_{2'6'} \\ Q_{1'6'} & Q_{2'6'} & Q_{6'6'} \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \tilde{\beta}_{1'1'} \\ \tilde{\beta}_{2'2'} \\ \tilde{\beta}_{1'2'} \end{Bmatrix} \left(T - T_{ref}\right), \quad \text{where}$$

$$\begin{Bmatrix} \tilde{\beta}_{1'1'} \\ \tilde{\beta}_{2'2'} \\ \tilde{\beta}_{1'2'} \end{Bmatrix} = -\begin{bmatrix} Q_{1'1'} & Q_{1'2'} & Q_{1'6'} \\ Q_{1'2'} & Q_{2'2'} & Q_{2'6'} \\ Q_{1'6'} & Q_{2'6'} & Q_{6'6'} \end{bmatrix} \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ 2\alpha_{1'2'} \end{Bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_{1'1'} & Q_{1'2'} & Q_{1'6'} \\ Q_{1'2'} & Q_{2'2'} & Q_{2'6'} \\ Q_{1'6'} & Q_{2'6'} & Q_{6'6'} \end{bmatrix} = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'6'} \\ S_{1'6'} & S_{2'6'} & S_{6'6'} \end{bmatrix}^{-1}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- **For convenience, let**

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \qquad [Q] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}$$

$$\{\alpha\} = \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \qquad \{\tilde{\beta}\} = \begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix}$$

$$[S'] = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'6'} \\ S_{1'6'} & S_{2'6'} & S_{6'6'} \end{bmatrix} \qquad [Q'] = \begin{bmatrix} Q_{1'1'} & Q_{1'2'} & Q_{1'6'} \\ Q_{1'2'} & Q_{2'2'} & Q_{2'6'} \\ Q_{1'6'} & Q_{2'6'} & Q_{6'6'} \end{bmatrix}$$

$$\{\alpha'\} = \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ 2\alpha_{1'2'} \end{Bmatrix} \qquad \{\tilde{\beta}'\} = \begin{Bmatrix} \tilde{\beta}_{1'1'} \\ \tilde{\beta}_{2'2'} \\ \tilde{\beta}_{1'2'} \end{Bmatrix}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- With this notation, the two sets of thermoelastic constitutive equations are expressed in symbolic form by

$$\{\mathbf{E}\} = [\mathbf{S}]\{\Sigma\} + \{\alpha\}\Theta \quad \text{and} \quad \{\Sigma\} = [\mathbf{Q}]\{\mathbf{E}\} + \{\tilde{\beta}\}\Theta$$

$$\{\mathbf{E}'\} = [\mathbf{S}']\{\Sigma'\} + \{\alpha'\}\Theta \quad \text{and} \quad \{\Sigma'\} = [\mathbf{Q}']\{\mathbf{E}'\} + \{\tilde{\beta}'\}\Theta$$

where $\Theta \equiv T - T_{ref}$

- By using the matrix form of the stress and strain transformation equations for plane stress, $\{\Sigma\} = [\mathbf{Q}]\{\mathbf{E}\} + \{\tilde{\beta}\}\Theta$ becomes

$$[\mathbf{T}_\sigma]^{-1}\{\Sigma'\} = [\mathbf{Q}][\mathbf{T}_\varepsilon]^{-1}\{\mathbf{E}'\} + \{\tilde{\beta}\}\Theta$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- Premultiplying by $\left[\mathbf{T}_\sigma\right]$ gives

$$\{\Sigma'\} = \left[\mathbf{T}_\sigma\right][\mathbf{Q}]\left[\mathbf{T}_\varepsilon\right]^{-1}\{\mathbf{E}'\} + \left[\mathbf{T}_\sigma\right]\{\tilde{\beta}\}\Theta$$

- Comparing this equation with $\{\Sigma'\} = [\mathbf{Q}']\{\mathbf{E}'\} + \{\tilde{\beta}'\}\Theta$ it follows that

$$[\mathbf{Q}'] = \left[\mathbf{T}_\sigma\right][\mathbf{Q}]\left[\mathbf{T}_\varepsilon\right]^{-1} \quad \text{and} \quad \{\tilde{\beta}'\} = \left[\mathbf{T}_\sigma\right]\{\tilde{\beta}\}$$

- Rearranging $[\mathbf{Q}'] = \left[\mathbf{T}_\sigma\right][\mathbf{Q}]\left[\mathbf{T}_\varepsilon\right]^{-1}$ gives

$$[\mathbf{Q}] = \left[\mathbf{T}_\sigma\right]^{-1}[\mathbf{Q}']\left[\mathbf{T}_\varepsilon\right]$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- Next, by using the matrix form of the stress and strain transformation equations for plane stress, $\{E\} = [S]\{\Sigma\} + \{\alpha\}\Theta$ becomes

$$[T_\varepsilon]^{-1}\{E'\} = [S][T_\sigma]^{-1}\{\Sigma'\} + \{\alpha\}\Theta$$

- Premultiplying by $[T_\varepsilon]$ gives

$$\{E'\} = [T_\varepsilon][S][T_\sigma]^{-1}\{\Sigma'\} + [T_\varepsilon]\{\alpha\}\Theta$$

- By comparing this equation with $\{E'\} = [S']\{\sigma'\} + \{\alpha'\}\Theta$ it follows that

$$[S'] = [T_\varepsilon][S][T_\sigma]^{-1} \quad \text{and} \quad \{\alpha'\} = [T_\varepsilon]\{\alpha\}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- **Rearranging** $[S'] = [T_\varepsilon][S][T_\sigma]^{-1}$ **gives** $[S] = [T_\varepsilon]^{-1}[S'][T_\sigma]$

- **Noting that for a dextral rotation about the x_3 axis,**

$[T_\varepsilon]^{-1} = [T_\sigma]^T$ **and** $[T_\sigma]^{-1} = [T_\varepsilon]^T$ **it follows that**

$$[S'] = [T_\varepsilon][S][T_\varepsilon]^T \qquad [S] = [T_\sigma]^T[S'][T_\sigma]$$

$$[Q'] = [T_\sigma][Q][T_\sigma]^T \qquad [Q] = [T_\varepsilon]^T[Q'][T_\varepsilon]$$

$$\{\alpha'\} = [T_\varepsilon]\{\alpha\} \qquad \{\alpha\} = [T_\sigma]^T\{\alpha'\}$$

$$\{\tilde{\beta}'\} = [T_\sigma]\{\tilde{\beta}\} \qquad \{\tilde{\beta}\} = [T_\varepsilon]^T\{\tilde{\beta}'\}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

with

$$\left[\mathbf{T}_{\sigma}(\theta_3)\right] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

and

$$\left[\mathbf{T}_{\varepsilon}(\theta_3)\right] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -\sin\theta_3\cos\theta_3 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- $[Q'] = [T_\sigma][Q][T_\sigma]^T$, with $m = \cos\theta_3$ and $n = \sin\theta_3$, yields

$$Q_{1'1'} = m^4 Q_{11} + 2m^2n^2(Q_{12} + 2Q_{66}) + 4mn(m^2Q_{16} + n^2Q_{26}) + n^4Q_{22}$$

$$Q_{1'2'} = m^2n^2(Q_{11} + Q_{22} - 4Q_{66}) - 2mn(m^2 - n^2)(Q_{16} - Q_{26}) + (m^4 + n^4)Q_{12}$$

$$Q_{1'6'} = m^2(m^2 - 3n^2)Q_{16} - m^3n(Q_{11} - Q_{12} - 2Q_{66}) + mn^3(Q_{22} - Q_{12} - 2Q_{66}) - n^2(n^2 - 3m^2)Q_{26}$$

$$Q_{2'2'} = m^4 Q_{22} + 2m^2n^2(Q_{12} + 2Q_{66}) - 4mn(m^2Q_{26} + n^2Q_{16}) + n^4Q_{11}$$

$$Q_{2'6'} = m^2(m^2 - 3n^2)Q_{26} + m^3n(Q_{22} - Q_{12} - 2Q_{66}) - mn^3(Q_{11} - Q_{12} - 2Q_{66}) - n^2(n^2 - 3m^2)Q_{16}$$

$$Q_{6'6'} = m^2n^2(Q_{11} + Q_{22} - 2Q_{12}) - 2mn(m^2 - n^2)(Q_{16} - Q_{26}) + (m^2 - n^2)^2Q_{66}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- $[\mathbf{Q}] = [\mathbf{T}_\varepsilon]^T [\mathbf{Q}'][\mathbf{T}_\varepsilon]$, with $m = \cos\theta_3$ and $n = \sin\theta_3$, yields

$$Q_{11} = m^4 Q_{1'1'} + 2m^2n^2(Q_{1'2'} + 2Q_{6'6'}) - 4mn(m^2Q_{1'6'} + n^2Q_{2'6'}) + n^4Q_{2'2'}$$

$$Q_{12} = m^2n^2(Q_{1'1'} + Q_{2'2'} - 4Q_{6'6'}) + 2mn(m^2 - n^2)(Q_{1'6'} - Q_{2'6'}) + (m^4 + n^4)Q_{1'2'}$$

$$\begin{aligned} Q_{16} = {} & m^2(m^2 - 3n^2)Q_{1'6'} + m^3n(Q_{1'1'} - Q_{1'2'} - 2Q_{6'6'}) \\ & - mn^3(Q_{2'2'} - Q_{1'2'} - 2Q_{6'6'}) - n^2(n^2 - 3m^2)Q_{2'6'} \end{aligned}$$

$$Q_{22} = m^4 Q_{2'2'} + 2m^2n^2(Q_{1'2'} + 2Q_{6'6'}) + 4mn(m^2Q_{2'6'} + n^2Q_{1'6'}) + n^4Q_{1'1'}$$

$$\begin{aligned} Q_{26} = {} & m^2(m^2 - 3n^2)Q_{2'6'} - m^3n(Q_{2'2'} - Q_{1'2'} - 2Q_{6'6'}) \\ & + mn^3(Q_{1'1'} - Q_{1'2'} - 2Q_{6'6'}) - n^2(n^2 - 3m^2)Q_{1'6'} \end{aligned}$$

$$Q_{66} = m^2n^2(Q_{1'1'} + Q_{2'2'} - 2Q_{1'2'}) + 2mn(m^2 - n^2)(Q_{1'6'} - Q_{2'6'}) + (m^2 - n^2)^2 Q_{6'6'}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- $[\mathbf{S}'] = [\mathbf{T}_\varepsilon][\mathbf{S}][\mathbf{T}_\varepsilon]^T$, with $m = \cos\theta_3$ and $n = \sin\theta_3$, yields

$$S_{1'1'} = m^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + 2mn(m^2 S_{16} + n^2 S_{26}) + n^4 S_{22}$$

$$S_{1'2'} = m^2 n^2 (S_{11} + S_{22} - S_{66}) - mn(m^2 - n^2)(S_{16} - S_{26}) + (m^4 + n^4) S_{12}$$

$$\begin{aligned} S_{1'6'} = {} & m^2(m^2 - 3n^2) S_{16} - m^3 n (2S_{11} - 2S_{12} - S_{66}) \\ & + mn^3(2S_{22} - 2S_{12} - S_{66}) - n^2(n^2 - 3m^2) S_{26} \end{aligned}$$

$$S_{2'2'} = m^4 S_{22} + m^2 n^2 (2S_{12} + S_{66}) - 2mn(m^2 S_{26} + n^2 S_{16}) + n^4 S_{11}$$

$$\begin{aligned} S_{2'6'} = {} & m^2(m^2 - 3n^2) S_{26} + m^3 n (2S_{22} - 2S_{12} - S_{66}) \\ & - mn^3(2S_{11} - 2S_{12} - S_{66}) - n^2(n^2 - 3m^2) S_{16} \end{aligned}$$

$$S_{6'6'} = 4m^2 n^2 (S_{11} + S_{22} - 2S_{12}) - 4mn(m^2 - n^2)(S_{16} - S_{26}) + (m^2 - n^2)^2 S_{66}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- $[S] = [T_\sigma]^T [S'][T_\sigma]$, with $m = \cos\theta_3$ and $n = \sin\theta_3$, yields

$$S_{11} = m^4 S_{1'1'} + m^2 n^2 (2S_{1'2'} + S_{6'6'}) - 2mn(m^2 S_{1'6'} + n^2 S_{2'6'}) + n^4 S_{2'2'}$$

$$S_{12} = m^2 n^2 (S_{1'1'} + S_{2'2'} - S_{6'6'}) + mn(m^2 - n^2)(S_{1'6'} - S_{2'6'}) + (m^4 + n^4) S_{1'2'}$$

$$\begin{aligned} S_{16} = {} & m^2(m^2 - 3n^2) S_{1'6'} + m^3 n (2S_{1'1'} - 2S_{1'2'} - S_{6'6'}) \\ & - mn^3 (2S_{2'2'} - 2S_{1'2'} - S_{6'6'}) - n^2(n^2 - 3m^2) S_{2'6'} \end{aligned}$$

$$S_{22} = m^4 S_{2'2'} + m^2 n^2 (2S_{1'2'} + S_{6'6'}) + 2mn(m^2 S_{2'6'} + n^2 S_{1'6'}) + n^4 S_{1'1'}$$

$$\begin{aligned} S_{26} = {} & m^2(m^2 - 3n^2) S_{2'6'} - m^3 n (2S_{2'2'} - 2S_{1'2'} - S_{6'6'}) \\ & + mn^3 (2S_{1'1'} - 2S_{1'2'} - S_{6'6'}) - n^2(n^2 - 3m^2) S_{1'6'} \end{aligned}$$

$$S_{66} = 4m^2 n^2 (S_{1'1'} + S_{2'2'} - 2S_{1'2'}) + 4mn(m^2 - n^2)(S_{1'6'} - S_{2'6'}) + (m^2 - n^2)^2 S_{6'6'}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- **Note that $[\mathbf{Q}']$ and $[\mathbf{Q}]$ can be expressed as**

$$[\mathbf{Q}'] = [\mathbf{T}_\sigma(\theta_3)][\mathbf{Q}][\mathbf{T}_\varepsilon(-\theta_3)] \quad \text{and} \quad [\mathbf{Q}] = [\mathbf{T}_\sigma(-\theta_3)][\mathbf{Q}'][\mathbf{T}_\varepsilon(\theta_3)]$$

  - **Thus, one set of transformed stiffness expressions can be obtained from the other by simply interchanging the primed and unprimed indices and replacing n with -n**

- **Likewise, $[\mathbf{S}']$ and $[\mathbf{S}]$ can be expressed as**

$$[\mathbf{S}'] = [\mathbf{T}_\varepsilon(\theta_3)][\mathbf{S}][\mathbf{T}_\sigma(-\theta_3)] \quad \text{and} \quad [\mathbf{S}] = [\mathbf{T}_\varepsilon(-\theta_3)][\mathbf{S}'][\mathbf{T}_\sigma(\theta_3)]$$

  - **Thus, one set of transformed compliance expressions can be obtained from the other by simply interchanging the primed and unprimed indices and replacing n with -n**

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- $\{\alpha'\} = [T_\varepsilon]\{\alpha\}$, with $m = \cos\theta_3$ and $n = \sin\theta_3$, yields

$$\alpha_{1'1'} = m^2\alpha_{11} + 2mn\alpha_{12} + n^2\alpha_{22} \qquad \alpha_{2'2'} = m^2\alpha_{22} - 2mn\alpha_{12} + n^2\alpha_{11}$$

$$\alpha_{1'2'} = (m^2 - n^2)\alpha_{12} + mn(\alpha_{22} - \alpha_{11})$$

- Similarly, $\{\alpha\} = [T_\sigma]^T\{\alpha'\}$ gives

$$\alpha_{11} = m^2\alpha_{1'1'} + 2mn\alpha_{1'2'} + n^2\alpha_{2'2'} \qquad \alpha_{22} = m^2\alpha_{2'2'} - 2mn\alpha_{1'2'} + n^2\alpha_{1'1'}$$

$$\alpha_{12} = (m^2 - n^2)\alpha_{1'2'} + mn(\alpha_{2'2'} - \alpha_{1'1'})$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONTINUED

- $\{\tilde{\beta}'\} = [\mathbf{T}_\sigma]\{\tilde{\beta}\}$, with $m = \cos\theta_3$ and $n = \sin\theta_3$, yields

$$\tilde{\beta}_{1'1'} = m^2\tilde{\beta}_{11} + 2mn\tilde{\beta}_{12} + n^2\tilde{\beta}_{22} \qquad \tilde{\beta}_{2'2'} = m^2\tilde{\beta}_{22} - 2mn\tilde{\beta}_{12} + n^2\tilde{\beta}_{11}$$

$$\tilde{\beta}_{1'2'} = \left(m^2 - n^2\right)\beta_{12} + mn\left(\tilde{\beta}_{22} - \tilde{\beta}_{11}\right)$$

- Similarly, $\{\tilde{\beta}\} = [\mathbf{T}_\varepsilon]^T\{\tilde{\beta}'\}$ gives

$$\tilde{\beta}_{11} = m^2\tilde{\beta}_{1'1'} - 2mn\tilde{\beta}_{1'2'} + n^2\tilde{\beta}_{2'2'} \qquad \tilde{\beta}_{22} = m^2\tilde{\beta}_{2'2'} + 2mn\tilde{\beta}_{1'2'} + n^2\tilde{\beta}_{1'1'}$$

$$\tilde{\beta}_{12} = \left(m^2 - n^2\right)\tilde{\beta}_{1'2'} - mn\left(\tilde{\beta}_{2'2'} - \tilde{\beta}_{1'1'}\right)$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRESS - CONCLUDED

- **Note that $\{\boldsymbol{\alpha}'\}$ and $\{\boldsymbol{\alpha}\}$ can be expressed as**

$$\{\boldsymbol{\alpha}'\} = [\mathbf{T}_\varepsilon(\theta_3)]\{\boldsymbol{\alpha}\} \quad \text{and} \quad \{\boldsymbol{\alpha}\} = [\mathbf{T}_\varepsilon(-\theta_3)]\{\boldsymbol{\alpha}'\}$$

  - **Thus, one set of transformed thermal-expansion expressions can be obtained from the other by simply interchanging the primed and unprimed indices and replacing n with -n**

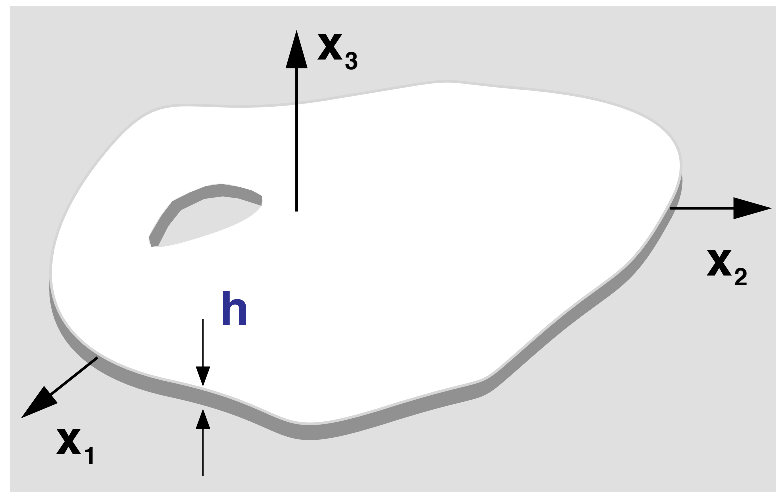
- **Likewise, $\{\tilde{\beta}'\}$ and $\{\tilde{\beta}\}$ can be expressed as**

$$\{\tilde{\beta}'\} = [\mathbf{T}_\sigma(\theta_3)]\{\tilde{\beta}\} \quad \text{and} \quad \{\tilde{\beta}\} = [\mathbf{T}_\sigma(-\theta_3)]\{\tilde{\beta}'\}$$

  - **Thus, one set of transformed thermal-compliance expressions can be obtained from the other by simply interchanging the primed and unprimed indices and replacing n with -n**

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS

- When relatively thin plates, with uniform thickness h, are supported and subjected to inplane loads such that the dominant stresses, strains, and displacements act only in planes parallel to the plane $x_3 = 0$ shown in the figure, significant simplifications can be made to the equations governing the elastic response



- Moreover, these dominant response quantities are presumed to vary symmetrically through the plate thickness, given by $-\frac{h}{2} \le x_3 \le +\frac{h}{2}$, such that no significant bending deformations are exhibited by the plate

- In contrast to the plane stress approximations previously presented herein, when these conditions exist, with respect to the through-the-thickness variations, the plate response is described as a state of ***generalized plane stress***

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

- For a state of ***generalized plane stress*** in an anisotropic solid, with respect to the plane $x_3 = 0$, the displacement fields in the x_1-, x_2-, and x_3-coordinate directions are approximated by averaging the through-the-thickness variations as follows

$$U_1(x_1, x_2) = \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} u_1(x_1, x_2, x_3)\, dx_3 \qquad U_2(x_1, x_2) = \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} u_2(x_1, x_2, x_3)\, dx_3$$

and

$$U_3(x_1, x_2) = \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} u_3(x_1, x_2, x_3)\, dx_3 = 0$$

- To exclude bending deformations, $U_3(x_1, x_2) = 0$ is required for the out-of-plane displacement field

- Note that allowing $u_3(x_1, x_2, x_3) = x_3\,\varepsilon$, where ε is a constant satisfies $U_3(x_1, x_2) = 0$

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

- The strain-displacement relations of the linear theory of elasticity, given by $\varepsilon_{ij}(x_1, x_2, x_3) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, are approximated as follows

- First, average strains are defined by $\bar{\varepsilon}_{ij}(x_1, x_2) = \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \varepsilon_{ij}(x_1, x_2, x_3)\, dx_3$

- Substituting $\varepsilon_{ij}(x_1, x_2, x_3) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ into this expression and using the definitions for the average displacements yields

$$\bar{\varepsilon}_{ij}(x_1, x_2) = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \quad 2\bar{\varepsilon}_{13} = 2\bar{\varepsilon}_{23} = 0 \quad \text{and} \quad \bar{\varepsilon}_{33} = \varepsilon$$

- Thus, ε represent a uniform through-the-thickness normal strain

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

- **Next, the stress field is approximated such that** $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$
  - **For this case, the stresses in a thin, flat body, that are normal to the plane $x_3 = 0$, are presumed negligible compared to the other stresses**

- **In addition, average stresses are defined by**

$$\bar{\sigma}_{ij}\left(x_1,\ x_2\right) = \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_{ij}\left(x_1,\ x_2,\ x_3\right)\, dx_3$$

- **The conditions on the presumed stress field are satisfied by the previously derived plane-stress constitutive equations given in the form**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right) \right\}$$ **and**

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref})$$

$$\begin{Bmatrix} \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ S_{14} & S_{24} & S_{46} \\ S_{15} & S_{25} & S_{56} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix} (T - T_{ref})$$

- **That is, substituting $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$ into the general form of the constitutive equations and simplifying yields the same plane stress constitutive equation given above**

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

- Restricting the plate to homogeneous construction, integration of the constitutive equations through the plate thickness yields

$$\begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_{11} \\ \bar{\varepsilon}_{22} \\ 2\bar{\varepsilon}_{12} \end{Bmatrix} - \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \frac{1}{h} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(T - T_{ref}\right) dx_3$$

- Next, the temperature change $T - T_{ref}$ is presumed to vary symmetrically through the plate thickness so as not to cause bending deformations

- The average temperature change is defined as

$$\overline{\Theta}\left(x_1, x_2\right) = \frac{1}{h} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left(T - T_{ref}\right) dx_3$$

such that the constitutive equations become

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

$$\begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \bar{\varepsilon}_{11} \\ \bar{\varepsilon}_{22} \\ 2\bar{\varepsilon}_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \overline{\Theta} \right\}$$

- Similarly,

$$\begin{Bmatrix} \bar{\varepsilon}_{11} \\ \bar{\varepsilon}_{22} \\ 2\bar{\varepsilon}_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \overline{\Theta}$$

and

$$\begin{Bmatrix} \bar{\varepsilon}_{33} \\ \bar{\varepsilon}_{23} \\ 2\bar{\varepsilon}_{13} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ S_{14} & S_{24} & S_{46} \\ S_{15} & S_{25} & S_{56} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix} \overline{\Theta}$$

- **Inspection of the last matrix equation reveals that the material must be monoclinic, with the plane $x_3 = 0$ being a plane of reflective symmetry, in order to satisfy the kinematic hypothesis** $2\bar{\varepsilon}_{13} = 2\bar{\varepsilon}_{23} = 0$ **and** $\bar{\varepsilon}_{33} = \varepsilon$

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

- **Note that the symmetry requirement on the material properties is consistent with the symmetry requirements imposed up front on the displacements, stresses, and strains**

- **Thus,** $\begin{Bmatrix} \bar{\varepsilon}_{33} \\ \bar{\varepsilon}_{23} \\ 2\bar{\varepsilon}_{13} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ S_{14} & S_{24} & S_{46} \\ S_{15} & S_{25} & S_{56} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \end{Bmatrix} \bar{\Theta}$ **becomes**

$\begin{Bmatrix} \bar{\varepsilon}_{33} \\ \bar{\varepsilon}_{23} \\ 2\bar{\varepsilon}_{13} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ 0 \\ 0 \end{Bmatrix} \bar{\Theta}$ **, which reduces to**

$$\varepsilon = S_{13}\bar{\sigma}_{11} + S_{23}\bar{\sigma}_{22} + S_{36}\bar{\sigma}_{12} + \alpha_{33}\bar{\Theta}$$

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

- In terms of engineering constants, $\varepsilon = -\frac{\nu_{13}}{E_1}\bar{\sigma}_{11} - \frac{\nu_{23}}{E_2}\bar{\sigma}_{22} + \frac{\eta_{12,3}}{E_3}\bar{\sigma}_{33} + \alpha_{33}\bar{\Theta}$ and

$$\begin{Bmatrix} \bar{\varepsilon}_{11} \\ \bar{\varepsilon}_{22} \\ 2\bar{\varepsilon}_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & \frac{\eta_{1,12}}{G_{12}} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{\eta_{2,12}}{G_{12}} \\ \frac{\eta_{12,1}}{E_1} & \frac{\eta_{12,2}}{E_2} & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \bar{\Theta}$$

- Now consider the previously derived **stress-transformation law** for a state of plane stress and a dextral rotation about the x_3 axis is given by

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

- This law was expressed symbolically by $\{\Sigma'\} = [T_\sigma(\theta_3)]\{\Sigma\}$

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

where $\{\Sigma'\} \equiv \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix}$, $\{\Sigma\} \equiv \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$, and

$$[\mathbf{T}_\sigma(\theta_3)] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

- Integrating the stress transformation law over the plate thickness yields the trans formation law $\{\bar{\Sigma}'\} = [\mathbf{T}_\sigma(\theta_3)]\{\bar{\Sigma}\}$, where

$$\{\bar{\Sigma}\} \equiv \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} dx_3 \quad \text{and} \quad \{\bar{\Sigma}'\} \equiv \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} dx_3$$

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONTINUED

- Likewise, the previously derived **strain-transformation law** for a state of plane stress and a dextral rotation about the x_3 axis is given by

$$\{\mathbf{E}'\} = [\mathbf{T}_\varepsilon(\theta_3)]\{\mathbf{E}\}$$

where

$$\{\mathbf{E}'\} \equiv \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix}, \quad \{\mathbf{E}\} \equiv \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}, \text{ and}$$

$$[\mathbf{T}_\varepsilon(\theta_3)] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -\sin\theta_3\cos\theta_3 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

CONSTITUTIVE EQUATIONS FOR GENERALIZED PLANE STRESS - CONCLUDED

- Integrating the strain transformation law over the plate thickness yields the transformation law $\left\{\overline{\mathbf{E}}'\right\} = \left[\mathbf{T}_{\varepsilon}(\theta_3)\right]\left\{\overline{\mathbf{E}}\right\}$, where

$$\left\{\overline{\mathbf{E}}\right\} \equiv \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \left\{\begin{matrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{matrix}\right\} dx_3 \quad \text{and} \quad \left\{\overline{\mathbf{E}}'\right\} \equiv \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} \left\{\begin{matrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{matrix}\right\} dx_3$$

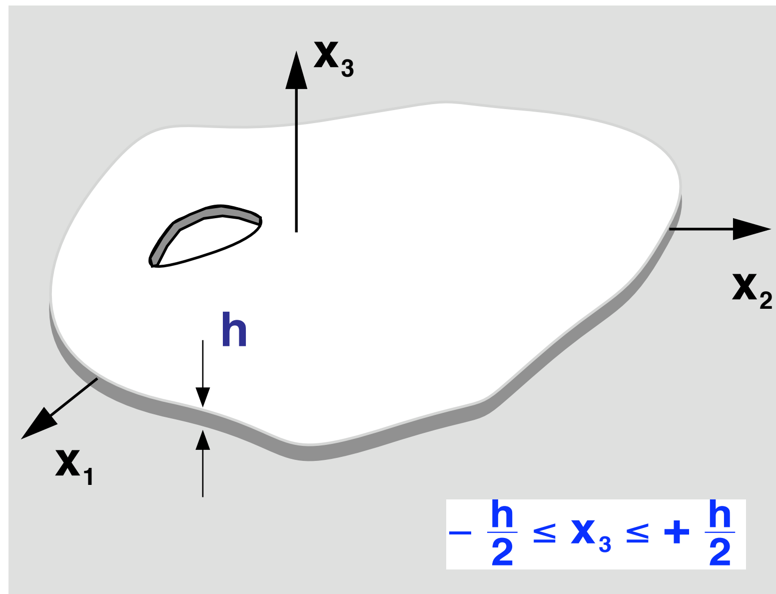
- Comparing the stress and strain transformation equations and the constitutive equations for ***generalized plane stress*** with the corresponding equations for **plane stress**, it is seen that they have identical structure

  - As a result, the constitutive equations for ***generalized plane stress*** transform in exactly the same way as those for **plane stress**

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES

- Another practical case of interest that is similar to generalized plane stress is the case of thin, nonhomogeneous plates that undergo inplane deformations without any bending deformations

- For this case, the stresses in a thin, flat body, that are normal to the x_1 - x_2 plane shown in the figure, are presumed negligible compared to the other stresses



- Thus, the stress field is approximated such that $\sigma_{13} = 0$, $\sigma_{23} = 0$, and $\sigma_{33} = 0$

- But, $\sigma_{11} = \sigma_{11}(x_1,x_2,x_3)$, $\sigma_{22} = \sigma_{22}(x_1,x_2,x_3)$, and $\sigma_{12} = \sigma_{12}(x_1,x_2,x_3)$ are permitted because of through-the thickness nonhomogeneity that is presumed to exist

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **In addition, the plate-like body is required to have a uniform thickness h and is not allowed to bend when subjected to inplane loads**

- **The conditions on the stress field are satisfied by the previously derived plane-stress constitutive equations given in the form**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\}$$

- **In the present formulation, the total strains are presumed to be uniform through the plate thickness; that is,**

$$\varepsilon_{11} = \varepsilon_{11}(x_1,x_2) \;, \quad \varepsilon_{22} = \varepsilon_{22}(x_1,x_2) \;, \text{ and } \varepsilon_{12} = \varepsilon_{12}(x_1,x_2)$$

- **However, the plate is allowed to be nonhomogeneous through the thickness such that the reduced stiffness coefficients and the coefficients of thermal expansion can vary with the x_3 coordinate**

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **Because of the plate's thinness, the temperature change $T - T_{ref}$ is presumed to be uniform through the thickness**

- **The requirement that the plate not bend is fulfilled by picking the middle surface of the plate to correspond to $x_3 = 0$ and then to require that the following integral be valid**

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} x_3 \, dx_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- **This matrix integral equation then requires the following through-the-thickness symmetry conditions on the stress field**

$$\sigma_{11}(x_1, x_2, -x_3) = \sigma_{11}(x_1, x_2, x_3), \quad \sigma_{22}(x_1, x_2, -x_3) = \sigma_{22}(x_1, x_2, x_3), \text{ and}$$

$$\sigma_{12}(x_1, x_2, -x_3) = \sigma_{12}(x_1, x_2, x_3)$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- The requirement that the plate not bend also places requirements on material properties

- First, note that substituting the plane stress constitutive equation into the previously stated integral equation yields

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} x_3 \, dx_3 \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} x_3 \, dx_3 \left(T - T_{ref}\right) = 0$$

- Next, note that for arbitrary strain and temperature fields, the two integrals in the above equation must vanish independently; that is,

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} x_3 \, dx_3 = 0 \quad \text{and} \quad \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} x_3 \, dx_3 = 0$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- The integral $\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} x_3 \, dx_3 = 0$ requires that the reduced stiffness coefficients be symmetric functions about the plane $x_3 = 0$

    - Thus, the full set of stiffness and compliance coefficients, C_{ij} and S_{ij}, must be symmetric about the plane $x_3 = 0$; i. e., **monoclinic**

- The integral $\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} x_3 \, dx_3 = 0$ and the symmetry requirement on the reduced stiffness coefficients require that the coefficients of thermal expansion be symmetric functions about the plane $x_3 = 0$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **Recall, that for a material that is monoclinic with respect to the plane $x_3 = 0$, the general form of the constitutive equations is**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 0 \\ 0 \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right)$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \beta_{11} \\ \beta_{22} \\ \beta_{33} \\ 0 \\ 0 \\ \beta_{12} \end{Bmatrix} \left(T - T_{ref}\right)$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **In addition,**

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} \left(C_{11} - \dfrac{C_{13}C_{13}}{C_{33}}\right) & \left(C_{12} - \dfrac{C_{13}C_{23}}{C_{33}}\right) & \left(C_{16} - \dfrac{C_{13}C_{36}}{C_{33}}\right) \\ \left(C_{12} - \dfrac{C_{13}C_{23}}{C_{33}}\right) & \left(C_{22} - \dfrac{C_{23}C_{23}}{C_{33}}\right) & \left(C_{26} - \dfrac{C_{23}C_{36}}{C_{33}}\right) \\ \left(C_{16} - \dfrac{C_{13}C_{36}}{C_{33}}\right) & \left(C_{26} - \dfrac{C_{23}C_{36}}{C_{33}}\right) & \left(C_{66} - \dfrac{C_{36}C_{36}}{C_{33}}\right) \end{bmatrix}$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{21}}{E_2} & \dfrac{\eta_{1,12}}{G_{12}} \\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & \dfrac{\eta_{2,12}}{G_{12}} \\ \dfrac{\eta_{12,1}}{E_1} & \dfrac{\eta_{12,2}}{E_2} & \dfrac{1}{G_{12}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix}^{-1}$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- To facilitate the formulation of a two-dimensional boundary-value problem, the through-the-thickness functional dependence of the stress field is eliminated by introducing the following **stress resultants**

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} dx_3$$

- Substituting

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\}$$

into the expressions for the stress resultants and performing the integration yields a two-dimensional constitutive equation

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **The two-dimensional constitutive equation is given by**

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \underset{\sim}{\beta}_{11} \\ \underset{\sim}{\beta}_{22} \\ \underset{\sim}{\beta}_{12} \end{Bmatrix} (T - T_{ref})$$

where

$$\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \equiv \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} dx_3 \quad \text{and} \quad \begin{Bmatrix} \underset{\sim}{\beta}_{11} \\ \underset{\sim}{\beta}_{22} \\ \underset{\sim}{\beta}_{12} \end{Bmatrix} \equiv -\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} dx_3$$

- **The A_{ij} and $\underset{\sim}{\beta}_{ij}$ are called the inplane, plate stiffness coefficients and thermal moduli, respectively**

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- A more convenient form of the two-dimensional constitutive equation is obtained by first inverting it to obtain

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^{-1} \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} - \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^{-1} \begin{Bmatrix} \underset{\sim}{\beta}_{11} \\ \underset{\sim}{\beta}_{22} \\ \underset{\sim}{\beta}_{12} \end{Bmatrix} (T - T_{ref})$$

- Next, the inverted matrix equation is manipulated to look like the plane-stress constitutive equations

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref})$$

- That is, a set of overall plate coefficients of thermal expansion are defined to make the construction of the equations parallel

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- The overall plate **coefficients of thermal expansion** are defined as

$$\begin{Bmatrix} \tilde{\alpha}_{11} \\ \tilde{\alpha}_{22} \\ 2\tilde{\alpha}_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^{-1} \begin{Bmatrix} \underset{\sim}{\beta}_{11} \\ \underset{\sim}{\beta}_{22} \\ \underset{\sim}{\beta}_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^{-1} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} dx_3$$

so that the two-dimensional constitutive equation is given by

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \tilde{\alpha}_{11} \\ \tilde{\alpha}_{22} \\ 2\tilde{\alpha}_{12} \end{Bmatrix} (T - T_{ref}) \right\}$$

or

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^{-1} \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} + \begin{Bmatrix} \tilde{\alpha}_{11} \\ \tilde{\alpha}_{22} \\ 2\tilde{\alpha}_{12} \end{Bmatrix} (T - T_{ref})$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- For convenience, let $\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix}$ so that

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} + \begin{Bmatrix} \tilde{\alpha}_{11} \\ \tilde{\alpha}_{22} \\ 2\tilde{\alpha}_{12} \end{Bmatrix} (T - T_{ref})$$

- The subscripted a terms are **plate compliances** that are given by

$$a_{11} = \frac{(A_{22}A_{66} - A_{26}^2)}{|A|} \qquad a_{12} = \frac{(A_{16}A_{26} - A_{12}A_{66})}{|A|} \qquad a_{16} = \frac{(A_{12}A_{26} - A_{16}A_{22})}{|A|}$$

$$a_{22} = \frac{(A_{11}A_{66} - A_{16}^2)}{|A|} \qquad a_{26} = \frac{(A_{12}A_{16} - A_{11}A_{26})}{|A|} \qquad a_{66} = \frac{(A_{11}A_{22} - A_{12}^2)}{|A|}$$

where $\quad |A| = (A_{11}A_{22} - A_{12}^2)A_{66} - A_{11}A_{26}^2 - A_{22}A_{16}^2 + 2A_{12}A_{16}A_{26}$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **Once a given *boundary-value problem* is solved, the strain fields $\varepsilon_{11}(x_1,x_2)$, $\varepsilon_{22}(x_1,x_2)$, and $2\varepsilon_{12}(x_1,x_2)$ are known**

- **The stresses at any point of the body are found by substituting the strain fields and the coordinates of the given point into**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\}$$

- **The other strains are given by**

$$\begin{Bmatrix} \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{33} \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref})$$ **which reduce to**

$2\varepsilon_{23} = 2\varepsilon_{13} = 0$ **and** $\varepsilon_{33} = S_{13}\sigma_{11} + S_{23}\sigma_{22} + S_{36}\sigma_{12} + \alpha_{33}(T - T_{ref})$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **Now consider a transformation of coordinates that corresponds to a dextral rotation about the x_3 axis**

- **The stress-transformation law for a state of plane stress has been given by**

$$\{\Sigma'\} = [T_\sigma(\theta_3)]\{\Sigma\}$$ **where** $$\{\Sigma'\} \equiv \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix}, \quad \{\Sigma\} \equiv \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$ **, and**

$$[T_\sigma(\theta_3)] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **Integrating the stress transformation law over the plate thickness yields the transformation law** $\{\mathbf{N}'\} = [\mathbf{T}_\sigma(\theta_3)]\{\mathbf{N}\}$**, where**

$$\{\mathbf{N}\} \equiv \begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} dx_3 \quad \text{and} \quad \{\mathbf{N}'\} \equiv \begin{Bmatrix} N_{1'1'} \\ N_{2'2'} \\ N_{1'2'} \end{Bmatrix} \equiv \int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} dx_3$$

- **Likewise, the previously derived strain-transformation law for a state of plane stress and a dextral rotation about the x_3 axis is given by**

$$\{\mathbf{E}'\} = [\mathbf{T}_\varepsilon(\theta_3)]\{\mathbf{E}\} \quad \text{where} \quad \{\mathbf{E}'\} \equiv \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix}, \quad \{\mathbf{E}\} \equiv \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix},$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

and $$\left[\mathbf{T}_{\varepsilon}(\theta_3)\right] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -\sin\theta_3\cos\theta_3 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$; and

remains unchanged for thin, nonhomogeneous plate

- For convenience, let $[\mathbf{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}$ and $[\mathbf{Q}] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}$

such that $$[\mathbf{A}] = \int_{-\frac{h}{2}}^{+\frac{h}{2}} [\mathbf{Q}]\, dx_3$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **Similarly, let** $[A'] \equiv \begin{bmatrix} A_{1'1'} & A_{1'2'} & A_{1'6'} \\ A_{1'2'} & A_{2'2'} & A_{2'6'} \\ A_{1'6'} & A_{2'6'} & A_{6'6'} \end{bmatrix}$ **and** $[Q'] \equiv \begin{bmatrix} Q_{1'1'} & Q_{1'2'} & Q_{1'6'} \\ Q_{1'2'} & Q_{2'2'} & Q_{2'6'} \\ Q_{1'6'} & Q_{2'6'} & Q_{6'6'} \end{bmatrix}$

  such that $$[A'] = \int_{-\frac{h}{2}}^{+\frac{h}{2}} [Q']\, dx_3$$

- **The transformation laws that relate $[A]$ and $[A']$ are obtained by using the following plane-stress transformation equations**

$$[Q'] = [T_\sigma][Q][T_\sigma]^T \qquad [Q] = [T_\varepsilon]^T[Q'][T_\varepsilon]$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- Integation of $[Q'] = [T_\sigma][Q][T_\sigma]^T$ and $[Q] = [T_\varepsilon]^T[Q'][T_\varepsilon]$ over the plate thickness yields $[A'] = [T_\sigma][A][T_\sigma]^T$ and $[A] = [T_\varepsilon]^T[A'][T_\varepsilon]$

- From $[A] = \int_{-\frac{h}{2}}^{+\frac{h}{2}} [Q]\, dx_3$, it follows that $[A]^{-1} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} [Q]^{-1}\, dx_3$

- Noting that $[a] = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \equiv [A]^{-1}$ and $[S] = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \equiv [Q]^{-1}$,

  it follows that $[a] = \int_{-\frac{h}{2}}^{+\frac{h}{2}} [S]\, dx_3$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **Similarly** $[a'] = \begin{bmatrix} a_{1'1'} & a_{1'2'} & a_{1'6'} \\ a_{1'2'} & a_{2'2'} & a_{2'6'} \\ a_{1'6'} & a_{2'6'} & a_{6'6'} \end{bmatrix} \equiv [A']^{-1}$, $[S'] = \begin{bmatrix} S_{1'1'} & S_{1'2'} & S_{1'6'} \\ S_{1'2'} & S_{2'2'} & S_{2'6'} \\ S_{1'6'} & S_{2'6'} & S_{6'6'} \end{bmatrix} \equiv [Q']^{-1}$,

  and $[a'] = \int_{-\frac{h}{2}}^{+\frac{h}{2}} [S']\, dx_3$

- **The transformation laws that relate $[a]$ and $[a']$ are obtained by using the following plane-stress transformation equations**

$$[S'] = [T_\varepsilon][S][T_\varepsilon]^T \qquad [S] = [T_\sigma]^T[S'][T_\sigma]$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- Integration of $[S'] = [T_\varepsilon][S][T_\varepsilon]^T$ and $[S] = [T_\sigma]^T[S'][T_\sigma]$ over the plate thickness yields $[a'] = [T_\varepsilon][a][T_\varepsilon]^T$ and $[a] = [T_\sigma]^T[a'][T_\sigma]$

- Now consider the thermal moduli of the plate given by

$$\{\underset{\sim}{\beta}\} \equiv \begin{Bmatrix} \underset{\sim}{\beta}_{11} \\ \underset{\sim}{\beta}_{22} \\ \underset{\sim}{\beta}_{12} \end{Bmatrix} \equiv -\int_{-\frac{h}{2}}^{+\frac{h}{2}} \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} dx_3$$

- Noting that $\{\tilde{\beta}\} \equiv \begin{Bmatrix} \tilde{\beta}_{11} \\ \tilde{\beta}_{22} \\ \tilde{\beta}_{12} \end{Bmatrix} = -\begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$ for plane stress, it follows that $\{\underset{\sim}{\beta}\} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \{\tilde{\beta}\}\, dx_3$ and that $\{\underset{\sim}{\beta}'\} = \int_{-\frac{h}{2}}^{+\frac{h}{2}} \{\tilde{\beta}'\}\, dx_3$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **The transformation laws that relate $\{\underset{\sim}{\beta}\}$ and $\{\underset{\sim}{\beta}'\}$ are obtained by using the following plane-stress transformation equations**

$$\{\tilde{\beta}'\} = [\mathbf{T}_\sigma]\{\tilde{\beta}\} \qquad \{\tilde{\beta}\} = [\mathbf{T}_\varepsilon]^T\{\tilde{\beta}'\}$$

- **Integration of these equations over the plate thickness yields**

$$\{\underset{\sim}{\beta}'\} = [\mathbf{T}_\sigma]\{\underset{\sim}{\beta}\} \quad \text{and} \quad \{\underset{\sim}{\beta}\} = [\mathbf{T}_\varepsilon]^T\{\underset{\sim}{\beta}'\}$$

- **Next, recall that the overall plate coefficients of thermal expansion have been given by**

$$\{\tilde{\alpha}\} \equiv \begin{Bmatrix} \tilde{\alpha}_{11} \\ \tilde{\alpha}_{22} \\ 2\tilde{\alpha}_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\beta}_{11} \\ \underset{\sim}{\beta}_{22} \\ \underset{\sim}{\beta}_{12} \end{Bmatrix} = [a]\{\underset{\sim}{\beta}\}$$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- It follows logically that $\{\tilde{\alpha}'\} = \begin{Bmatrix} \tilde{\alpha}_{1'1'} \\ \tilde{\alpha}_{2'2'} \\ 2\tilde{\alpha}_{1'2'} \end{Bmatrix} = \begin{bmatrix} a_{1'1'} & a_{1'2'} & a_{1'6'} \\ a_{1'2'} & a_{2'2'} & a_{2'6'} \\ a_{1'6'} & a_{2'6'} & a_{6'6'} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\beta}_{1'1'} \\ \underset{\sim}{\beta}_{2'2'} \\ \underset{\sim}{\beta}_{1'2'} \end{Bmatrix} = [a']\{\underset{\sim}{\beta}'\}$

- Substituting $[a'] = [T_\varepsilon][a][T_\varepsilon]^T$ and $\{\underset{\sim}{\beta}'\} = [T_\sigma]\{\underset{\sim}{\beta}\}$ into $\{\tilde{\alpha}'\} = [a']\{\underset{\sim}{\beta}'\}$ gives $\{\tilde{\alpha}'\} = [T_\varepsilon][a][T_\varepsilon]^T[T_\sigma]\{\underset{\sim}{\beta}\}$

- Next, using $[T_\varepsilon]^T = [T_\sigma]^{-1}$ gives $\{\tilde{\alpha}'\} = [T_\varepsilon][a]\{\underset{\sim}{\beta}\}$

- Then using $\{\tilde{\alpha}\} = [a]\{\underset{\sim}{\beta}\}$ gives the result $\{\tilde{\alpha}'\} = [T_\varepsilon]\{\tilde{\alpha}\}$

- Inverting this result and using $[T_\varepsilon]^{-1} = [T_\sigma]^T$ gives $\{\tilde{\alpha}\} = [T_\sigma]^T\{\tilde{\alpha}'\}$

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONTINUED

- **In summary:**

$$[A'] = [T_\sigma][A][T_\sigma]^T \qquad [A] = [T_\varepsilon]^T[A'][T_\varepsilon]$$

$$[a'] = [T_\varepsilon][a][T_\varepsilon]^T \qquad [a] = [T_\sigma]^T[a'][T_\sigma]$$

$$\{\underset{\sim}{\beta'}\} = [T_\sigma]\{\underset{\sim}{\beta}\} \qquad \{\underset{\sim}{\beta}\} = [T_\varepsilon]^T\{\underset{\sim}{\beta'}\}$$

$$\{\tilde{\alpha}'\} = [T_\varepsilon]\{\tilde{\alpha}\} \qquad \{\tilde{\alpha}\} = [T_\sigma]^T\{\tilde{\alpha}'\}$$

- **Comparing these equations with those of the plane stress case reveals that the specific transformation equations can be obtained from those given previously for plane stress as follows**

CONSTITUTIVE EQUATIONS FOR INPLANE DEFORMATIONS OF THIN PLATES - CONCLUDED

- **For plane stress, $[Q'] = [T_\sigma][Q][T_\sigma]^T$, with $m = \cos\theta_3$ and $n = \sin\theta_3$, gave**

$$Q_{1'1'} = m^4 Q_{11} + 2m^2n^2(Q_{12} + 2Q_{66}) + 4mn(m^2Q_{16} + n^2Q_{26}) + n^4Q_{22}$$

- **Thus, by similarity, $[A'] = [T_\sigma][A][T_\sigma]^T$ gives**

$$A_{1'1'} = m^4 A_{11} + 2m^2n^2(A_{12} + 2A_{66}) + 4mn(m^2A_{16} + n^2A_{26}) + n^4A_{22}$$

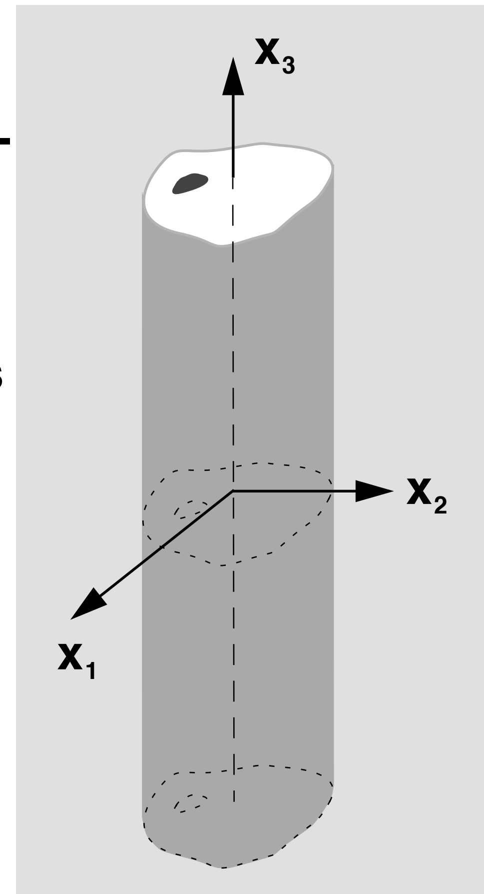
- **The other transformation equations are obtained in a similar manner**

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN

- When, analyzing solids that are ***relatively prismatic and slender***, simplifying assumptions are made about the strain state to facilitate analytical solution of practical problems

    - One such assumption is that the strains in a slender, prismatic body, that distort the cross-sectional planes, are negligible compared to the other strains

    - This simplification is commonly referred to as the **generalized plane-strain assumption**



- For a ***state of generalized plane strain*** in a homogeneous, anisotropic solid, with respect to the x_1 - x_2 plane, the strain field is approximated such that $\varepsilon_{33} - \alpha_{33}(T - T_{ref}) = \varepsilon$ and $2\varepsilon_{23} = 2\varepsilon_{13} = 0$, where ε is a constant

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **For this special case, the general constitutive equation**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\}$$

uncouples directly into

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\} + \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix} \varepsilon$$

and

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) \right\} + \begin{Bmatrix} C_{33} \\ C_{34} \\ C_{35} \end{Bmatrix} \varepsilon$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN

CONTINUED

- This equation is manipulated further by defining ***thermal moduli***

$$\begin{Bmatrix} \hat{\beta}_{11} \\ \hat{\beta}_{22} \\ \hat{\beta}_{12} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \hat{\beta}_{33} \\ \hat{\beta}_{23} \\ \hat{\beta}_{13} \end{Bmatrix} = -\begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

such that

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \hat{\beta}_{11} \\ \hat{\beta}_{22} \\ \hat{\beta}_{12} \end{Bmatrix} (T - T_{ref}) + \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix} \varepsilon \quad \text{and}$$

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ C_{14} & C_{24} & C_{46} \\ C_{15} & C_{25} & C_{56} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \hat{\beta}_{33} \\ \hat{\beta}_{23} \\ \hat{\beta}_{13} \end{Bmatrix} (T - T_{ref}) + \begin{Bmatrix} C_{33} \\ C_{34} \\ C_{35} \end{Bmatrix} \varepsilon$$

- Note that the stiffness coefficients are obtained by inverting the fully populated compliance matrix - a nontrivial task

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **Simplification of the following general constitutive equation is not as easy**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right)$$

- **First, the equation given above is expressed as**

$$\left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ \alpha_{33} \\ 2\alpha_{23} \\ 2\alpha_{13} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right) \right\} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN

CONTINUED

- Then, elimination of the **transverse-shearing strains** gives

$$\left\{\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 0\\ 0\\ 2\varepsilon_{12}\end{Bmatrix} - \begin{Bmatrix}\alpha_{11}\\ \alpha_{22}\\ \alpha_{33}\\ 0\\ 0\\ 2\alpha_{12}\end{Bmatrix}\left(T - T_{ref}\right)\right\} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16}\\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26}\\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36}\\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46}\\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56}\\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{Bmatrix}$$

- Rearranging the rows and columns into a convenient form gives

$$\left\{\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\\ \hline \varepsilon_{33}\\ 0\\ 0\end{Bmatrix} - \begin{Bmatrix}\alpha_{11}\\ \alpha_{22}\\ 2\alpha_{12}\\ \hline \alpha_{33}\\ 0\\ 0\end{Bmatrix}\left(T - T_{ref}\right)\right\} = \left[\begin{array}{ccc|ccc} S_{11} & S_{12} & S_{16} & S_{13} & S_{14} & S_{15}\\ S_{12} & S_{22} & S_{26} & S_{23} & S_{24} & S_{25}\\ S_{16} & S_{26} & S_{66} & S_{36} & S_{46} & S_{56}\\ \hline S_{13} & S_{23} & S_{36} & S_{33} & S_{34} & S_{35}\\ S_{14} & S_{24} & S_{46} & S_{34} & S_{44} & S_{45}\\ S_{15} & S_{25} & S_{56} & S_{35} & S_{45} & S_{55}\end{array}\right]\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\\ \hline \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- For convenience, let the mechanical strains be denoted by

$$\begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \\ \hline \varepsilon_{33}^{\sigma} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \\ \hline \varepsilon_{33} \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \\ \hline \alpha_{33} \\ 0 \\ 0 \end{Bmatrix} \left(T - T_{ref}\right)$$

- Using $\varepsilon_{33} - \alpha_{33}\left(T - T_{ref}\right) = \varepsilon$, the previous matrix constitutive equation becomes

$$\begin{Bmatrix} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \\ \hline \varepsilon \\ 0 \\ 0 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} S_{11} & S_{12} & S_{16} & S_{13} & S_{14} & S_{15} \\ S_{12} & S_{22} & S_{26} & S_{23} & S_{24} & S_{25} \\ S_{16} & S_{26} & S_{66} & S_{36} & S_{46} & S_{56} \\ \hline S_{13} & S_{23} & S_{36} & S_{33} & S_{34} & S_{35} \\ S_{14} & S_{24} & S_{46} & S_{34} & S_{44} & S_{45} \\ S_{15} & S_{25} & S_{56} & S_{35} & S_{45} & S_{55} \end{array}\right] \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \hline \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **Next, the matrix constitutive equation**

$$\left\{\begin{array}{c} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \\ \hline \varepsilon \\ 0 \\ 0 \end{array}\right\} = \left[\begin{array}{ccc|ccc} S_{11} & S_{12} & S_{16} & S_{13} & S_{14} & S_{15} \\ S_{12} & S_{22} & S_{26} & S_{23} & S_{24} & S_{25} \\ S_{16} & S_{26} & S_{66} & S_{36} & S_{46} & S_{56} \\ \hline S_{13} & S_{23} & S_{36} & S_{33} & S_{34} & S_{35} \\ S_{14} & S_{24} & S_{46} & S_{34} & S_{44} & S_{45} \\ S_{15} & S_{25} & S_{56} & S_{35} & S_{45} & S_{55} \end{array}\right] \left\{\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \hline \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{array}\right\}$$

is separated to get

$$\left\{\begin{array}{c} \varepsilon_{11}^{\sigma} \\ \varepsilon_{22}^{\sigma} \\ 2\varepsilon_{12}^{\sigma} \end{array}\right\} = \left[\begin{array}{ccc} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{array}\right] \left\{\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array}\right\} + \left[\begin{array}{ccc} S_{13} & S_{14} & S_{15} \\ S_{23} & S_{24} & S_{25} \\ S_{36} & S_{46} & S_{56} \end{array}\right] \left\{\begin{array}{c} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{array}\right\}$$

and

$$\left\{\begin{array}{c} \varepsilon \\ 0 \\ 0 \end{array}\right\} = \left[\begin{array}{ccc} S_{13} & S_{23} & S_{36} \\ S_{14} & S_{24} & S_{46} \\ S_{15} & S_{25} & S_{56} \end{array}\right] \left\{\begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array}\right\} + \left[\begin{array}{ccc} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{array}\right] \left\{\begin{array}{c} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{array}\right\}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **Solving the previous equation with ε for σ_{33}, σ_{23}, and σ_{13} gives**

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \varepsilon \\ 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ S_{14} & S_{24} & S_{46} \\ S_{15} & S_{25} & S_{56} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

- **Back substitution of the vector containing σ_{33}, σ_{23}, and σ_{13} into**

$$\begin{Bmatrix} \varepsilon^{\sigma}_{11} \\ \varepsilon^{\sigma}_{22} \\ 2\varepsilon^{\sigma}_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{bmatrix} S_{13} & S_{14} & S_{15} \\ S_{23} & S_{24} & S_{25} \\ S_{36} & S_{46} & S_{56} \end{bmatrix} \begin{Bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}$$

yields the result

$$\begin{Bmatrix} \varepsilon^{\sigma}_{11} \\ \varepsilon^{\sigma}_{22} \\ 2\varepsilon^{\sigma}_{12} \end{Bmatrix} = \begin{bmatrix} \mathscr{s}_{11} & \mathscr{s}_{12} & \mathscr{s}_{16} \\ \mathscr{s}_{12} & \mathscr{s}_{22} & \mathscr{s}_{26} \\ \mathscr{s}_{16} & \mathscr{s}_{26} & \mathscr{s}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} \varepsilon$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- The matrix with the subscripted $\mathscr{S}$ terms is given by

$$\begin{bmatrix} \mathscr{S}_{11} & \mathscr{S}_{12} & \mathscr{S}_{16} \\ \mathscr{S}_{12} & \mathscr{S}_{22} & \mathscr{S}_{26} \\ \mathscr{S}_{16} & \mathscr{S}_{26} & \mathscr{S}_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} - \begin{bmatrix} S_{13} & S_{14} & S_{15} \\ S_{23} & S_{24} & S_{25} \\ S_{36} & S_{46} & S_{56} \end{bmatrix} \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ S_{14} & S_{24} & S_{46} \\ S_{15} & S_{25} & S_{56} \end{bmatrix}$$

- The vector with the subscripted $\mathcal{S}$ terms is given by

$$\begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{14} & S_{15} \\ S_{23} & S_{24} & S_{25} \\ S_{36} & S_{46} & S_{56} \end{bmatrix} \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

- Next, expressing the mechanical strains in terms of the total strains and the strains caused by free thermal expansion results in

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \mathscr{S}_{11} & \mathscr{S}_{12} & \mathscr{S}_{16} \\ \mathscr{S}_{12} & \mathscr{S}_{22} & \mathscr{S}_{26} \\ \mathscr{S}_{16} & \mathscr{S}_{26} & \mathscr{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) + \begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} \varepsilon$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- Thus, the constitutive equations for generalized plane strain, in terms of compliance coefficients and thermal-expansion coefficients, become

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{16} \\ \mathcal{S}_{12} & \mathcal{S}_{22} & \mathcal{S}_{26} \\ \mathcal{S}_{16} & \mathcal{S}_{26} & \mathcal{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right) + \begin{Bmatrix} \mathscr{S}_{11} \\ \mathscr{S}_{22} \\ \mathscr{S}_{12} \end{Bmatrix} \varepsilon$$

where

$$\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{16} \\ \mathcal{S}_{12} & \mathcal{S}_{22} & \mathcal{S}_{26} \\ \mathcal{S}_{16} & \mathcal{S}_{26} & \mathcal{S}_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} - \begin{bmatrix} S_{13} & S_{14} & S_{15} \\ S_{23} & S_{24} & S_{25} \\ S_{36} & S_{46} & S_{56} \end{bmatrix} \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ S_{14} & S_{24} & S_{46} \\ S_{15} & S_{25} & S_{56} \end{bmatrix}$$

and

$$\begin{Bmatrix} \mathscr{S}_{11} \\ \mathscr{S}_{22} \\ \mathscr{S}_{12} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{14} & S_{15} \\ S_{23} & S_{24} & S_{25} \\ S_{36} & S_{46} & S_{56} \end{bmatrix} \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

- The $\mathcal{S}_{ij}$ are called the **reduced compliance coefficients**

- When $\varepsilon = 0$, the state of strain reduces to that known simply as ***plane strain***

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **Once a given generalized-plane-strain or regular-plane-strain boundary-value problem is solved by using the following equation, the stresses and strains in the following equation are known**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{16} \\ \mathcal{S}_{12} & \mathcal{S}_{22} & \mathcal{S}_{26} \\ \mathcal{S}_{16} & \mathcal{S}_{26} & \mathcal{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right) + \begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} \varepsilon$$

- **The other stresses are then given by**

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \varepsilon \\ 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ S_{14} & S_{24} & S_{46} \\ S_{15} & S_{25} & S_{56} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN

CONTINUED

- **The relationship between the reduced compliances and the stiffnesses is obtained by first considering the equation**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} - \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right) \right\} + \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix} \varepsilon$$

- **Inverting gives**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^{-1} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right) - \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^{-1} \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix} \varepsilon$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **Comparing**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^{-1} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) - \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^{-1} \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix} \varepsilon$$

with

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{16} \\ \mathcal{S}_{12} & \mathcal{S}_{22} & \mathcal{S}_{26} \\ \mathcal{S}_{16} & \mathcal{S}_{26} & \mathcal{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} (T - T_{ref}) + \begin{Bmatrix} \mathscr{S}_{11} \\ \mathscr{S}_{22} \\ \mathscr{S}_{12} \end{Bmatrix} \varepsilon$$

indicates that

$$\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{16} \\ \mathcal{S}_{12} & \mathcal{S}_{22} & \mathcal{S}_{26} \\ \mathcal{S}_{16} & \mathcal{S}_{26} & \mathcal{S}_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^{-1}$$

and

$$\begin{Bmatrix} \mathscr{S}_{11} \\ \mathscr{S}_{22} \\ \mathscr{S}_{12} \end{Bmatrix} = - \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^{-1} \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- For a material that is ***monoclinic** in the plane of the cross-section $x_3 = 0$* of the body, the *generalized-plane-strain* **thermal moduli** reduce to

$$\begin{Bmatrix} \widehat{\beta}_{11} \\ \widehat{\beta}_{22} \\ \widehat{\beta}_{12} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \widehat{\beta}_{33} \\ \widehat{\beta}_{23} \\ \widehat{\beta}_{13} \end{Bmatrix} = -\begin{bmatrix} C_{13} & C_{23} & C_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix}$$

- The *generalized-plane-strain* constitutive equations become

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \widehat{\beta}_{11} \\ \widehat{\beta}_{22} \\ \widehat{\beta}_{12} \end{Bmatrix} (T - T_{ref}) + \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix} \varepsilon \quad \text{and}$$

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{13} & C_{23} & C_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \widehat{\beta}_{33} \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) + \begin{Bmatrix} C_{33} \\ 0 \\ 0 \end{Bmatrix} \varepsilon$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **The last equation is reduced further to give** $\sigma_{23} = \sigma_{13} = 0$ **and**

$$\sigma_{33} = C_{13}\left(\varepsilon_{11} - \alpha_{11}\left(T - T_{ref}\right)\right) + C_{23}\left(\varepsilon_{22} - \alpha_{22}\left(T - T_{ref}\right)\right) + 2C_{36}\left(\varepsilon_{12} - \alpha_{12}\left(T - T_{ref}\right)\right) + C_{33}\varepsilon$$

- **Likewise, for a material that is *monoclinic in the plane of the cross-section* of the body,**

$$\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{16} \\ \mathcal{S}_{12} & \mathcal{S}_{22} & \mathcal{S}_{26} \\ \mathcal{S}_{16} & \mathcal{S}_{26} & \mathcal{S}_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} - \begin{bmatrix} S_{13} & 0 & 0 \\ S_{23} & 0 & 0 \\ S_{36} & 0 & 0 \end{bmatrix}\begin{bmatrix} S_{33} & 0 & 0 \\ 0 & S_{44} & S_{45} \\ 0 & S_{45} & S_{55} \end{bmatrix}^{-1}\begin{bmatrix} S_{13} & S_{23} & S_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{16} \\ \mathcal{S}_{12} & \mathcal{S}_{22} & \mathcal{S}_{26} \\ \mathcal{S}_{16} & \mathcal{S}_{26} & \mathcal{S}_{66} \end{bmatrix} = \begin{bmatrix} \left(S_{11} - \frac{S_{13}S_{13}}{S_{33}}\right) & \left(S_{12} - \frac{S_{13}S_{23}}{S_{33}}\right) & \left(S_{16} - \frac{S_{13}S_{36}}{S_{33}}\right) \\ \left(S_{12} - \frac{S_{13}S_{23}}{S_{33}}\right) & \left(S_{22} - \frac{S_{23}S_{23}}{S_{33}}\right) & \left(S_{26} - \frac{S_{23}S_{36}}{S_{33}}\right) \\ \left(S_{16} - \frac{S_{13}S_{36}}{S_{33}}\right) & \left(S_{26} - \frac{S_{23}S_{36}}{S_{33}}\right) & \left(S_{66} - \frac{S_{36}S_{36}}{S_{33}}\right) \end{bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **Similarly,**

$$\begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{14} & S_{15} \\ S_{23} & S_{24} & S_{25} \\ S_{36} & S_{46} & S_{56} \end{bmatrix} \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

simplifies to

$$\begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} = \begin{bmatrix} S_{13} & 0 & 0 \\ S_{23} & 0 & 0 \\ S_{36} & 0 & 0 \end{bmatrix} \begin{bmatrix} S_{33} & 0 & 0 \\ 0 & S_{44} & S_{45} \\ 0 & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

or

$$\left\{ \mathcal{S}_{11} \ \mathcal{S}_{22} \ \mathcal{S}_{12} \right\}^T = \left\{ \frac{S_{13}}{S_{33}} \ \frac{S_{23}}{S_{33}} \ \frac{S_{36}}{S_{33}} \right\}^T$$

- **The other stresses are then given by**

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} S_{33} & 0 & 0 \\ 0 & S_{44} & S_{45} \\ 0 & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \varepsilon \\ 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} S_{33} & 0 & 0 \\ 0 & S_{44} & S_{45} \\ 0 & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{bmatrix} S_{13} & S_{23} & S_{36} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

which reduce to $\sigma_{33} = \frac{1}{S_{33}}\left[\varepsilon - S_{13}\sigma_{11} - S_{23}\sigma_{22} - S_{36}\sigma_{12}\right]$ **and** $\sigma_{23} = \sigma_{13} = 0$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- For a specially orthotropic material, the ***generalized-plane-strain*** thermal moduli reduce to

$$\begin{Bmatrix} \widehat{\beta}_{11} \\ \widehat{\beta}_{22} \\ \widehat{\beta}_{12} \end{Bmatrix} = -\begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \widehat{\beta}_{33} \\ \widehat{\beta}_{23} \\ \widehat{\beta}_{13} \end{Bmatrix} = -\begin{bmatrix} C_{13} & C_{23} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{Bmatrix}$$

- The ***generalized-plane-strain*** constitutive equations become

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \widehat{\beta}_{11} \\ \widehat{\beta}_{22} \\ 0 \end{Bmatrix} (T - T_{ref}) + \begin{Bmatrix} C_{13} \\ C_{23} \\ 0 \end{Bmatrix} \varepsilon \quad \text{and}$$

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} C_{13} & C_{23} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \widehat{\beta}_{33} \\ 0 \\ 0 \end{Bmatrix} (T - T_{ref}) + \begin{Bmatrix} C_{33} \\ 0 \\ 0 \end{Bmatrix} \varepsilon$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- The last equation is reduced further to give $\sigma_{23} = \sigma_{13} = 0$ and

$$\sigma_{33} = C_{13}\left(\varepsilon_{11} - \alpha_{11}\left(T - T_{ref}\right)\right) + C_{23}\left(\varepsilon_{22} - \alpha_{22}\left(T - T_{ref}\right)\right) + C_{33}\varepsilon$$

- In terms of **engineering constants**,

$$C_{11} = \frac{E_1}{\Delta}\left(1 - \nu_{23}\,\nu_{32}\right) \qquad C_{22} = \frac{E_2}{\Delta}\left(1 - \nu_{13}\,\nu_{31}\right) \qquad C_{33} = \frac{E_3}{\Delta}\left(1 - \nu_{12}\,\nu_{21}\right)$$

$$C_{12} = \frac{E_1}{\Delta}\left(\nu_{21} + \nu_{23}\,\nu_{31}\right) = \frac{E_2}{\Delta}\left(\nu_{12} + \nu_{13}\,\nu_{32}\right) \qquad C_{66} = G_{12}$$

$$C_{13} = \frac{E_1}{\Delta}\left(\nu_{31} + \nu_{21}\,\nu_{32}\right) = \frac{E_3}{\Delta}\left(\nu_{13} + \nu_{12}\,\nu_{23}\right)$$

$$C_{23} = \frac{E_2}{\Delta}\left(\nu_{32} + \nu_{12}\,\nu_{31}\right) = \frac{E_3}{\Delta}\left(\nu_{23} + \nu_{13}\,\nu_{21}\right)$$

where $\Delta = 1 - \nu_{12}\,\nu_{21} - \nu_{23}\,\nu_{32} - \nu_{13}\,\nu_{31} - 2\nu_{21}\,\nu_{32}\,\nu_{13}$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN

CONTINUED

- The nonzero **thermal moduli** are expressed as

$$\widehat{\beta}_{11} = -\frac{E_1\left[\left(1 - \nu_{23}\,\nu_{32}\right)\alpha_{11} + \left(\nu_{21} + \nu_{23}\,\nu_{31}\right)\alpha_{22}\right]}{1 - \nu_{12}\,\nu_{21} - \nu_{23}\,\nu_{32} - \nu_{13}\,\nu_{31} - 2\nu_{21}\,\nu_{32}\,\nu_{13}}$$

$$\widehat{\beta}_{22} = -\frac{E_2\left[\left(\nu_{12} + \nu_{13}\,\nu_{32}\right)\alpha_{11} + \left(1 - \nu_{13}\,\nu_{31}\right)\alpha_{22}\right]}{1 - \nu_{12}\,\nu_{21} - \nu_{23}\,\nu_{32} - \nu_{13}\,\nu_{31} - 2\nu_{21}\,\nu_{32}\,\nu_{13}}$$

$$\widehat{\beta}_{33} = -\frac{E_3\left[\left(\nu_{31} + \nu_{21}\,\nu_{32}\right)\alpha_{11} + \left(\nu_{23} + \nu_{13}\,\nu_{21}\right)\alpha_{22}\right]}{1 - \nu_{12}\,\nu_{21} - \nu_{23}\,\nu_{32} - \nu_{13}\,\nu_{31} - 2\nu_{21}\,\nu_{32}\,\nu_{13}}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- Similarly, for a specially orthotropic material, the *generalized-plane-strain* constitutive matrix becomes

$$\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \mathcal{S}_{16} \\ \mathcal{S}_{12} & \mathcal{S}_{22} & \mathcal{S}_{26} \\ \mathcal{S}_{16} & \mathcal{S}_{26} & \mathcal{S}_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} - \begin{bmatrix} S_{13} & 0 & 0 \\ S_{23} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_{33} & 0 & 0 \\ 0 & S_{44} & 0 \\ 0 & 0 & S_{55} \end{bmatrix}^{-1} \begin{bmatrix} S_{13} & S_{23} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & 0 \\ \mathcal{S}_{12} & \mathcal{S}_{22} & 0 \\ 0 & 0 & \mathcal{S}_{66} \end{bmatrix} = \begin{bmatrix} \left(S_{11} - \dfrac{S_{13}S_{13}}{S_{33}}\right) & \left(S_{12} - \dfrac{S_{13}S_{23}}{S_{33}}\right) & 0 \\ \left(S_{12} - \dfrac{S_{13}S_{23}}{S_{33}}\right) & \left(S_{22} - \dfrac{S_{23}S_{23}}{S_{33}}\right) & 0 \\ 0 & 0 & S_{66} \end{bmatrix}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **Also**

$$\begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & 0 \\ \mathcal{S}_{12} & \mathcal{S}_{22} & 0 \\ 0 & 0 & \mathcal{S}_{66} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}^{-1}$$

- **Similarly,**

$$\begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} = \begin{bmatrix} S_{13} & S_{14} & S_{15} \\ S_{23} & S_{24} & S_{25} \\ S_{36} & S_{46} & S_{56} \end{bmatrix} \begin{bmatrix} S_{33} & S_{34} & S_{35} \\ S_{34} & S_{44} & S_{45} \\ S_{35} & S_{45} & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

simplifies to

$$\begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} = \begin{bmatrix} S_{13} & 0 & 0 \\ S_{23} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} S_{33} & 0 & 0 \\ 0 & S_{44} & 0 \\ 0 & 0 & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

or

$$\begin{Bmatrix} \mathcal{S}_{11} & \mathcal{S}_{22} & \mathcal{S}_{12} \end{Bmatrix}^{T} = \begin{Bmatrix} \dfrac{S_{13}}{S_{33}} & \dfrac{S_{23}}{S_{33}} & 0 \end{Bmatrix}^{T}$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- Finally, the ***generalized-plane-strain*** constitutive equations for a **specially orthotropic material** become

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \mathscr{S}_{11} & \mathscr{S}_{12} & 0 \\ \mathscr{S}_{12} & \mathscr{S}_{22} & 0 \\ 0 & 0 & \mathscr{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 0 \end{Bmatrix} (T - T_{ref}) + \begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ 0 \end{Bmatrix} \varepsilon$$

$$\begin{bmatrix} \mathscr{S}_{11} & \mathscr{S}_{12} & 0 \\ \mathscr{S}_{12} & \mathscr{S}_{22} & 0 \\ 0 & 0 & \mathscr{S}_{66} \end{bmatrix} = \begin{bmatrix} \left(S_{11} - \dfrac{S_{13}S_{13}}{S_{33}}\right) & \left(S_{12} - \dfrac{S_{13}S_{23}}{S_{33}}\right) & 0 \\ \left(S_{12} - \dfrac{S_{13}S_{23}}{S_{33}}\right) & \left(S_{22} - \dfrac{S_{23}S_{23}}{S_{33}}\right) & 0 \\ 0 & 0 & S_{66} \end{bmatrix}$$

$$\begin{Bmatrix} \mathcal{S}_{11} & \mathcal{S}_{22} & \mathcal{S}_{12} \end{Bmatrix}^T = \begin{Bmatrix} \dfrac{S_{13}}{S_{33}} & \dfrac{S_{23}}{S_{33}} & 0 \end{Bmatrix}^T$$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONTINUED

- **The matrix equation for the other stresses becomes**

$$\begin{Bmatrix} \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} S_{33} & 0 & 0 \\ 0 & S_{44} & 0 \\ 0 & 0 & S_{55} \end{bmatrix}^{-1} \begin{Bmatrix} \varepsilon \\ 0 \\ 0 \end{Bmatrix} - \begin{bmatrix} S_{33} & 0 & 0 \\ 0 & S_{44} & 0 \\ 0 & 0 & S_{55} \end{bmatrix}^{-1} \begin{bmatrix} S_{13} & S_{23} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

which reduces to $\sigma_{33} = \frac{1}{S_{33}}\left[\varepsilon - S_{13}\sigma_{11} - S_{23}\sigma_{22}\right]$ **and** $\sigma_{23} = \sigma_{13} = 0$

CONSTITUTIVE EQUATIONS FOR PLANE STRAIN
CONCLUDED

- In terms of the engineering constants, the **reduced compliance coefficients** and are given by

$$s_{11} = \frac{1 - \nu_{13}\nu_{31}}{E_1} \qquad s_{12} = -\frac{\left(\nu_{12} + \nu_{13}\nu_{32}\right)}{E_1} = -\frac{\left(\nu_{21} + \nu_{23}\nu_{31}\right)}{E_2}$$

$$s_{22} = \frac{1 - \nu_{23}\nu_{32}}{E_2} \qquad s_{66} = \frac{1}{G_{12}}$$

- Similarly,

$$\mathcal{S}_{11} = -\nu_{31} \qquad \mathcal{S}_{22} = -\nu_{32} \qquad \mathcal{S}_{12} = 0 \qquad \text{and}$$

$$\sigma_{33} = E_3\varepsilon + \nu_{31}\sigma_{11} + \nu_{32}\sigma_{22}$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRAIN

- **The two primary constitutive equations of generalized plane strain are given by**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{Bmatrix} \widehat{\beta}_{11} \\ \widehat{\beta}_{22} \\ \widehat{\beta}_{12} \end{Bmatrix} \left(T - T_{ref}\right) + \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix} \varepsilon \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{16} \\ s_{12} & s_{22} & s_{26} \\ s_{16} & s_{26} & s_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \left(T - T_{ref}\right) + \begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} \varepsilon$$

- **To obtain transformation equations for the constitutive terms appearing in these equations, transformation equations that relate**

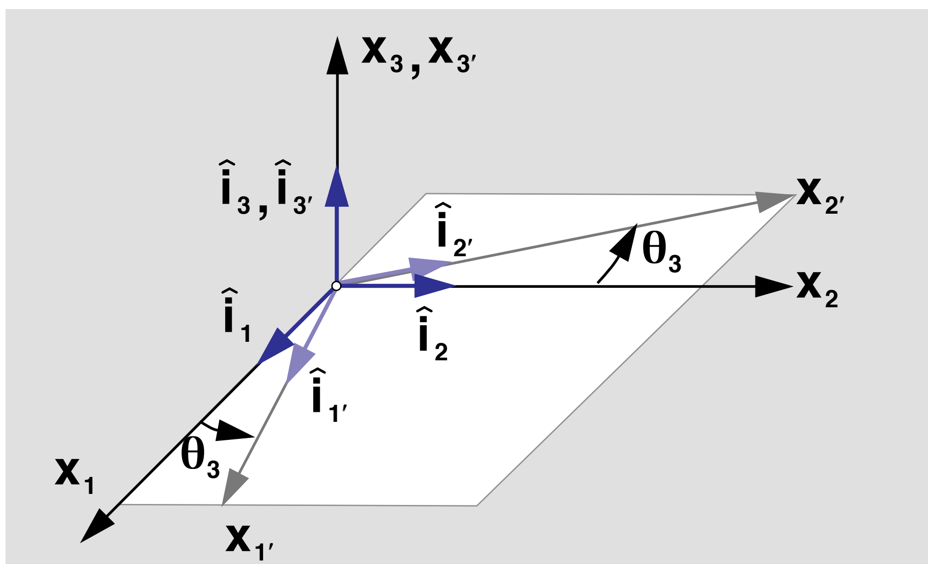
$$\{\Sigma'\} \equiv \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} \quad \text{to} \quad \{\Sigma\} \equiv \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \quad \text{are needed}$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRAIN - CONTINUED

- Likewise transformation equations that relate

$$\{\mathbf{E}'\} \equiv \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} \text{ to } \{\mathbf{E}\} \equiv \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} \text{ are needed}$$

- Consider the dextral (right-handed) rotation of coordinate frames shown in the figure



STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRAIN - CONTINUED

- Previously, the matrix form of the stress-transformation law for this specific transformation was given as

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{3'3'} \\ \sigma_{2'3'} \\ \sigma_{1'3'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -2\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

- By inspection, it follows that

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRAIN - CONTINUED

- Thus, the **stress-transformation law** for a state of generalized plane strain and a dextral rotation about the x_3 axis is identical to that for the corresponding plane-stress case and is given by

$$\{\Sigma'\} = [T_\sigma(\theta_3)]\{\Sigma\}$$ where

$$[T_\sigma(\theta_3)] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

- Likewise, the inverse is given by $\{\Sigma\} = [T_\sigma(\theta_3)]^{-1}\{\Sigma'\}$ where

$$[T_\sigma(\theta_3)]^{-1} = [T_\sigma(-\theta_3)] = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin\theta_3\cos\theta_3 & -\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRAIN - CONTINUED

- Previously, the matrix form of the strain-transformation law for this specific transformation was given as

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ \varepsilon_{3'3'} \\ 2\varepsilon_{2'3'} \\ 2\varepsilon_{1'3'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 0 & 0 & 0 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & 0 & 0 & 0 & -\sin\theta_3\cos\theta_3 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & 0 & 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & 0 & 0 & 0 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

- Inspection of this matrix equation reveals

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -\sin\theta_3\cos\theta_3 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}$$

STRESS AND STRAIN TRANSFORMATION EQUATIONS FOR PLANE STRAIN - CONCLUDED

- The last matrix equation is expressed symbolically by

$$\left\{\mathbf{E}'\right\} = \left[\mathbf{T}_{\varepsilon}(\theta_3)\right]\left\{\mathbf{E}\right\}$$ where

$$\left[\mathbf{T}_{\varepsilon}(\theta_3)\right] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -\sin\theta_3\cos\theta_3 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

- Similarly, the matrix form of the inverse is given by

$$\left\{\mathbf{E}\right\} = \left[\mathbf{T}_{\varepsilon}(\theta_3)\right]^{-1}\left\{\mathbf{E}'\right\}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRAIN

- In terms of another set of coordinates $\left(x_{1'}, x_{2'}, x_{3'}\right)$ that correspond to a dextral rotation about the x_3 axis, the constitutive equations must also have the forms given as

$$\begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} = \begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'6'} \\ C_{1'2'} & C_{2'2'} & C_{2'6'} \\ C_{1'6'} & C_{2'6'} & C_{6'6'} \end{bmatrix} \begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \widehat{\beta}_{1'1'} \\ \widehat{\beta}_{2'2'} \\ \widehat{\beta}_{1'2'} \end{Bmatrix} \left(T - T_{ref}\right) + \begin{Bmatrix} C_{1'3'} \\ C_{2'3'} \\ C_{3'6'} \end{Bmatrix} \varepsilon \quad \text{and}$$

$$\begin{Bmatrix} \varepsilon_{1'1'} \\ \varepsilon_{2'2'} \\ 2\varepsilon_{1'2'} \end{Bmatrix} = \begin{bmatrix} \mathcal{S}_{1'1'} & \mathcal{S}_{1'2'} & \mathcal{S}_{1'6'} \\ \mathcal{S}_{1'2'} & \mathcal{S}_{2'2'} & \mathcal{S}_{2'6'} \\ \mathcal{S}_{1'6'} & \mathcal{S}_{2'6'} & \mathcal{S}_{6'6'} \end{bmatrix} \begin{Bmatrix} \sigma_{1'1'} \\ \sigma_{2'2'} \\ \sigma_{1'2'} \end{Bmatrix} + \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ 2\alpha_{1'2'} \end{Bmatrix} \left(T - T_{ref}\right) + \begin{Bmatrix} S_{1'1'} \\ S_{2'2'} \\ S_{1'2'} \end{Bmatrix} \varepsilon$$

$$\begin{Bmatrix} \widehat{\beta}_{1'1'} \\ \widehat{\beta}_{2'2'} \\ \widehat{\beta}_{1'2'} \end{Bmatrix} = - \begin{bmatrix} C_{1'1'} & Q_{1'2'} & C_{1'6'} \\ C_{1'2'} & C_{2'2'} & C_{2'6'} \\ C_{1'6'} & C_{2'6'} & C_{6'6'} \end{bmatrix} \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ 2\alpha_{1'2'} \end{Bmatrix} \qquad \begin{bmatrix} \mathcal{S}_{1'1'} & \mathcal{S}_{1'2'} & \mathcal{S}_{1'6'} \\ \mathcal{S}_{1'2'} & \mathcal{S}_{2'2'} & \mathcal{S}_{2'6'} \\ \mathcal{S}_{1'6'} & \mathcal{S}_{2'6'} & \mathcal{S}_{6'6'} \end{bmatrix} = \begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'6'} \\ C_{1'2'} & C_{2'2'} & C_{2'6'} \\ C_{1'6'} & C_{2'6'} & C_{6'6'} \end{bmatrix}^{-1}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRAIN - CONTINUED

and

$$\begin{Bmatrix} \mathcal{S}_{1'1'} \\ \mathcal{S}_{2'2'} \\ \mathcal{S}_{1'2'} \end{Bmatrix} = -\begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'6'} \\ C_{1'2'} & C_{2'2'} & C_{2'6'} \\ C_{1'6'} & C_{2'6'} & C_{6'6'} \end{bmatrix}^{-1} \begin{Bmatrix} C_{1'3'} \\ C_{2'3'} \\ C_{3'6'} \end{Bmatrix}$$

- For convenience, let

$$[\mathscr{s}] = \begin{bmatrix} \mathscr{s}_{11} & \mathscr{s}_{12} & \mathscr{s}_{16} \\ \mathscr{s}_{12} & \mathscr{s}_{22} & \mathscr{s}_{26} \\ \mathscr{s}_{16} & \mathscr{s}_{26} & \mathscr{s}_{66} \end{bmatrix} \qquad [\mathrm{C}] = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \qquad \{\mathcal{C}\} = \begin{Bmatrix} C_{13} \\ C_{23} \\ C_{36} \end{Bmatrix}$$

$$\{\alpha\} = \begin{Bmatrix} \alpha_{11} \\ \alpha_{22} \\ 2\alpha_{12} \end{Bmatrix} \qquad \{\widehat{\beta}\} = \begin{Bmatrix} \widehat{\beta}_{11} \\ \widehat{\beta}_{22} \\ \widehat{\beta}_{12} \end{Bmatrix} \qquad \{\mathcal{S}\} = \begin{Bmatrix} \mathcal{S}_{11} \\ \mathcal{S}_{22} \\ \mathcal{S}_{12} \end{Bmatrix} \qquad \Theta \equiv T - T_{ref}$$

such that the corresponding constitutive equations are given by

$$\{\mathrm{E}\} = [\mathscr{s}]\{\Sigma\} + \{\alpha\}\Theta + \{\mathcal{S}\}\varepsilon$$ and $$\{\Sigma\} = [\mathrm{C}]\{\mathrm{E}\} + \{\widehat{\beta}\}\Theta + \{\mathcal{C}\}\varepsilon$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRAIN - CONTINUED

- **Additionally, let**

$$[\mathcal{s}'] = \begin{bmatrix} s_{1'1'} & s_{1'2'} & s_{1'6'} \\ s_{1'2'} & s_{2'2'} & s_{2'6'} \\ s_{1'6'} & s_{2'6'} & s_{6'6'} \end{bmatrix} \qquad [C'] = \begin{bmatrix} C_{1'1'} & C_{1'2'} & C_{1'6'} \\ C_{1'2'} & C_{2'2'} & C_{2'6'} \\ C_{1'6'} & C_{2'6'} & C_{6'6'} \end{bmatrix} \qquad \{\mathcal{C}'\} = \begin{Bmatrix} C_{1'3'} \\ C_{2'3'} \\ C_{3'6'} \end{Bmatrix}$$

$$\{\alpha'\} = \begin{Bmatrix} \alpha_{1'1'} \\ \alpha_{2'2'} \\ 2\alpha_{1'2'} \end{Bmatrix} \qquad \{\tilde{\beta}'\} = \begin{Bmatrix} \tilde{\beta}_{1'1'} \\ \tilde{\beta}_{2'2'} \\ \tilde{\beta}_{1'2'} \end{Bmatrix} \qquad \{\mathcal{S}'\} = \begin{Bmatrix} \mathcal{S}_{1'1'} \\ \mathcal{S}_{2'2'} \\ \mathcal{S}_{1'2'} \end{Bmatrix}$$

such that the corresponding constitutive equations are given by

$$\{E'\} = [\mathcal{s}']\{\Sigma'\} + \{\alpha'\}\Theta + \{\mathcal{S}'\}\varepsilon$$ **and**

$$\{\Sigma'\} = [C']\{E'\} + \{\hat{\beta}'\}\Theta + \{\mathcal{C}'\}\varepsilon$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRAIN - CONTINUED

- **Substituting** $\{\Sigma\} = [T_\sigma]^{-1}\{\Sigma'\}$ **and** $\{E\} = [T_\varepsilon]^{-1}\{E'\}$ **into** $\{\Sigma\} = [C]\{E\} + \{\hat{\beta}\}\Theta + \{C\}\varepsilon$ **gives**

$$[T_\sigma]^{-1}\{\Sigma'\} = [C][T_\varepsilon]^{-1}\{E'\} + \{\hat{\beta}\}\Theta + \{C\}\varepsilon$$

- **Premultiplying by** $[T_\sigma]$ **gives**

$$\{\Sigma'\} = [T_\sigma][C][T_\varepsilon]^{-1}\{E'\} + [T_\sigma]\{\hat{\beta}\}\Theta + [T_\sigma]\{C\}\varepsilon$$

- **Comparing this equation with** $\{\Sigma'\} = [C']\{E'\} + \{\hat{\beta}'\}\Theta + \{C'\}\varepsilon$ **it follows that**

$[C'] = [T_\sigma][C][T_\varepsilon]^{-1}$, $\{\hat{\beta}'\} = [T_\sigma]\{\hat{\beta}\}$, **and** $\{C'\} = [T_\sigma]\{C\}$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRAIN - CONTINUED

- **Rearranging** $[\mathbf{C}'] = [\mathbf{T}_\sigma][\mathbf{C}][\mathbf{T}_\varepsilon]^{-1}$ **gives** $[\mathbf{C}] = [\mathbf{T}_\sigma]^{-1}[\mathbf{C}'][\mathbf{T}_\varepsilon]$

- **Likewise,** $\{\widehat{\beta}\} = [\mathbf{T}_\sigma]^{-1}\{\widehat{\beta}'\}$ **and** $\{C\} = [\mathbf{T}_\sigma]^{-1}\{C'\}$

- **Next, substituting** $\{\Sigma\} = [\mathbf{T}_\sigma]^{-1}\{\Sigma'\}$ **and** $\{\mathbf{E}\} = [\mathbf{T}_\varepsilon]^{-1}\{\mathbf{E}'\}$ **into**

$\{\mathbf{E}\} = [\mathscr{s}]\{\Sigma\} + \{\alpha\}\Theta + \{\mathcal{S}\}\varepsilon$ **gives**

$$[\mathbf{T}_\varepsilon]^{-1}\{\mathbf{E}'\} = [\mathscr{s}][\mathbf{T}_\sigma]^{-1}\{\Sigma'\} + \{\alpha\}\Theta + \{\mathcal{S}\}\varepsilon$$

- **Premultiplying by** $[\mathbf{T}_\varepsilon]$ **gives**

$$\{\mathbf{E}'\} = [\mathbf{T}_\varepsilon][\mathscr{s}][\mathbf{T}_\sigma]^{-1}\{\Sigma'\} + [\mathbf{T}_\varepsilon]\{\alpha\}\Theta + [\mathbf{T}_\varepsilon]\{\mathcal{S}\}\varepsilon$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRAIN - CONTINUED

- **Comparing the last equation with** $\{E'\} = [\mathcal{s}']\{\Sigma'\} + \{\alpha'\}\Theta + \{\mathcal{S}'\}\varepsilon$ **reveals that**

$$[\mathcal{s}'] = [T_\varepsilon][\mathcal{s}][T_\sigma]^{-1}, \quad \{\alpha'\} = [T_\varepsilon]\{\alpha\}, \text{ and } \{\mathcal{S}'\} = [T_\varepsilon]\{\mathcal{S}\}$$

- **Rearranging** $[\mathcal{s}'] = [T_\varepsilon][\mathcal{s}][T_\sigma]^{-1}$ **gives** $[\mathcal{s}] = [T_\varepsilon]^{-1}[\mathcal{s}'][T_\sigma]$

- **In addition,** $\{\alpha\} = [T_\varepsilon]^{-1}\{\alpha'\}$ **and** $\{\mathcal{S}\} = [T_\varepsilon]^{-1}\{\mathcal{S}'\}$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRAIN - CONTINUED

- Noting that for a dextral rotation about the x_3 axis,

$$[\mathbf{T}_\varepsilon]^{-1} = [\mathbf{T}_\sigma]^{\mathsf{T}} \quad \text{and} \quad [\mathbf{T}_\sigma]^{-1} = [\mathbf{T}_\varepsilon]^{\mathsf{T}} \quad \text{it follows that}$$

$$[\mathscr{s}'] = [\mathbf{T}_\varepsilon][\mathscr{s}][\mathbf{T}_\varepsilon]^{\mathsf{T}} \qquad [\mathscr{s}] = [\mathbf{T}_\sigma]^{\mathsf{T}}[\mathscr{s}'][\mathbf{T}_\sigma]$$

$$[\mathbf{C}'] = [\mathbf{T}_\sigma][\mathbf{C}][\mathbf{T}_\sigma]^{\mathsf{T}} \qquad [\mathbf{C}] = [\mathbf{T}_\varepsilon]^{\mathsf{T}}[\mathbf{C}'][\mathbf{T}_\varepsilon]$$

$$\{\alpha'\} = [\mathbf{T}_\varepsilon]\{\alpha\} \qquad \{\alpha\} = [\mathbf{T}_\sigma]^{\mathsf{T}}\{\alpha'\}$$

$$\{\hat{\beta}'\} = [\mathbf{T}_\sigma]\{\hat{\beta}\} \qquad \{\hat{\beta}\} = [\mathbf{T}_\varepsilon]^{\mathsf{T}}\{\hat{\beta}'\}$$

$$\{C'\} = [\mathbf{T}_\sigma]\{C\} \qquad \{C\} = [\mathbf{T}_\varepsilon]^{\mathsf{T}}\{C'\}$$

$$\{\mathcal{S}'\} = [\mathbf{T}_\varepsilon]\{\mathcal{S}\} \qquad \{\mathcal{S}\} = [\mathbf{T}_\sigma]^{\mathsf{T}}\{\mathcal{S}'\}$$

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRAIN - CONTINUED

with

$$\left[\mathbf{T}_{\sigma}(\theta_3)\right] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & 2\sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -2\sin\theta_3\cos\theta_3 \\ -\sin\theta_3\cos\theta_3 & \sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

and

$$\left[\mathbf{T}_{\varepsilon}(\theta_3)\right] \equiv \begin{bmatrix} \cos^2\theta_3 & \sin^2\theta_3 & \sin\theta_3\cos\theta_3 \\ \sin^2\theta_3 & \cos^2\theta_3 & -\sin\theta_3\cos\theta_3 \\ -2\sin\theta_3\cos\theta_3 & 2\sin\theta_3\cos\theta_3 & \cos^2\theta_3 - \sin^2\theta_3 \end{bmatrix}$$

- Comparing these equations with those of the plane stress case reveals that the specific transformation equations can be obtained from those given previously for plane stress as follows

TRANSFORMED CONSTITUTIVE EQUATIONS FOR PLANE STRAIN - CONCLUDED

- For plane stress, $[Q'] = [T_\sigma][Q][T_\sigma]^T$, with $m = \cos\theta_3$ and $n = \sin\theta_3$, gave

$$Q_{1'1'} = m^4 Q_{11} + 2m^2n^2(Q_{12} + 2Q_{66}) + 4mn(m^2 Q_{16} + n^2 Q_{26}) + n^4 Q_{22}$$

- Thus, by similarity, $[C'] = [T_\sigma][C][T_\sigma]^T$ gives

$$C_{1'1'} = m^4 C_{11} + 2m^2n^2(C_{12} + 2C_{66}) + 4mn(m^2 C_{16} + n^2 C_{26}) + n^4 C_{22}$$

- The other transformation equations are obtained in a similar manner

LINES AND CURVES
OF
MATERIAL SYMMETRY

LINES AND CURVES OF MATERIAL SYMMETRY

- Up to this point of the present study, a local view of material symmetry at a specific point **P** of a material body has been examined

- The conditions for the existence of various types of material symmetries have been ascertained by using reflective-symmetry transformations based on rectangular Cartesian coordinate frames that are local to the point **P**

- For some homogeneous materials, the corresponding planes of material symmetry for every point of the material body are aligned such that it is possible to define at least one straight line whose tangent is perpendicular to each corresponding symmetry plane

    - This line is called a **principal material direction**

- When more than one distinct line connects sets of contiguous material points with identical planes of reflective symmetry, more than one principal material direction exists

LINES AND CURVES OF MATERIAL SYMMETRY
CONTINUED

- For example, a homogeneous orthotropic material possesses three principal material directions that are mutually perpendicular

- One important point illustrated herein is that the constitutive equations become simpler when a material symmetry plane exists, and that the inherent simplicity becomes hidden when the constitutive equations are expressed in terms of another rectangular Cartesian coordinate frame whose axes are oriented differently

    - For example, one coordinate frame exists for a generally orthotropic material in which the constitutive equations correspond to a specially orthotropic material

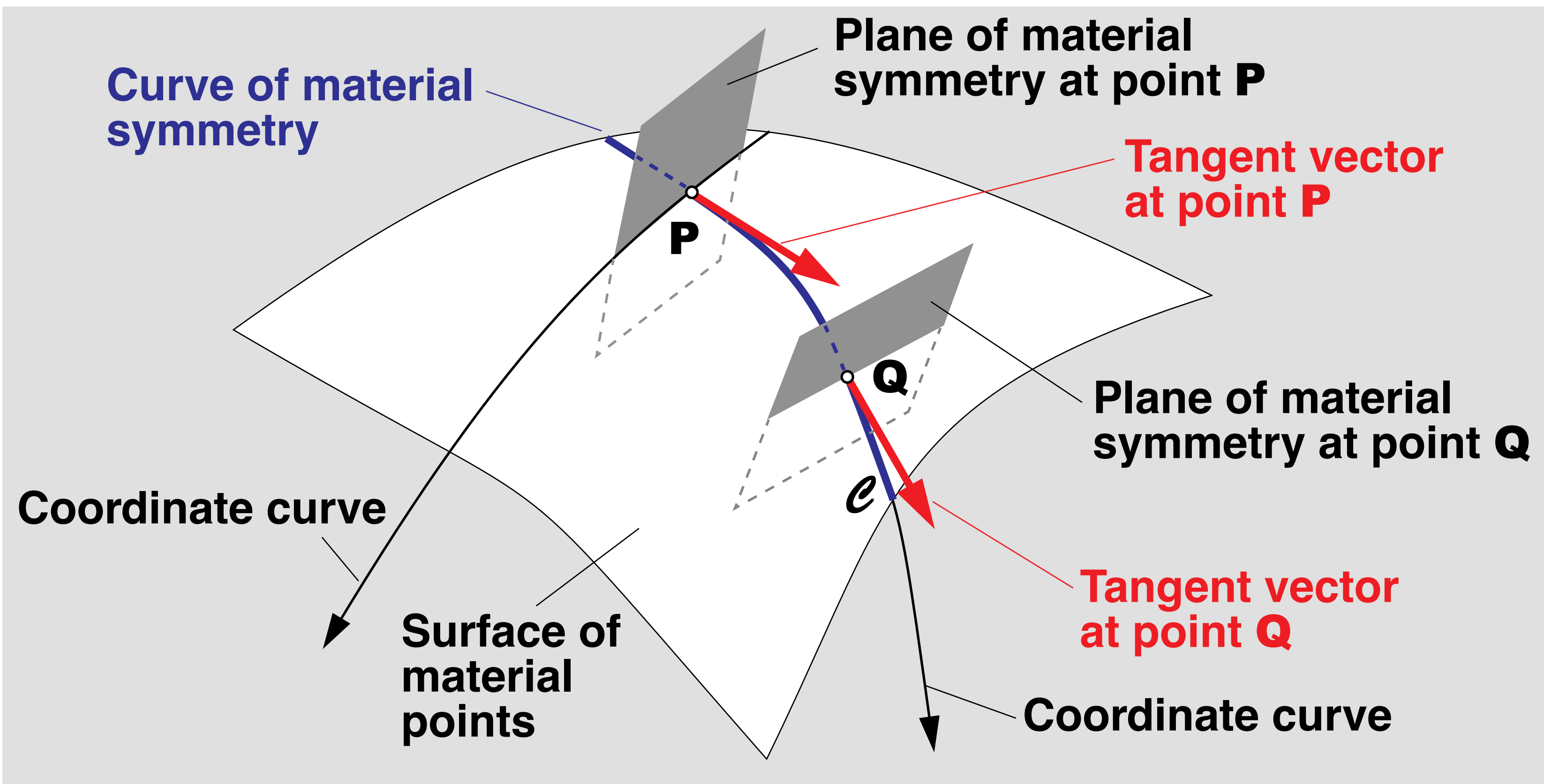
- In addition, by defining a global rectangular Cartesian coordinate frame with at least one axis parallel to a principal material direction, the simplicity of the constitutive equations can be exploited to simplify the corresponding boundary-value problem

LINES AND CURVES OF MATERIAL SYMMETRY
CONTINUED

- In a more general scenario, a contiguous set of material points may exist with a plane of elastic symmetry, at a given point, that is perpendicular to the tangent to a curve, at that point
    - For this case, the principal material direction follows a smooth curve, referred to herein as a **material symmetry curve**
- For this case, a curvilinear coordinate system can be defined in which one coordinate curve coincides with a material symmetry curve
    - For example, consider an orthotropic material in which the three perpendicular principal directions at a given point of the body coincide with radial, circumferential, and axial directions of a cylindrical coordinate system
- It is important to emphasize that the rectangular Cartesian coordinate frames used to define symmetry transformations are local frames associated with a material point and not a global coordinate frame

LINES AND CURVES OF MATERIAL SYMMETRY
CONCLUDED

- Thus, at a given point of a material symmetry curve, one should envision a local rectangular Cartesian coordinate frames upon which all symmetry conditions associated with that point are deduced



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14. ABSTRACT
An in-depth tutorial on the constitutive equations for elastic, anisotropic materials is presented. Basic concepts are introduced that are used to characterize materials, and notions about how anisotropic material deform are presented. Hooke's law and the Duhamel-Neuman law for isotropic materials are presented and discussed. Then, the most general form of Hooke's law for elastic anisotropic materials is presented and symmetry requirements are given. A similar presentation is also given for the generalized Duhamel-Neuman law for elastic, anisotropic materials that includes thermal effects. Transformation equations for stress and strains are presented and the most general form of the transformation equations for the constitutive matrices are given. Then, specialized transformation equations are presented for dextral roatations about the coordinate axes. Next, concepts of material symmetry are introduced and criteria for material symmetries are presesented. Additionally, engineering constants of fully anisotropic, elastic materials are derived from first principles and the specialized to several cases of practical importance.

15. SUBJECT TERMS
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