



Probabilistic Analysis for Comparing Fatigue Data Based on Johnson-Weibull Parameters

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Summary

Leonard Johnson published a methodology for establishing the confidence that two populations of data are different. Johnson's methodology is dependent on limited combinations of test parameters (Weibull slope, mean life ratio, and degrees of freedom) and a set of complex mathematical equations. In this report, a simplified algebraic equation for confidence numbers is derived based on the original work of Johnson. The confidence numbers calculated with this equation are compared to those obtained graphically by Johnson. Using the ratios of mean life, the resultant values of confidence numbers at the 99 percent level deviate less than 1 percent from those of Johnson. At a 90 percent confidence level, the calculated values differ between +2 and -4 percent. The simplified equation is used to rank the experimental lives of three aluminum alloys (AL 2024, AL 6061, and AL 7075), each tested at three stress levels in rotating beam fatigue, analyzed using the Johnson-Weibull method, and compared to the ASTM Standard (E739-91) method of comparison. The ASTM Standard did not statistically distinguish between AL 6061 and AL 7075. However, it is possible to rank the fatigue lives of different materials with a reasonable degree of statistical certainty based on combined confidence numbers using the Johnson-Weibull analysis. AL 2024 was found to have the longest fatigue life, followed by AL 7075, and then AL 6061. The ASTM Standard and the Johnson-Weibull analysis result in the same stress-life exponent p for each of the three aluminum alloys at the median, or L_{50} , lives.

Introduction

If a machine or structural component is properly designed and manufactured, a phenomenon that may limit its usable life is fatigue. Fatigue failure is not deterministic, but probabilistic. Within a given population, the variability in fatigue lives can be one or two orders of magnitude. This variability is often the result of material manufacture and use.

Fatigue is a failure mode that occurs at a stress level below the ultimate tensile stress, and is a direct result of cyclic loading. Under fatigue loading, cracks can develop at surface defects, finish marks, and inclusions (Refs. 1 and 2) or in

high-stress regions of a structural body that are initially free from microscopic defects. Once cracking occurs, a dominant crack usually propagates progressively to fracture (Ref. 1).

For high-cycle components such as shafts and other high-speed rotating machinery components, the concept of a fatigue limit is employed in their design. That is, the machine element is designed where the stress to which the component is subjected is below that at which fatigue will occur. However, there are problems in defining a fatigue limit for some materials because factors such as heat treatment, surface condition, residual stresses, temperature, and environment affect fatigue life. As a further complication, aluminum and other nonferrous alloys do not exhibit a fatigue limit (Ref. 1). If an aluminum component is run long enough at any stress level it will eventually fail from fatigue.

In the real world there is always a probability that a part will fail. Instead of designing a component to never fail, machine components can be designed for a "safe" life. Hence, it becomes necessary to design for finite life with an acceptable risk or probability of failure. A knowledge and understanding of a material's fatigue life, or probability of fatigue failure, becomes essential to a good engineering design.

As a result of the expense or time associated with fatigue testing, it has been practice to analyze fatigue data based on limited sample sizes—often, 10 or fewer fatigue tests are analyzed. It is essential to maximize the information gained from a limited amount of data to ensure that data distribution is factored into the analysis. Because of a limited understanding of probabilistic failure analysis, the first inclination of most practicing engineers is to determine a statistical mean or median life. The distribution of failures or scatter in the life is not necessarily taken into account. No consideration is given to the distribution in the fatigue lives. At best, error brackets based upon the standard deviation might be considered.

If the applied stresses corresponding to the fatigue failures are known, engineers often plot the data on a stress-life ($S-N$) curve and try to draw conclusions by comparing these curves. This would be considered as the next level in complexity of a fatigue life analysis. In general, stress-life data are analyzed according to the ASTM E739-91, "Standard Practice for Statistical Analysis of Linear or Linearized Stress-Life ($S-N$) and Strain-Life ($\epsilon-N$) Fatigue Data" (Ref. 3). Unfortunately, the

standard does not provide a method to rank the confidence that one $S-N$ curve is superior to another. At best, the standard can be used to define 95-percent confidence bands. This pair of bands brackets the linearized curve-fit and establishes limits which 95 percent of the test data would fall. The only fact that can be inferred is whether or not competing 95-percent confidence bands bracket data sets with or without overlap. If there is no overlap, it is reasonable to infer that the data sets are different from each other at a 95-percent level of confidence.

The authors of the ASTM standard (Ref. 3) recognized the inherent limitations of the standard when stating that, “As alternate fatigue models and statistical analysis are continually being developed, later revisions of this practice may subsequently present analyses that permit more complete interpretation of $S-N$ and $\epsilon-N$ data.” In fact, the ASTM standard refers to the Weibull distribution function as a possible alternative statistical analysis for such data. There are many examples in the literature of the use of the Weibull distribution function to determine the life and strength of materials, structural components, and systems/machines (Refs. 4 to 9). It is worth noting that Weibull analysis has also been used to evaluate preventative maintenance practices (Refs. 10 and 11).

Leonard G. Johnson (Ref. 12), while with the GM Research Center in the 1950s and 1960s, is credited with the first practical engineering analysis technique for evaluating fatigue data based on the Weibull distribution function. Although Johnson developed a practical method for establishing confidence that two or more populations are different, it requires difficult interpolation between or extrapolation beyond his published plots. Additionally, the mathematical technique Johnson presents is somewhat difficult to follow for someone not familiar with statistics or gamma function solutions. Finally, Johnson’s work is documented in the 1964 text (Ref. 12) that is somewhat rare and difficult to find. As a result of these factors, his technique of ranking fatigue data populations based upon confidence numbers has fallen out of widespread use.

In view of the aforementioned, the objectives of the work reported herein are to (1) develop a simplified relationship to calculate confidence numbers independent of interpolation of Johnson’s graphical plots, (2) develop a simplified relation to calculate confidence numbers beyond those combinations of parameters used by Johnson (Ref. 12), and (3) apply these relations to rank the experimental lives of three aluminum alloys (AL 2024, AL 6061, and AL 7075), each tested at three stress levels in rotating beam fatigue.

Nomenclature

A_0	parameter defined by Equation (3b)
B_0	parameter defined by Equation (3c)
C	confidence number
D	parameter defined by Equation (5a)
DOF	total degrees of freedom

L	life, number of stress cycles or hr
L_{10}	10 percent life or life at which 90 percent of a population survives, number of stress cycles or hr
L_{50}	50 percent life or life at which 50 percent of a population survives, number of stress cycles of hr
L_x	life at which $(1-S)$ samples have failed
L_β	characteristic life or life at which 63.2 percent of population fails, number of stress cycles or hr
m	Weibull slope
MLR	mean life ratio
$MLR_{@99\text{percent}}$	mean life ratio at a confidence number of 99 percent
MLR_{exp}	experimental mean life ratio
n	number of data points in a data set
N	life, number of stress cycles
p	stress-life exponent
S	probability of survival, fraction or percent
Y_1	correction parameter
ϵ	strain
σ	bending stress, MPa (ksi)

Probabilistic Analysis

Weibull Distribution Function

Weibull (Refs. 13 to 15) is credited with being the first to suggest a reasonable way to estimate fracture life with a statistical distribution function. The probability distribution function empirically identified by Weibull in 1939 is as follows:

$$\ln \ln \frac{1}{S} = m \ln \left(\frac{L_x}{L_\beta} \right); \quad 0 < L_x < \infty; 0 < S < 1 \quad (1)$$

This form is referred to as the two-parameter Weibull distribution function. Weibull empirically determined this function by fitting curves to fracture data. The variable S is the level of survivability (probability of survival) being considered.

For example, if 15 percent of the samples have failed, then the survivability would be 0.85. L_x is the life in cycles or hours at which $1-S$ percent of samples have failed. In the case of S equaling 90 percent, L_x is the life at which 10 percent of the samples have failed—this is typically referred to as the L_{10} life. L_β is the characteristic life of the material, which is defined as the life at which 63.2 percent of the samples have failed. Finally, m is the Weibull parameter or slope, which is an indicator of the scatter or distribution in the data—the larger the number the smaller the amount of scatter. A Weibull slope of 1.0 is indicative of an exponential distribution, 2.0 is Rayleigh distribution, and 3.57 is a normal, or Gaussian, distribution of the data.

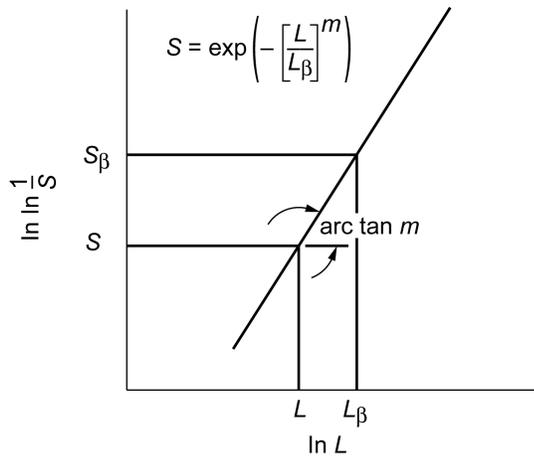


Figure 1.—Weibull plot, where m is the Weibull slope or tangent of line; S , probability of survival; L , life; L_β , life at which 63.2 percent of population fails; and S_β , probability of survival of 36.8 percent at which $L = L_\beta$, or $L/L_\beta = 1$.

Figure 1 is a generic Weibull plot. When plotting the $\ln \ln [1/S]$ as the ordinate against the $\ln L$ as the abscissa, fatigue data are assumed to plot as a straight line. The ordinate $\ln \ln [1/S]$ is graduated in statistical percent of components failed or removed for cause as a function of $\ln L$, the natural logarithm of the time or cycles to failure. The tangent of the line is designated the Weibull parameter or slope m , which is indicative of the shape of the cumulative distribution, or the amount of scatter of the data previously discussed.

Johnson (Ref. 12) provided the first practical engineering method for data analysis and projections using the Weibull distribution function. Effectively, a distribution of a small number of fatigue lives is ranked, plotted, and fitted with a straight line. From this Weibull plot, the life at any probability of survival can be determined.

The relative ranking of lives of competing materials can be ranked with a basic comparison of L_{10} lives determined with Weibull techniques (Ref. 16). Johnson (Ref. 12) carried this concept further, by establishing confidence numbers that established the probability of whether two data sets being compared were different. These confidence numbers were dependent upon knowing the Weibull slope, total degrees of freedom, and either a mean life or L_{10} life ratio.

Confidence Numbers

Johnson (Ref. 12) developed a method based on Weibull parameters for establishing the probability, which he termed “confidence number,” that two values being compared were statistically different from each other. He defined the confidence number as being “the probability that the true mean life

ratio of the populations is greater than unity.” In other words, the reasonable statistical certainty that population A is better than population B. The confidence number is calculated from a set of complex mathematical equations derived by Johnson and is a function of the degrees of freedom of the two data sets being compared, the ratio of the lives of the two data sets, and the Weibull slope m . Confidence numbers greater than 90 percent are considered statistically significant. This means that if the tests were to be repeated 100 times, 90 out of 100 times population A would be better than population B.

Johnson (Ref. 12) published 10 graphs based on his equations for Weibull slopes of 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 2.5, 3.0, 3.5, and 4.0 in order to graphically determine confidence numbers. Each graph incorporates total degrees of freedom of 12, 24, 48, 96, 400, and 3000 and mean life ratios from 1.0 to at most 3.0.

Figure 2, reproduced from Johnson (Ref. 12) for Weibull slope $m = 1$, is representative of the 10 confidence number graphs. To determine the confidence number, the appropriate graph is selected based upon the Weibull slope. At the appropriate mean life ratio, the intersection with the linear curve corresponding to the desired total degrees of freedom corresponds to the confidence number.

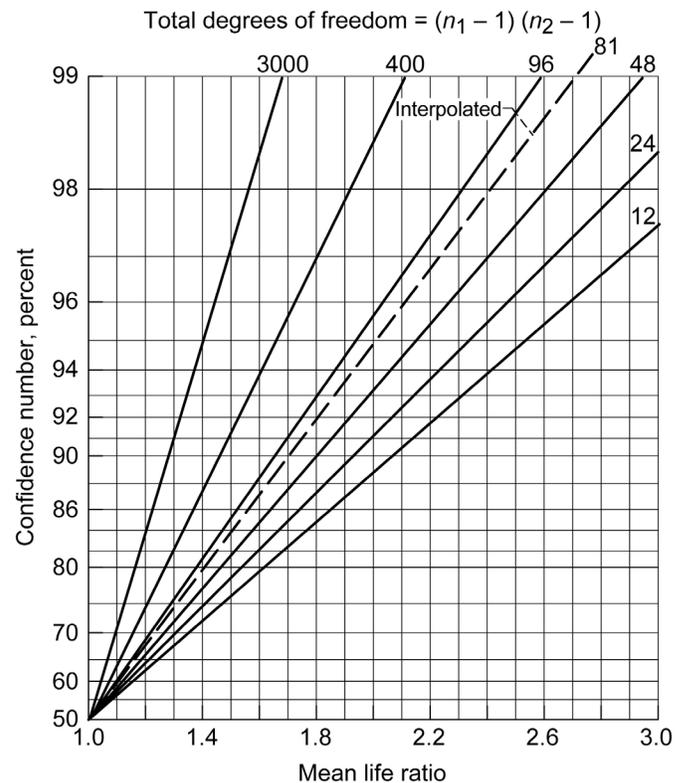


Figure 2.—Confidence numbers from Johnson (Ref. 12) as a function of mean life ratio, total degrees of freedom, and Weibull slope, m ($m = 1$).

Total Degrees of Freedom

The number of degrees of freedom is considered to be the number of parameters that may be independently varied. That is, the degree of freedom is an expression of the number of options available. For a population with a fixed mean containing n_i failures, $(n_i - 1)$ members can be chosen arbitrarily (Ref. 12). Once $(n_i - 1)$ members have been selected, there is no choice of the final selection since only one member remains. Thus for this case the number of degrees of freedom is said to $(n_i - 1)$.

When trying to rank or compare two populations, Johnson (Ref. 12) identifies the total degrees of freedom DOF as

$$DOF = (n_1 - 1)(n_2 - 1) \quad (2)$$

where n_1 and n_2 are the total numbers of fatigue lives in the two data sets being compared. For example, if two data sets being compared have 10 data points each, the DOF will be 81.

Looking at each of Johnson's graphs, the limitations of the select cases for which the Johnson graphs are applicable become apparent. For example, intersection with a linear curve with a total degree of freedom of 81 requires interpolation between 48 and 96. This interpolation for 81 degrees of freedom is shown in Figure 2. Weibull slopes that either fall between or above those for which graphs exist require interpolation between or extrapolation beyond the range covered by Johnson's graphs. Alternately, use of Johnson's mathematical solution for confidence numbers requires an understanding of Johnson's mathematical methodology. Additionally, the mathematical technique Johnson presents is somewhat difficult to follow for someone not familiar with statistics or gamma function solutions.

Mean Life Ratio

In addition to being a function of DOF and Weibull slope, use of Johnson's plots require knowledge of the mean life ratio (MLR) if the confidence number is to be determined. If the entire data set comprises fatigue failures (i.e., there are no suspended tests), then the mean life and average life are the same. Since Weibull distributions are generally skewed, the percentage failed or location of the mean typically differs from the 50 percent value (unless the Weibull slope is 3.57; i.e., a normal distribution), and decreases with increasing Weibull slope. For example, at a Weibull slope of 1.0 the mean life occurs at 63.2 percent probability of failure (36.8 percent probability of survival). For a Weibull slope of 2.0 the mean life is found at 54.4 percent probability of failure (45.6 percent probability of survival). Thus, care should be given since mean life and average life cannot be used interchangeably in most cases. The mean life must be determined from an appropriate Weibull analysis. When comparing two data sets, the MLR is the larger of the two mean lives, divided by the smaller.

Determining Mean Life Ratios at a Confidence Number of 99 Percent

Because the limitations for the select cases for which the Johnson graphs are applicable have been established, it is desirable to have a simplified relationship for confidence number that is a function of the Weibull parameters and DOF . To do this, Johnson's confidence number graphs were scaled and a single mathematical equation that reproduced Johnson's linear plots was derived.

First, Johnson's 10 confidence number graphs (Ref. 12) were enlarged. The MLR corresponding to a confidence number of 0.99 was read from each figure for each of the six DOF . By empirically optimizing linear parameters to minimize the percent difference between the calculated and graphically determined MLR at 99 percent ($MLR_{@99\text{percent}}$), Equation (3a) was derived to provide a calculated value of $MLR_{@99\text{percent}}$:

$$MLR_{@99\text{percent}} = [A_0 \ln(DOF) + B_0]^2 + 1 \quad (3a)$$

where

$$A_0 = \frac{-0.0844}{m} - 0.05584 \quad (3b)$$

and

$$B_0 = \frac{1.2796}{m} + 0.6729 \quad (3c)$$

where m is the Weibull slope, and Equations (3b) and (3c) are functions of m and used in Equation (3a). Equation (3a) fits Johnson's graphs for $1 \leq m \leq 4$ values with a deviation in MLR of at most 6.5 percent with the potential for extrapolation. This calculated value of $MLR_{@99\text{percent}}$ will be used in the following section to determine the confidence number—the value of ultimate interest. While B_0 is linear in $1/m$, A_0 deviates somewhat from linearity in the range $2 < m < 3$ and is possibly better fitted as a cubic function, but then extrapolation would become highly questionable.

The intercept values of the DOF locus at a confidence number of 99 percent were used to determine a calculated MLR to within 6.5 percent (Eq. (3a)). For those interpolating values within the range of Johnson's figures and needing $MLR_{@99\text{percent}}$ to a deviation within 2.1 percent, Equation (3a) can be corrected as follows:

$$MLR_{@99\text{percent, corrected}} = \frac{MLR_{@99\text{percent, from Eq. (3a)}}}{1 - Y_1} \quad (4a)$$

where

$$Y_1 = \frac{0.5588 \left[\ln(DOF)^2 - 5.8506 \ln(DOF) + 12.923 \right]}{100} \quad (4b)$$

Equation (4b) is used to calculate a constant used in Equation (4a). Equation (4a) results in a smaller percent difference between the calculated and graphically determined *MLR*, but should not be used to extrapolate values beyond the combination of Weibull parameters used by Johnson.

It is not known whether Johnson limited his design curves to the 10 slopes from 1.0 to 4.0 and 6 degrees of freedom from 12 to 3000 for technical or practical reasons. Nevertheless, Johnson does not place any limits on Weibull slope or *DOF* when using his mathematical methodology to determine a confidence number, and there are ample examples throughout his text where he interpolates values from his figures.

Simplified Expression for Confidence Numbers at Any Reasonable *DOF* and Weibull Slope

Ultimately what is desired is a simplified equation that provides the confidence number C_i at any reasonable combination of Weibull slope, *DOF*, and *MLR*. Since all of the lines of a Johnson plot pass through a point at $MLR = 1.0$ and $C_i = 0.5$, the confidence number plots in Johnson (Ref. 12) can be recreated (Fig. 2) by drawing lines through this common point and points at the intersection of the 99 percent confidence number and *MLR* calculated using Equation (3a). These lines were fitted with an equation that minimized the deviation from the Johnson lines, and Equation (5a) is the resulting equation:

$$C_i = 1 - 0.5 \exp[-D(MLR_{\text{exp}} - 1)] \quad (5a)$$

where MLR_{exp} equals the experimental *MLR*. The variable D is given by Equation (5b), where

$$D = \frac{\ln\left[\frac{0.5}{1 - 0.99}\right]}{(MLR_{@99\text{percent}} - 1)} \quad (5b)$$

$$= \frac{3.912}{(MLR_{@99\text{percent}} - 1)}$$

where $MLR_{@99\text{percent}}$ is found using Equation (3a) (or Eq. (4a) if trying to interpolate with greater accuracy). Equation (5b) is the calculation of a constant used in Equation (5a).

Equation (5a) was used to calculate the confidence for a mean life ratio's *DOF* and Weibull slopes corresponding to confidence numbers of 99 percent as read from Johnson's 10 figures. The percent deviation of the calculated confidence number from that graphically determined is reported in Tables I through III.

The calculated values of confidence numbers in Table I differ from the graphically determined value of 0.99 by less than 1.0 percent. At a 95 percent (0.95) confidence number (Table II), the calculated value differs between +1.5 and -2.0 percent from the graphically determined value. At a 90 percent (0.90) confidence number (Table III), the calculated value differs

TABLE I.—PERCENT DIFFERENCE BETWEEN THE CALCULATED CONFIDENCE NUMBER AND 0.99 CONFIDENCE NUMBER (REF. 12) FOR 10 WEIBULL SLOPES AND 6 DEGREES OF FREEDOM

Weibull slope	Degrees of freedom					
	12	24	48	96	400	3000
4.0	0.22	0.42	0.54	0.81	0.82	0.06
3.5	0.03	0.25	0.46	0.59	0.52	-0.03
3.0	-0.21	0.02	0.14	0.39	0.48	0.05
2.5	-0.41	-0.22	-0.04	0.08	0.32	-0.03
2.0	-0.09	-0.10	-0.12	-0.06	-0.10	-0.64
1.8	0.04	-0.04	-0.14	0.03	-0.06	-0.56
1.6	-0.25	-0.14	0.03	0.02	-0.02	-0.65
1.4	-0.24	-0.09	-0.05	0.31	0.18	-0.36
1.2	-0.03	0.07	0.06	0.35	0.31	-0.06
1.0	-0.04	0.09	0.10	0.35	0.55	0.05

TABLE II.—PERCENT DIFFERENCE BETWEEN THE CALCULATED CONFIDENCE NUMBER AND 0.95 CONFIDENCE NUMBER (REF. 12) FOR 10 WEIBULL SLOPES AND 6 DEGREES OF FREEDOM

Weibull slope	Degrees of freedom					
	12	24	48	96	400	3000
4.0	-0.24	0.13	0.54	0.71	0.49	-1.53
3.5	-0.80	-0.26	-0.09	0.38	0.36	-1.41
3.0	-1.40	-0.99	-0.71	-0.32	-0.32	-1.66
2.5	-1.89	-1.39	-0.98	-0.54	-0.28	-1.48
2.0	-0.55	-0.40	-0.50	-0.25	-0.39	-2.09
1.8	-0.01	-0.22	-0.55	0.09	-0.09	-2.04
1.6	-0.95	-0.49	-0.15	-0.17	-0.07	-2.37
1.4	-0.77	-0.36	-0.21	0.63	0.33	-1.10
1.2	-0.09	0.11	0.08	0.94	0.78	-0.31
1.0	-0.12	0.01	0.15	0.93	1.29	0.16

TABLE III.—PERCENT DIFFERENCE BETWEEN THE CALCULATED CONFIDENCE NUMBER AND 0.90 CONFIDENCE NUMBER (REF. 12) FOR 10 WEIBULL SLOPES AND 6 DEGREES OF FREEDOM

Weibull slope	Degrees of freedom					
	12	24	48	96	400	3000
4.0	-1.31	-0.80	-0.47	-0.27	-0.69	-4.08
3.5	-1.95	-1.13	-0.89	-0.59	-0.44	-3.38
3.0	-2.37	-1.98	-1.87	-1.13	-1.56	-3.87
2.5	-3.55	-2.87	-2.28	-1.87	-1.65	-3.36
2.0	-1.01	-0.62	-0.81	-0.57	-0.37	-3.34
1.8	-0.04	-0.46	-0.82	-0.03	-0.23	-3.30
1.6	-1.53	-0.88	-0.36	-0.10	-0.25	-4.06
1.4	-1.25	-0.57	-0.42	0.70	0.09	-2.03
1.2	-0.19	0.17	0.24	1.25	1.32	-0.27
1.0	-0.19	0.08	0.16	1.30	1.66	-0.11

between +2.0 and -4.0 percent from the graphically determined value. Thus, confidence numbers can be determined within acceptable variances using Equation (5a) for any reasonable values of *DOF* and Weibull slope.

Combined or Composite Confidence Numbers

When more than two related data sets are available for comparison, a composite confidence number can be calculated that incorporates significantly more data in its calculation using the following equation from Johnson (Ref. 12):

$$C = 1 - 2^{n-1} \prod_{i=1}^n (1 - C_i) \quad (6)$$

For example, if two *S-N* curves are being compared where the fatigue life (number of cycles) is known for each of the materials at seven stresses, Equation (5a) can be used to determine the confidence number that the two lives were different at each of the seven stresses. Rather than try to interpret the significance of each of the seven confidence number values separately, each of the individual confidence numbers (C_i) at each of the individual stress levels are incorporated into the combined confidence number using Equation (6) from Johnson (Ref. 12). For this example, $n = 7$ in Equation (6).

Apparatus, Specimens, and Experimental Procedure

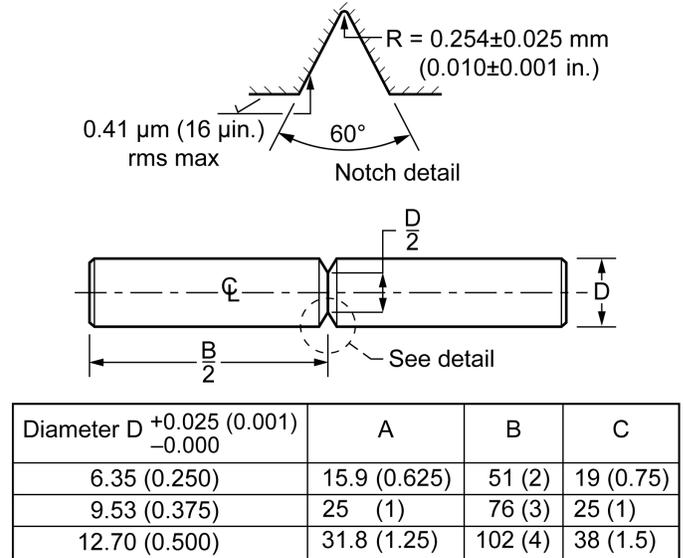
Experimental fatigue lives of three rotating-beam aluminum alloys (AL 2024, AL 6061, and AL 7075), were determined experimentally and initially reported in Vlcek et al. (Ref. 17). The relative fatigue lives of the three aluminum alloys are ranked herein using the simplified expression (Eq. (5a)) to determine Johnson's confidence numbers.

Fatigue lives of notched, rotating 12.7-mm- (0.5-in.-) diameter by 101-mm- (4-in.-) long aluminum shafts (Fig. 3), were determined using a commercial rotational fatigue tester. This test is commonly referred to as the Krause, Budd, or Vishay fatigue test. Figure 4(a) is a sketch of the primary components of the rotational fatigue tester.

The notches in the test sample were cut on a lathe using a tungsten carbide tool insert. No surface finish or polish was performed on the cut surface. The notches were located midway (50.8 mm, 2 in.) along the length of the test sample (see Fig. 3). The reduced diameters of the shafts at the notches were 6.35 mm (0.25 in.), and the walls of the notch were at a 60° angle.

For each test, one end of the test sample was mounted in a 12.7-mm- (0.5-in.-) diameter rotating collet. The test sample extended from the collet like a horizontal cantilever (Fig. 4 (b)). The free end was then mounted in a collet that was part of a symmetric, tapered shaft extension. The collet was rotated by a variable-speed electric motor.

A bearing from which a constant load was suspended was placed on the free end of the tapered shaft. As a result, a constant, sinusoidal load that is totally reversing each rotation of the shaft was applied to the free end of the rotating shaft. By



Dimensions are in mm (in.)

Figure 3.—12.7-mm (0.5-in.) aluminum alloy test sample used in rotating fatigue tester. A is the collet grip length and C is the exposed sample length ($C = B - 2A$).

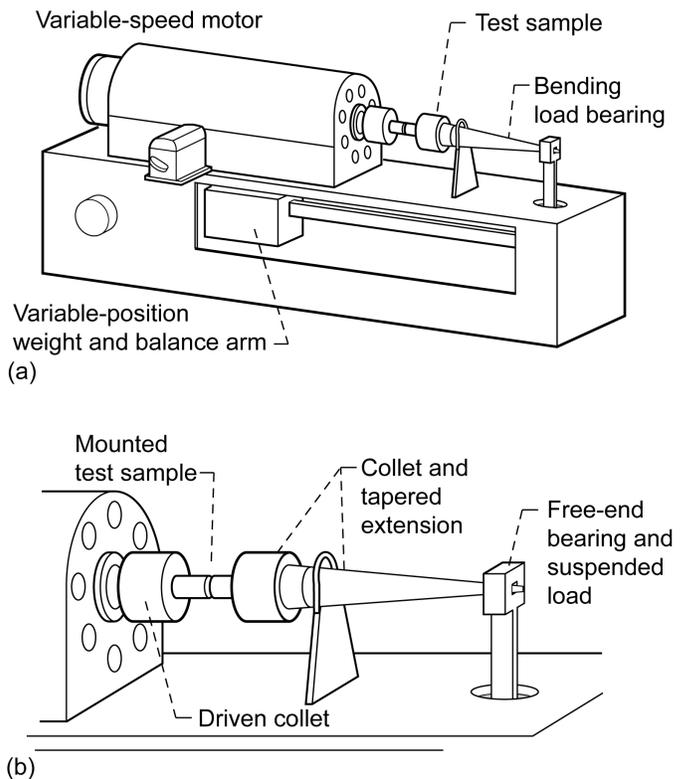


Figure 4.—Rotational bending fatigue tester. (a) Primary components. (b) Closeup of test section.

sliding a weight along a balance arm on the apparatus, the bending stress at the minimum sample cross section can be varied from 0 to 900 MPa (130.3 ksi). For this axially symmetrically loaded specimen, a fatigue crack can initiate at any point around the periphery of the shaft.

All tests were performed at room temperature and 7500 rpm. Each material was tested at three different bending stresses: 157.2, 180, and 202 MPa (22.8, 26.1, and 29.3 ksi). Ten tests were conducted to failure at each stress level.

Test Materials

Three aluminum alloys, AL 2024, AL 6061, and AL 7075, were used in this study. Without alloying elements, aluminum is too soft or ductile for most engineering applications (Ref. 18).

Aluminum 2024 is used widely in the aircraft industry. At 3.8 to 4.9 percent, copper is the principal alloying element in this material. The maximum yield stress of 324 MPa (47 ksi) and Brinell hardness of 120 were reported by the material supplier for AL 2024. It is worth noting that unprotected, this alloy does not have as good corrosion resistance as other aluminum alloys.

Aluminum 6061 is among the series of alloys that are heat treatable because of the presence of magnesium silicide. The supplier provided that the yield stress of the AL 6061 alloy was 276 MPa (40 ksi) and the Brinell hardness was 95. While considered not as strong as other aluminum alloys, AL 6061 is one of the most versatile and widely used of the aluminum alloys because of good formability and corrosion resistance.

The major alloying element in AL 7075 is zinc at 5.1 to 6.1 percent. Magnesium, chromium, and copper are three other alloying elements found in AL 7075. With a yield stress of 503 MPa (73 ksi) and a Brinell hardness of 150, AL 7075 is considered to be among the highest strength aluminum alloys.

It is used widely in the fabrication of highly stressed components, and it is a material of choice for airframes.

Results and Discussions

Rotating beam fatigue tests were carried out at three stress levels 157.2, 180, and 202 MPa (22.8, 26.1, and 29.3 ksi) on three aluminum shaft alloys—AL 2024, AL 6061, and AL 7075 (Ref. 17)—and are presented in Table IV. These test results are analyzed and discussed herein.

Test Results

Figure 5 shows stress-life ($S-N$) plots of the data of Table IV, plotted according to ASTM Standard E739–91 (Ref. 3). There is a lot of scatter in the data at each of the three stress levels. From curve fits of the life data shown in Figure 5, the median life L_{50} is determined at the various stress levels. These results are summarized in Table V. From these plots, the life L is inversely proportional to the stress σ raised to the stress-life exponent p , where

$$L \sim \frac{1}{\sigma^p} \quad (7)$$

The values for stress-life exponent p are shown in Figure 5(d) and summarized in Table VI.

From Figure 5, AL 2024 again has the longest fatigue life of the three alloys followed by AL 7075 and AL 6061. However, computing the 95-percent confidence bands from the ASTM standard suggests that there is no statistical significance between the lives of AL 7075 and AL 6061.

TABLE IV.—NUMBER OF STRESS CYCLES TO FAILURE AND STATISTICAL RESULTS OF 10 SAMPLES OF THREE ALUMINUM ALLOYS AT THREE BENDING STRESSES

Sample number	AL 2024			AL 6061			AL 7075		
	157.2 MPa (22.8 ksi)	180.0 MPa (26.1 ksi)	202.0 MPa (29.3 ksi)	157.2 MPa (22.8 ksi)	180.0 MPa (26.1 ksi)	202.0 MPa (29.3 ksi)	157.2 MPa (22.8 ksi)	180.0 MPa (26.1 ksi)	202.0 MPa (29.3 ksi)
1	142 000	219 700	78 300	38 900	16 700	7 200	38 800	46 100	21 700
2	180 800	207 700	144 400	46 100	20 900	14 800	104 800	39 800	21 900
3	273 200	146 600	113 300	57 000	23 500	20 800	51 600	40 300	17 300
4	188 000	141 300	152 600	36 400	17 500	10 400	43 100	52 300	24 200
5	226 400	190 500	135 100	40 200	13 800	8 600	64 700	52 600	24 200
6	224 000	116 400	59 800	80 700	28 800	9 300	53 700	42 500	12 200
7	220 400	205 700	86 200	123 300	23 500	5 600	94 100	27 400	11 200
8	204 500	157 300	75 200	43 100	27 100	8 800	100 900	16 800	10 200
9	226 400	215 500	68 500	54 400	27 600	9 600	47 300	28 100	10 300
10	203 600	180 800	61 600	51 400	32 700	7 600	76 900	37 300	10 500
Average	208 930	178 150	97 500	57 150	23 210	10 270	67 590	38 320	16 370
Standard deviation	34 732	35 768	35 658	26 545	6 007	4 424	24 902	11 408	6 110
Maximum	273 200	219 700	152 600	123 300	32 700	20 800	104 800	52 600	24 200
Minimum	14 200	116 400	59 800	36 400	13 800	5 600	38 800	16 800	10 200

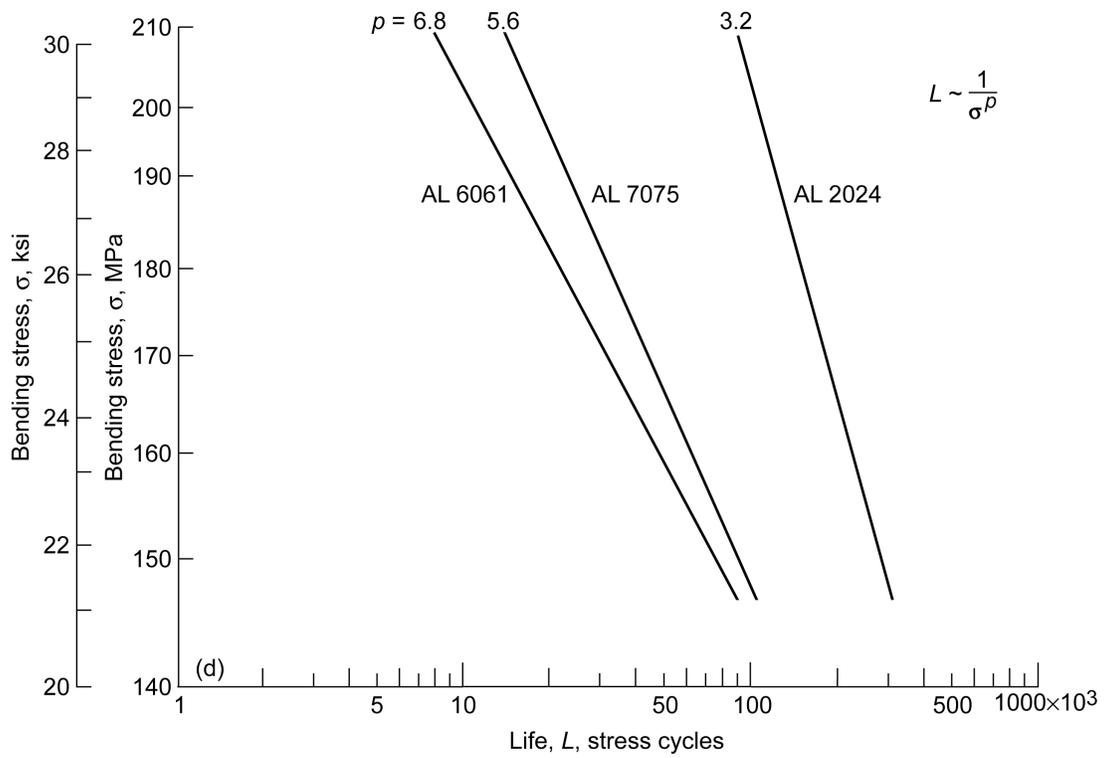
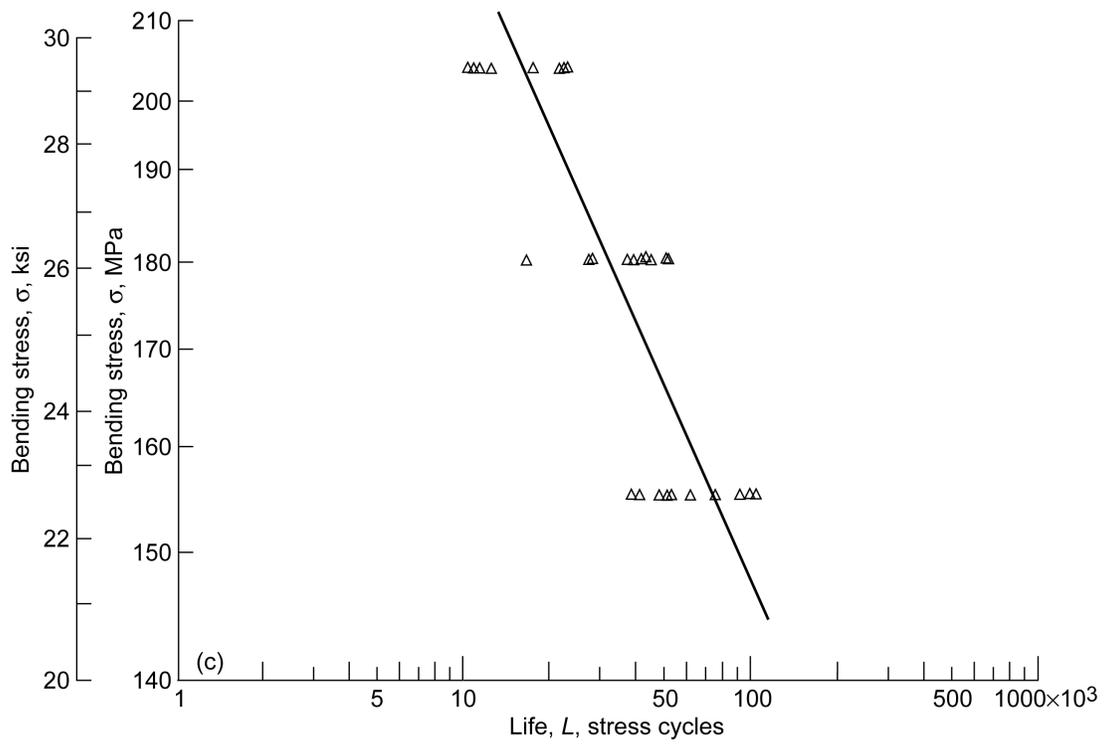


Figure 5.—Concluded.

TABLE V.—MEDIAN LIFE
[ASTM Standard E739–91.]

Stress level, MPa (ksi)	Calculated life (mean life)		
	AL 2024	AL 6061	AL 7075
157.2 (22.8)	223 951	54 147	67 848
180.0 (26.1)	146 016	21 567	31 719
202.0 (29.3)	101 268	9 811	16 550

TABLE VI.—STRESS-LIFE COMPONENTS DETERMINED FOR THREE ALUMINUM ALLOYS USING FOUR DIFFERENT *S-N* CURVES

<i>S-N</i> curve method	Stress-life exponent, <i>p</i>		
	AL 2024	AL 6061	AL 7075
ASTM standard (Fig. 5)	3.2	6.8	5.6
Johnson-Weibull (Fig. 6(a)), L_{10}	4.4	6.6	5.8
Johnson-Weibull (Fig. 6(b)), L_{50}	3.1	6.8	5.6

TABLE VII.—WEIBULL PARAMETERS FROM FATIGUE LIVES OF THREE ALUMINUM ALLOYS AT THREE BENDING STRESSES
[From Ref. 17.]

Alloy	Weibull parameter	Bending stress		
		157.2 MPa (22.8 ksi)	180.0 MPa (26.1 ksi)	202.0 MPa (29.3 ksi)
AL 2024	Slope	6.220	5.122	2.901
	L_{10}	156 207	124 678	50 701
	L_{50}	211 468	180 100	97 063
	Mean	208 503	177 876	98 208
	Characteristic life	224 304	193 459	110 136
AL 6061	Slope	2.549	4.028	2.747
	L_{10}	26 987	14 635	5 118
	L_{50}	56 510	23 362	10 162
	Mean	57 922	23 201	10 333
	Characteristic life	65 249	25 587	11 612
AL 7075	Slope	2.894	3.100	2.640
	L_{10}	35 029	20 898	7 980
	L_{50}	67 160	38 373	16 290
	Mean	67 965	38 624	16 631
	Characteristic life	76 226	43 189	18 716

In aerospace applications, early lives at high reliability is of primary interest and not necessarily the average or median life, even for comparison purposes. Hence, it may be reasonably concluded that where there are differences in lives that are relatively small, the ASTM method does not provide a method to differentiate with any reasonable engineering or statistical certainty whether a material is superior to another based on a fatigue criteria.

Weibull Analysis

Weibull analysis is ideally suited for determining the probable fatigue life of a larger population with a limited amount of data, and methodologies exist to rank the populations. Each data set in Table I was ranked and the Weibull parameters were determined according to Johnson (Ref. 12). These Weibull parameters are reported in Table VII. The L_{10} and L_{50} lives at each stress level for each material are plotted as function of stress in *S-N* plots of Figure 6. The stress-life exponents *p* are given in Figure 6 and summarize in Table VI. The ASTM Standard and the Johnson-Weibull analysis result in the same stress-life exponent *p* for each of the three aluminum alloys at the median or L_{50} lives.

The ranking of the fatigue lives of AL 6061 to AL 7075, AL 6061 to AL 2024, and AL 7075 to AL 2024 was determined using Equation (5a). Then the combined confidence number, which is the probability that the 2 sets of 10 fatigue lives at each of the three stresses is different, is determined.

First, the probability that the mean life of the AL 7075 is greater than that of the AL 6061 is determined by calculating the confidence number using Equation (5a). The necessary Weibull parameters (slope and mean life ratio, *MLR*) can be found in Table IV. Both data sets contain 10 fatigue failures; thus the *DOF* is still 81. If the Weibull slopes for both sets of data were equal, only one confidence number would need to be determined. Since the AL 6061 and AL 7075 data sets have different Weibull slopes (Table VII) at a bending stress of 157.2 MPa (22.8 ksi), the confidence numbers must be determined at slopes of both 2.894 and 2.549, and then the average confidence number is calculated. From Equations (3) and (5), and Table VII with $MLR_{exp} = 67965/57922 \sim 1.17$, for a Weibull slope of 2.894 the confidence number is found to be 0.851, and for a Weibull slope of 2.549 the confidence number is found to be 0.831. The confidence number (0.841) reported in Table VII is the average of these two values.

If the method is repeated at the 180.0 and 202.0 MPa (26.1 and 29.3 ksi) bending stresses, the confidence numbers are found to be 0.998 and 0.992, respectively. Thus, the life of the AL 7075 aluminum is greater than that of the AL 6061 at 180.0 and 202.0 MPa (26.1 and 29.3 ksi) with greater than a 99 percent confidence. However, the difference in the fatigue lives of the two materials at 157.2 MPa (22.8 ksi) are not statistically significant, with an 84 percent confidence.

Since multiple related data sets are available, a combined or composite confidence number can be determined. According to Johnson (Ref. 12), a “composite” or combined confidence number *C* can be determined that incorporates and compares all data sets (six in this case).

Substituting in the values of *C_i* at the three bending stresses (0.831, 0.998, and 0.992) and solving for *C* using Equation (6) where in this case *n* = 3,

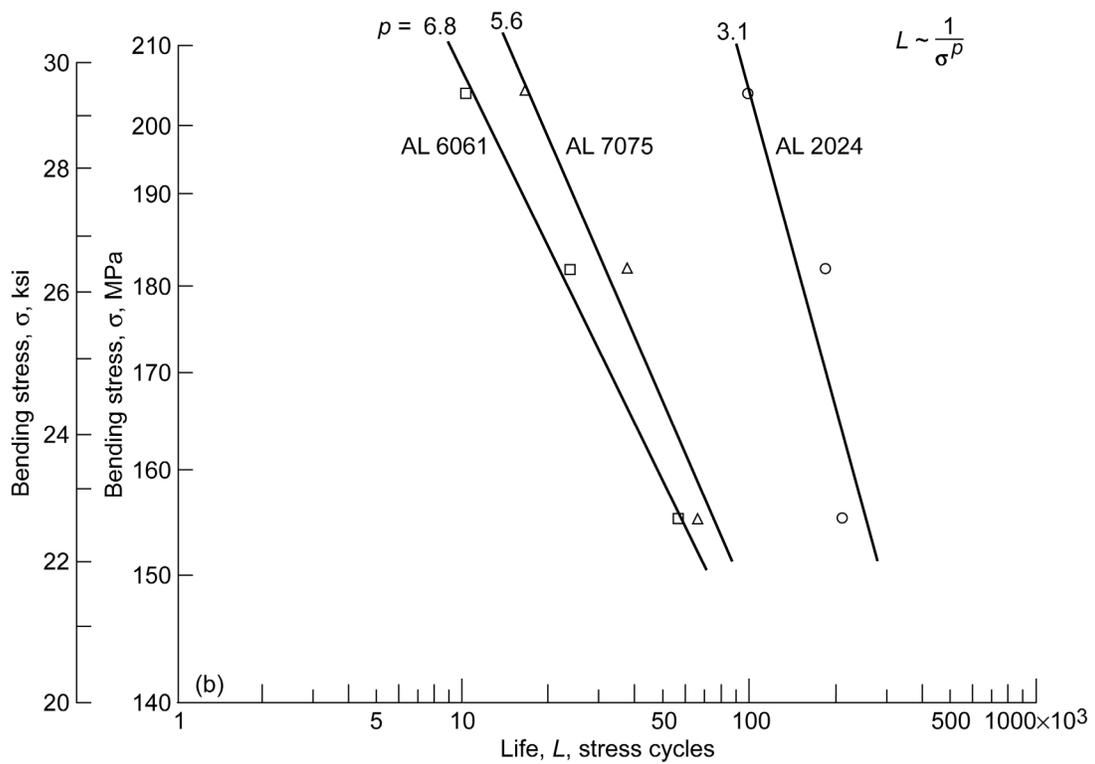
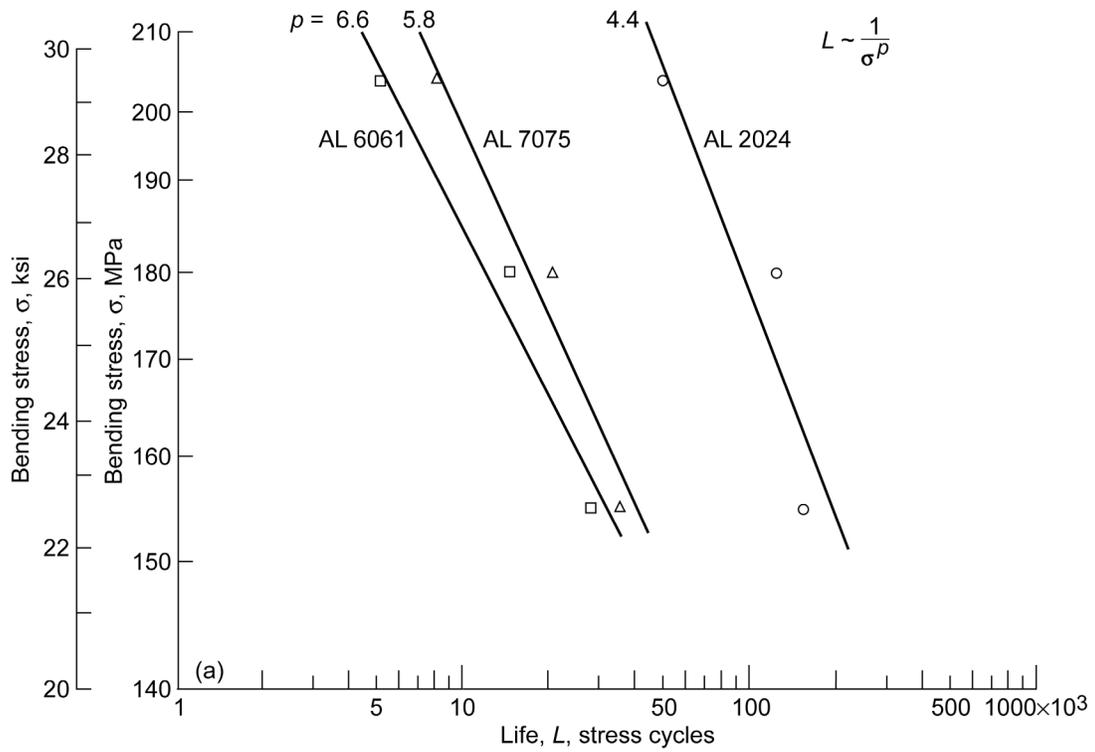


Figure 6.—Stress-life (S-N) curves for aluminum alloys AL 2024, AL 6061, and AL 7075 in rotational bending fatigue using Johnson-Weibull analysis, where p is the stress-life exponent. (a) 10-percent lives, L_{10} . (b) 50-percent (median) lives, L_{50} .

TABLE VIII.—CALCULATED CONFIDENCE NUMBERS FOR THREE ALUMINUM ALLOYS
AT THREE BENDING STRESSES

Bending stress, MPa (ksi)	Aluminum alloys compared	Mean life ratio	Weibull slope			Degrees of freedom	Calculated confidence number
			m_{AL7075}	m_{AL6061}	m_{AL2024}		
157.2	AL 6061 and AL 7075	1.17	2.894	2.549	-----	81	0.841
180.0	AL 6061 and AL 7075	1.66	3.10	4.028	-----	81	0.998
202.0	AL 6061 and AL 7075	1.61	2.64	2.747	-----	81	0.992
157.2	AL 6061 and AL 2024	3.60	-----	2.549	6.220	81	0.999
180.0	AL 6061 and AL 2024	7.67	-----	4.028	5.122	81	0.999
202.0	AL 6061 and AL 2024	9.50	-----	2.747	2.901	81	0.999
157.2	AL 7075 and AL 2024	3.07	2.894	-----	6.220	81	0.999
180.0	AL 7075 and AL 2024	4.61	3.10	-----	5.122	81	0.999
202.0	AL 7075 and AL 2024	5.91	2.64	-----	2.901	81	0.999

$$C = 1 - 2^{(3-1)}(1 - 0.831)(1 - 0.998)(1 - 0.992) \quad (8)$$

$$= 0.999989$$

As a result of incorporating two-thirds more data, greater confidence is achieved in stating that one material is longer lived than another at these test conditions. In this case, the AL 7075 is longer lived than the AL 6061.

The process is repeated to rank AL 2024 and AL 7075 and rank AL 2024 and AL 6061 by determining the probability and comparing the *MLR* of each. The individual confidence numbers for both of these comparisons at each of three bending stresses are reported in Table VIII. The combined confidence number in both of these cases is significantly greater than 0.99. The AL 7075 mean fatigue life is greater than that of AL 6061 with a probability more than 99 percent. The fatigue life of the AL 2024 is greater than both the AL 7075 and AL 6061 with a probability or more than 99 percent. It is thus possible to rank the fatigue lives of different materials with a reasonable degree of statistical certainty.

Summary of Results

Probabilistic failure analysis is essential when analysis of *S-N* curves is inconclusive in determining the relative ranking of two or more materials. In 1964, Leonard Johnson published a methodology for establishing the confidence that two populations of data are different. Johnson's methodology is dependent on a limited combination of test parameters (Weibull slope, mean life ratio, and degrees of freedom) and a set of complex mathematical equations. To create a simplified method, Johnson's original confidence number plots were fitted, and an equation was found for confidence number as a function of Weibull slope, degree of freedom, and mean life ratio. The confidence numbers calculated with this equation were compared to those obtained graphically from Johnson. The simplified equation was used to rank the experimental lives of three aluminum alloys (AL 2024, AL 6061, and AL 7075), each tested at three stress levels in rotating beam fatigue analyzed using the Johnson-Weibull method and compared to the

ASTM Standard (E739-91) method of comparison. The following results were obtained:

1. Using the ratios of mean life, the resultant values of confidence numbers at 99 percent deviated less than 1 percent from those of Johnson using the simplified algebraic equation. At a 90 percent (0.90) confidence number, the calculated values differed between +2 and -4 percent.
2. The ASTM Standard (E739-91) method did not statistically distinguish between AL 6061 and AL 7075. However, it is possible to rank the fatigue lives of different materials with a reasonable degree of statistical certainty based on combined confidence numbers using Johnson-Weibull analysis. AL 2024 was found to have the longest fatigue life, followed by AL 7075, and then AL 6061.
3. The ASTM Standard and the Johnson-Weibull analysis result in the same stress-life exponent *p* for each of the three aluminum alloys at the median or *L*₅₀ lives.

Glenn Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, April 29, 2013

References

1. Tipton, S.M.: Static and Fatigue Design. Mechanical Design Handbook, Harold A. Rothbart, ed., McGraw-Hill, New York, NY, sec. 7, 1996, pp. 7.3-7.73.
2. Zaretsky, E.V.: STLE Life Factors for Rolling Bearings. STLE SP-34, 1992.
3. Standard Practice for Statistical Analysis of Linear or Linearized Stress-Life (*S-N*) and Strain-Life (*ε-N*) Fatigue Data. Active Standard E739-91, vol. 03.01, ASTM International, 2004.
4. Burrow, M.F. et al.: Analysis of Tensile Bond Strengths Using Weibull Statistics. Biomaterials, vol 25, no. 20, 2004, pp. 5031-5035.
5. Ellis, F.V.; and Tordonato, S.: Failure Analysis and Life Assessment Studies for Boiler Tubes. ASME Pressure Vessels and Piping Division Publication, vol. 392, 1999, pp. 3-13.
6. Tomimatsu, M.; Sakai, M.; and Kikuchi, M.: Fracture Toughness Evaluation Based on the Master Curve Procedure. ASME Pressure Vessels and Piping Division Publication, 1997, pp. 343-348.

7. Osborne, Nora; Graves, George G.; and Ferber, Matt K.: Dynamic Fatigue Testing of Candidate Ceramic Materials for Turbine Engines to Determine Slow-Crack-Growth Parameters. ASME Paper, 1995.
8. Ostojic, P.; and Berndt, C.C.: Variability in Strength of Thermally Sprayed Coatings. *Surf. Coat. Technol.*, vol. 34, no. 1, 1987, pp. 43–50.
9. Holland, F.A.; and Zaretsky, E.V.: Investigation of Weibull Statistics in Fracture-Analysis of Cast-Aluminum. *J. Mech. Des.*, vol. 112, no. 2, 1990, pp. 246–254.
10. Williams, Samuel R.; and Fec, Michael C.: Weibull Analysis of Reconditioned Railroad Roller Bearing Life Test Data. ASME Rail Transportation Division Publication, vol. 5, 1992, pp. 83–87.
11. Summers-Smith, J.D.: Fault Diagnosis as an Aid to Process Machine Reliability. *Quality and Reliability Engineering International*, vol 5, no 3, 1989, pp. 203–205.
12. Johnson, L.G.: *The Statistical Treatment of Fatigue Experiments*. Elsevier Publishing Co., Amsterdam, The Netherlands, 1964.
13. Weibull, W.: *A Statistical Theory of the Strength of Materials*. Ingenioersvetenskapsakad. Handl., no. 151, Stockholm, Sweden, 1939.
14. Weibull, W.: *The Phenomenon of Rupture in Solids*. Ingenioersvetenskapsakad. Handl., no. 153, Stockholm, Sweden, 1939.
15. Weibull, W.: A Statistical Distribution Function of Wide Applicability. *J. Appl. Mech. Trans. ASME*, vol. 18, no. 3, 1951, pp. 293–297.
16. Vlcek, B.L.; Hendricks, R.C.; and Zaretsky, E.V.: Comparative Fatigue Lives of Rubber and PVC Wiper Cylindrical Coatings. *Trib. Trans.*, vol. 46, no. 1, 2003, pp. 101–110.
17. Vlcek, B.L., et al.: Experimental Determination of Fatigue Lives of Rotating Aluminum Alloys Using Weibull Analysis. 125–ISROMAC–11, Paper presented at the 11th International Symposium on Transport Phenomena and Dynamics of Rotating Machinery Honolulu, HI, 2006.
18. *Aluminum Standards and Data*. Sixth ed., Aluminum Association, Washington, DC, 1979.

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