A MULTI-SCALE REFINED ZIGZAG THEORY FOR MULTILAYERED COMPOSITE AND SANDWICH PLATES WITH IMPROVED TRANSVERSE SHEAR STRESSES

LUIGI IURLARO[*], MARCO GHERLONE[*], MARCO DI SCIUVA[*] AND ALEXANDER TESSLER[†]

[*] Department of Mechanical and Aerospace Engineering, Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
e-mail: {luigi.iurlaro, marco.gherlone, marco.disciuva}@polito.it, web: http://www.aesdo.polito.it

[†] Structural Mechanics and Concepts Branch, NASA Langley Research Center
Mail Stop 190, Hampton, Virginia, 23681-2199, U.S.A.
email: alexander.tessler-1@nasa.gov - web: http://www.nasa.gov

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Abstract. The Refined Zigzag Theory (RZT) enables accurate predictions of the in-plane displacements, strains, and stresses. The transverse shear stresses obtained from constitutive equations are layer-wise constant. Although these transverse shear stresses are generally accurate in the average, layer-wise sense, they are nevertheless discontinuous at layer interfaces, and thus they violate the requisite interlaminar continuity of transverse stresses. Recently, Tessler applied Reissner's mixed variational theorem and RZT kinematic assumptions to derive an accurate and efficient shear-deformation theory for homogeneous, laminated composite, and sandwich beams, called RZT$^{(m)}$, where "m" stands for "mixed". Herein, the RZT$^{(m)}$ for beams is extended to plate analysis, where two alternative assumptions for the transverse shear stresses field are examined: the first follows Tessler's formulation, whereas the second is based on Murakami's polynomial approach. Results for elasto-static simply supported and cantilever plates demonstrate that Tessler's formulation results in a powerful and efficient structural theory that is well-suited for the analysis of multilayered composite and sandwich panels.

1 INTRODUCTION

Over the past two decades, composite materials have been increasingly used in civil, automotive, and aerospace applications. Numerous structural theories have been explored for the analysis of multilayered composite and sandwich structures. Since transverse shear deformations govern damage mechanisms that contribute to delamination initiation and propagation, efforts to develop accurate predictions of transverse shear strains and stresses have been extensive [1].

In the framework of displacement-based lamination theories, a distinction is usually made between Equivalent Single Layer (ESL) and Layer-Wise (LW) theories [2]. The former assume a *coarse* approximation for the displacement components, thus reducing a multilayered plate to a single-layer plate whose behavior is governed by some average constitutive properties. In contrast to ESL theories, LW theories employ kinematic assumptions for every layer, hence the number of kinematic variables increases with the number of layers. Although the LW theories are sufficiently accurate, they become computationally intensive for application to laminates that have a large number of layers.

Generally, the transverse shear stresses obtained from the constitutive equations, using either ESL and ZZ theories, suffer from the lack of interlaminar continuity and often exhibit rather inaccurate magnitudes across the laminate thickness. A post-processing analysis, which usually results in improved transverse shear stresses, is integration of equilibrium equations of elasticity theory. Since these computations involve second-order partial derivatives of kinematic variables, their application in finite element analysis is often associated with significant errors. Alternatively, mixed-field variational principles have been used [3-5], wherein the displacement and transverse shear stress assumptions are made independently.

A viable compromise between adequate accuracy and computational efficiency is offered by the so-called ZigZag (ZZ) theories, wherein the assumed kinematic field is a superposition of the *coarse* and *fine* distributions through the thickness. The *fine* distribution, also known as *zigzag*, is commonly piecewise linear, and it enables the inplane-displacement partial derivative taken with respect to the thickness coordinate to be discontinuous at the layer interfaces. The ZZ theories are computationally efficient because they have a fixed number of variables regardless of the number of layers. These theories also achieve global response predictions comparable to those of the LW theories [6,7]. Recently, Tessler et al. [8,9] developed the so-called Refined Zigzag Theory (RZT), wherein the displacement field of First-order Shear Deformation Theory (FSDT) is enriched by the addition of suitable zigzag functions that have C^0-continuous distributions through the thickness. Each of the in-plane zigzag displacements is controlled by a single amplitude variable. Thus, the FSDT five-variables plate kinematics is increased to seven variables. Efficient, C^0-continuous, RZT-based finite elements have also been developed for beams, plates, and shells [10-12].

In this paper, a mixed-field formulation is undertaken in the framework of RZT plate kinematic assumptions and Reissner's mixed variational theorem [13]. Two alternative strategies for the approximation of transverse shear stresses are examined. The first strategy adopts Tessler's methodology in [14] and is denoted as $RZT_1^{(m)}$. The transverse shear stresses are derived using three-dimensional equilibrium equations of elasticity theory. The second strategy, $RZT_2^{(m)}$, is based on Murakami's [3] polynomial approach. The two approaches are assessed by way of elasto-static problems for simply supported and cantilever plates subjected to bi-sinusoidal and uniformly distributed loading. Analytic solutions are obtained and compared with three-dimensional exact elasticity solutions, high-fidelity finite element solutions, and RZT (displacement theory) solutions. The comparisons indicate that $RZT_1^{(m)}$ is consistently more accurate than $RZT_2^{(m)}$, hence it can be used effectively in the analysis of multilayered composite and sandwich plates.

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2 RZT$^{(m)}$ ASSUMPTIONS

Consider a laminated plate of uniform thickness *2h* made by *N* perfectly bonded orthotropic layers. The plate is referred to the orthogonal Cartesian coordinate system $(\mathbf{x} \equiv (x_1, x_2), z)$, where the plate midplane Ω is placed on the **x**-plane, whereas the through-the-thickness coordinate *z* ranges from *–h* to *h*. Throughout this paper, Greek subscripts take the values 1, 2.

Reissner's mixed variational theorem [13] permits independent assumptions to be used for displacements and transverse shear stresses. This theorem enforces, as a constraint, the compatibility between the transverse shear strains resulting from the strain-displacement relations and those that are assumed a priori, i.e.,

$$\int_{\Omega} \int_{-h}^{h} \left[\delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} + \delta \boldsymbol{\tau}_{\mathbf{a}}^T \left(\boldsymbol{\gamma} - \boldsymbol{\gamma}_{\mathbf{a}} \right) \right] dz \, d\Omega = \delta W_e \tag{1}$$

where δ is the variational operator and W_e represents the work of the external loads. The strain vector $\boldsymbol{\varepsilon}^T \equiv \{\varepsilon_{11}, \varepsilon_{22}, \gamma_{12}, \gamma_{1z}, \gamma_{2z}\}$ contains the in-plane and transverse-shear, $\boldsymbol{\gamma}^T \equiv \{\gamma_{1z}, \gamma_{2z}\}$, strains obtained by means of the linear strain-displacement relations. The stress vector $\boldsymbol{\sigma}^T \equiv \{\sigma_{11}, \sigma_{22}, \tau_{12}, \tau_{\mathbf{a}1z}, \tau_{\mathbf{a}2z}\}$ contains the in-plane stresses obtained by means of Hooke's law and the transverse-shear stresses, $\boldsymbol{\tau}_{\mathbf{a}}^T \equiv \{\tau_{\mathbf{a}1z}, \tau_{\mathbf{a}2z}\}$, that are assumed independently. The vector $\boldsymbol{\gamma}_{\mathbf{a}}^T \equiv \{\gamma_{\mathbf{a}1z}, \gamma_{\mathbf{a}2z}\}$ is obtained from the assumed stresses, $\boldsymbol{\tau}_{\mathbf{a}}$, using Hooke's law.

In what follows, the assumed displacement and transverse-shear stress fields are described in detail.

2.1 Displacement field

The kinematic assumptions of RZT are adopted, having demonstrated superior predictive capabilities to model both the global and local response quantities [8-12,15]. The displacement components are defined in the Cartesian coordinate system as [9]

$$\begin{cases} u_\alpha^{(k)}(\mathbf{x}, z) \equiv u_\alpha(\mathbf{x}) + z\theta_\alpha(\mathbf{x}) + \phi_\alpha^{(k)}(z)\psi_\alpha(\mathbf{x}) \\ u_z(\mathbf{x}, z) \equiv w(\mathbf{x}) \end{cases} \tag{2}$$

In Eq. (2), and in the following, the superscript (*k*) indicates quantities corresponding to the *k*th lamina, u_α is a uniform displacement component along the x_α-axis, θ_α is the average bending rotation of the transverse normal about x_β-axis ($\alpha \neq \beta$), and *w* is the transverse deflection. The kinematic variable ψ_α represents the amplitude of the zigzag rotation, whereas $\phi_\alpha^{(k)}$, derived in [9], is a piecewise linear zigzag function that is independent of the state of deformation. The RZT kinematic variables include those of FSDT, u_α, w, and θ_α, as well as the two additional variables called the zigzag rotations, ψ_α.

RZT is characterized by a multi-scale kinematic description since the displacement field is given by the superposition of the *coarse* and *fine* contributions: the former, modeling the kinematics with the resolution on the scale of the entire plate thickness, corresponds to the FSDT, whereas the latter models the mechanical behavior with the resolution on the scale of the *k*th material layer.

The strains are obtained using the linear strain-displacement relations

$$2\varepsilon_{\alpha\beta}^{(k)} = u_{\alpha,\beta} + u_{\beta,\alpha} + z\left(\theta_{\alpha,\beta} + \theta_{\beta,\alpha}\right) + \phi_\alpha^{(k)}\psi_{\alpha,\beta} + \phi_\beta^{(k)}\psi_{\beta,\alpha}; \quad \gamma_{\alpha z}^{(k)} = \gamma_\alpha + \beta_\alpha^{(k)}\psi_\alpha \tag{3}$$

with $\gamma_\alpha \equiv w_{,\alpha} + \theta_\alpha$ and $\beta_\alpha^{(k)} \equiv \phi_{\alpha,z}^{(k)}$. Hooke's constitutive relations are then invoked to compute the stresses

$$\sigma_{\alpha\beta}^{(k)} = C_{\alpha\beta\gamma\delta}^{(k)}\varepsilon_{\gamma\delta}^{(k)}; \quad \tau_{\alpha z}^{(k)} = Q_{\alpha\beta}^{(k)}\gamma_{\beta z}^{(k)} \tag{4}$$

where $C_{\alpha\beta\gamma\delta}^{(k)}$ and $Q_{\alpha\beta}^{(k)}$ are the transformed elastic stiffness coefficients referred to the $(\mathbf{x}, z)$ coordinate system and relative to the plane-stress condition that assumes that transverse normal stress is negligibly small in relation to the in-plane stresses.

2.2 Transverse shear stresses

In this study, two thickness-wise distributions of the transverse shear stresses that are continuous along the layer interfaces and satisfy traction conditions at the top and bottom plate surfaces are considered: (i) a distribution derived by way of three-dimensional elasticity equilibrium equations, and (ii) an assumed polynomial distribution. In both cases, the transverse shear stresses can be expressed as

$$\boldsymbol{\tau}_\mathbf{a} = \mathbf{Z_f}(z)\mathbf{f_v}(\mathbf{x}) + \mathbf{Z_n}(z)\mathbf{n_v}(\mathbf{x}) \tag{5}$$

where the matrices $\mathbf{Z_f}$, $\mathbf{Z_n}$ are dependent on the thickness coordinate, and the vector $\mathbf{f_v}$ is a stress function of the in-plane coordinates; $\mathbf{n_v}^T = \left\{\bar{p}_1^{(b)}, \bar{p}_1^{(t)}, \bar{p}_2^{(b)}, \bar{p}_2^{(t)}\right\}$ is a vector containing prescribed surface tractions that act along the x_α-direction; $\bar{p}_\alpha^{(t)}$ are prescribed on the top surface, S_t, and $\bar{p}_\alpha^{(b)}$ on the bottom surface, S_b.

2.2.1 Integrated transverse shear stresses (version $\mathrm{RZT}_1^{(m)}$)

Three-dimensional equilibrium equations of elasticity are commonly used in an attempt to derive improved, layer interface-continuous transverse shear stresses. Auricchio and Sacco [4] used an equilibrium-integration approach to derive transverse shear stresses for the FSDT-based plate analysis. It was recognized that due to a large number of independent stress parameters, the mixed-field variational formulation tends to fit the constitutive-based stresses very closely, yielding only insignificant improvements for either the equilibrium- or polynomial-based stresses. Moreover, an ad hoc function had to be added to the integrated transverse shear stresses in order to satisfy the traction-free boundary conditions on the top bounding surface, whereas the bottom zero-traction condition was enforced a priori.

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Recently, Tessler [14] presented a mixed-field formulation for RZT beams, which derives the transverse shear stress from the two-dimensional elasticity equilibrium equations. A key step in the formulation is that the transverse shear stress is made to satisfy exactly the first (axial) equilibrium equation, and hence it satisfies a priori the top and bottom traction conditions of arbitrary distributions, including the special cases of zero-traction conditions. The derived stress is also fully continuous along layer interfaces. The problem is reduced to replacing two second-order derivatives of the kinematic variables with two unknown stress functions that are determined using Reissner's mixed-field theorem.

Herein, Tessler's [14] methodology is used to derive an RZT mixed-field formulation for plate analysis. By neglecting the body forces, the first two equilibrium equations of elasticity theory may be written as

$$\tau_{\alpha z,z} = -\left(\sigma_{\alpha\alpha,\alpha} + \tau_{\alpha\beta,\beta}\right), \qquad \beta \neq \alpha \tag{6}$$

Integrating with respect to the z-coordinate and enforcing the traction conditions at the bottom plate surface ($z=-h$) yields

$$\tau_{\alpha z} = -\bar{p}_\alpha^{(b)} - \int\limits_{-h}^{z} \left(\sigma_{\alpha\alpha,\alpha} + \tau_{\alpha\beta,\beta}\right) dz, \qquad \beta \neq \alpha \tag{7}$$

Introducing Eq. (4) into Eq. (7), and after some straightforward manipulations, the transverse shear stresses involving eighteen second-order partial derivatives of u_α, θ_α and ψ_α are obtained. To circumvent the over fitting deficiency encountered in [4], a simple strategy is pursued herein that leads to the functional simplicity associated with the mixed-field RZT beam formulation [14]. To simplify Eq. (7), cylindrical bending is considered for each of the x_α directions, resulting only in the second partial derivatives of u_α, θ_α and ψ_α (with respect to x_α in the (x_α,z)-plane) in the expressions for the transverse shear stresses. Thus, the integrated shear stresses become

$$\tau_{\alpha z} = -\bar{p}_\alpha^{(b)} - \left(\int\limits_{-h}^{z} C_{\alpha\alpha}^{(k)} dz\right) u_{\alpha,\alpha\alpha} - \left(\int\limits_{-h}^{z} z C_{\alpha\alpha}^{(k)} dz\right) \theta_{\alpha,\alpha\alpha} - \left(\int\limits_{-h}^{z} \phi_\alpha^{(k)} C_{\alpha\alpha}^{(k)} dz\right) \psi_{\alpha,\alpha\alpha} \tag{8}$$

To fulfill the full shear traction equilibrium on the top plate face, the first two RZT equilibrium equations, describing the in-plane equilibrium, (see [9]), are written for cylindrical bending case as

$$\left(\int\limits_{-h}^{h} C_{\alpha\alpha}^{(k)} dz\right) u_{\alpha,\alpha\alpha} + \left(\int\limits_{-h}^{h} z C_{\alpha\alpha}^{(k)} dz\right) \theta_{\alpha,\alpha\alpha} + \left(\int\limits_{-h}^{h} \phi_\alpha^{(k)} C_{\alpha\alpha}^{(k)} dz\right) \psi_{\alpha,\alpha\alpha} + \bar{p}_\alpha = 0 \tag{9}$$

where $\bar{p}_\alpha \equiv \bar{p}_\alpha^{(t)} + \bar{p}_\alpha^{(b)}$. Solving Eq. (9) for $u_{\alpha,\alpha\alpha}$ gives

$$u_{\alpha,\alpha\alpha} = \left(\int_{-h}^{h} C_{\alpha\alpha}^{(k)} dz \right)^{-1} \left[-\left(\int_{-h}^{h} z C_{\alpha\alpha}^{(k)} dz \right) \theta_{\alpha,\alpha\alpha} - \left(\int_{-h}^{h} \phi_{\alpha}^{(k)} C_{\alpha\alpha}^{(k)} dz \right) \psi_{\alpha,\alpha\alpha} - \bar{p}_{\alpha} \right] \tag{10}$$

Substituting Eq. (10) into Eq. (8) results in the transverse shear stresses that have only two second-order partial derivatives, $\theta_{\alpha,\alpha\alpha}$ and $\psi_{\alpha,\alpha\alpha}$, per stress component, and the thickness-distribution functions that satisfy all traction equilibrium conditions exactly, including the layer-interface equilibrium conditions. The final form of the transverse shear stresses is given by Eq. (5) in which the surface tractions are grouped in $\mathbf{n}_v$, and the second-order partial derivatives are replaced by independent functions of $\mathbf{x}$ and are grouped in $\mathbf{f}_v$. The corresponding version of the theory is herein denoted by $RZT_1^{(m)}$.

2.2.2 Polynomial approximation of shear stresses (version $RZT_2^{(m)}$)

The use of polynomial approximation for the transverse shear stresses in a mixed-field formulation appears for the first time in [3], and since has been adopted by many investigators. For each material layer, a polynomial thickness distribution is assumed and is expressed as

$$\tau_{\mathbf{a}\alpha z}^{(k)} = \tau_{\alpha z}^{b(k)} F_b^{(k)}(z) + \tau_{\alpha z}^{t(k)} F_t^{(k)}(z) + T_{\alpha}^{(k)} F_m^{(k)}(z) \tag{11}$$

where $\tau_{\alpha z}^{b(k)}$ and $\tau_{\alpha z}^{t(k)}$ are the values of transverse shear stress $\tau_{\mathbf{a}\alpha z}^{(k)}$ at the bottom and top interface of the kth layer, respectively; $T_{\alpha}^{(k)}$ stands for the average shear stress in the kth layer of thickness $2h^{(k)}$

$$T_{\alpha}^{(k)} \equiv \frac{1}{2h^{(k)}} \int_{z_{(k-1)}}^{z_{(k)}} \tau_{\mathbf{a}\alpha z}^{(k)} dz \tag{12}$$

Moreover

$$F_b^{(k)} = \frac{3}{4}\left(\zeta^{(k)}\right)^2 - \frac{1}{2}\zeta^{(k)} - \frac{1}{4}; \; F_t^{(k)} = \frac{3}{4}\left(\zeta^{(k)}\right)^2 + \frac{1}{2}\zeta^{(k)} - \frac{1}{4}; \; F_m^{(k)} = \frac{3}{2}\left(1 - \left(\zeta^{(k)}\right)^2\right) \tag{13}$$

with $\zeta^{(k)} \equiv \left(z - z_m^{(k)}\right)/h^{(k)} \in [-1;+1]$, $z_m^{(k)}$ representing the coordinate of the kth midplane.

The mixed-field formulation which is based on the RZT kinematic assumptions, Eq. (2), and the assumed transverse shear stresses, Eq. (11), is herein denoted as $RZT_2^{(m)}$.

3 $RZT^{(m)}$ GOVERNING EQUATIONS

The $RZT^{(m)}$ plate equilibrium equations are derived from Reissner's mixed variational theorem, wherein the body forces are neglected. The plate is subjected to a transversely distributed pressure $\bar{q}$, acting on the midplane Ω, to tangential surface tractions $\bar{p}_{\alpha}^{(t)}$ and

$\bar{p}_\alpha^{(b)}$, acting along the x_α-direction and applied at the top, S_t, and bottom, S_b, plate surfaces, respectively, and to the tractions $\left(\bar{T}_\alpha, \bar{T}_z\right)$ applied on a part of the midplane boundary.

The left-hand side of Eq. (1) has two contributions: the first term is the variation of the strain energy, the second term enforces compatibility between the γ and γ_a strain fields. These two terms are decoupled since the variational operator is applied to independent variables; hence, the compatibility term may be treated separately, that is

$$\int_\Omega \int_{-h}^h \delta\tau_a^T \left(\gamma - \gamma_a\right) dz d\Omega = 0 \tag{14}$$

Regardless of the type of approximation, the assumed transverse shear stresses have the general form given by Eq. (5). Substituting Eq. (5) into Eq. (14) and performing the variational operation and through-the-thickness integration yields the following expression for the transverse shear stresses

$$\tau_a = Z_g(z)g + Z_\psi(z)\psi + Z_e(z)n_v \tag{15}$$

where $g^T \equiv \{\gamma_1, \gamma_2\}$, $\psi^T \equiv \{\psi_1, \psi_2\}$, and where $Z_g(z)$, $Z_\psi(z)$ and $Z_e(z)$ are matrices of functions of the z-coordinate.

Performing the integration by parts on the remaining part of Eq. (1)

$$\int_\Omega \int_{-h}^h \left(\delta\varepsilon^T \sigma\right) dz d\Omega = \delta W_e \tag{16}$$

gives rise to the equilibrium equations of RZT$^{(m)}$

$$N_{\alpha\beta,\beta} + \bar{p}_\alpha = 0; \quad Q_{\alpha,\alpha} + \bar{q} = 0; \quad M_{\alpha\beta,\beta} - Q_\alpha + \bar{m}_\alpha = 0; \quad M_{\alpha\beta,\beta}^\phi - Q_\alpha^\phi = 0 \tag{17}$$

and a set of variationally consistent boundary conditions that are identical to those of RZT (refer to [9]). In Eq. (17), $\bar{p}_\alpha = \bar{p}_\alpha^{(t)} + \bar{p}_\alpha^{(b)}$, $\bar{m}_\alpha = h\left(\bar{p}_\alpha^{(t)} - \bar{p}_\alpha^{(b)}\right)$; $N_{\alpha\beta}$, $M_{\alpha\beta}$, and $M_{\alpha\beta}^\phi$ are the same stress resultants as those defined in RZT [9]. For this new theory, the transverse shear stress resultants are given by

$$\begin{aligned}
\left\{\begin{matrix} Q_1 \\ Q_2 \end{matrix}\right\} &= \int_{-h}^h \left\{\begin{matrix} \tau_{a1z}^{(k)} \\ \tau_{a2z}^{(k)} \end{matrix}\right\} dz = A_T g + B_T \psi + E_e n_v \\
\left\{\begin{matrix} Q_1^\phi \\ Q_2^\phi \end{matrix}\right\} &= \int_{-h}^h \left\{\begin{matrix} \beta_1^{(k)} \tau_{a1z}^{(k)} \\ \beta_2^{(k)} \tau_{a2z}^{(k)} \end{matrix}\right\} dz = C_T g + D_T \psi + F_e n_v
\end{aligned} \tag{18}$$

where A_T, B_T, C_T, D_T, E_T, and F_T are the resulting shear-stiffness coefficient matrices.

4   NUMERICAL RESULTS

To assess the accuracy of both $\text{RZT}_1^{(m)}$ and $\text{RZT}_2^{(m)}$, exact and approximate solutions are developed for the bending of rectangular plates defined on the domain $x_1 \in [0,a]$, $x_2 \in [0,b]$, $z \in [-h,h]$: (1) A plate that is simply supported along the four edges and subjected to a bi-sinusoidal transverse pressure, and (2) A cantilever plate subjected to a uniform transverse pressure. For problem (1), an exact solution is readily obtained using trigonometric expansions for the kinematic variables. For problem (2), an approximate solution is obtained using the Rayleigh-Ritz method, where the kinematic variables are approximated using the Gram-Schmidt polynomials. For details on these solutions, refer to [9].

The mechanical material properties and the stacking sequences are summarized in Tables 1 and 2.

Table 1: Mechanical properties of orthotropic materials

Material	$E_1^{(k)} / E_2^{(k)} / E_3^{(k)}$ (GPa)	$G_{12}^{(k)} / G_{13}^{(k)} / G_{23}^{(k)}$ (GPa)	$\nu_{12}^{(k)} / \nu_{13}^{(k)} / \nu_{23}^{(k)}$
A	157.9/9.584/9.584	5.93/5.93/3.227	0.32/0.32/0.49
F	50/10/10	5/5/5	0.25/ 0.25/ 0.25
N	$10^{-5}/10^{-5}/75.85\times10^{-3}$	$22.5\times10^{-3}/22.5\times10^{-3}/22.5\times10^{-3}$	0.01/0.01/0.01
Q	525/21/21	10.5/10.5/10.5	0.25/0.25/0.25

Table 2: Laminate stacking sequences (layers sequence is in the positive z-direction)

Laminate	Normalized lamina thickness, $h^{(k)}/h$	Lamina materials	Lamina orientation (°)
L1	(0.5/0.5)	(A/A)	(0/90)
L2	(0.25/0.25/0.25/0.25)	(A/A/A/A)	(0/90/0/90)
L3	(0.1/0.7/0.2)	(F/N/Q)	(0/0/0)

***Problem 1*. A simply supported rectangular plate ($b=3a$) subjected to a bi-sinusoidal transverse pressure,** $q(x_1, x_2) = q_0 \sin(\pi x_1 / a) \sin(\pi x_2 / b)$ **.**

For comparison purposes, both the three-dimensional elasticity [16] and exact RZT [9] solutions are used. Two cross-ply laminates (L1 and L2) are considered, and the non-dimensional maximum deflections corresponding to various values of the span-to-thickness ratio, $a/2h$, are compared. Results in Table 3 demonstrate that the use of a mixed-field formulation can lead to slight enhancements of the deflection predictions when compared with the displacement-based RZT, since the improvements in the transverse shear stresses contribute directly to the transverse-shear stiffness. In addition, it is evident that $\text{RZT}_1^{(m)}$ yields consistently more accurate results than the $\text{RZT}_2^{(m)}$ formulation. Similar observations in relation to the RZT beam analysis were made by Gherlone [15], where it was also pointed out that the mixed-field zigzag formulations retain the same level of accuracy for the in-plane response quantities as those obtained using the displacement-based formulations.

Subsequently, to focus on the transverse shear stresses, their through-the-thickness distributions are examined in detail.

Table 3: Problem 1, Laminates L1, L2: Normalized maximum (central) deflection, $\bar{w} = (10^2 D_{11} / q_0 a^4) w(a/2, b/2)$.

Laminate	$a/2h$	3D Elasticity	RZT	$RZT_1^{(m)}$	$RZT_2^{(m)}$
L1	8	2.546	2.512	2.547	2.545
	10	2.449	2.427	2.449	2.448
	20	2.319	2.314	2.319	2.319
	50	2.283	2.282	2.283	2.283
	100	2.278	2.277	2.278	2.278
L2	8	1.505	1.472	1.507	1.487
	10	1.375	1.353	1.375	1.362
	20	1.198	1.193	1.198	1.195
	50	1.148	1.147	1.148	1.148
	100	1.141	1.141	1.141	1.141

Figure 1 shows a comparison of the through-the-thickness distributions of normalized transverse shear stresses for laminate L1 ($a/2h = 8$). The $RZT_1^{(m)}$ solution is highly accurate as evidenced by the comparison with the reference solution, labeled as 3D Elasticity. Moreover, the RZT and RZT-Integrated solutions refer to the transverse shear stresses based on, respectively, the constitutive equations and the integration of three-dimensional equilibrium equations. The integrated shear stresses are also very accurate. However, in contrast to the $RZT_1^{(m)}$ solution, the integrated stresses are obtained by means of a post-processing scheme that makes use of second-order partial derivatives of the kinematic variables. On the other hand, the $RZT_2^{(m)}$ stresses are significantly less accurate because they follow closely the piecewise-constant distributions of the RZT stresses that are derived from constitutive relations. This behavior is due to the fact that the number of stress variables inherent in the polynomial scheme is proportional to the number of layers. In contrast, the $RZT_1^{(m)}$ formulation uses only two stress variables for each of the transverse-shear stress components, and these variables are independent of the number of layers (see details in Section 2.2.)

***Problem 2***. **A cantilever square plate subjected to a uniform transverse pressure,** $q(x_1, x_2) = q_0$**.**

For this problem, the exact solution does not exist. To assess the accuracy of $RZT^{(m)}$, a high-fidelity FEM (MSC/MD-NASTRAN® [17]) solution is used. The model is regularly discretized using linear-strain solid elements, HEXA8. There are sixty-five elements along each span direction, five elements through the thickness of the bottom face, eight elements along the top face and fifteen elements through the core thickness.

In Figure 2, five through-the-thickness distributions of the transverse shear stress are compared, with the $RZT_1^{(m)}$ distribution demonstrating the closest correlation with the three-dimensional FEM solution, that is comparable in accuracy only to the RZT-Integrated solution.
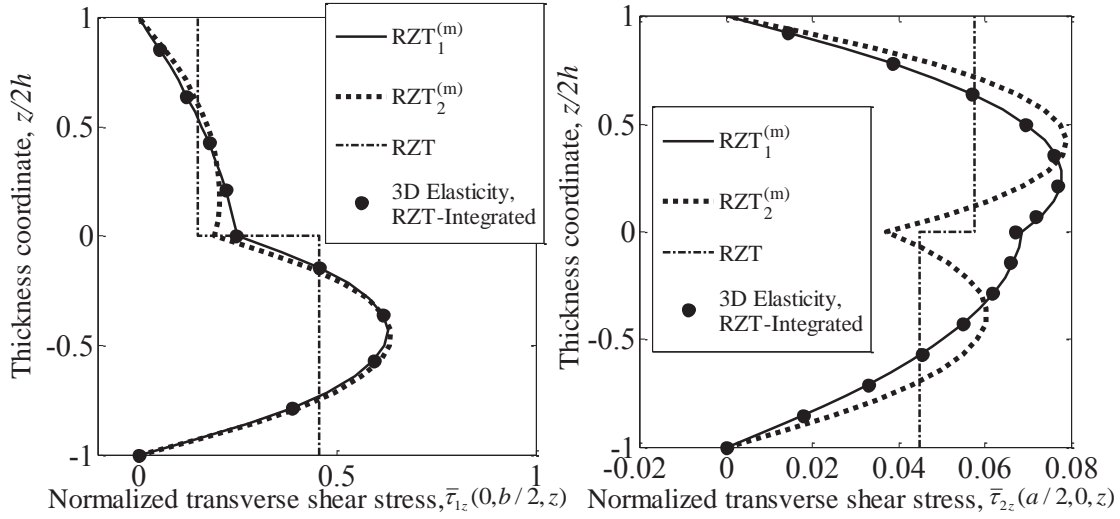
Figure 1: Problem 1, Laminate L1: Through-the-thickness distribution of normalized transverse shear stresses $\overline{\tau}_{\alpha z} = \left(2h / q_0 a^2\right) \tau_{\alpha z}^{(k)}$.
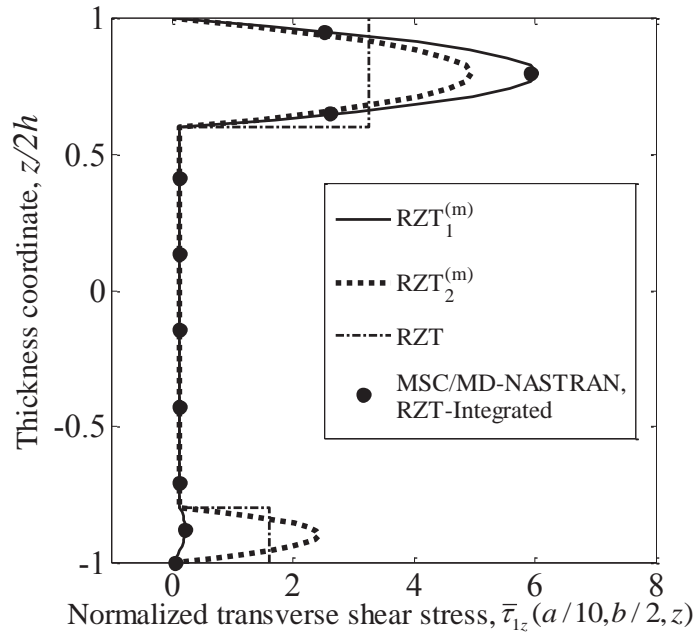
Figure 2: Problem 2, Laminate L3: Through-the-thickness distribution of normalized transverse shear stresses $\overline{\tau}_{1z} = \left(2h / q_0 a^2\right) \tau_{1z}^{(k)}$.

5   CONCLUSIONS

In this paper, a multi-scale refined zigzag theory for the analysis of multilayered composite and sandwich plates has been presented. The new theory, called RZT$^{(m)}$ (Refined Zigzag Theory, Mixed-formulation), is developed following the RZT$^{(m)}$ formulation for beam analysis recently proposed by Tessler. The approach is based on Reissner's mixed variational theorem and on the kinematic assumptions of the Refined Zigzag Theory (RZT). Two alternative formulations for the transverse-shear stresses were examined. The first, RZT$_1^{(m)}$, follows the RZT$^{(m)}$ beam formulation by Tessler, which is based on a closed-form integration procedure of elasticity-theory equilibrium equations. The second formulation, RZT$_2^{(m)}$, incorporates Murakami's polynomial approach.

Results for simply supported and cantilevered rectangular laminated composite and sandwich plates in bending have been examined using the RZT$_1^{(m)}$, RZT$_2^{(m)}$ and the original RZT (displacement-based) formulation. For comparison purposes, corresponding results were obtained using three-dimensional elasticity and high-fidelity FEM solutions. Both RZT$^{(m)}$ formulations have demonstrated improved modeling of the transverse shear stiffness and stresses, achieving slightly more accurate deflection predictions when compared with those of RZT. Of the two formulations, however, RZT$_1^{(m)}$ produced consistently superior results. In contrast, the polynomial-based formulation, RZT$_2^{(m)}$, has shown to exhibit serious deficiencies: the assumed transverse shear stresses tend to approximate closely those of their constitutive counterparts, leading to unsatisfactory results.

In this paper it has been clearly demonstrated that a mixed-field formulation, based on Reissner's variational theorem and coupled with the refined zigzag theory for plates, may offer substantial improvements in predicting the transverse-shear stresses and stiffness. The accuracy of a mixed-field formulation depends on the choice of the assumed transverse shear stresses. The transverse shear stresses obtained by integration of three-dimensional equilibrium equations produced an improved theory, RZT$_1^{(m)}$, that offers superior predictions of the in-plane behavior, typical of RZT, along with the enhanced predictions for the transverse shear stresses and stiffness. This new formulation is thus perfectly suited for the analysis of multilayered composite and sandwich structures. The RZT$_1^{(m)}$ formulation may also be used for developing simple and efficient C^0-continous plate and shell finite elements.

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