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Edge Influence Coefficients for Cylinders With Linearly Varying Wall Thickness

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ABSTRACT

Influence coefficients for the edge deformations of tapered transition cylinders that are subjected to edge-bending loads are given in this Report. The coefficients are obtained by integrating the classical small-deflection differential equations of equilibrium on a high-speed computer.

The following parameters were investigated for 3:1 transitions: the ratio of the large-end wall thickness of the tapered cylinder to its mean radius (ranging from 0.01 to 0.36), for ratio ρ of large-end wall thickness to small-end wall thickness (ranging from 1.5 to 5.0). A table of large- and small-end real arguments is included for determining transitions different from 3:1.

I. INTRODUCTION

The use of transitions for connecting two shells of different wall thicknesses to minimize discontinuity stresses at the junction of the shells has been recognized and is used in the design of pressure vessels. The A.S.M.E. pressure vessel code specifies how and where these transitions are to be used for coded vessels. The designer, when analyzing coded or non-coded pressure vessels requiring a detailed stress analysis, must depend on code recommendations or attempt to approximate the effects of these tapered cylinders as rings. This Report presents a method for obtaining the edge-bending loads at the discontinuity junctions.

The differential equations describing the behavior of cylindrical shells with linearly varying wall thickness can be obtained from Ref. 1 and 2. However, to the author's knowledge, a solution of these equations in the usable form of tabulated influence coefficients has not been obtained. The differential equations presented by Hetenyi (Ref. 1) and Timoshenko (Ref. 2) have been modified to solve for these coefficients for a 3:1 transition and Poisson's ratio of 0.3. A method is presented for modifying the resultant influence coefficients for other parameters.

In view of the desirability of tabulated coefficients, calculations were performed for a large number of shells. The equations used in the analysis, the geometric characteristics of these various shells, and the method used to obtain the influence coefficients are discussed in this Report. The large- and small-end real arguments and the final coefficients are given in Tables 1 and 2. A method for determining the effect of eccentric loading for uniform internal pressure has been presented.

II. LOADING AND GEOMETRIC CHARACTERISTICS OF THE SHELLS INVESTIGATED

The free-body diagram of a typical pressure vessel configuration is shown in Fig. 1. The changes in cross section introduce discontinuity stresses at junctions I and II. The structures are identified for the purpose of clarity as follows: structure (2) is a short or long cylindrical shell, or surface of revolution; structure (3) is a tapered shell; and structure (4) is a short or long cylindrical shell or surface of revolution.

In this analysis, positive radial displacements are considered radially outward from the centerline. Positive rotations are considered counterclockwise. At the upper and lower ends of structure (3), M_0, Q_0 and M_L, Q_L are the axisymmetrical moments [in. (lb/in.)] and shears (lb/in.) respectively, and y_0, θ_0 and y_L, θ_L are the displacements and angular rotations, respectively.

The ranges of the various parameters are as follows (see Fig. 1 for applicable notation):

$$\frac{t_4}{R_3} = 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.10, 0.12, 0.14, 0.16, 0.18, 0.20, 0.22, 0.24, 0.26, 0.28, 0.30, 0.32, 0.34 \text{ and } 0.36$$

$$\rho = \frac{t_4}{t_2} = 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5 \text{ and } 5.0$$

III. DETERMINATION OF INFLUENCE COEFFICIENTS

In the case of structure (3), bent by edge moments and shears distributed along both edges, the displacement and rotation at the upper edge due to a unit moment or shear at the lower edge must be considered. The classical differential equations of equilibrium for cylinders with linearly varying wall thickness are adequately described in Ref. 1 and 2 and therefore are only partially derived in this Report. Equations (1) through (8) inclusive are the modified forms of these equations.

From Fig. 2, for small angles,

$$t_x = \alpha x$$

Therefore

$$t_2 = \alpha x_0$$

and

$$t_4 = \alpha (x_0 + l_3)$$

where

$$\lambda^4 = \frac{12(1 - \nu^2)}{\alpha^2 R^2}$$

and Z, Z' are the Schleicher functions of the real argument ξ. The prime denotes differentiation with respect to the argument ξ.

The radial displacement, rotation, moment, and shear at the upper end of structure (3) ($x = x_0$) for the real argument

$$\xi_0 = 2 \lambda \sqrt{x_0}$$

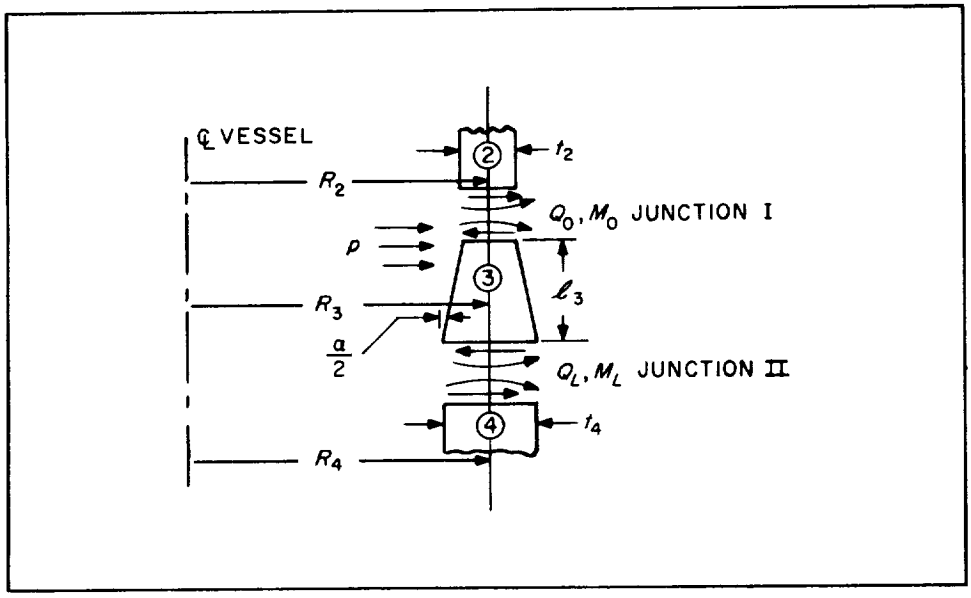

Fig. 1. Free-body diagram of a cylinder of linearly varying wall thickness

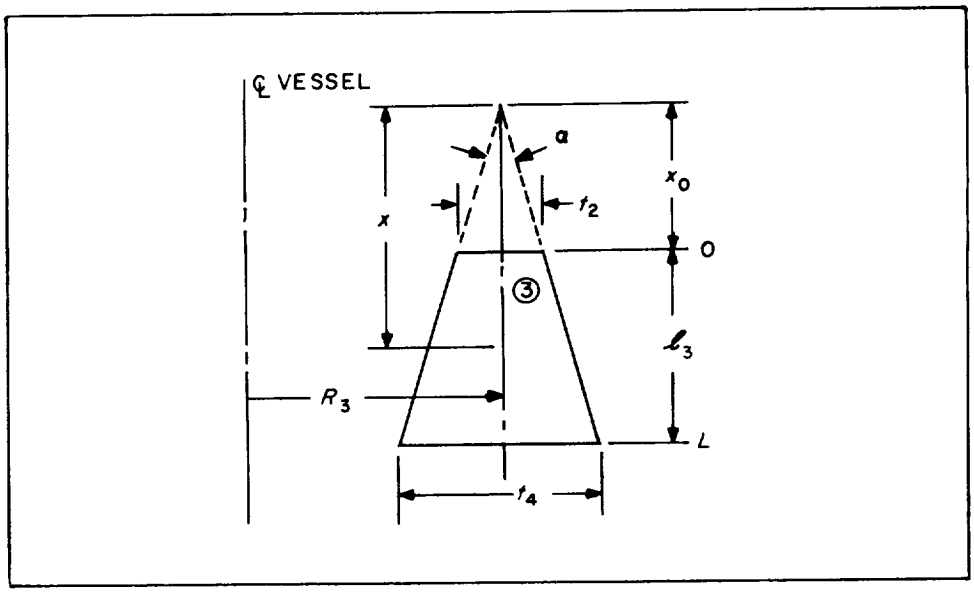

Fig. 2. Nomenclature of a cylinder of linearly varying wall thickness

are

$$K_1 y_0 = \{ C_1 Z_1'(\xi_0) + C_2 Z_2'(\xi_0) + C_3 Z_3'(\xi_0) + C_4 Z_4'(\xi_0) \} \tag{1}$$

$$\begin{aligned} K_3 \theta_0 = \{ & C_1 [\xi_0 Z_2(\xi_0) - 2 Z_1'(\xi_0)] - C_2 [\xi_0 Z_1(\xi_0) + 2 Z_2'(\xi_0)] \\ & + C_3 [\xi_0 Z_4(\xi_0) - 2 Z_3'(\xi_0)] - C_4 [\xi_0 Z_3(\xi_0) + 2 Z_4'(\xi_0)] \} \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{M_0}{E K_5} = \{ & C_1 [\xi_0^2 Z_2'(\xi_0) - 4 \xi_0 Z_2(\xi_0) + 8 Z_1'(\xi_0)] \\ & - C_2 [\xi_0^2 Z_1'(\xi_0) - 4 \xi_0 Z_1(\xi_0) - 8 Z_2'(\xi_0)] \\ & + C_3 [\xi_0^2 Z_4'(\xi_0) - 4 \xi_0 Z_4(\xi_0) + 8 Z_3'(\xi_0)] \\ & - C_4 [\xi_0^2 Z_3'(\xi_0) - 4 \xi_0 Z_3(\xi_0) - 8 Z_4'(\xi_0)] \} \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{Q_0}{E K_7} = \{ & C_1 [\xi_0 Z_1(\xi_0) + 2 Z_2'(\xi_0)] + C_2 [\xi_0 Z_2(\xi_0) - 2 Z_1'(\xi_0)] \\ & + C_3 [\xi_0 Z_3(\xi_0) + 2 Z_4'(\xi_0)] + C_4 [\xi_0 Z_4(\xi_0) - 2 Z_3'(\xi_0)] \} \end{aligned} \tag{4}$$

The radial displacement, rotation, moment, and shear at the lower end ($x = x_0 + l_3$) for the real argument

$$\xi_L = 2 \lambda \sqrt{x_0 + l_3}$$

are

$$K_2 y_L = \{ C_1 Z_1'(\xi_L) + C_2 Z_2'(\xi_L) + C_3 Z_3'(\xi_L) + C_4 Z_4'(\xi_L) \} \tag{5}$$

$$\begin{aligned} K_4 \theta_L = \{ & C_1 [\xi_L Z_2(\xi_L) - 2 Z_1'(\xi_L)] - C_2 [\xi_L Z_1(\xi_L) + 2 Z_2'(\xi_L)] \\ & + C_3 [\xi_L Z_4(\xi_L) - 2 Z_3'(\xi_L)] - C_4 [\xi_L Z_3(\xi_L) + 2 Z_4'(\xi_L)] \} \end{aligned} \tag{6}$$

$$\frac{M_L}{E K_6} = \{ C_1 [\xi_L^2 Z_2'(\xi_L) - 4 \xi_L Z_2(\xi_L) + 8 Z_1'(\xi_L)]$$
$$- C_2 [\xi_L^2 Z_1'(\xi_L) - 4 \xi_L Z_1(\xi_L) - 8 Z_2'(\xi_L)]$$
$$+ C_3 [\xi_L^2 Z_4'(\xi_L) - 4 \xi_L Z_4(\xi_L) + 8 Z_3'(\xi_L)]$$
$$- C_4 [\xi_L^2 Z_3'(\xi_L) - 4 \xi_L Z_3(\xi_L) - 8 Z_4'(\xi_L)] \} \tag{7}$$

$$\frac{Q_L}{E K_8} = \{ C_1 [\xi_L Z_1(\xi_L) + 2 Z_2'(\xi_L)] + C_2 [\xi_L Z_2(\xi_L) - 2 Z_1'(\xi_L)]$$
$$+ C_3 [\xi_L Z_3(\xi_L) + 2 Z_4'(\xi_L)] + C_4 [\xi_L Z_4(\xi_L) - 2 Z_3'(\xi_L)] \} \tag{8}$$

where $K_1 \cdots K_8$ are defined as follows:

$$\left.\begin{aligned}
K_1 &= (x_0)^{1/2} & K_5 &= \frac{\alpha^3 (x_0)^{1/2}}{48(1-\nu^2)} \\
K_2 &= (x_0 + l_3)^{1/2} & K_6 &= \frac{\alpha^3 (x_0 + l_3)^{1/2}}{48(1-\nu^2)} \\
K_3 &= 2(x_0)^{3/2} & K_7 &= \frac{\alpha^3 \lambda^2 (x_0)^{1/2}}{24(1-\nu^2)} \\
K_4 &= 2(x_0 + l_3)^{3/2} & -K_8 &= \frac{\alpha^3 \lambda^2 (x_0 + l_3)^{1/2}}{24(1-\nu^2)}
\end{aligned}\right\} \tag{9}$$

For small angles of α

$$t_x \cong \alpha x$$

For 3:1 transitions (i.e., $\alpha \cong 1/3$)

$$x_0 = 3t_2$$

and

$$(x_0 + l_3) = 3t_4$$

and substituting the above expressions in Eq. (9) we arrive at Eq. (10):

$$\left.\begin{aligned}
K_1 &= 1.732\,(t_2)^{1/2} & K_5 &= 0.001469\,(t_2)^{1/2} \\
K_2 &= 1.732\,(t_4)^{1/2} & K_6 &= 0.001469\,(t_4)^{1/2} \\
K_3 &= 6\,(t_2)^{3/2} & K_7 &= \frac{0.02912\,(t_2)^{1/2}}{R_3} \\
K_4 &= 6\,(t_4)^{3/2} & K_8 &= -\frac{0.02912\,(t_4)^{1/2}}{R_3}
\end{aligned}\right\} \tag{10}$$

The solutions of y_0, y_L and θ_0, θ_L in terms of Q_0, Q_L and M_0, M_L are obtained by solving for the constants of integration in Eq. (1) through (8) and arranging the results into the following form:

$$E \begin{vmatrix} K_1 y_0 \\ K_3 \theta_0 \\ K_2 y_L \\ -K_4 \theta_L \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \\ b_9 & b_{10} & b_{11} & b_{12} \\ b_{13} & b_{14} & b_{15} & b_{16} \end{vmatrix} \begin{vmatrix} \dfrac{Q_0}{K_7} \\ \dfrac{M_0}{K_5} \\ \dfrac{Q_L}{K_8} \\ \dfrac{M_L}{K_6} \end{vmatrix} \tag{11}$$

where $b_1 \cdots b_{16}$ are the influence coefficients obtained from Table 2.

IV. RADIAL DISPLACEMENT AND ROTATION DUE TO INTERNAL PRESSURE LOADING

Since structure (3) is considered as a tapered beam on elastic supports, it must satisfy the following differential equation:

$$\frac{d^2}{dx^2}\left(E\,I\,\frac{d^2y}{dx^2}\right) + k\,y = q \tag{12}$$

For a tapered cylinder with a moment of inertia

$$I_{(\text{per unit width})} = \frac{t\,x^3}{12\,(1-\nu^2)}$$

and a modulus of foundation

$$k = \frac{E\,t_x}{R\,x^2}$$

loaded by internal pressure

$$q = p\left(1 - \frac{\nu}{2}\right)$$

and a variable wall thickness

$$t_x = a\,x$$

substitution of the above into Eq. (12) results in Eq. (13).

$$\frac{d^2}{dx^2}\left[x^3\,\frac{d^2y}{dx^2}\right] + \frac{12(1-\nu^2)}{a^2\,R_x^2}\,xy = q\left[\frac{12(1-\nu^2)}{E\,a^3}\right] \tag{13}$$

The particular integral of Eq. (13) results in the displacement

$$y_x = \frac{R_x^2 p \left(1 - \frac{\nu}{2}\right)}{E \alpha x}$$

and the rotation

$$\frac{dy}{dx} = \theta_x = - \frac{R_x^2 p \left(1 - \frac{\nu}{2}\right)}{E \alpha x^2}$$

The change of slope dy/dx is of higher order effect and is considered negligible (i.e., $dy/dx = 0$).

At the upper end of structure 3

$$x = x_0$$

Therefore

$$y_0 = \frac{R_2^2 p \left(1 - \frac{\nu}{2}\right)}{E \alpha x_0}$$

and

$$E \theta_0 = 0$$

At the lower end of structure 3

$$x = (x_0 + l_3)$$

Therefore

$$y_L = \frac{R_4^2 p \left(1 - \frac{\nu}{2}\right)}{E \alpha (x_0 + l_3)}$$

and

$$E\theta_L = 0$$

For $\alpha = 1/3$

$$x_0 = 3t_2$$

$$(x_0 + l_3) = 3t_4$$

$$\left.\begin{aligned} Ey_0 &= \frac{R_2^2 p \left(1 - \frac{\nu}{2}\right)}{t_2} \quad ; \quad E\theta_0 = 0 \\ Ey_L &= \frac{R_4^2 p \left(1 - \frac{\nu}{2}\right)}{t_4} \quad ; \quad E\theta_L = 0 \end{aligned}\right\} \tag{14}$$

and the final form of Eq. (11) under internal pressure loading is

$$\left.\begin{aligned} Ey_0 &= + \frac{b_1 Q_0}{K_1 K_7} + \frac{b_2 M_0}{K_1 K_5} + \frac{b_3 Q_L}{K_1 K_8} + \frac{b_4 M_L}{K_1 K_6} + \frac{p R_2^2 \left(1 - \frac{\nu}{2}\right)}{t_2} \\ E\theta_0 &= + \frac{b_5 Q_0}{K_3 K_7} + \frac{b_6 M_0}{K_3 K_5} + \frac{b_7 Q_L}{K_3 K_8} + \frac{b_8 M_L}{K_3 K_6} + 0 \\ Ey_L &= + \frac{b_9 Q_0}{K_2 K_7} + \frac{b_{10} M_0}{K_2 K_5} + \frac{b_{11} Q_L}{K_2 K_8} + \frac{b_{12} M_L}{K_2 K_6} + \frac{p R_4^2 \left(1 - \frac{\nu}{2}\right)}{t_4} \\ -E\theta_L &= + \frac{b_{13} Q_0}{K_4 K_7} + \frac{b_{14} M_0}{K_4 K_5} + \frac{b_{15} Q_L}{K_4 K_8} + \frac{b_{16} M_L}{K_4 K_6} + 0 \end{aligned}\right\} \tag{15}$$

Table 1. Table of real arguments ξ_L, ξ_0, for determining b_1 ---- b_{16} (Table 2)

$\frac{t_4}{R_3}$	ξ_L	$\rho = 1.5$ ξ_0	$\rho = 2.0$ ξ_0	$\rho = 2.5$ ξ_0	$\rho = 3.0$ ξ_0	$\rho = 3.5$ ξ_0	$\rho = 4.0$ ξ_0	$\rho = 4.5$ ξ_0	$\rho = 5.0$ ξ_0
0.01	1.09	0.89	0.77	0.69	0.63	0.59	0.54	0.51	0.49
0.02	1.54	1.26	1.09	0.97	0.89	0.82	0.77	0.72	0.69
0.03	1.89	1.54	1.33	1.14	1.09	1.01	0.94	0.89	0.84
0.04	2.18	1.78	1.54	1.38	1.26	1.15	1.09	1.03	0.97
0.05	2.44	1.99	1.72	1.54	1.41	1.30	1.22	1.15	1.09
0.06	2.67	2.18	1.89	1.69	1.54	1.42	1.33	1.26	1.19
0.07	2.88	2.35	2.04	1.82	1.66	1.54	1.44	1.36	1.29
0.08	3.08	2.52	2.18	1.98	1.78	1.65	1.54	1.45	1.38
0.09	3.27	2.67	2.31	2.07	1.89	1.75	1.63	1.54	1.46
0.10	3.44	2.81	2.44	2.18	1.99	1.84	1.72	1.62	1.54
0.12	3.77	3.08	2.67	2.39	2.18	2.02	1.89	1.78	1.69
0.14	4.08	3.33	2.88	2.58	2.35	2.18	2.04	1.92	1.82
0.16	4.36	3.56	3.08	2.76	2.52	2.33	2.18	2.05	1.95
0.18	4.62	3.77	3.27	2.92	2.67	2.47	2.31	2.18	2.07
0.20	4.87	3.98	3.44	3.08	2.81	2.60	2.44	2.30	2.18
0.22	5.11	4.17	3.61	3.23	2.95	2.72	2.55	2.41	2.29
0.24	5.34	4.36	3.77	3.37	3.08	2.85	2.67	2.52	2.39
0.26	5.55	4.53	3.93	3.51	3.21	2.97	2.78	2.62	2.48
0.28	5.76	4.70	4.08	3.62	3.33	3.08	2.88	2.72	2.58
0.30	5.96	4.87	4.22	3.77	3.44	3.19	2.98	2.81	2.67
0.32	6.16	5.03	4.36	3.90	3.56	3.29	3.08	2.90	2.76
0.34	6.35	5.18	4.49	4.02	3.67	3.39	3.17	2.99	2.84
0.36	6.53	5.34	4.62	4.13	3.77	3.50	3.27	3.08	2.92

Table 2. Edge influence coefficients for tapered shells

$$\rho = \frac{t_4}{t_2} = 1.5$$

$\frac{t_4}{R_3}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
0.01	18.2411	−207.673	27.9753	254.422	−164.236	1530.85	−169.766	−1875.54	−27.9053	214.011	−17.6739	−262.180	−301.658	2811.88	−311.807	−3444.69
0.02	−7.75357	26.9818	−3.44467	−32.9788	42.7945	−202.321	44.5663	247.211	3.43511	−28.0508	9.23791	34.2933	78.1176	−369.202	81.3679	451.427
0.03	−6.37583	15.8289	−3.8256	−19.4245	37.4275	−117.497	40.4892	144.110	3.80179	−17.0490	7.88214	20.9300	69.1770	−217.054	74.8656	266.534
0.04	−5.38533	9.34259	−4.39143	−11.7454	30.6518	−74.6034	25.4711	93.8934	2.88982	−11.3692	5.40020	14.29692	53.2344	−137.519	46.8949	172.747
0.05	−3.54075	5.14063	−1.94680	−6.29834	20.4282	−38.3811	21.8405	46.9475	1.96212	−5.52476	4.35656	6.77679	37.6295	−70.5826	40.2773	86.6507
0.06	−1.45889	1.90690	−0.62963	−2.27764	5.09442	−13.3162	10.3099	15.8278	−0.20151	−1.75153	2.40553	2.10248	9.30161	−24.2943	18.9564	29.1699
0.07	−2.60239	2.73439	−1.45270	−3.34515	15.1303	−20.4695	16.1206	24.9642	1.46013	−2.92403	3.17971	3.58479	27.8192	−37.5195	29.7034	46.0708
0.08	−2.21583	2.05185	−1.21242	−2.50230	12.9452	−15.6286	13.9468	18.9845	1.19476	−2.18966	2.73005	2.67813	23.5913	−28.3701	25.4907	34.7683
0.09	−2.02525	1.65827	−1.13268	−2.02551	11.8198	−12.4789	12.6035	15.1661	1.13310	−1.76909	2.47371	2.16864	21.6708	−22.7646	23.1908	27.9775
0.10	−1.83613	1.36232	−1.03236	−1.66412	10.7732	−10.3102	11.5155	12.5180	1.03570	−1.45928	2.25381	1.79030	19.7197	−18.7576	21.1698	23.0841
0.12	−1.46160	0.90102	−0.81234	−1.09741	8.47717	−6.84790	9.25293	8.26481	0.79693	−0.97108	1.83855	1.19072	15.4644	−12.3800	16.9939	15.2490
0.14	−1.32059	0.68841	−0.73599	−0.84170	7.75575	−5.21540	8.15084	6.29743	0.76194	−0.74049	1.60889	0.91252	14.2265	−9.44555	15.0682	11.7161
0.16	−0.91721	0.42587	−0.48308	−0.51804	4.89414	−3.17422	5.83377	3.79807	0.36712	−0.43115	1.20029	0.53577	8.76762	−5.60976	10.6878	7.01673
0.18	−1.02403	0.41870	−0.57188	−0.50977	6.02304	−3.19387	6.33840	3.81241	0.58709	−0.44837	1.25250	0.55371	11.0054	−5.72001	11.7466	7.13677
0.20	−0.90172	0.35590	−0.52372	−0.41889	5.02063	−2.63525	5.62933	3.02855	0.40964	−0.34541	1.07555	0.41395	9.13057	−4.69615	10.4345	5.68693
0.22	−0.45020	0.17043	−0.23096	−0.20007	1.88923	−1.28588	3.30742	1.43394	0.00470	−0.15657	0.72402	0.19379	3.27042	−2.19020	6.13456	2.72737
0.24	−0.26753	0.09934	−0.11529	−0.11667	0.55814	−0.74533	2.27275	0.79497	−0.16482	−0.07641	0.54777	0.10152	0.75851	−1.19836	4.22507	1.54465
0.26	−0.17425	0.06553	−0.06156	−0.07053	0.03634	−0.49509	1.76624	0.51450	−0.21262	−0.03761	0.45320	0.06203	−0.35115	−0.70426	3.21896	0.98918
0.28	−0.13960	0.04621	−0.03133	−0.05598	−0.14234	−0.36042	1.47480	0.36945	−0.21317	−0.02006	0.40279	0.04503	−0.73237	−0.45102	2.65718	0.71216
0.30	−0.11622	0.03374	−0.01972	−0.04465	−0.21575	−0.28625	1.32792	0.30014	−0.19175	−0.01208	0.37101	0.03896	−0.87123	−0.31412	2.39122	0.58984
0.32	−0.10426	0.02647	0.02087	−0.03366	−0.21864	−0.21265	1.34126	0.28106	−0.16542	−0.00304	0.35701	0.03287	−1.04411	−0.11662	2.13549	0.44218
0.34	−0.10317	0.02067	0.07524	−0.02378	−0.05681	−0.09277	1.36325	0.26272	−0.08648	0.01844	0.35292	0.03174	−0.94737	−0.10167	1.89107	0.30099
0.36	−0.10959	0.02500	0.12013	−0.01823	−0.10846	−0.08195	1.39108	0.26850	−0.09530	0.02165	0.34547	0.02712	−1.09193	0.12978	1.97796	0.27479

Table 2 (Cont'd)

$$\rho = \frac{t_4}{t_2} = 2.0$$

$\frac{t_4}{R_3}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
0.01	52.1290	-331.956	65.9996	470.233	-196.720	1201.10	-227.367	-1701.57	-65.9803	383.552	-68.1157	-543.295	558.623	3411.04	-645.618	-4831.55
0.02	-4.55331	10.3724	-2.23517	-14.6349	12.3131	-37.3590	13.7578	52.5698	2.23039	-11.5740	6.53081	16.3543	34.6714	-104.912	38.7963	148.371
0.03	-3.55059	5.62488	-2.14949	-7.97949	9.93255	-19.9124	11.2507	28.1043	2.14198	-6.35624	5.10070	9.04274	28.4508	-56.7477	32.3187	80.8591
0.04	-5.00761	5.19754	-2.90724	-7.54779	16.3613	-19.0940	13.4864	27.5769	4.51975	-6.13263	5.46952	8.92324	46.4744	-53.8810	38.3710	78.5989
0.05	-2.15135	2.05440	-1.30059	-2.89971	6.08452	-7.39044	6.85140	10.2890	1.3065	-2.31714	3.08216	3.29591	17.2863	-20.7097	19.6158	29.5841
0.06	-1.81964	1.46735	-1.09878	-2.05875	5.24114	-5.39629	5.88620	7.43120	1.09897	-1.64931	2.59507	2.33734	14.6802	-14.8353	16.6630	21.1614
0.07	-1.56727	1.08868	-0.94464	-1.52288	4.53051	-4.03644	5.07280	5.50682	0.94482	-1.21948	2.23206	1.73035	12.6366	-10.9804	14.3523	15.7049
0.08	-1.16422	0.76206	-0.72276	-1.03879	3.03700	-2.76578	3.95120	3.62850	0.50033	-0.80759	1.78505	1.12373	8.44409	-7.44460	11.2447	10.4562
0.09	-1.21231	0.65219	-0.73125	-0.90899	3.47972	-2.42646	3.87186	3.24204	0.73160	-0.72610	1.72790	1.03682	9.72565	-6.50140	11.0870	9.40924
0.10	-1.11146	0.54612	-0.67067	-0.75649	3.25447	-2.08166	3.60905	2.74593	0.67221	-0.60662	1.57957	0.86449	8.96433	-5.45996	10.2269	7.90509
0.12	-0.90898	0.37307	-0.54948	-0.51222	2.64676	-1.44169	2.94756	1,84267	0.54439	-0.41300	1.31156	0.59166	7.24697	-3.67586	8.42200	5.38842
0.14	-0.77714	0.27197	-0.46593	-0.37091	2.24735	-1.06054	2.45471	1.30922	0.46311	-0.29606	1.10954	0.42840	6.15980	-2.63214	7.13981	3.93406
0.16	-0.63927	0.20168	-0.38556	-0.27138	1.82122	-0.80580	2.06382	0.95145	0.95145	0.34329	-0.21208	0.94099	0.31128	4.84637	-1.89449	6.00928
0.18	-0.61389	0.169577	-0.36333	-0.22648	1.81771	-0.70152	1.93339	0.80309	0.36473	-0.18072	0.87131	0.26517	4.84071	-1.60071	5.65553	2.47474
0.20	-0.55409	0.14277	-0.32935	-0.18349	1.61494	-0.59649	1.71354	0.63592	0.30994	-0.14794	0.77921	0.21256	4.29730	-1.32808	5.11567	2.02740
0.22	-0.41624	0.10714	-0.25926	-0.13356	1.17225	-0.47088	1.41018	0.45857	0.16902	-0.10387	0.65577	0.15317	2.89854	-0.95678	4.22353	1.50059
0.24	-0.30408	0.08376	-0.20456	-0.09854	0.80599	-0.38553	1.16774	0.33361	0.04187	-0.07137	0.55274	0.10975	1.67588	-0.70842	3.5142	1.12624
0.26	-0.23684	0.06727	-0.16885	-0.07279	0.59836	-0.32810	1.01786	0.26090	-0.02571	-0.04968	0.48333	0.08417	0.93129	-0.53427	3.03309	0.89589
0.28	-0.19507	0.05398	-0.13980	-0.06087	0.47923	-0.28204	0.89335	0.20504	-0.05774	-0.03426	0.43365	0.06692	0.47190	-0.39326	2.63965	0.71953
0.30	-0.16850	0.04583	-0.12788	-0.05203	0.41107	-0.25105	0.80955	0.17117	-0.07470	-0.02388	0.39606	0.05683	0.16595	-0.30496	2.42402	0.62726
0.32	-0.14164	0.03402	-0.10185	-0.04366	0.29895	-0.20771	0.81200	0.15935	-0.09432	-0.00844	0.37257	0.04843	-0.33198	-0.1518	2.18572	0.51433
0.34	-0.11588	0.02074	-0.06584	-0.03436	0.22349	-0.14726	0.81505	0.14918	-0.07849	0.01076	0.35452	0.04231	-0.77269	0.04422	1.90072	0.39653
0.36	-0.11411	0.01918	-0.04167	-0.03035	0.18209	-0.13763	0.85788	0.15526	-0.07817	0.01201	0.34049	0.03839	-0.77236	0.05624	1.86436	0.36727

Table 2 (Cont'd)

$$\rho = \frac{t_4}{t_2} = 2.5$$

$\frac{t_4}{R_3}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
0.01	−13.6828	59.5723	−11.8700	−94.1191	28.3714	−140.055	33.8457	221.079	11.8756	−71.0952	21.2115	112.3718	111.866	−551.737	133.508	872.230
0.02	−3.69902	7.09162	−2.16983	−11.2277	6.67380	−16.3869	7.86960	25.7447	2.17015	−8.36236	6.06020	13.2868	26.6268	−64.8765	31.5097	103.251
0.03	−2.52992	3.15306	−1.71775	−5.19672	4.09426	−6.3673	4.93789	10.2725	1.71500	−3.79844	4.32137	6.31913	18.5448	−28.2314	22.5760	47.1499
0.04	−2.78256	2.41270	−1.65978	−3.89445	5.67965	−5.81746	5.40725	9.18621	2.28964	−2.90917	3.77753	4.73636	22.3398	−22.3206	21.4242	36.5703
0.05	−1.65072	1.31036	−1.07146	−2.04961	3.10919	−3.15104	3.6438	4.73248	1.07262	−1.53688	2.67665	2.45089	12.2110	−11.8824	14.5901	19.1167
0.06	−1.38656	0.92649	−0.89668	−1.43788	2.64313	−2.28157	3.07914	3.34704	0.89485	−1.07787	2.23905	1.71894	10.2390	−8.35484	12.2572	13.4902
0.07	−1.19289	0.68523	−0.76987	−1.05839	2.27182	−1.70833	2.62441	2.44505	0.77124	−0.79256	1.92467	1.27032	8.78800	−6.12316	10.5345	9.98192
0.08	−1.07885	0.55702	−0.68633	−0.84217	2.18126	−1.48859	2.49571	2.06653	0.68525	−0.63678	1.70902	1.00497	7.98417	−5.00346	9.53601	8.05724
0.09	−0.93163	0.42028	−0.59744	−0.63783	1.79965	−1.09718	2.04497	1.47520	0.59731	−0.47768	1.49698	0.77007	6.82256	−3.68529	8.23549	6.10814
0.10	−0.81384	0.34792	−0.53135	−0.51306	1.50539	−0.91686	1.80632	1.16442	0.45392	−0.38261	1.32562	0.60523	5.64795	−2.99385	7.31013	4.88413
0.12	−0.70159	0.24349	−0.44874	−0.35773	1.38164	−0.68616	1.53994	0.82454	0.44358	−0.26957	1.13634	0.44025	5.05473	−2.05557	6.26645	3.52107
0.14	−0.60272	0.18013	−0.37940	−0.25892	1.19685	−0.53366	1.28558	0.58635	0.37871	−0.19340	0.96511	0.32153	4.30903	−1.46926	5.35454	2.61076
0.16	−0.51192	0.13917	−0.32320	−0.19372	1.02488	−0.43736	1.10182	0.43489	0.29853	−0.14255	0.83434	0.24190	3.51469	−1.08355	4.63374	1.99625
0.18	−0.45585	0.10812	−0.25575	−0.14411	0.95371	−0.37229	0.93504	0.33205	0.27277	−0.10916	0.72127	0.18707	3.16549	−0.81472	3.96105	1.56851
0.20	−0.43401	0.09757	−0.26340	−0.12549	0.89950	−0.34018	0.87831	0.27436	0.25671	−0.09561	0.68142	0.16195	2.98601	−0.71841	3.86696	1.37990
0.22	−0.36116	0.08152	−0.22651	−0.09818	0.76177	−0.30280	0.76310	0.21141	0.17625	−0.07377	0.60223	0.12774	2.28644	−0.56889	3.42430	1.11947
0.24	−0.29554	0.06921	−0.19433	−0.07722	0.63069	−0.27407	0.65830	0.16178	0.09393	−0.05638	0.53157	0.10066	1.59762	−0.46172	3.03026	0.91549
0.26	−0.25233	0.06046	−0.17159	−0.06345	0.55419	−0.25317	0.59124	0.13134	0.04224	−0.04343	0.48045	0.08309	1.12217	−0.37878	2.74987	0.78012
0.28	−0.22195	0.05170	−0.14959	−0.05094	0.49916	−0.23172	0.51211	0.10095	0.012185	−0.03216	0.43899	0.06908	0.76640	−0.29108	2.47042	0.65548
0.30	−0.19915	0.04636	−0.13885	−0.04459	0.47376	−0.22031	0.48259	0.08391	0.012276	−0.02456	0.40619	0.06058	0.50110	−0.24432	2.33728	0.59668
0.32	−0.16842	0.03856	−0.12662	−0.03995	0.42090	−0.20351	0.51329	0.08964	−0.04792	−0.01145	0.38700	0.05384	−0.00933	−0.14152	2.20868	0.52419
0.34	−0.13672	0.02730	−0.10765	−0.03473	0.34383	−0.17249	0.53941	0.08971	−0.06883	0.00632	0.36773	0.04752	−0.54903	0.01123	2.01336	0.44118
0.36	−0.13216	0.02574	−0.09481	−0.03240	0.32710	−0.16803	0.57731	0.09570	−0.07062	0.00735	0.35195	0.04354	−0.59636	0.01935	1.96838	0.41441

Table 2 (Cont'd)

$$\rho = \frac{t_4}{t_2} = 3.0$$

$\frac{t_4}{t_3}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
0.01	−12.4437	48.4195	−11.6067	−83.7962	19.2377	−83.4406	23.7352	144.158	11.6249	−59.8193	20.8425	103.5961	99.6757	−431.589	123.063	747.633
0.02	− 3.17035	5.50146	− 2.00972	− 9.47554	4.35475	− 9.55598	5.31871	16.2124	2.00698	− 6.71302	5.70983	11.63200	22.4538	− 48.5344	27.5895	84.2999
0.03	− 2.28284	2.69448	− 1.57490	− 4.63311	3.19741	− 4.73200	3.92267	7.89045	1.57131	− 3.30034	4.10780	5.74608	16.5264	− 23.7187	20.5296	41.4962
0.04	− 2.12641	1.67183	− 1.35187	− 2.93848	2.37204	− 3.02637	3.27639	5.06715	1.75043	− 2.07174	3.3302	3.70733	16.30234	− 14.7488	17.1009	26.6519
0.05	− 1.38157	0.98859	− 0.94783	− 1.67128	1.96740	− 1.83247	2.37323	2.85643	0.94962	− 1.19408	2.47831	2.08852	9.96189	− 8.55500	12.4323	15.1635
0.06	− 1.15254	0.69258	− 0.79019	− 1.16093	1.64173	− 1.31625	1.96238	1.96627	0.78978	− 0.82766	2.06941	1.45722	8.27472	− 5.91252	10.38922	10.61077
0.07	− 0.99175	0.51509	− 0.67698	− 0.85408	1.42002	− 1.01102	1.67540	1.43854	0.67758	− 0.60810	1.77864	1.07770	7.08806	− 4.3303	8.93824	7.87606
0.08	− 0.87096	0.40004	− 0.58980	− 0.65238	1.26556	− 0.81996	1.47242	1.10390	0.58874	− 0.46504	1.55600	0.82623	6.18453	− 3.30854	7.83990	6.07858
0.09	− 0.77882	0.32013	− 0.52523	− 0.51447	1.14347	− 0.68268	1.30695	0.86711	0.52571	− 0.36626	1.38526	0.65558	5.50657	− 2.59838	7.01014	4.85219
0.10	− 0.70824	0.26558	− 0.47561	− 0.42034	1.05279	− 0.59008	1.18399	0.70751	0.47667	− 0.29913	1.25678	0.53947	4.98134	− 2.11473	6.38281	4.01527
0.12	− 0.58942	0.19176	− 0.39401	− 0.28748	0.85522	− 0.45752	0.96246	0.46935	0.35773	− 0.20670	1.04307	0.36992	3.91352	− 1.45933	5.31381	2.79634
0.14	− 0.50673	0.14163	− 0.33045	− 0.20620	0.78076	− 0.37320	0.80317	0.33308	0.33004	− 0.14575	0.89393	0.27502	3.43332	− 1.00679	4.57951	2.10393
0.16	− 0.43878	0.11271	− 0.28427	− 0.15566	0.69740	− 0.32459	0.69703	0.24980	0.26656	− 0.10872	0.77978	0.21061	2.84379	− 0.74922	4.02208	1.64524
0.18	− 0.40569	0.09347	− 0.25402	− 0.12334	0.66716	− 0.29202	0.61684	0.19470	0.25454	− 0.08648	0.70344	0.17183	2.63489	− 0.58221	3.66672	1.36326
0.20	− 0.37092	0.08057	− 0.22636	− 0.09847	0.62277	− 0.26900	0.54061	0.15074	0.22359	− 0.07114	0.63459	0.14059	2.35910	− 0.47714	3.34790	1.14303
0.22	− 0.32014	0.06940	− 0.18977	− 0.07703	0.56870	− 0.25163	0.47670	0.11960	0.16329	− 0.05589	0.56283	0.11303	1.87563	− 0.38476	2.98683	0.95169
0.24	− 0.27927	0.06278	− 0.17533	− 0.06313	0.51031	− 0.23754	0.41678	0.09108	0.10521	− 0.04553	0.51404	0.09404	1.42116	− 0.33233	2.76676	0.81576
0.26	− 0.24895	0.05637	− 0.15752	− 0.05252	0.47954	− 0.22651	0.37717	0.07414	0.06449	−0. 03630	0.47137	0.07991	1.07217	− 0.28062	2.56283	0.71460
0.28	− 0.22573	0.05022	− 0.13972	− 0.04296	0.45505	− 0.21580	0.33659	0.05856	0.03520	− 0.02834	0.43541	0.06793	0.78638	− 0.22883	2.35570	0.61837
0.30	− 0.20884	0.04617	− 0.12906	− 0.03696	0.44555	− 0.20827	0.30196	0.04758	0.01467	− 0.02225	0.40571	0.06046	0.57644	− 0.19408	2.24837	0.56983
0.32	− 0.18646	0.04115	− 0.12502	− 0.03453	0.42979	− 0.20011	0.33198	0.05095	−0.02179	− 0.01206	0.39258	0.05550	0.13991	− 0.12549	2.19766	0.52101
0.34	− 0.15491	0.03226	− 0.11597	− 0.03160	0.37924	− 0.18380	0.36017	0.05360	−0.05715	0.00334	0.37742	0.05040	−0.41268	− 0.00812	2.08745	0.46318
0.36	− 0.15053	0.03081	− 0.10844	− 0.03015	0.37427	− 0.18102	0.39336	0.05894	−0.05973	0.00418	0.36154	0.04648	−0.45459	− 0.00256	2.04558	0.43810

Table 2 (Cont'd)

$$\rho = \frac{t_4}{t_2} = 3.5$$

$\frac{t_4}{R_3}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
0.01	−9.18583	32.7147	−8.40307	−60.4315	11.3947	−46.1030	14.6027	84.8782	8.41133	−41.9586	17.0291	77.6007	71.8542	−289.750	92.1945	536.103
0.02	−2.70729	4.34614	−1.81707	− 8.10200	2.9223	− 5.93116	3.71440	10.7637	1.81671	− 5.52305	5.41454	10.3933	19.2112	− 37.9576	24.6490	71.6594
0.03	−1.97371	2.17002	−1.42959	− 3.99922	2.20907	− 3.07342	2.80181	5.37635	1.42712	− 2.75058	3.90156	5.16602	14.2798	− 18.8588	18.4581	35.6464
0.04	−1.70521	1.24594	−1.15428	− 2.37528	1.92533	− 1.77298	2.12216	3.07714	1.43511	− 1.59231	3.05669	3.13340	12.8973	− 10.7407	14.6136	21.4249
0.05	−1.19244	0.79602	−0.86432	− 1.43971	1.34624	− 1.21176	1.66763	1.90772	0.86549	− 0.98697	2.35788	1.88107	8.57908	− 6.72129	11.19818	13.03944
0.06	−0.99533	0.56019	−0.71878	− 0.99857	1.12883	− 0.89236	1.37639	1.31015	0.71813	− 0.68271	1.96773	1.31230	7.11530	− 4.63319	9.35597	9.13593
0.07	−0.86715	0.42535	−0.61974	− 0.74187	1.00915	− 0.72138	1.20487	0.98510	0.62040	− 0.50825	1.69818	0.97862	6.15823	− 3.44644	8.11664	6.86478
0.08	−0.76174	0.33148	−0.53946	− 0.56529	0.90176	− 0.59657	1.05359	0.75066	0.53910	− 0.38724	1.48535	0.75039	5.36195	− 2.61766	7.11919	5.30072
0.09	−0.67999	0.26602	−0.47845	− 0.44376	0.81445	− 0.50654	0.92760	0.58352	0.47864	− 0.30316	1.32098	0.59431	4.74795	− 2.03937	6.35508	4.22740
0.10	−0.61783	0.22147	−0.43193	− 0.36102	0.75041	− 0.44585	0.83333	0.47098	0.43276	− 0.24625	1.19779	0.48868	4.27787	− 1.64666	5.78225	3.49821
0.12	−0.51878	0.16072	−0.35729	− 0.24683	0.65427	− 0.36438	0.68905	0.31854	0.35583	− 0.16893	1.00551	0.34284	3.50020	− 1.11079	4.87426	2.48280
0.14	−0.45699	0.12466	−0.30219	− 0.17844	0.56815	− 0.31139	0.56401	0.21928	0.28517	− 0.12240	0.85268	0.25013	2.90806	− 0.80563	4.18816	1.86619
0.16	−0.39118	0.09945	−0.25713	− 0.13275	0.52906	− 0.27937	0.48394	0.16180	0.24363	− 0.08865	0.74809	0.19375	2.4388	− 0.56966	3.69394	1.47054
0.18	−0.36131	0.08365	−0.22781	− 0.10428	0.51077	− 0.25828	0.42431	0.12393	0.22816	− 0.06952	0.67355	0.15772	2.22745	− 0.43314	3.36603	1.22135
0.20	−0.33249	0.07322	−0.20187	− 0.08274	0.48691	− 0.24359	0.36900	0.09438	0.20109	− 0.05698	0.60878	0.12969	1.99160	− 0.35096	3.08380	1.03137
0.22	−0.29562	0.06523	−0.17771	− 0.06570	0.47538	− 0.23199	0.31830	0.07115	0.15404	− 0.04553	0.55087	0.10710	1.61520	− 0.28555	2.82391	0.87759
0.24	−0.27710	0.06351	−0.16182	− 0.05448	0.46666	− 0.22388	0.28936	0.05766	0.10048	− 0.03560	0.50022	0.08930	1.19877	− 0.24036	2.59566	0.75686
0.26	−0.24172	0.05484	−0.13960	− 0.04431	0.42387	− 0.21778	0.25916	0.04630	0.07085	− 0.03105	0.46165	0.07716	0.99092	− 0.22122	2.43675	0.67732
0.28	−0.22385	0.05024	−0.12694	− 0.03629	0.41384	− 0.21047	0.22564	0.03426	0.04539	− 0.02468	0.43096	0.06662	0.74958	− 0.18283	2.27300	0.59384
0.30	−0.21154	0.04667	−0.11710	− 0.03122	0.41219	− 0.20490	0.20172	0.02715	0.02694	− 0.01987	0.40358	0.05987	0.57838	− 0.15866	2.18931	0.55350
0.32	−0.19327	0.04264	−0.11539	− 0.02938	0.40530	− 0.19978	0.20000	0.02910	−0.00744	− 0.01170	0.39414	0.05916	0.20252	− 0.10980	2.17804	0.51582
0.34	−0.16740	0.03627	−0.11231	− 0.02789	0.38235	− 0.19090	0.24689	0.03259	−0.04701	0.00141	0.38327	0.05191	− 0.31248	− 0.01916	2.12766	0.47362
0.36	−0.16351	0.03466	−0.10843	− 0.02716	0.38405	− 0.18816	0.27836	0.03757	−0.05030	0.00219	0.36743	0.04803	− 0.35406	− 0.01476	2.08863	0.44947

Table 2 (Cont'd)

$$\rho = \frac{t_4}{t_2} = 4.0$$

$\frac{t_4}{R_3}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
0.01	−6.72423	21.7843	−6.12138	−43.9274	6.35882	−23.8352	8.57971	47.7248	6.13009	−29.4252	14.2681	59.4659	52.2427	−194.447	70.6445	393.212
0.02	−2.48968	3.83747	−1.80119	− 7.60204	2.27413	− 4.41293	2.99103	8.41141	1.79949	− 5.04382	5.38830	10.11486	18.01779	− 33.6414	23.9898	67.7157
0.03	−1.76099	1.83774	−1.35666	− 3.62959	1.62314	− 2.17014	2.12856	3.95194	1.35551	− 2.40848	3.80791	4.87828	12.9468	− 15.9739	17.4268	32.6072
0.04	−1.52547	1.07862	−1.07453	− 2.15336	1.48233	− 1.36021	1.68906	2.38198	1.31200	− 1.40389	2.95181	2.92247	11.5809	− 9.24723	13.6880	19.5564
0.05	−1.06788	0.68399	−0.80750	− 1.29980	1.01879	− 0.91818	1.28688	1.42766	0.80847	− 0.86465	2.28532	1.76302	7.74439	− 5.71046	10.4956	11.89488
0.06	−0.89068	0.48254	−0.6796	− 0.89964	0.85319	− 0.68887	1.05414	0.97184	0.67063	− 0.59606	1.90760	1.23044	6.41193	− 3.91771	8.77203	8.34018
0.07	−0.77452	0.36666	−0.57686	− 0.66535	0.76078	− 0.50516	0.91417	0.72221	0.57763	− 0.44098	1.64439	0.91560	5.52379	− 2.88933	7.59388	6.25159
0.08	−0.67939	0.28659	−0.50072	− 0.50480	0.67903	− 0.47610	0.79194	0.54389	0.50034	− 0.33422	1.43758	0.70132	4.78818	− 2.17786	6.65367	4.82254
0.09	−0.60531	0.23083	−0.44252	− 0.39446	0.61299	− 0.41229	0.69015	0.41751	0.43346	− 0.27330	1.32000	0.57200	4.4335	− 1.72100	5.93485	4.06205
0.10	−0.55432	0.19494	−0.40064	− 0.32238	0.57707	− 0.37411	0.62784	0.34137	0.40134	− 0.21234	1.16063	0.45822	3.81980	− 1.36603	5.42201	3.20280
0.12	−0.46876	0.14405	−0.33115	− 0.22015	0.51356	− 0.31814	0.51864	0.22990	0.33014	− 0.14533	0.97559	0.32255	3.12407	− 0.91620	4.58535	2.28685
0.14	−0.40719	0.11223	−0.27684	− 0.15736	0.46642	− 0.28159	0.42534	0.15823	0.27657	− 0.10244	0.83110	0.23754	2.62011	− 0.63480	3.95486	1.73099
0.16	−0.36842	0.09293	−0.23950	− 0.11871	0.42215	− 0.25755	0.35794	0.11318	0.21920	− 0.07778	0.72741	0.18332	2.16378	− 0.48155	3.50425	1.37430
0.18	−0.33116	0.07881	−0.20875	− 0.09170	0.42085	− 0.24379	0.31216	0.08572	0.20903	− 0.05845	0.65529	0.14953	1.95822	− 0.34229	3.19129	1.14295
0.20	−0.30774	0.07004	−0.18486	− 0.07278	0.41147	− 0.23372	0.27333	0.06529	0.18511	− 0.04810	0.59373	0.12364	1.75761	− 0.27718	2.93776	0.97266
0.22	−0.27778	0.06342	−0.16250	− 0.05760	0.39395	− 0.22542	0.23306	0.04804	0.14405	− 0.03852	0.53949	0.10298	1.43480	− 0.22519	2.70754	0.83544
0.24	−0.25366	0.05893	−0.14458	− 0.04666	0.38489	− 0.21958	0.20467	0.03655	0.10297	− 0.03254	0.49411	0.08723	1.14605	− 0.20350	2.52275	0.73232
0.26	−0.23564	0.05483	−0.13013	− 0.03865	0.38113	− 0.21437	0.18155	0.02820	0.07298	− 0.02709	0.45836	0.07582	0.91427	− 0.17994	2.37803	0.65621
0.28	−0.22110	0.05089	−0.11548	− 0.03135	0.38003	− 0.20942	0.15867	0.02082	0.04962	− 0.02170	0.42750	0.06566	0.70323	− 0.15001	2.22039	0.57917
0.30	−0.21144	0.04778	−0.10479	− 0.02688	0.38370	− 0.20541	0.14320	0.01671	0.03284	− 0.01764	0.40077	0.05925	0.55593	− 0.13141	2.14542	0.54319
0.32	−0.19723	0.04442	−0.10586	− 0.02550	0.38341	− 0.20129	0.15402	0.01732	0.00079	− 0.01103	0.39438	0.05598	0.22897	− 0.09665	2.16197	0.51184
0.34	−0.17575	0.039417	−0.10547	− 0.02461	0.37281	− 0.19570	0.17481	0.20145	−0.03932	0.00017	0.38668	0.05273	− 0.24324	− 0.02528	2.14960	0.47890
0.36	−0.17267	0.03790	−0.10335	− 0.02410	0.37818	− 0.19343	0.19840	0.23738	−0.04222	0.00079	0.37127	0.04896	− 0.27675	− 0.02346	2.11475	0.45579

Table 2 (Cont'd)

$$\rho = \frac{t_4}{t_2} = 4.5$$

$\frac{t_4}{R_3}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
0.01	-11.2908	37.5023	-13.0401	-80.2043	9.76039	-34.9492	13.6256	74.3683	-13.0513	-52.3940	24.0886	112.212	95.3609	-339.743	133.314	727.895
0.02	- 2.2368	3.30767	- 1.71325	- 6.98509	1.71468	- 3.21161	2.33469	6.40988	1.71304	- 4.50315	5.28236	9.66423	16.5640	- 29.3212	22.9197	63.1997
0.03	- 1.60124	1.61564	- 1.28040	- 3.34870	1.27844	- 1.68512	1.71842	3.12938	1.27814	- 2.16898	3.71483	4.64596	11.9477	- 14.1107	16.5971	30.4954
0.04	- 1.37398	0.94559	- 1.01561	- 1.97941	1.15085	- 1.06687	1.35676	1.86700	1.22168	- 1.25821	2.88581	2.78005	10.56769	- 8.11214	13.0642	18.2481
0.05	- 0.96831	0.60253	- 0.76293	- 1.19555	0.79701	- 0.73495	1.02408	1.11040	0.76327	- 0.77445	2.23403	1.68162	7.12015	- 4.99928	10.0115	11.1260
0.06	- 0.81564	0.43207	- 0.63716	- 0.83395	0.68567	- 0.57539	0.85532	0.77173	0.63677	- 0.53879	1.87010	1.18031	5.93901	- 3.46576	8.41459	7.86341
0.07	- 0.70598	0.32783	- 0.54486	- 0.61229	0.60661	- 0.47719	0.73078	0.56370	0.54536	- 0.39518	1.60874	0.87517	5.08191	- 2.52818	7.25874	5.87096
0.08	- 0.61786	0.25678	- 0.47165	- 0.46245	0.53943	- 0.40817	0.62501	0.41815	0.47136	- 0.29752	1.40592	0.66986	4.38661	- 1.88854	6.35497	4.52545
0.09	- 0.55448	0.20964	- 0.41766	- 0.36266	0.49705	- 0.36390	0.55103	0.32480	0.41792	- 0.23259	1.25157	0.53191	3.88091	- 1.46606	5.68778	3.62756
0.10	- 0.50595	0.17728	- 0.37632	- 0.29449	0.46549	- 0.33338	0.49380	0.26081	0.37685	- 0.18823	1.13529	0.43812	3.48847	- 1.17631	5.18435	3.01427
0.12	- 0.40394	0.133128	- 0.31037	- 0.20049	0.41976	- 0.29210	0.40573	0.17401	0.30172	- 0.12820	0.95483	0.30894	3.01720	- 0.78203	4.39180	2.16000
0.14	- 0.37485	0.10544	- 0.25844	- 0.14262	0.38828	- 0.26507	0.32992	0.11809	0.25822	- 0.08967	0.81407	0.22800	2.37440	- 0.53460	3.79573	1.64184
0.16	- 0.33215	0.08832	- 0.22085	- 0.10620	0.36589	- 0.24833	0.27677	0.08386	0.21116	- 0.06583	0.71466	0.17707	1.9608	- 0.38389	3.37125	1.30947
0.18	- 0.32070	0.07634	- 0.19615	- 0.08355	0.35617	- 0.23523	0.23968	0.62286	0.19201	- 0.05261	0.64329	0.14433	1.78494	- 0.29829	3.08731	1.09770
0.20	- 0.28886	0.06864	- 0.17091	- 0.06517	0.35943	- 0.22926	0.20822	0.04666	0.17179	- 0.04135	0.58320	0.11955	1.57766	- 0.22405	2.83935	0.93452
0.22	- 0.26453	0.06299	- 0.15046	- 0.05167	0.35242	- 0.22311	0.17815	0.03407	0.13543	- 0.03343	0.53188	0.10033	1.29960	- 0.18380	2.63320	0.80949
0.24	- 0.24471	0.05903	- 0.13334	- 0.04158	0.34950	- 0.21837	0.15389	0.02496	0.09090	- 0.02846	0.48863	0.08549	1.04773	- 0.16765	2.46324	0.71343
0.26	- 0.22996	0.05421	- 0.11944	- 0.03419	0.35004	- 0.21417	0.13395	0.01835	0.07233	- 0.02385	0.45444	0.07467	0.84442	- 0.14923	2.33051	0.64235
0.28	- 0.21840	0.05193	- 0.10607	- 0.02772	0.35384	- 0.21015	0.11695	0.01309	0.05102	- 0.01938	0.42501	0.06499	0.65959	- 0.12652	2.18661	0.57017
0.30	- 0.21024	0.04910	- 0.09622	- 0.02334	0.35807	- 0.20663	0.09939	0.00877	0.03552	- 0.01585	0.39968	0.05888	0.52696	- 0.11130	2.12111	0.53611
0.32	- 0.19888	0.04632	- 0.09676	- 0.02232	0.36070	- 0.20371	0.11000	0.01008	0.00593	- 0.01031	0.39419	0.05593	0.23878	- 0.08554	2.14917	0.50883
0.34	- 0.18131	0.04220	- 0.09748	- 0.02201	0.36211	- 0.19969	0.13206	0.01361	- 0.0335	- 0.00063	0.38851	0.053130	- 0.19406	- 0.02855	2.15951	0.48115
0.36	- 0.17907	0.04075	- 0.09694	- 0.02147	0.36739	- 0.19762	0.14522	0.01516	- 0.03597	- 0.00016	0.37363	0.04950	- 0.2207	- 0.0268	2.12980	0.45921

Table 2 (Cont'd)

$$\rho = \frac{t_4}{t_2} = 5.0$$

$\frac{t_4}{R_3}$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
0.01	-23.9140	80.3531	27.5161	-179.102	19.3017	-66.9874	27.5161	148.907	31.5318	-114.599	50.7362	255.614	212.878	-736.804	303.689	1643.74
0.02	- 2.07058	2.98948	- 1.63151	- 6.56979	1.42296	- 2.62796	1.98185	5.37579	1.63068	- 4.16207	5.18506	9.32371	15.5751	- 26.7750	22.1124	60.2671
0.03	- 1.45525	1.42646	- 1.21642	- 3.11103	1.00553	- 1.32307	1.38987	2.48700	1.21449	- 1.96956	3.65084	4.47600	11.1011	- 12.5900	15.9896	28.9000
0.04	- 1.22371	0.82238	- 0.94345	- 1.80497	0.88026	- 0.84080	1.07098	1.44429	1.11994	- 1.11611	2.80127	2.62648	9.56464	- 7.06952	12.37560	16.97919
0.05	- 0.88790	0.54170	- 0.72634	- 1.11525	0.64370	- 0.61666	0.83886	0.89424	0.72652	- 0.70605	2.19639	1.62362	6.64083	- 4.48095	9.66625	10.59232
0.06	- 0.74424	0.38790	- 0.60339	- 0.77241	0.54913	- 0.48945	0.68988	0.61072	0.60320	- 0.48726	1.83579	1.13640	5.50567	- 3.07515	8.10146	7.46156
0.07	- 0.65024	0.29883	- 0.51872	- 0.57091	0.49748	- 0.41966	0.59888	0.45342	0.51925	- 0.35999	1.58298	0.84636	4.73901	- 2.26056	7.01986	5.60413
0.08	- 0.57252	0.23693	- 0.44963	- 0.43244	0.45087	- 0.36882	0.51773	0.34005	0.44938	- 0.27209	1.38458	0.64921	4.10209	- 1.69563	6.15898	4.33566
0.09	- 0.51211	0.19375	- 0.39654	- 0.33718	0.41305	- 0.33216	0.44912	0.25923	0.39703	- 0.21091	1.23168	0.51485	3.60966	- 1.30170	5.50523	3.47082
0.10	- 0.46985	0.16564	- 0.35777	- 0.27439	0.39300	- 0.30974	0.40582	0.20980	0.35829	- 0.17120	1.11827	0.42491	3.25068	- 1.04727	5.02805	2.89305
0.12	- 0.40023	0.12605	- 0.29411	- 0.18599	0.35938	- 0.27713	0.33028	0.13809	0.29315	- 0.11575	0.94059	0.29984	2.63866	- 0.68835	4.26228	2.07720
0.14	- 0.35029	0.10134	- 0.24388	- 0.13161	0.33537	- 0.25591	0.26542	0.09207	0.24378	- 0.08299	0.80231	0.22158	2.19176	- 0.46396	3.68888	1.58347
0.16	- 0.31466	0.08681	- 0.20894	- 0.09831	0.32497	- 0.24299	0.22465	0.06572	0.19685	- 0.05885	0.70525	0.17261	1.79405	- 0.33283	3.28639	1.26823
0.18	- 0.29258	0.07546	- 0.18236	- 0.07593	0.32349	- 0.23411	0.19263	0.04799	0.18256	- 0.04492	0.63513	0.14090	1.62147	- 0.23871	3.00720	1.06412
0.20	- 0.28507	0.06783	- 0.16102	- 0.05984	0.31509	- 0.22538	0.16207	0.03403	0.15997	- 0.03774	0.57589	0.11680	1.46060	- 0.19805	2.77730	0.91098
0.22	- 0.25425	0.06330	- 0.14050	- 0.04705	0.32179	- 0.22262	0.14005	0.02489	0.12793	- 0.02946	0.52647	0.09849	1.19197	- 0.15285	2.58196	0.79211
0.24	- 0.23777	0.05973	- 0.12386	- 0.03757	0.32204	- 0.21859	0.11853	0.01737	0.09501	- 0.02523	0.48464	0.08426	0.96663	- 0.14073	2.42200	0.70074
0.26	- 0.22515	0.05651	- 0.11023	- 0.03061	0.32506	- 0.21516	0.10087	0.01197	0.07070	- 0.02120	0.45151	0.07385	0.78336	- 0.12551	2.29721	0.63296
0.28	- 0.21544	0.05324	- 0.09782	- 0.02476	0.33175	- 0.21174	0.08781	0.00809	0.05103	- 0.01740	0.42307	0.06449	0.61752	- 0.10774	2.16237	0.56393
0.30	- 0.20885	0.05066	- 0.08848	- 0.02078	0.33887	- 0.20869	0.07389	0.00481	0.03661	- 0.01439	0.39841	0.05857	0.49959	- 0.09625	2.10295	0.53177
0.32	- 0.19963	0.04805	- 0.08956	- 0.01999	0.34568	- 0.20601	0.08268	0.005936	0.00894	- 0.00968	0.39391	0.05587	0.24014	- 0.07701	2.14031	0.50681
0.34	- 0.19947	0.04871	- 0.09392	- 0.02016	0.38093	- 0.20321	0.10010	0.00864	-0.02742	- 0.00096	0.39019	0.05352	- 0.14899	- 0.02754	2.16987	0.48345
0.36	- 0.18317	0.04327	- 0.08972	- 0.01936	0.35432	- 0.20125	0.11267	0.01057	-0.03101	- 0.00079	0.37494	0.04977	- 0.17928	- 0.02924	2.13662	0.46069

NOMENCLATURE

C	constants of integration
E	modulus of elasticity
I_r	moment of inertia of ring
$K_1 \cdots K_8$	shape coefficients for tapered shell
M	axisymmetric moment
N	normal force
Q	axisymmetric shear
R	radius of shell
R_a	radius to center of pressure
R_r	radius to centroid of tapered ring
Z	Schleicher function
Z'	first derivative of Schleicher function
k	modulus of foundation
l	length
p	pressure
t	thickness of shell
y	radial deflection
$b_1 \cdots b_{16}$	influence coefficients for tapered shell
x_1, x_2	distance from centroid of tapered ring to discontinuity junction
α	included angle of taper
θ	rotation
λ^4	$\dfrac{12(1-\nu^2)}{\alpha^2 R^2}$
ν	Poisson's ratio
ξ_0, ξ_L	upper- and lower-end real arguments for Schleicher functions

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APPENDIX

The solution (Ref. 3) of a cylinder of linearly varying wall thickness having parameters and geometric characteristics different from $\nu = 0.3$, $\alpha = 1/3$, and $R_2 \neq R_3 \neq R_4$ requires modification of the influence coefficients and the radial displacements and rotations due to internal pressure. Figures A-1 and A-2 are free-body diagrams of a typical pressure vessel, where $R_2 \neq R_3 \neq R_4$.

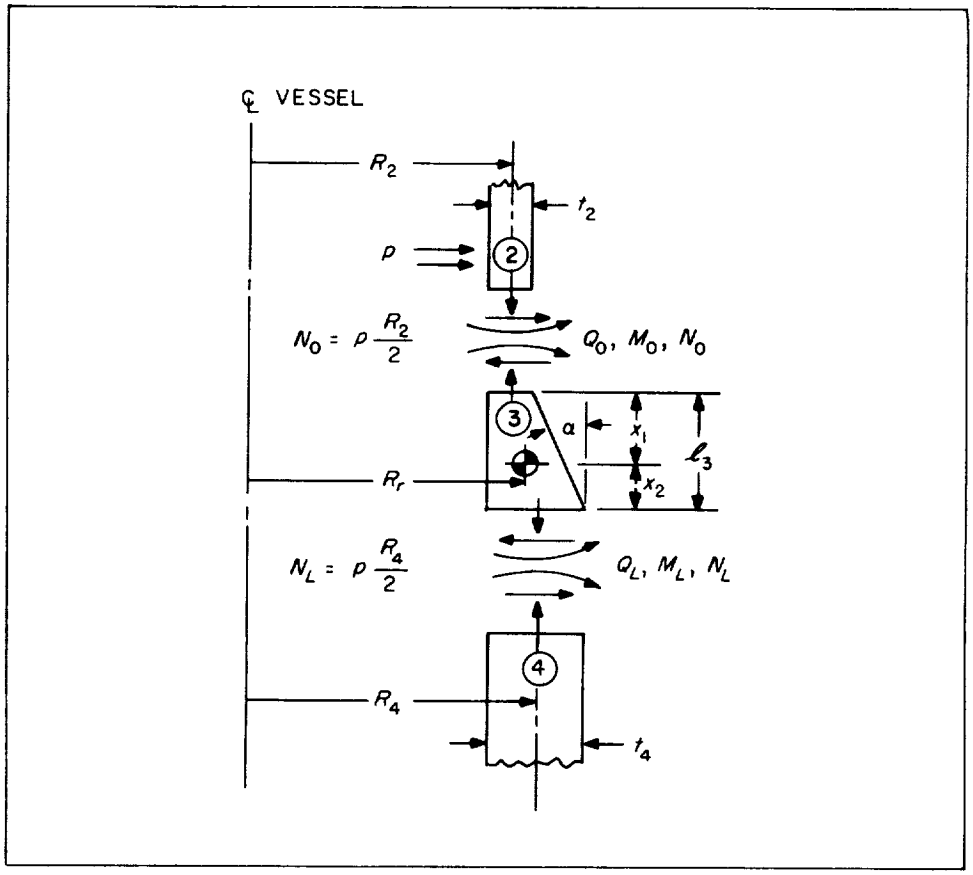

Fig. A-1. Free-body diagram of a transition R_4 greater than R_2

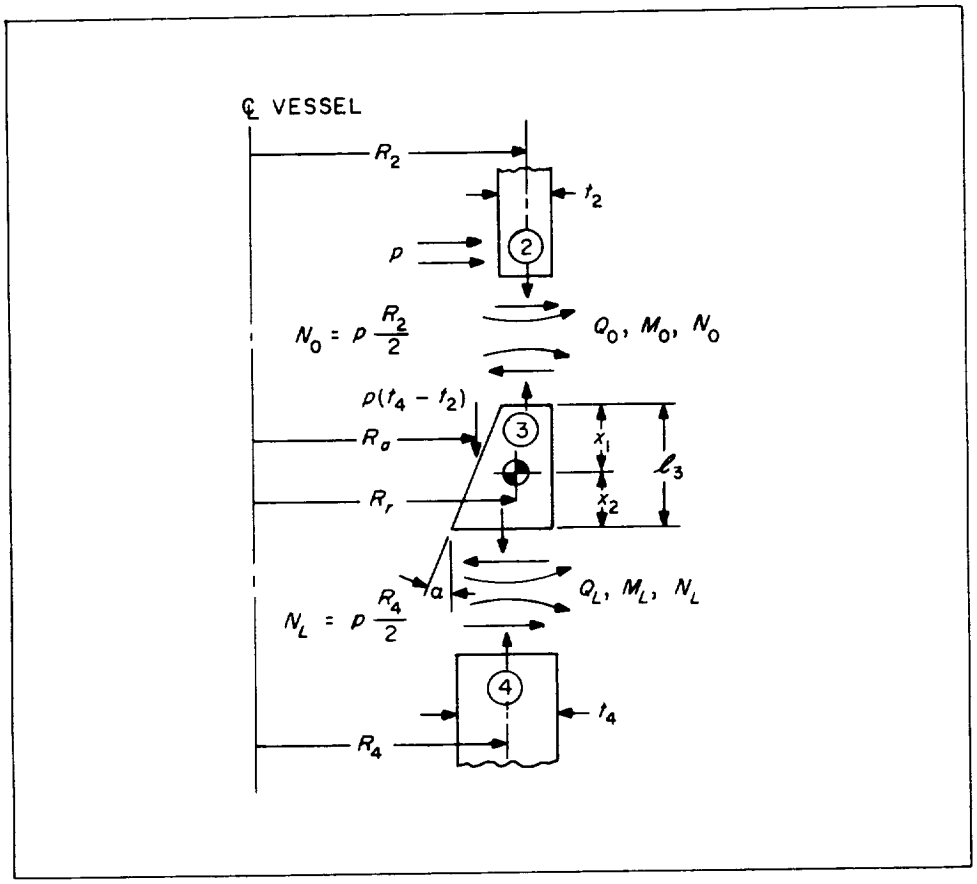

Fig. A-2. Free-body diagram of a transition R_4 less than R_2

I. RADIAL DISPLACEMENT AND ROTATION DUE TO INTERNAL PRESSURE LOADING

From Eq. (14) (geometry of Fig. 1)

$$E y_0 = \frac{p R_2^2 \left(1 - \frac{\nu}{2}\right)}{t_2}$$

and

$$E y_L = \frac{p R_4^2 \left(1 - \frac{\nu}{2}\right)}{t_4}$$

The increments of slopes produced by unequal displacements at the edges of structure (3) of length l_3 (Fig. A-1 and A-2) are

$$E \theta_0 = E \theta_L = - \frac{1}{l_3} \left[y_0 - y_L\right] \tag{A-1}$$

The total twisting moment (Mt) per unit length or circumference due to internal pressure about the centroidal axis of structure (3) from Fig. A-1 is

$$2\pi R_r Mt_3 = -2\pi p \left[R_2 N_0 (R_r - R_2) + R_4 N_L (R_4 - R_r)\right]$$

from Fig. A-2:

$$2\pi R_r Mt_3 = 2\pi p \left[R_2 N_0 (R_2 - R_r) + R_4 N_0 (R_r - R_4) + R_a (t_4 - t_2)(R_r - R_a)\right]$$

or

$$\left.\begin{aligned} Mt_3 &= -\frac{p}{R_r}\left[\frac{R_2^2}{2}(R_r - R_2) + \frac{R_4^2}{2}(R_4 - R_r)\right] \\ Mt_3 &= \frac{p}{R_r}\left[\frac{R_2^2}{2}(R_2 - R_r) + \frac{R_4^2}{2}(R_r - R_4) + R_a(t_4 - t_2)(R_r - R_a)\right] \end{aligned}\right\} \tag{A-2}$$

The radial displacement due to the twisting moment Mt_3 is

$$Ey_0 = -\frac{Mt_3 R_r^2 x_1}{I_r} \quad ; \quad Ey_L = \frac{Mt_3 R_r^2 x_2}{I_r} \tag{A-3}$$

The rotation due to the twisting moment Mt_3 is

$$E\theta_0 = \frac{Mt_3 R_r^2}{I_r} \quad ; \quad E\theta_L = \frac{Mt_3 R_r^2}{I_r} \tag{A-4}$$

The total radial displacement due to pressure at the upper and lower ends of structure 3 (Fig. A-1) is

$$\left.\begin{aligned} Ey_0 &= \frac{pR_2^2\left(1 - \frac{\nu}{2}\right)}{t_2} + \frac{pR_r}{I_r}\left[\frac{R_2^2}{2}(R_r - R_2)\right]x_1 + \frac{pR_r}{I_r}\left[\frac{R_4^2}{2}(R_4 - R_r)\right]x_1 \\ Ey_L &= \frac{pR_4^2\left(1 - \frac{\nu}{2}\right)}{t_4} - \frac{pR_r}{I_r}\left[\frac{R_2^2}{2}(R_r - R_2)\right]x_2 - \frac{pR_r}{I_r}\left[\frac{R_4^2}{2}(R_4 - R_r)\right]x_2 \end{aligned}\right\} \tag{A-5}$$

The total rotation due to pressure at the upper and lower ends of structure 3 (Fig. A-1) is

$$E\theta_0 = E\theta_L = -\frac{pR_r}{I_r}\left[\frac{R_2^2}{2}(R_r - R_2)\right] - \frac{pR_r}{I_r}\left[\frac{R_4^2}{2}(R_4 - R_r) - \frac{p\left(1 - \frac{\nu}{2}\right)}{l_3}\left(\frac{R_2^2}{t_2} - \frac{R_4^2}{t_4}\right)\right] \tag{A-6}$$

The total radial displacement due to pressure at the upper and lower ends of structure 4 (Fig. A-2) is

$$
\left.
\begin{aligned}
E y_0 &= \frac{p R_2^2 \left(1 - \frac{\nu}{2}\right)}{t_2} - \frac{p R_r}{l_r} \left[\frac{R_2^2}{2} (R_2 - R_r)\right] x_1 \\
&\quad - \frac{p R_r}{l_r} \left[\frac{R_4^2}{2} (R_r - R_4)\right] x_1 - \frac{p R_r}{l_r} \left[R_a (t_4 - t_2)(R_r - R_a)\right] x_1 \\
E y_L &= \frac{p R_4^2 \left(1 - \frac{\nu}{2}\right)}{t_4} + \frac{p R_r}{l_r} \left[\frac{R_2^2}{2} (R_2 - R_r)\right] x_2 \\
&\quad + \frac{p R_r}{l_r} \left[\frac{R_4^2}{2} (R_r - R_4)\right] x_2 + \frac{p R_r}{l_r} \left[R_a (t_4 - t_2)(R_r - R_a)\right] x_2
\end{aligned}
\right\} \quad \text{(A-7)}
$$

The total rotation due to pressure at the upper and lower ends of structure 3 (Fig. A-2) is

$$
\left.
\begin{aligned}
E \theta_0 = E \theta_L &= \frac{p R_r}{l_r} \left[\frac{R_2^2}{2} (R_2 - R_r)\right] + \frac{p R_r}{l_r} \left[\frac{R_4^2}{2} (R_4 - R_r)\right] \\
&\quad + \frac{p R_r}{l_r} \left[R_a (t_4 - t_2)(R_r - R_a)\right] + \frac{p \left(1 - \frac{\nu}{2}\right)}{l_3} \left[\frac{R_4^2}{t_4} - \frac{R_2^2}{t_2}\right]
\end{aligned}
\right\} \quad \text{(A-8)}
$$

II. MODIFICATION OF THE INFLUENCE COEFFICIENTS FOR POISSON'S RATIO DIFFERENT FROM 0.30 AND α DIFFERENT FROM 1/3

It is evident that an exact solution for the modified influence coefficients requires the evaluation of the actual upper and lower real arguments, ξ_0 and ξ_L, substitution of these arguments into Eq. (1) through (8), then computation of $b_1 \cdots b_{16}$.

However, a solution for the influence coefficients may be closely approximated by evaluating the actual arguments

$$\xi_0 = 2\lambda (x_0)^{1/2}$$

and

$$\xi_L = 2\lambda (x_0 + l_3)^{1/2}$$

for the modified geometric configuration, and by the use of Table 1, determining the nearest t_4/R_3 and $\rho = t_4/t_2$ relationship.

The influence coefficients $b_1 \cdots b_{16}$ may then be obtained from Table 2. The shape coefficients $k_1 \cdots k_8$ are obtained by evaluating Eq. (9), and the displacements and rotations by Eq. (11).

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