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LAPLACE COEFFICIENTS AND THEIR NEWCOMB DERIVATIVES

by

Imre G. Izsak

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LAPLACE COEFFICIENTS AND THEIR NEWCOMB DERIVATIVES[1]

by

Imre G. Izsak[2]

The computation of the Laplace coefficients and their Newcomb derivatives, which play such an important role in the literal development of the planetary disturbing function, is an elaborate task in numerical analysis. Anticipating our future needs, we thought it advisable to construct an IBM-7090 computer program for the evaluation of these transcendental functions. This report describes the adopted computational procedure, with the derivation of the basic equations and graphs, obtained on an EAI data-plotter, that illustrate the behavior of the functions in question.

The Laplace coefficients $b_k^s(\alpha)$ are defined here as the coefficients in the Fourier expansion of the generating function

$$\alpha^{s-\frac{1}{2}} (1-2\alpha \cos \psi + \alpha^2)^{-s} = b_0^s(\alpha) + 2 \sum_{k=1}^{\infty} b_k^s(\alpha) \cos k\psi \; ; \tag{1}$$

on the left side of this equation the symbol s means an exponent, but on the right side, just an upper index; the range of the argument α is supposed to be $0 < \alpha < 1$. According to this definition,

$$b_k^s(\alpha) = \frac{\alpha^{s-\frac{1}{2}}}{\pi} \int_0^{\pi} (1-2\alpha \cos \psi + \alpha^2)^{-s} \cos k\psi \, d\psi \quad \text{for } k = 0,1,2,\ldots \tag{2}$$

Alternatively, one writes the expansion (1) in the complex form. With the notations $\zeta = \exp i\psi$, $b_{-k}^s(\alpha) = b_k^s(\alpha)$ and

$$f = 1 - 2\alpha \cos \psi + \alpha^2 = 1 + \alpha^2 - \alpha(\zeta + \zeta^{-1}) = (1 - \alpha\zeta)(1 - \alpha\zeta^{-1})$$

it becomes

$$\alpha^{s-\frac{1}{2}} f^{-s} = \sum_{-\infty}^{\infty} b_k^s(\alpha)\zeta^k \; .$$

---

[1]This work was supported in part by Grant No. NsG 87-60 from the National Aeronautics and Space Administration.

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In the special case of circular and coplanar planetary orbits one needs the Laplace coefficients with $s = 1/2$ only. If the orbits are circular but not coplanar, one can choose, among other things, to develop the Fourier coefficients of the disturbing function into powers of $\nu = \sin^2 \frac{J}{2}$, in which case Laplace coefficients with $s = 1/2, 3/2, 5/2 \ldots$ are needed; or to develop them into powers of the two variables $\mu = \cos^2 \frac{J}{2}$ and $\nu = \sin^2 \frac{J}{2}$, in which case one uses Tisserand polynomials and Laplace coefficients with $s = 1/2$ only. Both methods were considered by Leverrier. The first is more convenient if the mutual inclination J of the orbits is small, the second holds for all inclinations.

More involved is the development of the Fourier coefficients of the disturbing function into powers of the orbital eccentricities. In order to accomplish this development, one needs the Newcomb derivatives

$$D^j b_k^s(\alpha)$$

of the Laplace coefficients. The differential operator D is defined as $D = \alpha d/d\alpha$, so that for example $D^2 = \alpha d/d\alpha + \alpha^2 d^2/d\alpha^2$.

It is well known that the Laplace coefficients are representable by hypergeometric series, as

$$b_k^s(\alpha) = \begin{bmatrix} s \\ k \end{bmatrix} \alpha^{k+s-\frac{1}{2}} F(s, k+s, k+1 \,|\, \alpha^2) \,, \tag{3}$$

$$\text{where} \quad \begin{bmatrix} s \\ k \end{bmatrix} = (-1)^k \binom{-s}{k} = \frac{s.s+1\ldots s+k-1}{1.2\ldots k} = \frac{\Gamma(s+k)}{\Gamma(s)\,\Gamma(k+1)} \,.$$

The derivatives of the Laplace coefficients in turn can be expressed as linear combinations of hypergeometric series. For small values of the argument α, for $\alpha^2 < 0.1$ say, the evaluation of these series does not have to be considered an impractical process, especially if one applies certain convergence-improving transformations. For larger values of the argument, however, the computation of Laplace coefficients is much better achieved by the use of properly chosen recurrence relations. Such a method is described in the following.

First of all, Innes (1909, 1910) has observed that the Laplace coefficients and their derivatives belonging to the index s+1 can be easily obtained from those belonging to the index s. Therefore we are concerned with the special case $s = 1/2$ only, where we omit this index altogether. Then the functions to be computed are up to a certain order:

$$
\begin{array}{ccccc}
b_0 , & Db_0 , & D^2 b_0 , & \dots , & D^J b_0 \\
b_1 , & Db_1 , & D^2 b_1 , & \dots , & D^J b_1 \\
b_2 , & Db_2 , & D^2 b_2 , & \dots , & D^J b_2 \\
\vdots & \vdots & \vdots & & \vdots \\
b_K , & Db_K , & D^2 b_K , & \dots , & D^J b_K
\end{array}
$$

The first column in this scheme we compute by using a transformation of the argument α proposed by Andoyer (1923); it is similar to the Landen transformation in the theory of elliptic integrals. To obtain the first row we compute, by essentially the same transformation, a set of hypergeometric functions that in turn determine the values of the $D^j b_0$ for $j = 1,2,\dots,J$. Following Innes (1910), the second column is derived by the help of a simple recurrence relation, and the second row is determined by the elements in the first one. As to the rest of the scheme, we are then all set to apply a recurrence relation of Innes (1909) that is admirably well suited for the computation of high-order derivatives of the Laplace coefficients. Let us give a short proof of the equations on which this recursive process is based.

1. Any hypergeometric function $F(a,b,c|z)$ with $Re(c) > Re(b) > 0$ and $|z| < 1$ is represented by the integral

$$F(a,b,c|z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}du \tag{4}$$

(Whittaker and Watson, 1958). Application of this formula to the Laplace coefficients (3) belonging to the index $s = 1/2$ gives

$$
\begin{aligned}
b_k(\alpha) &= \left[{1/2 \atop k}\right] \alpha^k F(1/2,k+1/2,k+1|\alpha^2) \\
&= \frac{\alpha^k}{\pi} \int_0^1 u^{k-\frac{1}{2}}(1-u)^{-\frac{1}{2}}(1-\alpha^2 u)^{-\frac{1}{2}}du = \frac{2\alpha^k}{\pi} \int_0^{\pi/2} \frac{\cos^{2k}\psi\, d\psi}{\sqrt{1-\alpha^2\cos^2\psi}} .
\end{aligned}
$$

In the last integrand the function under the square-root sign can be written as

$$1-\alpha^2 \cos^2 \psi = \frac{1-2\alpha_1 \cos 2\psi + \alpha^2}{(1+\alpha_1}$$

with the substitution

$$\alpha = \frac{2\sqrt{\alpha_1}}{1+\alpha_1} \quad , \qquad \alpha_1 = \left(\frac{\alpha}{1+\sqrt{1-\alpha^2}}\right)^2 \quad ; \tag{5}$$

the Fourier expansion of the numerator is

$$\cos^{2k}\psi = 4^{-k}\left\{\binom{2k}{k} + 2\cdot\sum_{h=1}^{k}\binom{2h}{k-h}\cos 2h\psi\right\} .$$

Therefore, recalling the definition (2), we see that the Laplace coefficients with argument α can be expressed by those with argument α_1 as

$$\boxed{b_k(\alpha) = \alpha^k\,(1+\alpha_1)\sum_{h=0}^{k} C_{kh}\,b_h(\alpha_1)} , \tag{6}$$

where $C_{ko} = 4^{-k}\binom{2k}{k}$ and $C_{kh} = 2.4^{-k}\binom{2k}{k-h}$ for $h = 1,2,\ldots,k$.

2. Consider the functions

$$\beta_\ell = \begin{bmatrix}1/2\\ \ell\end{bmatrix}\alpha^{2\ell}\,F(1/2+\ell, 1/2+\ell,\ \ell+1\,|\,\alpha^2) , \qquad (\ell = 0,1,\ldots,J). \tag{7}$$

Because the derivative of any hypergeometric function with respect to its argument is

$$\frac{d}{dz}F(a,b,c\,|\,z) = \frac{ab}{c}F(a+1,b+1,c+1\,|\,z)$$

and $\quad D = \alpha d/d\alpha = 2\alpha^2 d/d\alpha^2$,

the first Newcomb derivative of the functions β_ℓ appears in the form

$$D\beta_\ell = 2\ell\beta_\ell + (2\ell+1)\,\beta_{\ell+1} ;$$

in general, $$\boxed{D^j\beta_\ell = 2\ell\,D^{j-1}\beta_\ell + (2\ell+1)D^{j-1}\beta_{\ell+1}} . \tag{8}$$

Thus if the functions

$$b_o = \beta_o, \ \beta_1, \ \dots, \ \beta_{J-2}, \ \beta_{J-1}, \ \beta_J$$

were known, we could easily obtain the functions

$$Db_o = D\beta_o, \ D\beta_1, \ \dots, \ D\beta_{J-2}, \ D\beta_{J-1},$$

then the functions

$$D^2 b_o = D^2\beta_o, \ D^2\beta_1, \ \dots, \ D^2\beta_{J-2},$$

and finally

$$D^J b_o = D^J\beta_o = D^{J-1}\beta_1 .$$

But the computation of the functions β_ℓ with argument α is at once reduced to that of the functions b_h with argument α_1 as follows:

Apply the transformation

$$F(a,b,c|z) = (1-z)^{c-a-b}F(c-a,c-b,c|z)$$

of hypergeometric functions to definition (7), and then use the integral representation (4):

$$\beta_\ell = \begin{bmatrix} 1/2 \\ \ell \end{bmatrix} \alpha^{2\ell} (1-\alpha^2)^{-\ell} F(1/2,1/2,\ell+1|\alpha^2)$$

$$= \frac{\alpha^{2\ell}}{\pi(1-\alpha^2)^\ell} \int_0^1 u^{-\frac{1}{2}}(1-u)^{\ell-\frac{1}{2}}(1-\alpha^2 u)^{-\frac{1}{2}}du = \frac{2\alpha^{2\ell}}{\pi(1-\alpha^2)^\ell}\int_0^{\pi/2} \frac{\sin^{2\ell}\psi \, d\psi}{\sqrt{1-\alpha^2\cos^2\psi}} .$$

The Fourier expansion of the numerator in the last integral is

$$\sin^{2\ell}\psi = 4^{-\ell}\left\{\binom{2\ell}{\ell} + 2\sum_{h=1}^{\ell}(-1)^h\binom{2\ell}{\ell-h}\cos 2h\psi\right\};$$

therefore by the above transformation of the argument α we have

$$\boxed{\beta_\ell(\alpha) = A^\ell(1+\alpha_1)\sum_{h=0}^{\ell}(-1)^h c_{\ell h} b_h(\alpha_1)} , \qquad (9)$$

with $A = \alpha^2(1-\alpha^2)^{-1}$.

If α_1 is not small enough for an easy evaluation of the functions $b_h(\alpha_1)$, we can repeat the transformation (5) by introducing the argument

$$\alpha_2 = \left(\frac{\alpha_1}{1+\sqrt{1-\alpha_1^2}}\right)^2 , \text{ so that } b_k(\alpha_1) = \alpha_1^k (1+\alpha_2) \sum_{h=0}^{k} C_{kh} b_h(\alpha_2) .$$

The convergence of the null-sequence α, α_1, α_2, ... is very fast indeed. For instance, if $\alpha = 0.99$, then $\alpha_1 = 0.75274$, $\alpha_2 = 0.20605$, $\alpha_3 = 0.01085$, $\alpha_4 = 0.00003$, .

and $\alpha^2 = 0.9801$, $\alpha_1^2 = 0.56663$, $\alpha_2^2 = 0.04245$, $\alpha_3^2 = 0.00012$, ...

It is expedient to observe that the computation of a more general class of hypergeometric functions, namely that of

$$F(1/2+\ell, 1/2+k+\ell, k+\ell+1 \,|\, \alpha^2) ,$$

can be reduced to the evaluation of the Laplace coefficients b_h with the argument α_1.

3. Now we turn our attention to the recurrence relations of Innes; doing so, we again consider Laplace coefficients belonging to the general index s. In addition to the differential operator $D = \alpha\partial/\partial\alpha$ it will be convenient to use the operator $d = \zeta\partial/\partial\zeta = -i\partial/\partial\psi$.

Simple rules of differentiation are

$$df = -\alpha(\zeta-\zeta^{-1}) \qquad d^2f = -\alpha(\zeta+\zeta^{-1})$$

$$Df = 2\alpha^2 - \alpha(\zeta+\zeta^{-1}) = f - (1-\alpha^2) \qquad D^2f = 4\alpha^2 - \alpha(\zeta+\zeta^{-1}) ;$$

note that $[Df]^2 - [df]^2 = 4\alpha^2 f$ and $D^2f - d^2f = 4\alpha^2$.

As to the derivatives of the function f^{-s}, we have

$$df^{-s} = -sf^{-s-1}df = s\alpha(\zeta-\zeta^{-1})\, f^{-s-1}$$

$$Df^{-s} = -sf^{-s-1}Df = -sf^{-s} + s(1-\alpha^2)\, f^{-s-1}$$

$$d^2f^{-s} = s(1+s)\, f^{-s-2}[df]^2 - sf^{-s-1}d^2f$$

$$D^2f^{-s} = s(1+s)\, f^{-s-2}[Df]^2 - sf^{-s-1}D^2f ;$$

note that $$\{D+d\}f^{-s} = 2s\alpha\zeta(1-\alpha\zeta^{-1})\ f^{-s-1} \tag{10}$$

$$\{D-d\}\ f^{-s} = 2s\alpha\zeta^{-1}(1-\alpha\zeta)\ f^{-s-1} \tag{11}$$

and

$$\{D^2 - d^2\}\ f^{-s} = 4s^2\alpha^2\ f^{-s-1} \tag{12}$$

Recurrence relations for the functions b_k^s and $D^j b_k^s$ are obtained by substituting the series

$$f^{-s} = \alpha^{-s+\frac{1}{2}} \sum_{-\infty}^{\infty} b_k^s \zeta^k$$

and its derivatives into combinations of these equations and comparing the corresponding coefficients of ζ^k. Application of the differential operator D^j to any function of the form $\alpha^r\ \Phi(\alpha)$ obviously results in

$$D^j[\alpha^r\ \Phi(\alpha)] = \alpha^r\{D+r\}^j\ \Phi(\alpha)\ . \tag{13}$$

Eliminating f^{-s-1} between equation (10) and the identity

$$f^{-s} = (1-\alpha\zeta)(1-\alpha\zeta^{-1})\ f^{-s-1}\ ,$$

we get $$\{D+d\}f^{-s} = \alpha\zeta\{D+d+2s\}f^{-s}\ ,$$

that is $$\sum_{-\infty}^{\infty}\{D-s+1/2+k\}\ b_k^s\zeta^k = \alpha\sum_{-\infty}^{\infty}\{D+s-1/2+k\}\ b_{k-1}^s\zeta^k\ .$$

Thus for $s = 1/2$ $$\boxed{\{D+k\}\ b_k = \alpha\{D+k\}\ b_{k-1}}\ . \tag{14}$$

Writing $-k$ for k in this equation and remembering that by definition $b_{-k} = b_k$, we have

$$\{D-k\}\ b_k = \alpha\{D-k\}\ b_{k+1}\ .$$

In the special case $k = 0$ this becomes $Db_1 = \alpha^{-1}\ Db_o$, and according to rule (13) there results

$$\boxed{D^j b_1 = \alpha^{-1}\{D-1\}^{j-1}Db_o}\ . \tag{15}$$

4. Combination of equations (11) and (12) gives

$$\{D^2 + 2s(D-d) - d^2\}\, \varrho^{-s} = 4s^2 \alpha \zeta^{-1} f^{-s-1} ,$$

from which there follows

$$\boxed{\{D^2 + D - (k+s+1/2)(k+s-1/2)\} b_k^s = 4s^2 b_{k+1}^{s+1} .} \qquad (16)$$

This most remarkable recurrence relation permits us to compute the Laplace coefficients and their Newcomb derivatives belonging to the index $s+1$ from those belonging to the index s with the greatest ease.

Writing $-k-2$ for k in equation (16) and using the relation $b_{-k}^s = b_k^s$ we also have

$$\{D^2 + D - (k-s+3/2)(k-s+5/2)\} b_{k+2}^s = 4s^2 b_{k+1}^{s+1} . \qquad (17)$$

Comparison of equations (16) and (17) yields a relation among functions belonging to the same index s, which for $s = 1/2$ becomes

$$\{D^2 + D - (k+1)k\} b_k = \{D^2 + D - (k+1)(k+2)\} b_{k+2} ,$$

or what is the same thing,

$$\{D^2 + D - k(k-1)\} b_k = \{D^2 + D - (k-1)(k-2)\}\, b_{k-2} .$$

This equation, like equation (16) does not contain the argument α explicitly; therefore, for derivatives of any order, we can immediately write

$$\boxed{\{D^j + D^{j-1} - k(k-1)D^{j-2}\} b_k = \{D^j + D^{j-1} - (k-1)(k-2)D^{j-2}\} b_{k-2} .} \qquad (18)$$

Equations (6), (9), (8), (15), (14), (18) -- and (16) -- solve the problem of computing the Laplace coefficients and their Newcomb derivatives in a very efficient way.

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The figures on the following pages show the behavior of the functions $D^j b_k$ in the interval $0.1 \leq \alpha \leq 0.9$, for $k = 0,1,\ldots, 16$ and $j = 0,1,\ldots,12$. The scale on the vertical axis is logarithmic.

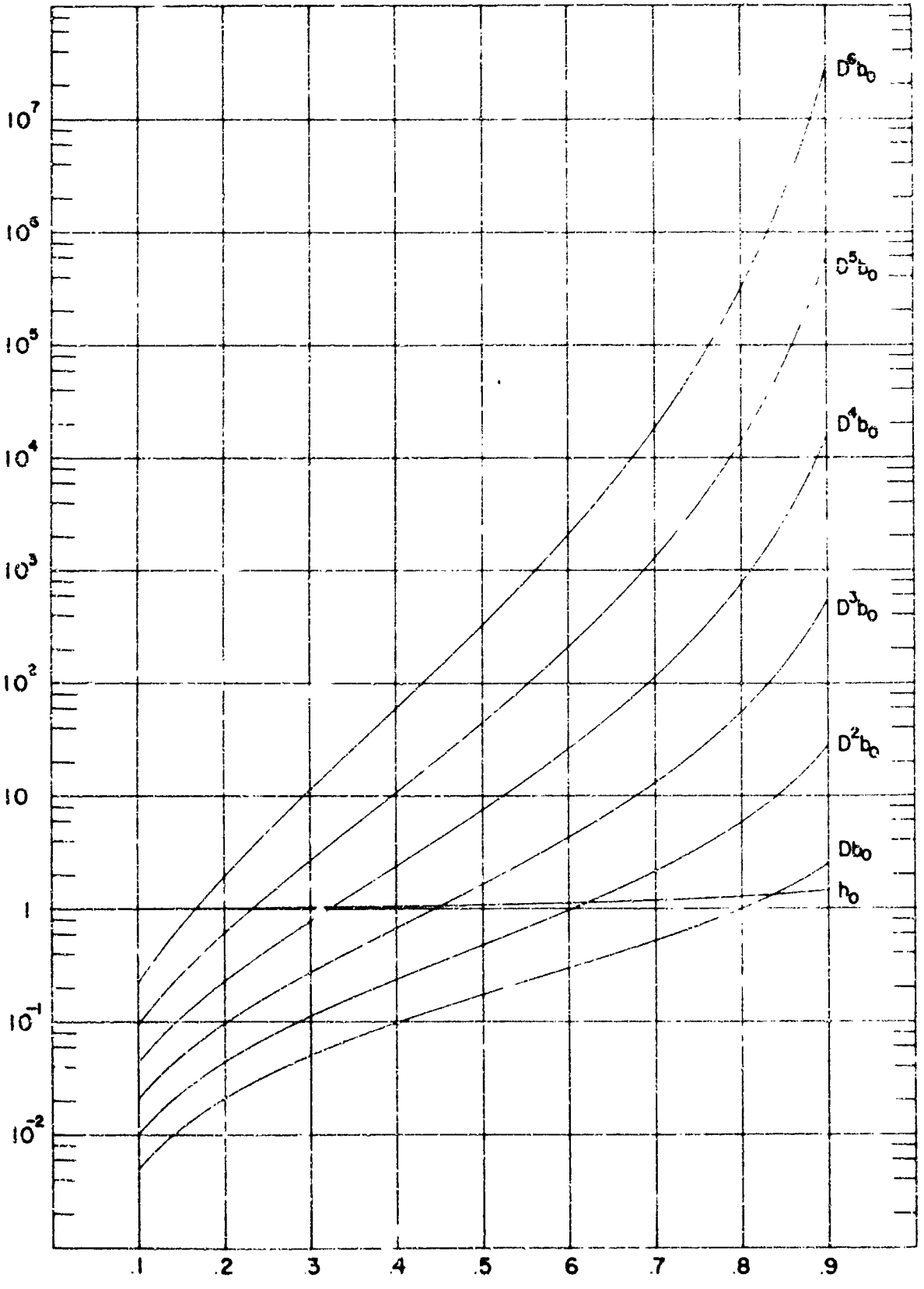
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Figure 1

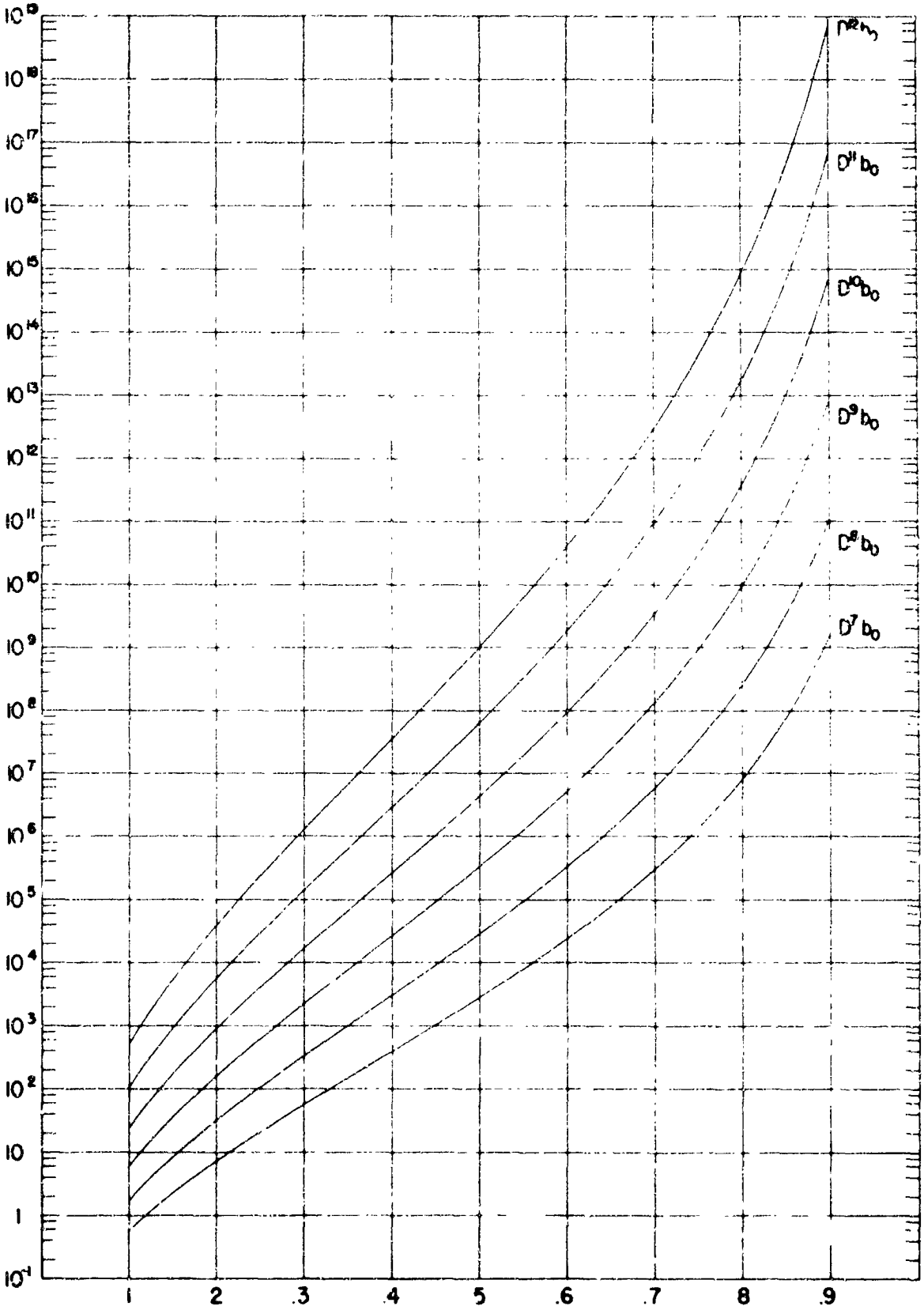
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Figure 2

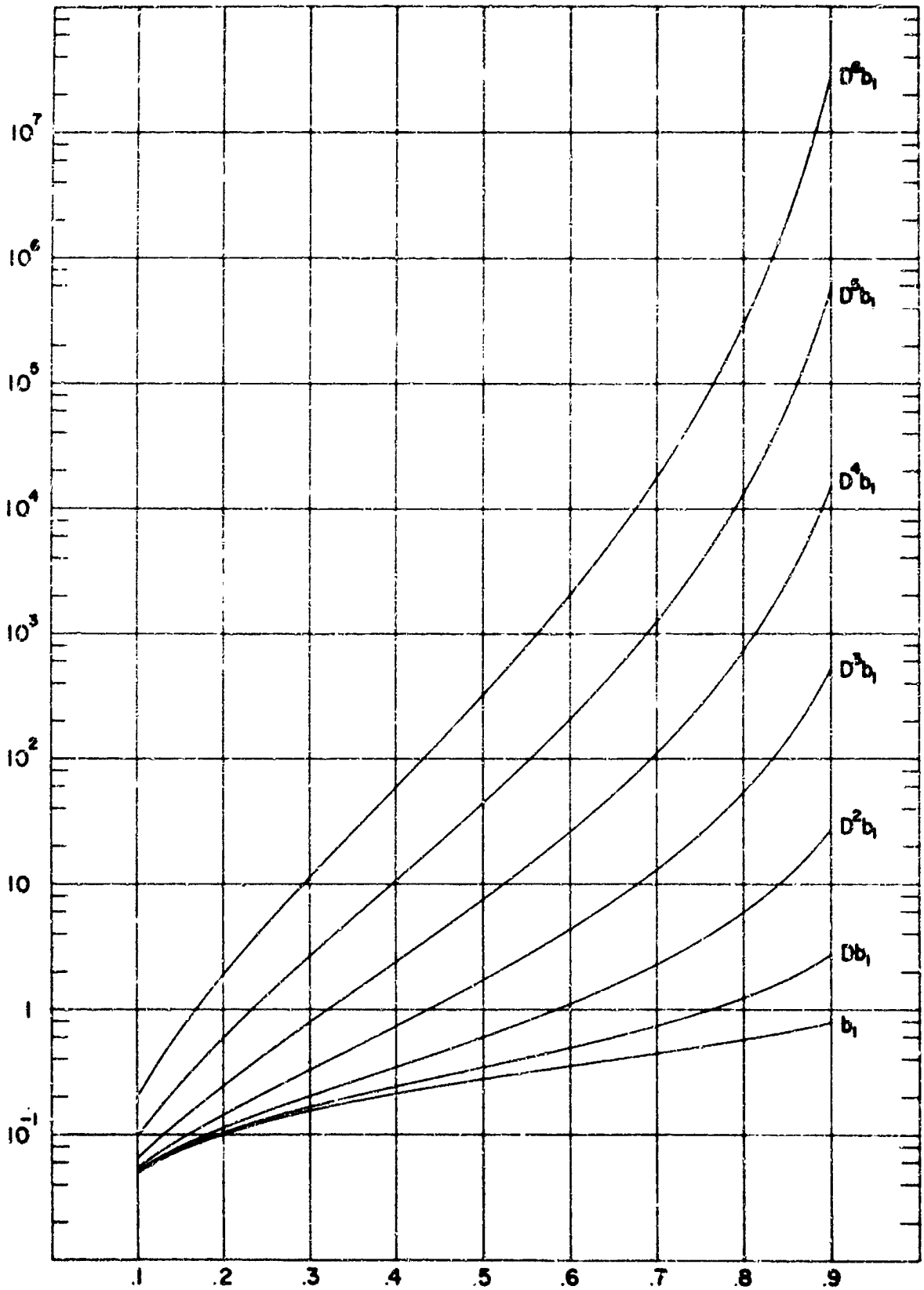
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Figure 3

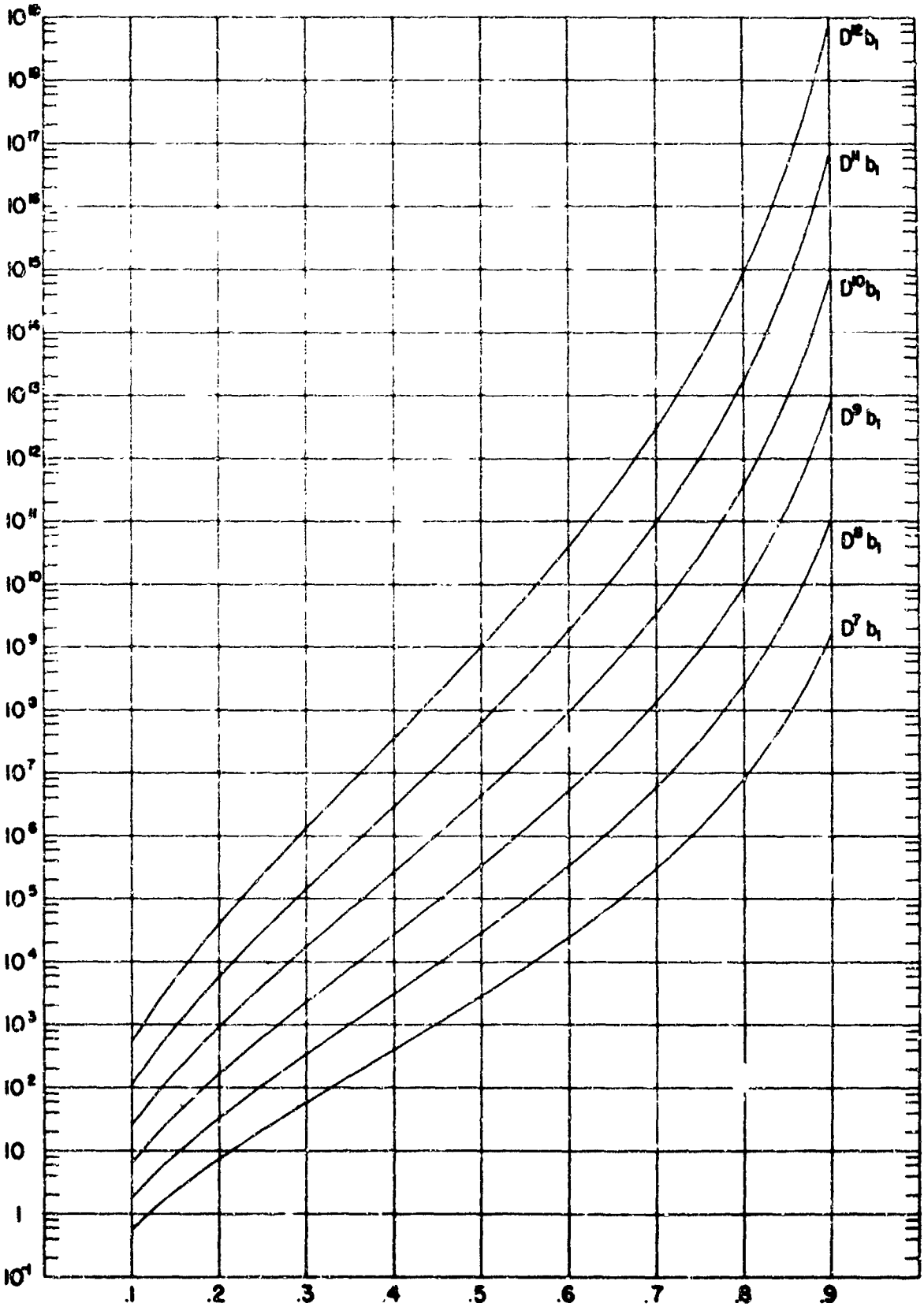
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Figure 4

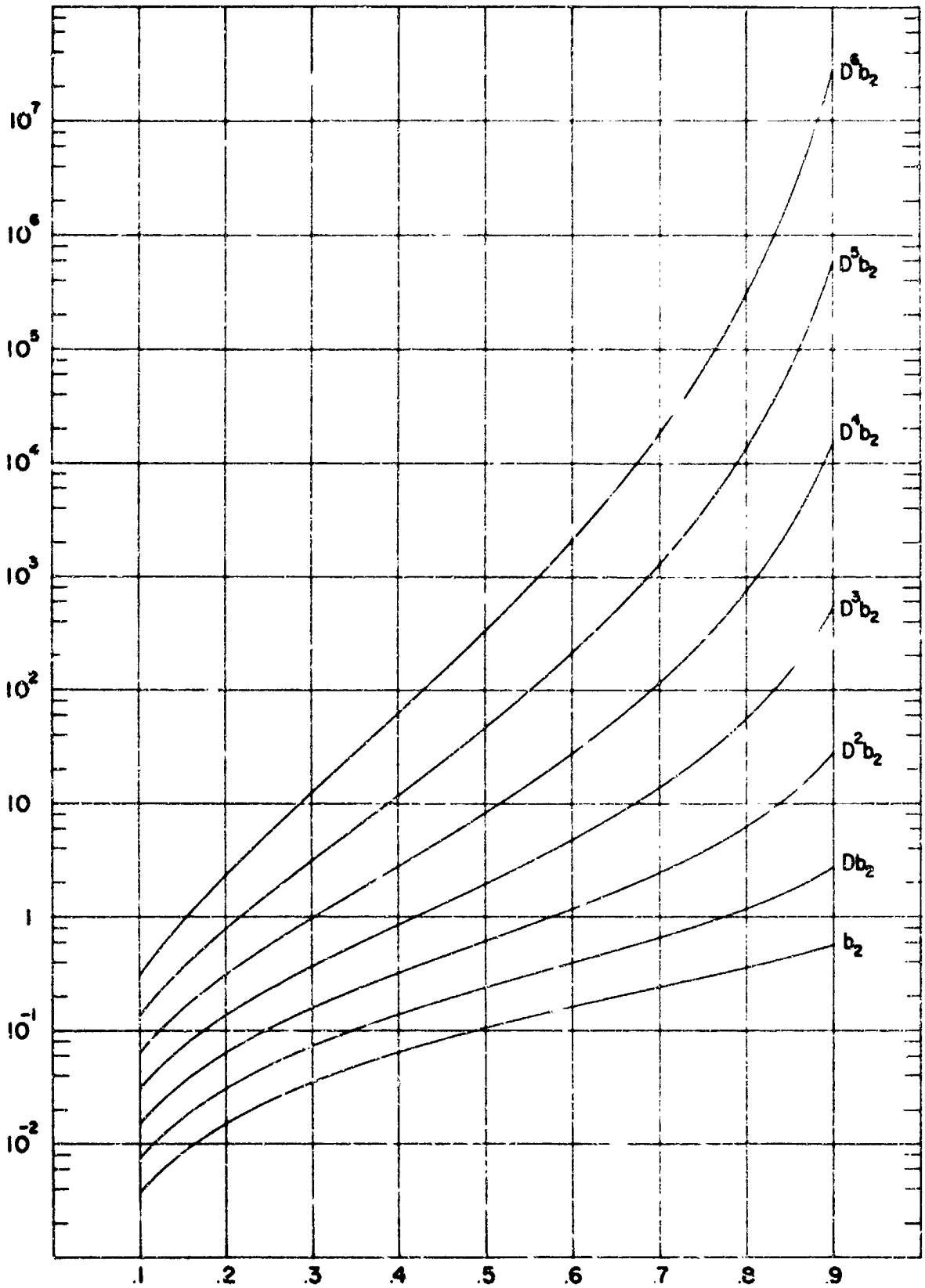
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Figure 5

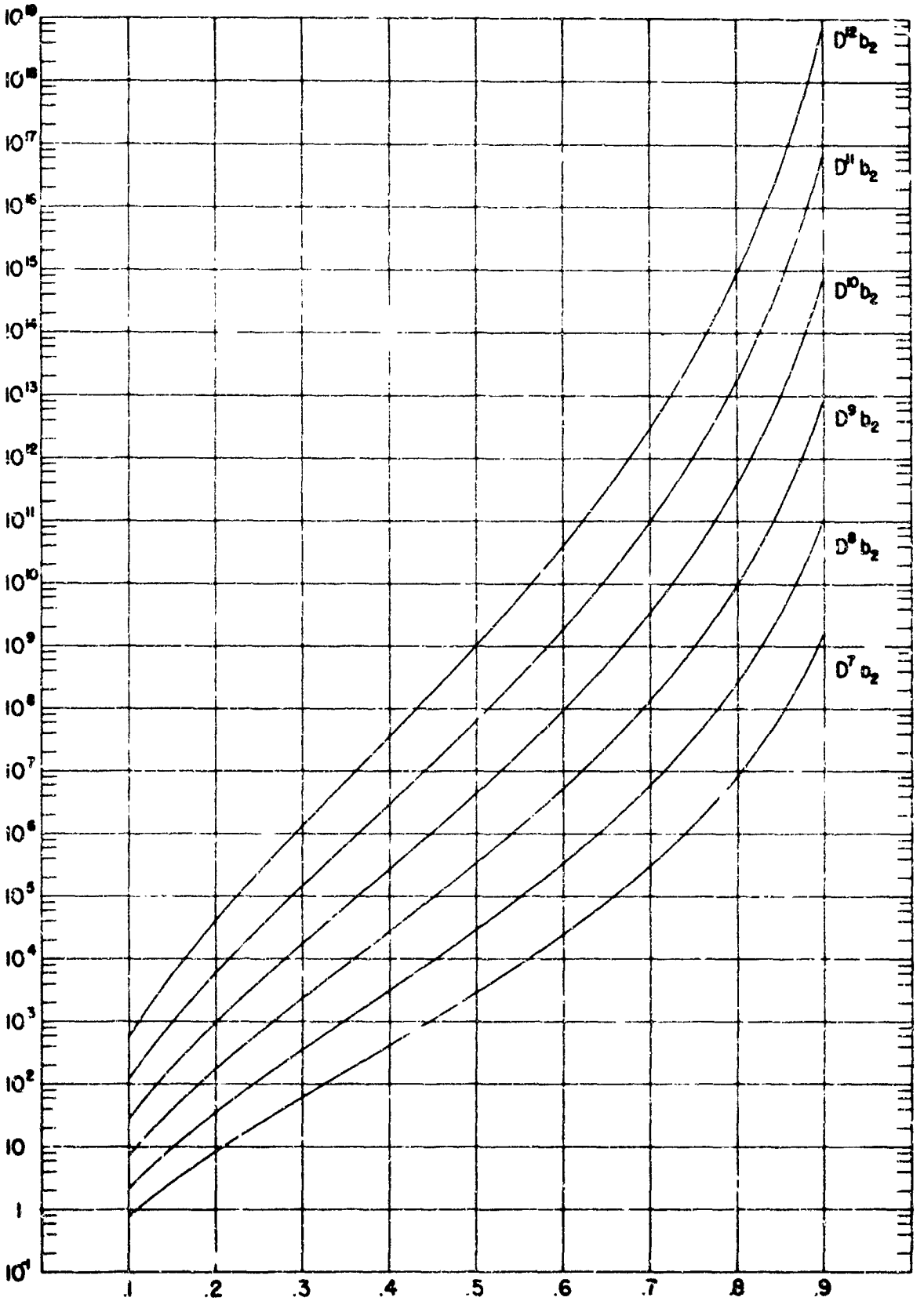
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Figure 6

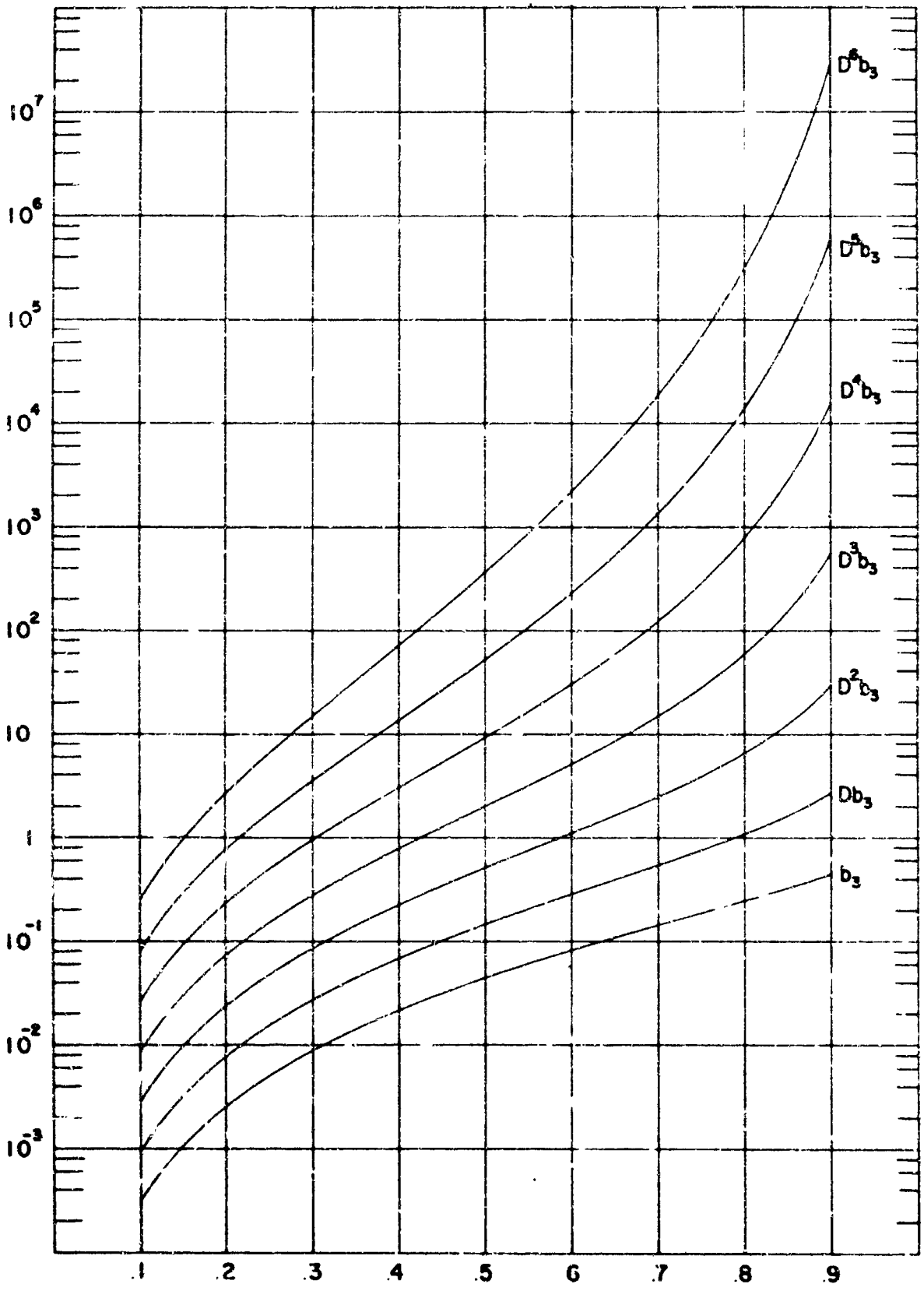
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Figure 7

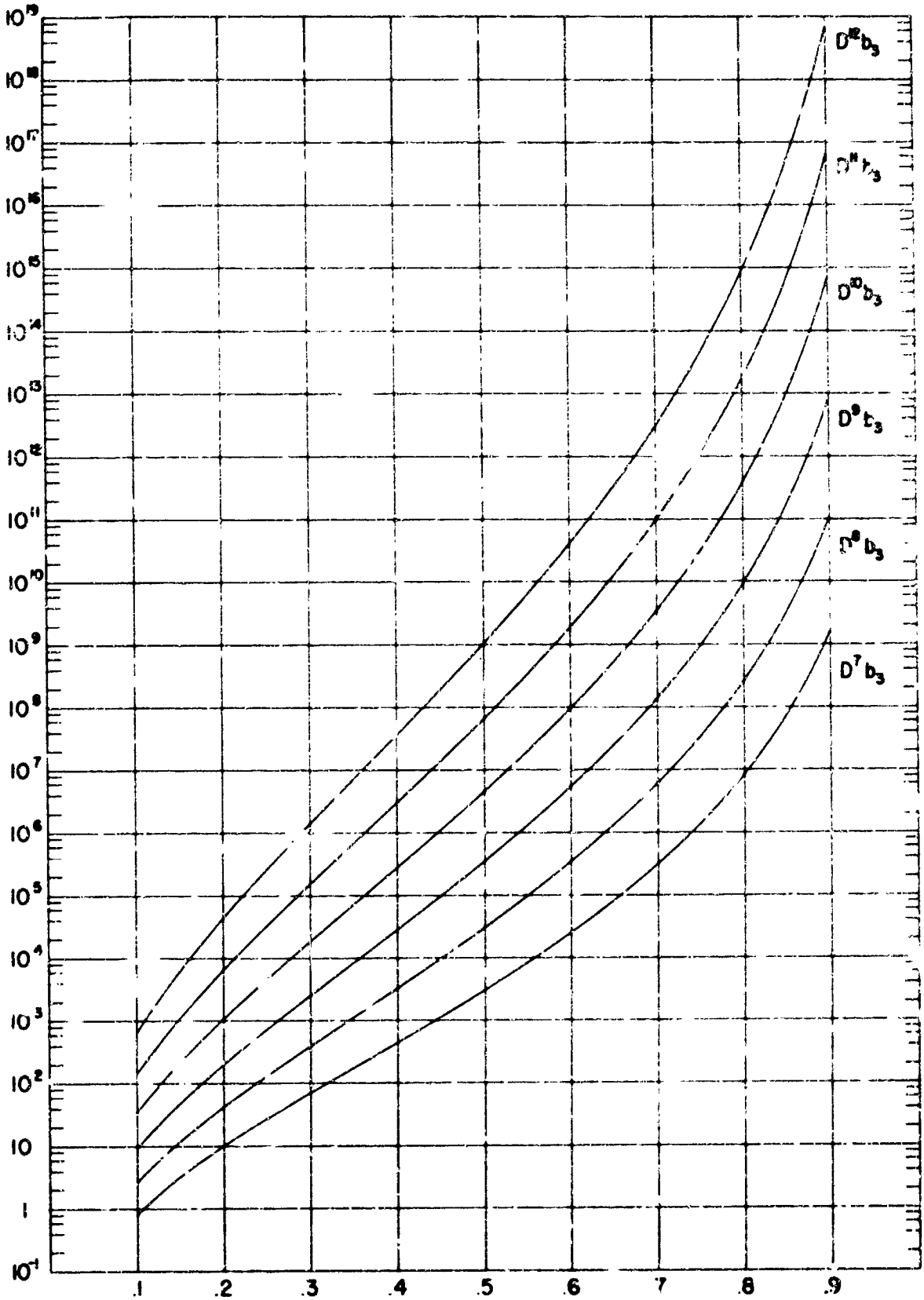
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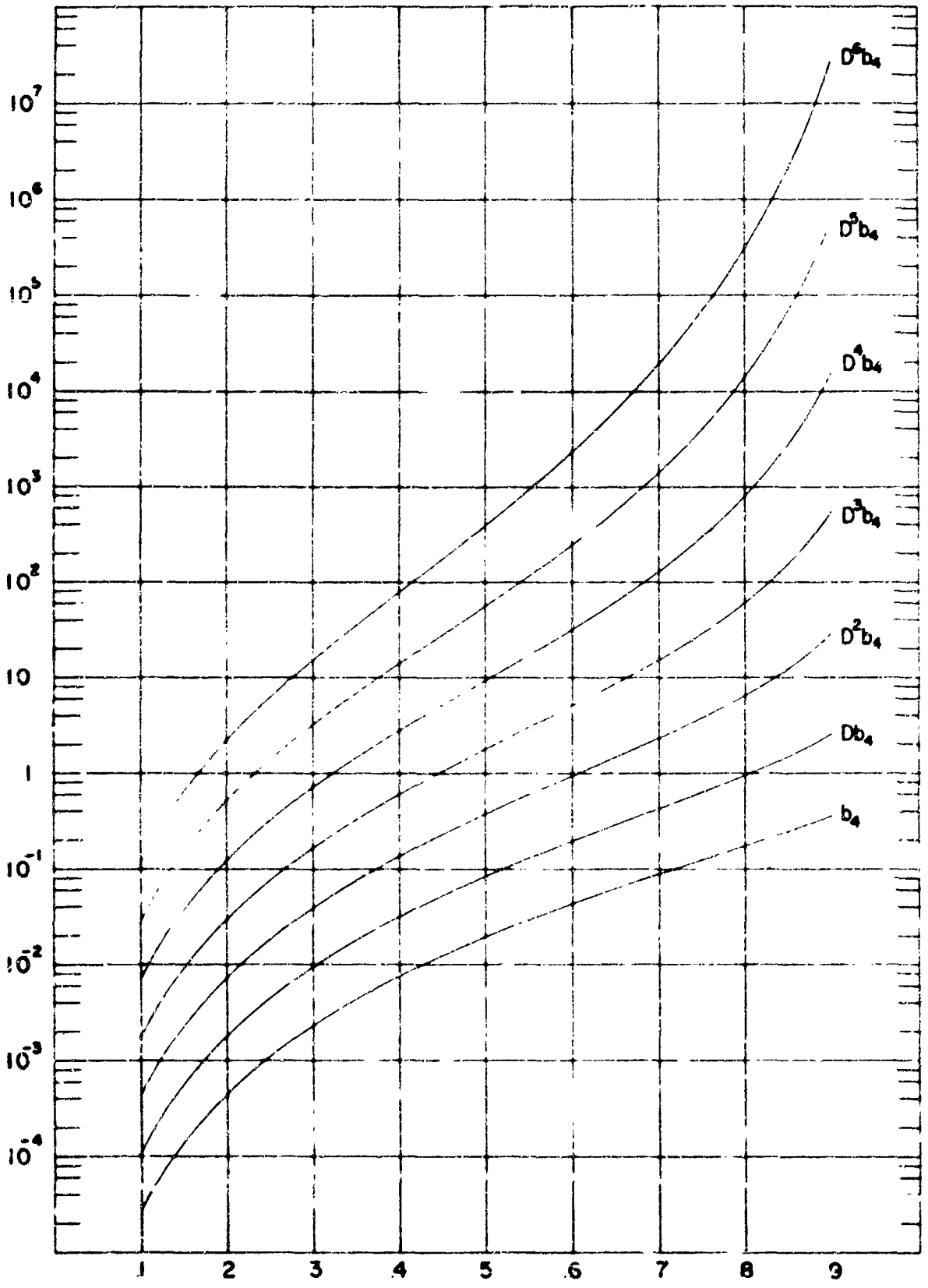
Figure 8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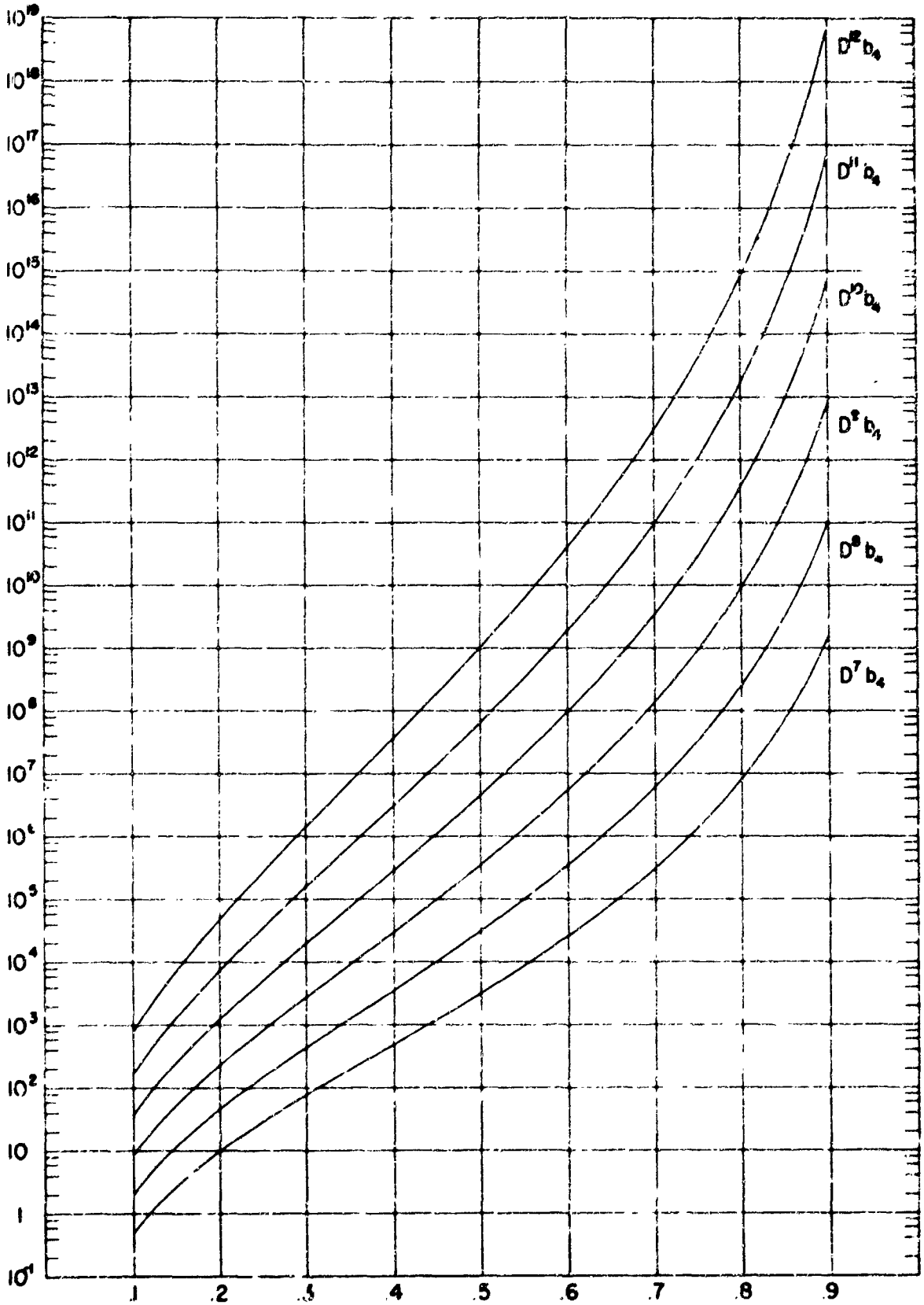
5
6
7
8
9
D^6b_4
D^5b_4
D^4b_4
D^3b_4
D^2b_4
Db_4
b_4

Figure 9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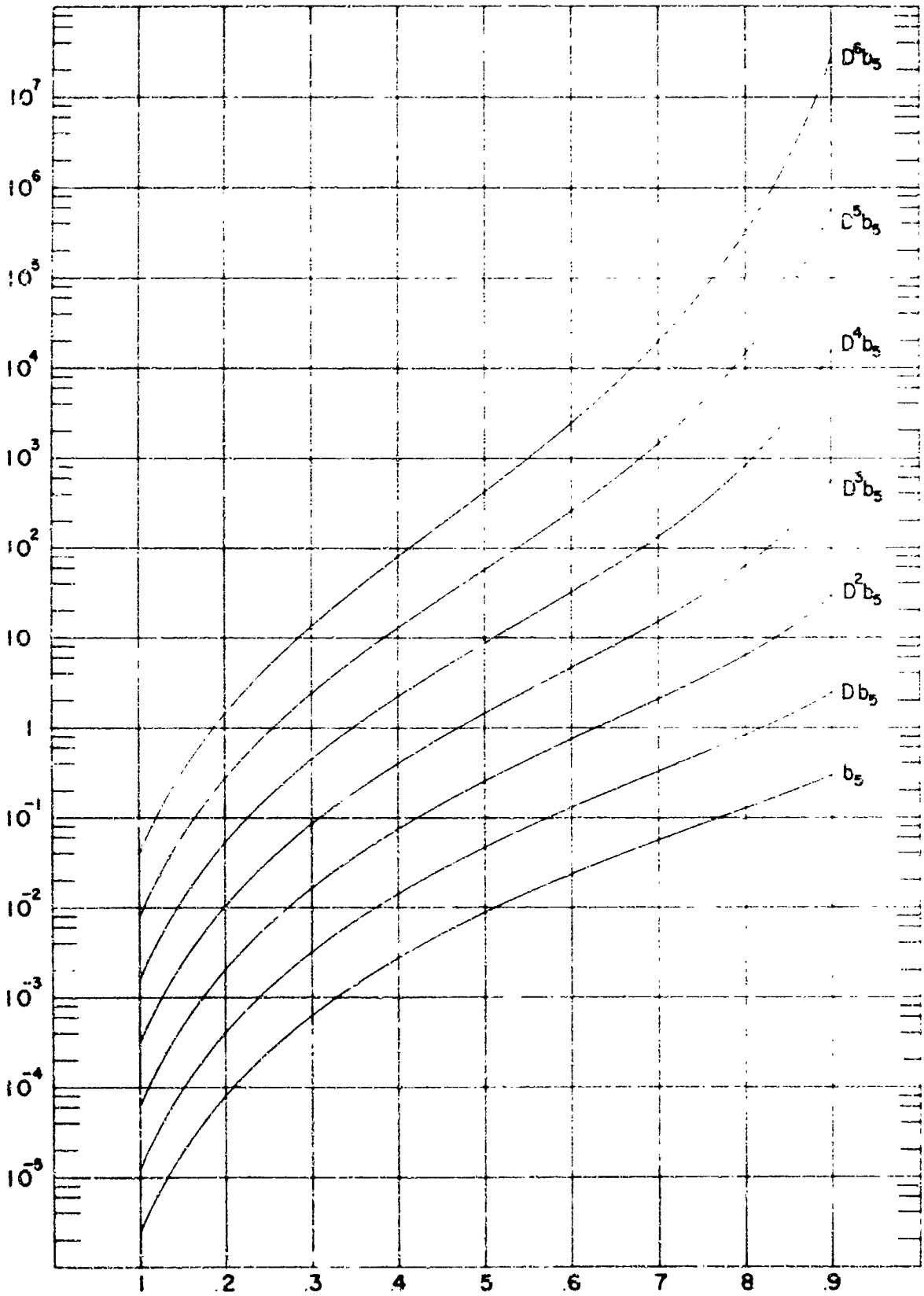
10^{17}
10^{16}
10^{15}
10^{14}
10^{13}
10^{12}
10^{11}
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10
1
10^{-1}
.1
.2
.3
.4
.5
.6
.7
.8
.9
$D^{12} b_4$
$D^{11} b_4$
$D^{10} b_4$
$D^{9} b_4$
$D^{8} b_4$
$D^{7} b_4$

Figure 10


10^7
10^6
10^5
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
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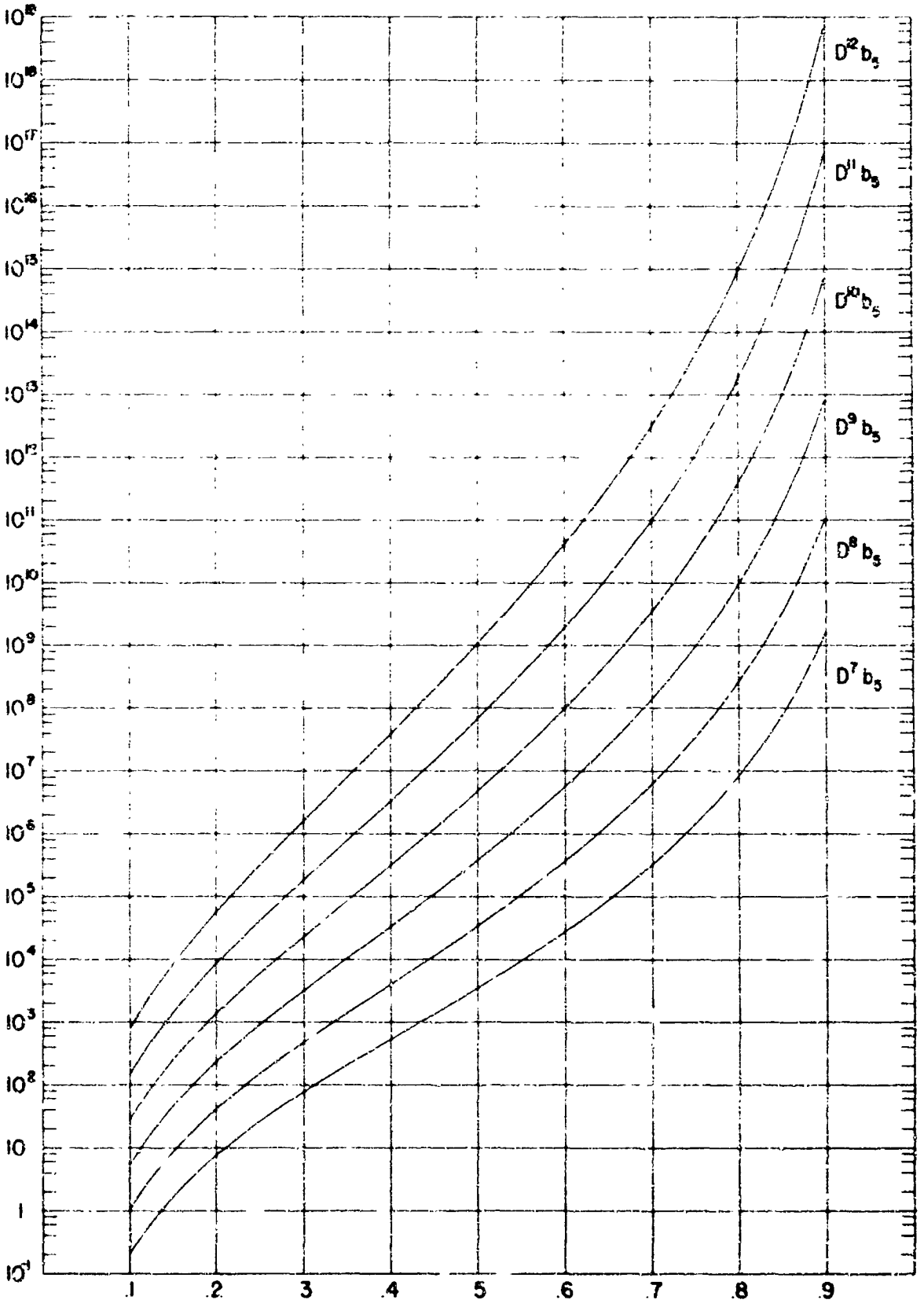
.2
.3
.4
5
.6
7
8
.9
D^6b_5
D^5b_5
D^4b_5
D^3b_5
D^2b_5
Db_5
b_5

Figure 11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10^{10}
10^{9}
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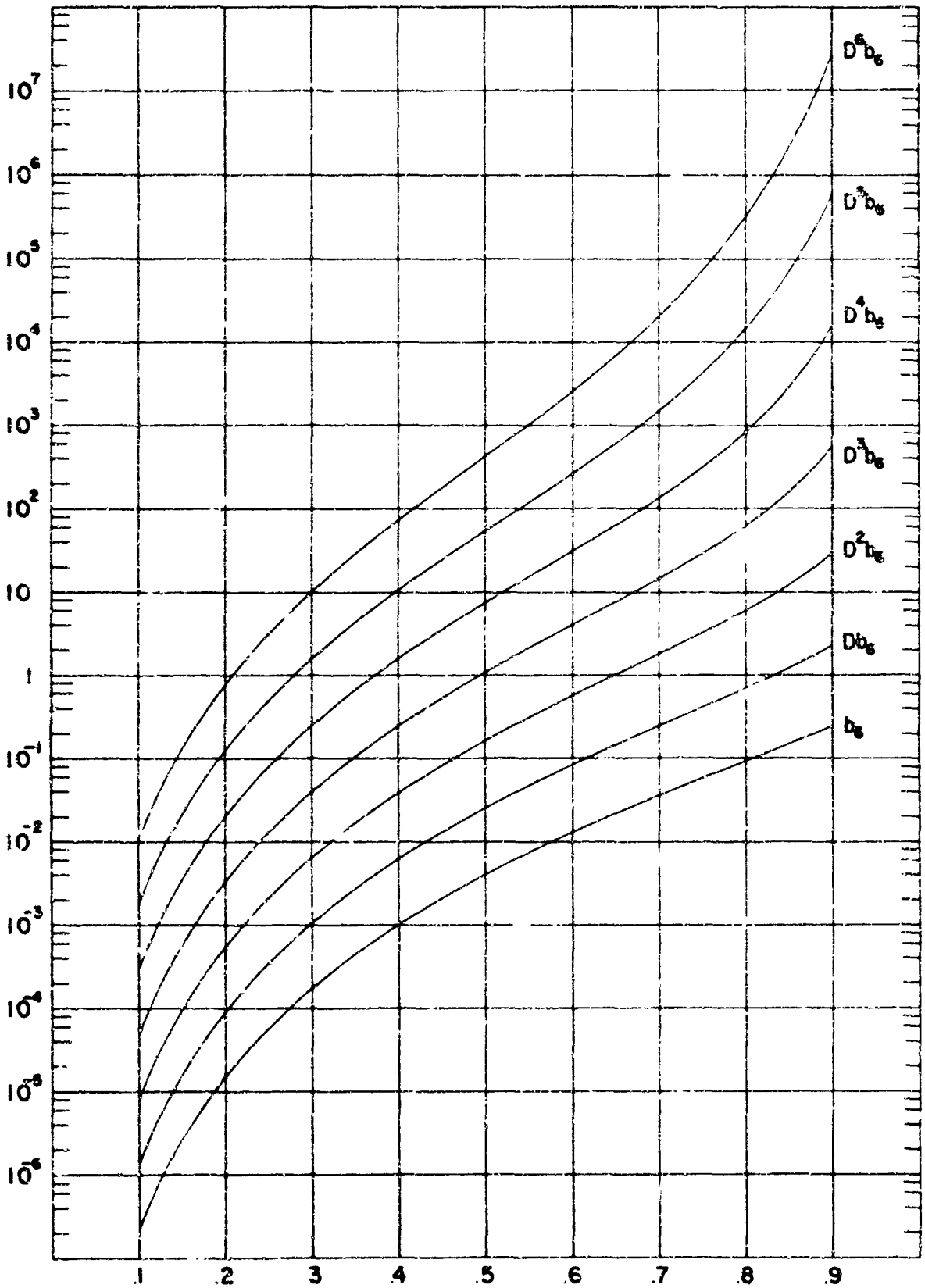
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10
1
10^{-1}
.1
.2
.3
.4
.5
.6
.7
.8
.9
$D^{12} b_5$
$D^{11} b_5$
$D^{10} b_5$
$D^{9} b_5$
$D^{8} b_5$
$D^{7} b_5$

Figure 12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10^5
10^4
10^3
10^2
10
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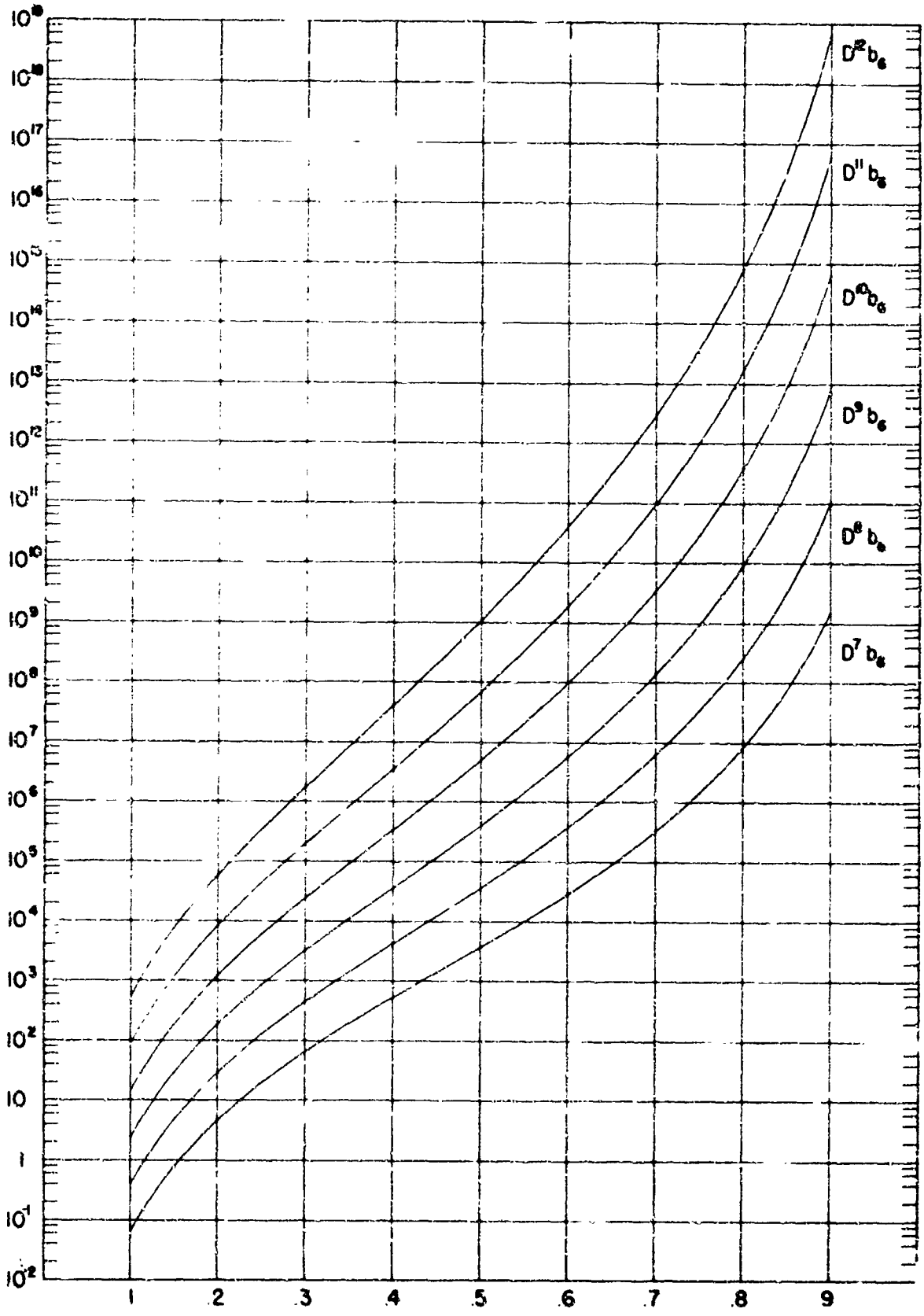
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
.1
.2
.3
.4
.5
.6
.7
.8
.9
D^6b_s
D^5b_s
D^4b_s
D^3b_s
D^2b_s
Db_s
b_s

Figure 13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10^{11}
10^{10}
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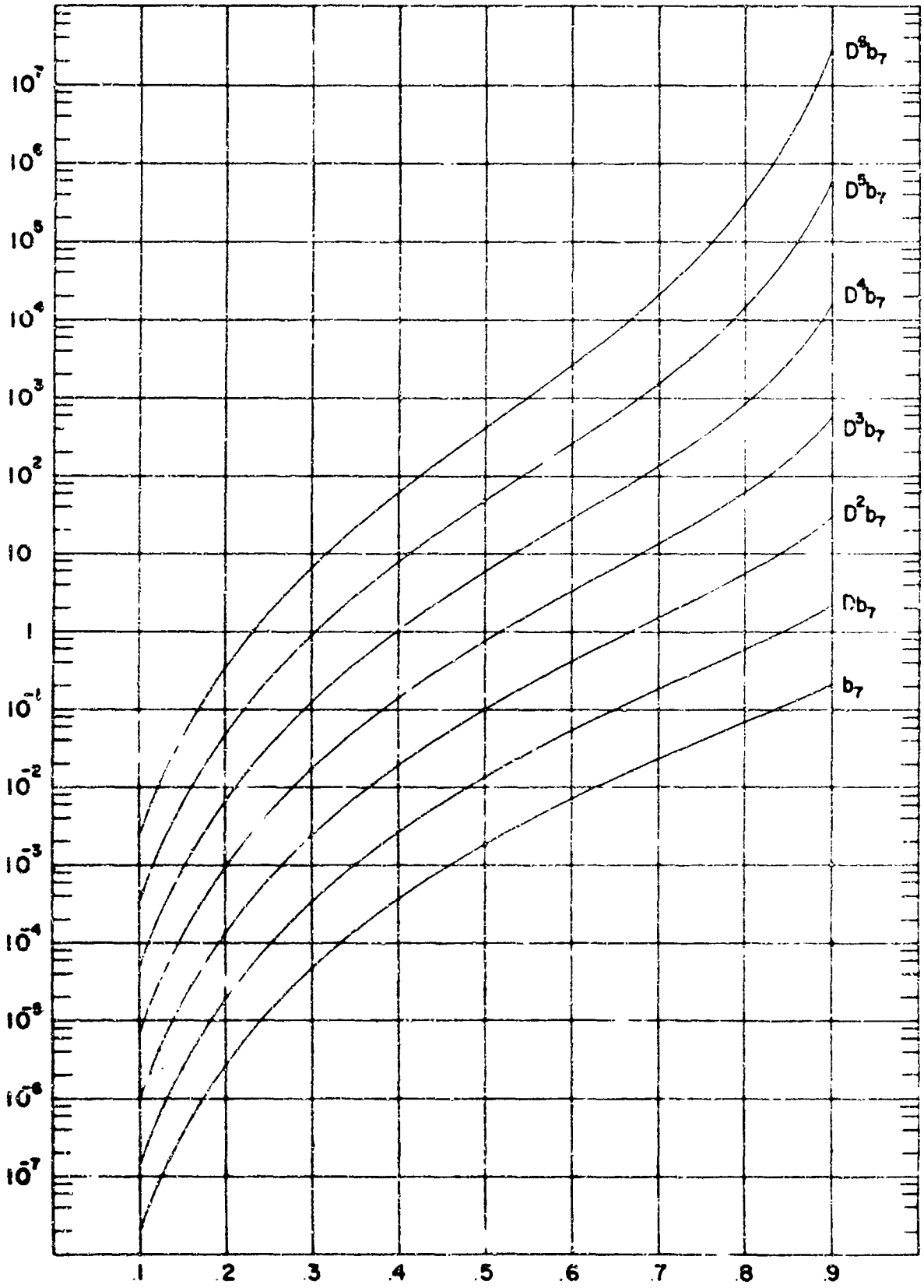
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10
1
10^{-1}
10^{-2}
1
.2
.3
.4
.5
.6
.7
.8
9
$D^{12}b_6$
$D^{11}b_6$
$D^{10}b_6$
$D^{9}b_6$
$D^{8}b_6$
$D^{7}b_6$

Figure 14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10^5
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
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10^{-6}
10^{-7}
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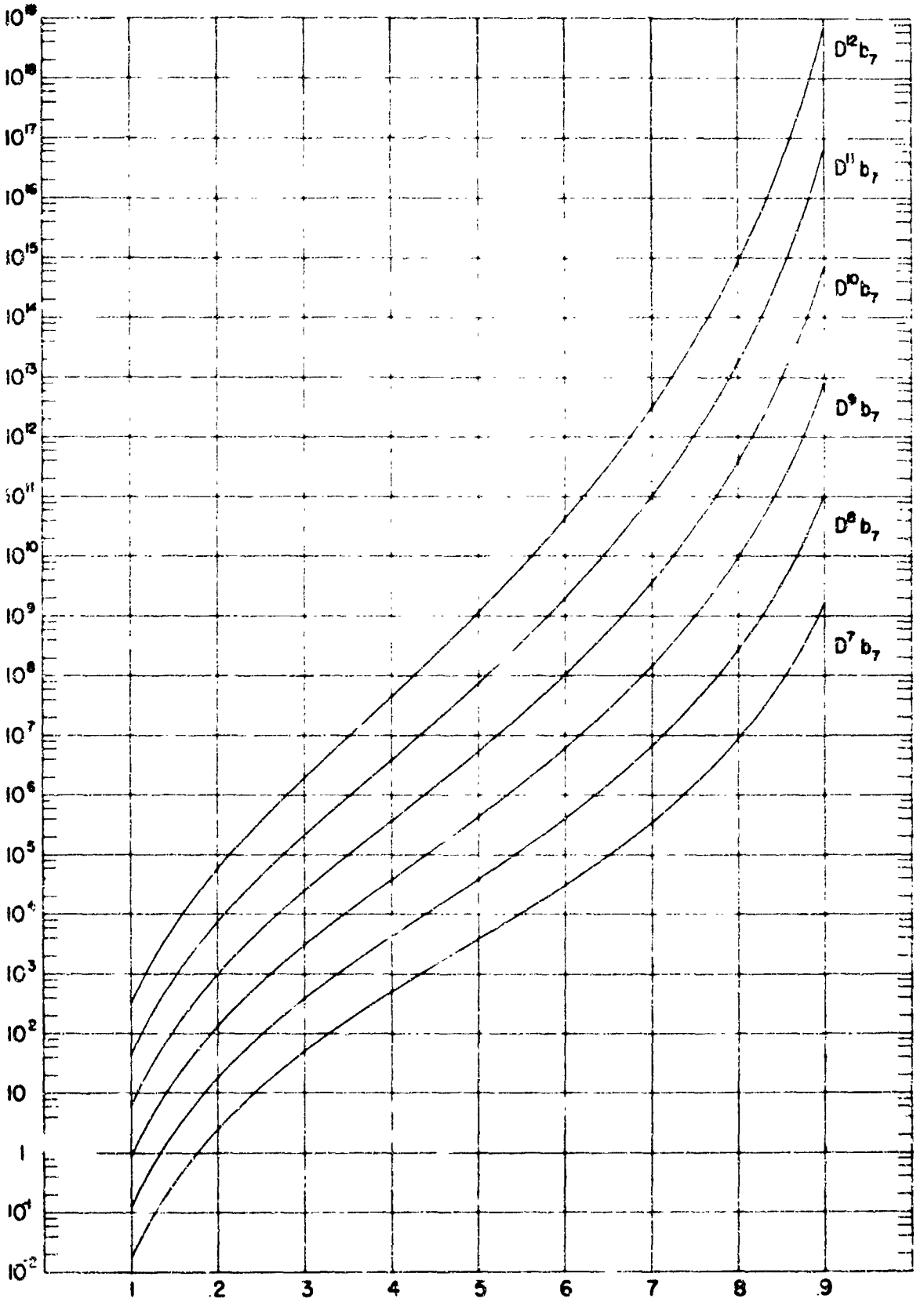
.2
.3
.4
.5
.6
.7
.8
.9
D^6b_7
D^5b_7
D^4b_7
D^3b_7
D^2b_7
Db_7
b_7

Figure 15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10^{10}
10^{9}
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10^{5}
10^{4}
10^{3}
10^{2}
10
1
10^{-1}
10^{-2}
1
2
3
4
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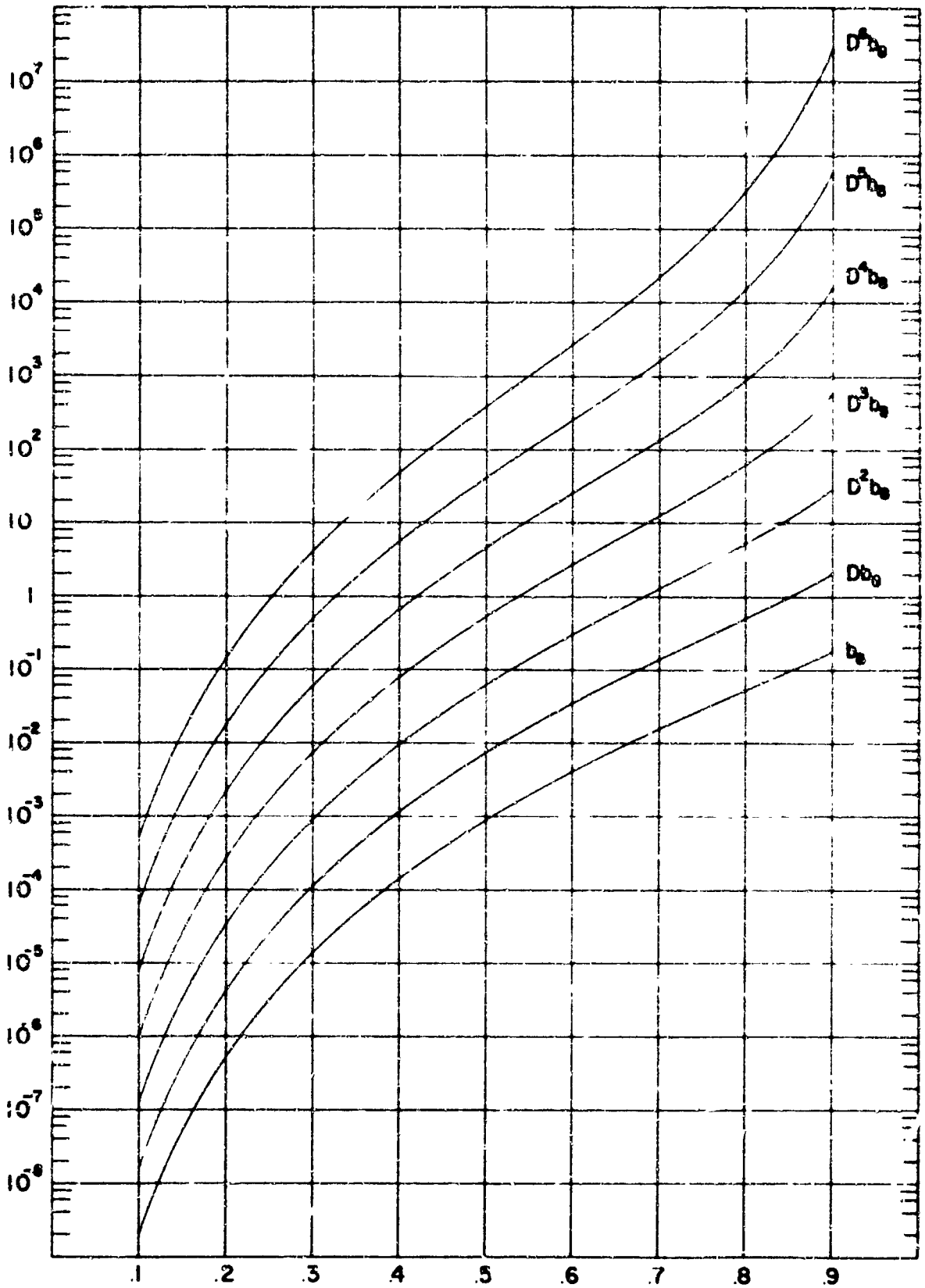
6
7
8
9
$D^{12}b_7$
$D^{11}b_7$
$D^{10}b_7$
$D^{9}b_7$
$D^{8}b_7$
$D^{7}b_7$

Figure 16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10^5
10^4
10^3
10^2
10
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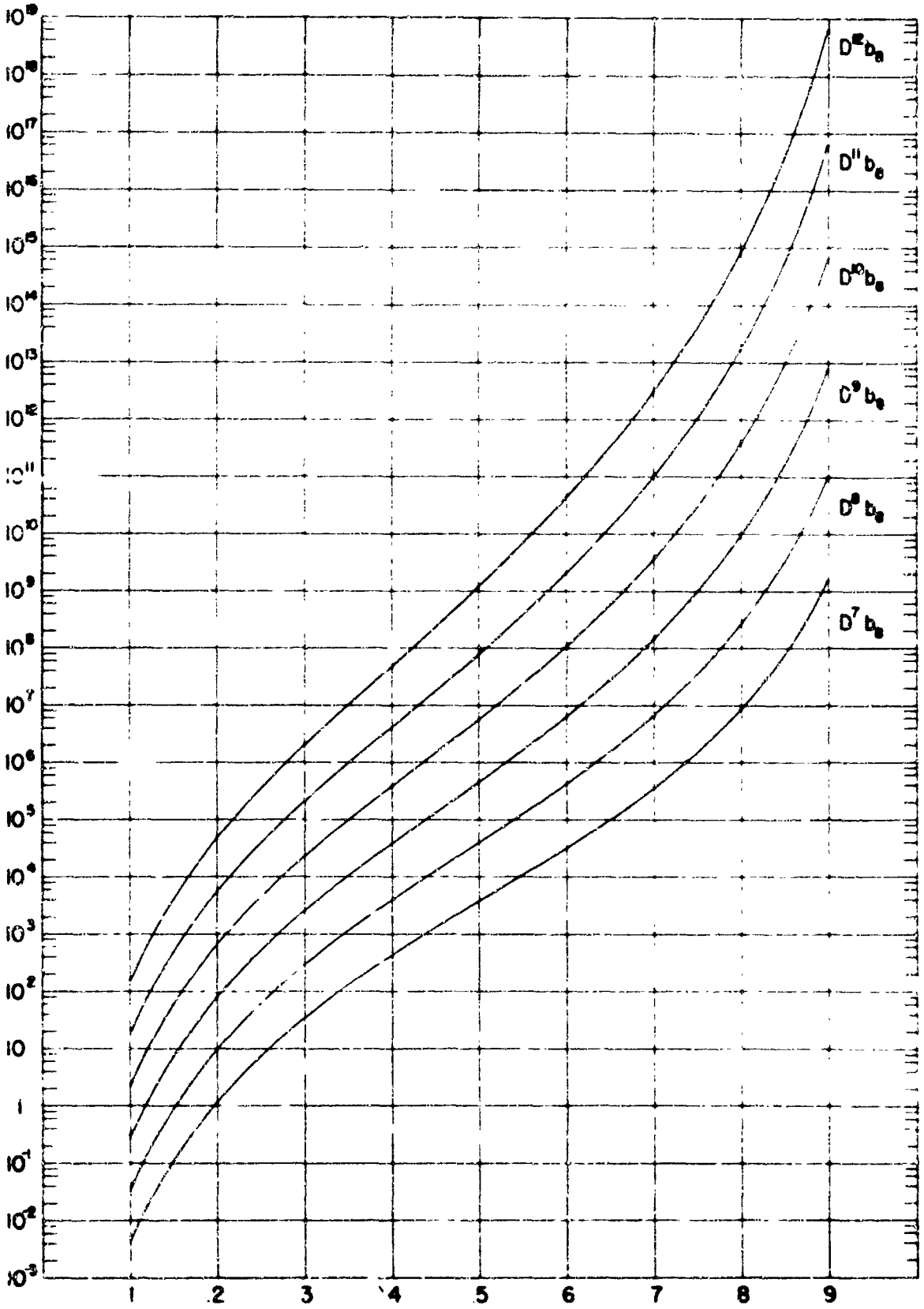
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
.1
.2
.3
.4
.5
.6
.7
.8
.9
D^6b_0
D^5b_0
D^4b_0
D^3b_0
D^2b_0
Db_0
b_0

Figure 17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10^{15}
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10^{11}
10^{10}
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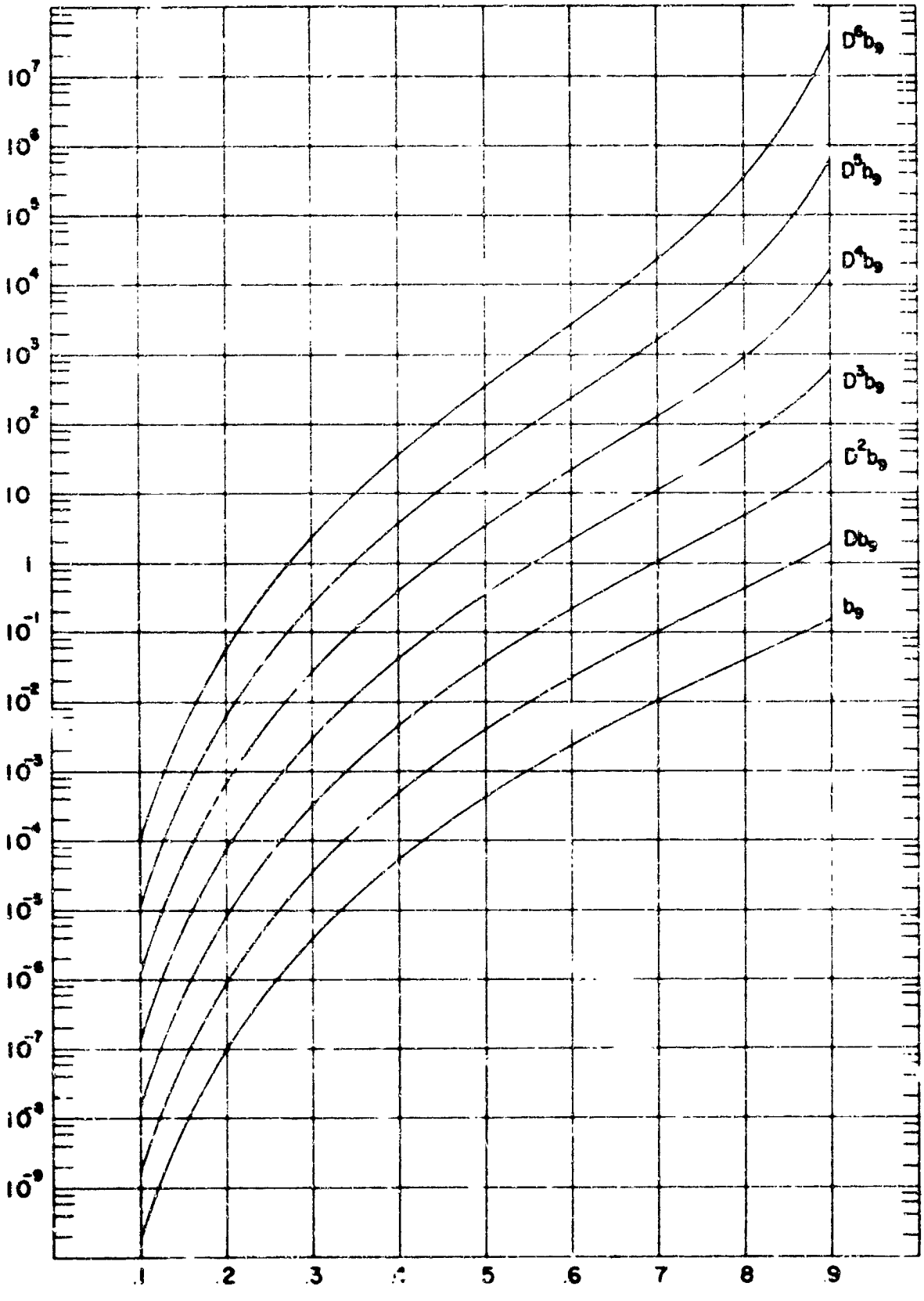
10^{2}
10
1
10^{-1}
10^{-2}
10^{-3}
1
.2
3
.4
.5
6
7
8
9
$D^{12}b_e$
$D^{11}b_e$
$D^{10}b_e$
$D^{9}b_e$
$D^{8}b_e$
$D^{7}b_e$

Figure 18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10^5
10^4
10^3
10^2
10
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10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
.1
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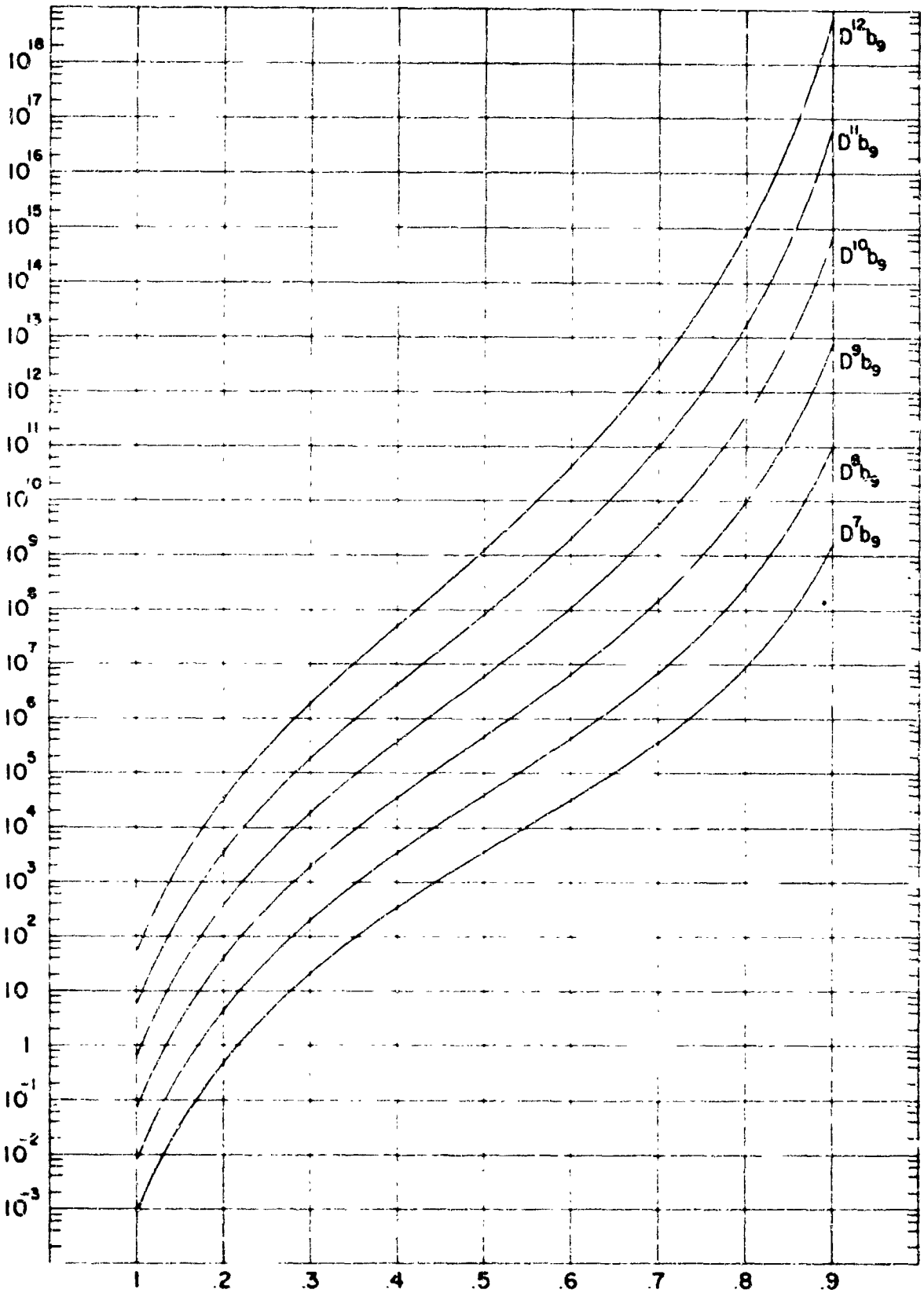
.5
.6
.7
.8
.9
D^6b_9
D^5b_9
D^4b_9
D^3b_9
D^2b_9
Db_9
b_9

Figure 19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10^{13}
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10^{10}
10^{9}
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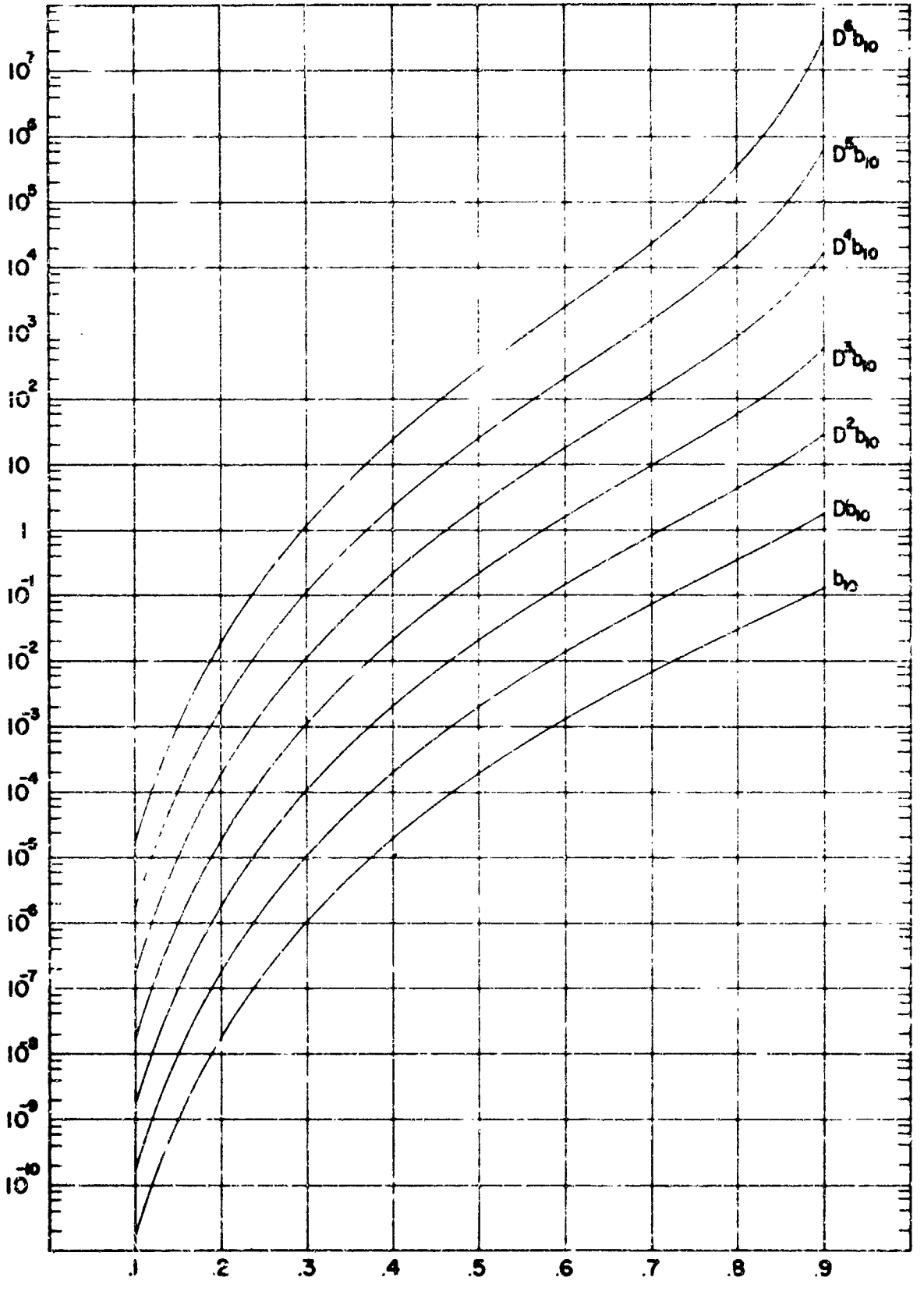
10^{5}
10^{4}
10^{3}
10^{2}
10
1
10^{-1}
10^{-2}
10^{-3}
1
.2
.3
.4
.5
.6
.7
.8
.9
$D^{12}b_9$
$D^{11}b_9$
$D^{10}b_9$
$D^{9}b_9$
$D^{8}b_9$
$D^{7}b_9$

Figure 20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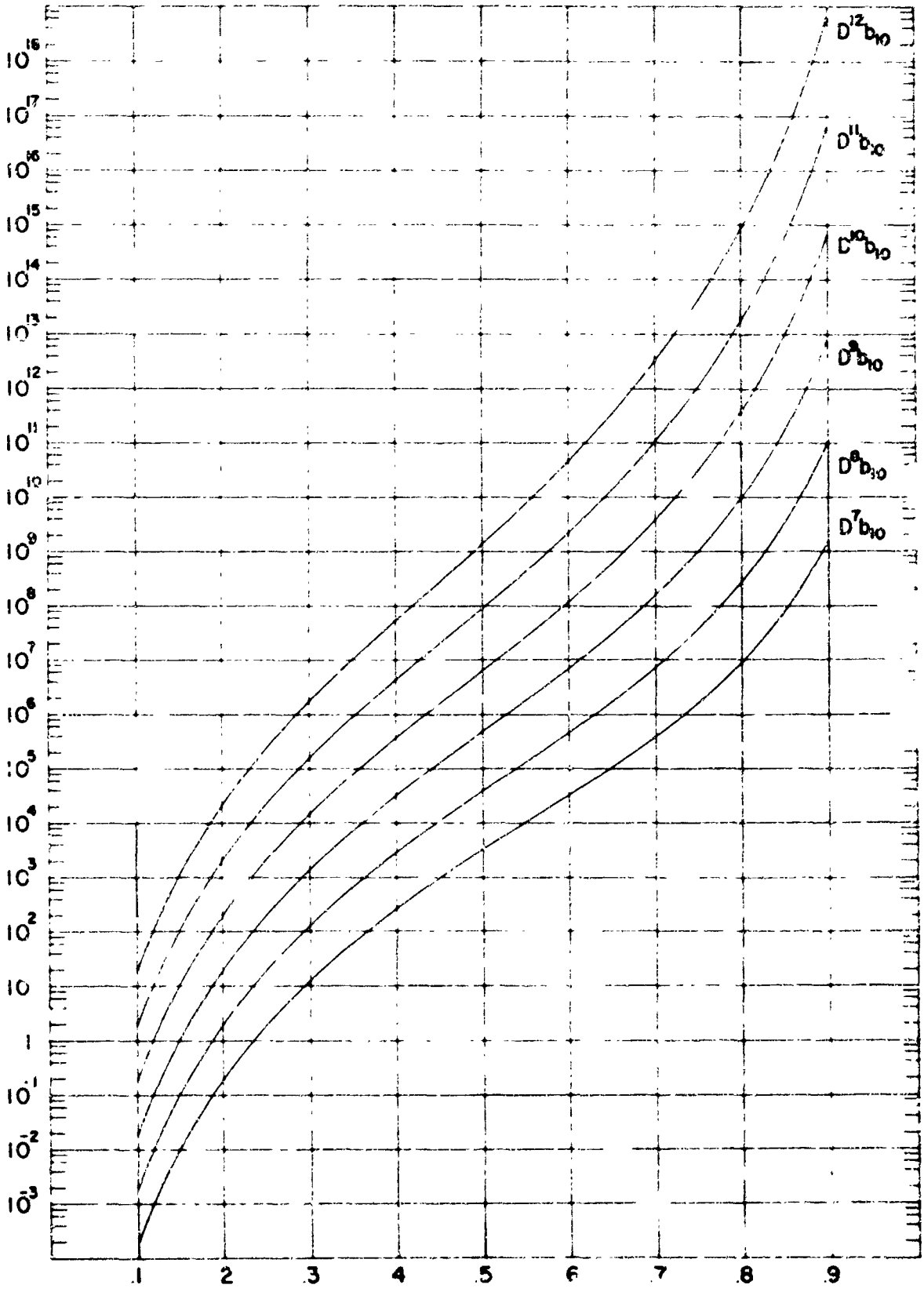
10^6
10^5
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
.1
.2
.3
.4
.5
.6
.7
.8
.9
D^6b_{10}
D^5b_{10}
D^4b_{10}
D^3b_{10}
D^2b_{10}
Db_{10}
b_{10}

Figure 21


$D^{12}b_{10}$
$D^{11}b_{10}$
$D^{10}b_{10}$
$D^{9}b_{10}$
$D^{8}b_{10}$
$D^{7}b_{10}$
10^{18}
10^{17}
10^{16}
10^{15}
10^{14}
10^{13}
10^{12}
10^{11}
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10
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10^{-1}
10^{-2}
10^{-3}
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.2
.3
.4
.5
.6
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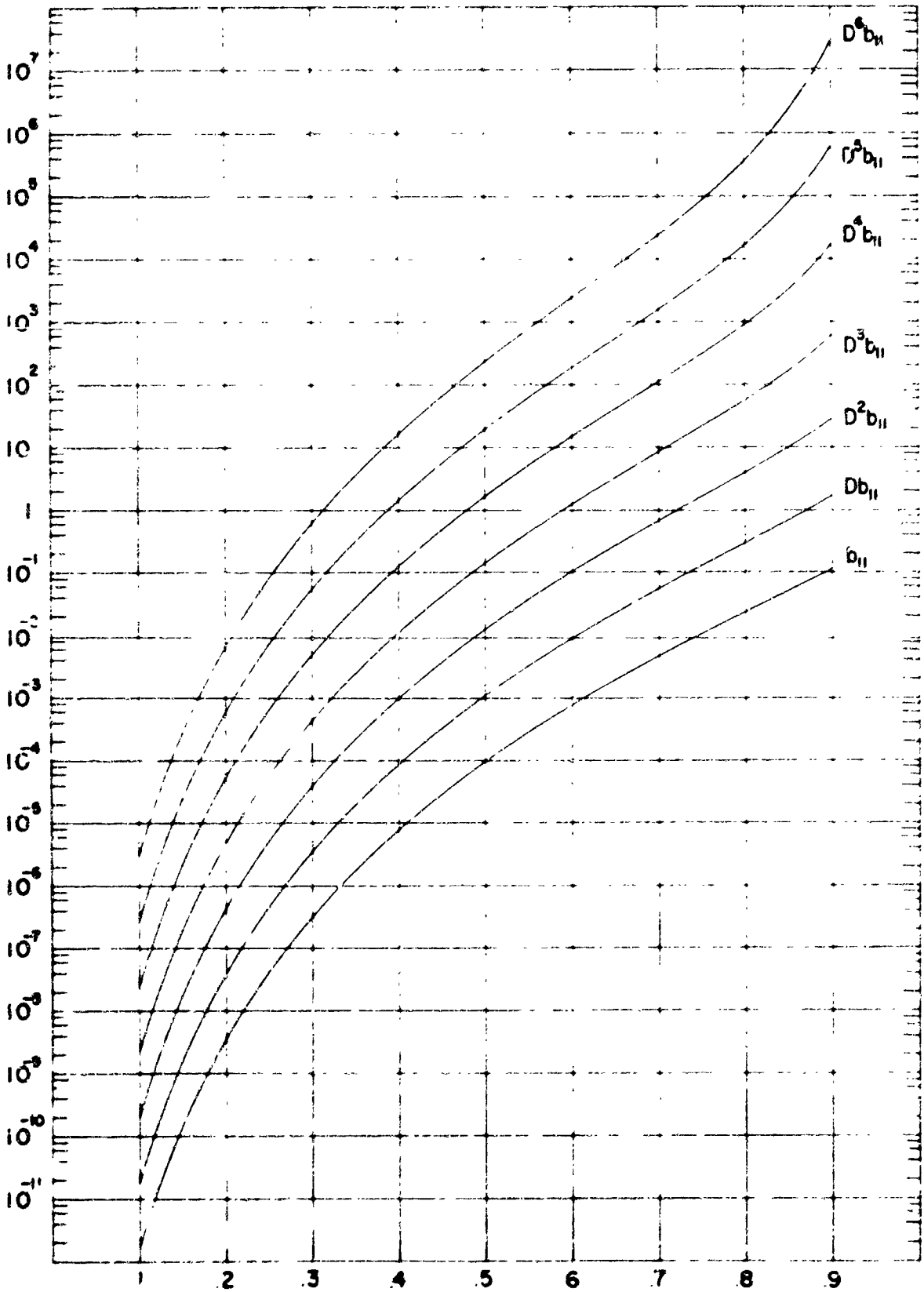
.8
.9

Figure 22


D^6b_{11}
D^5b_{11}
D^4b_{11}
D^3b_{11}
D^2b_{11}
Db_{11}
b_{11}
10^7
10^6
10^5
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
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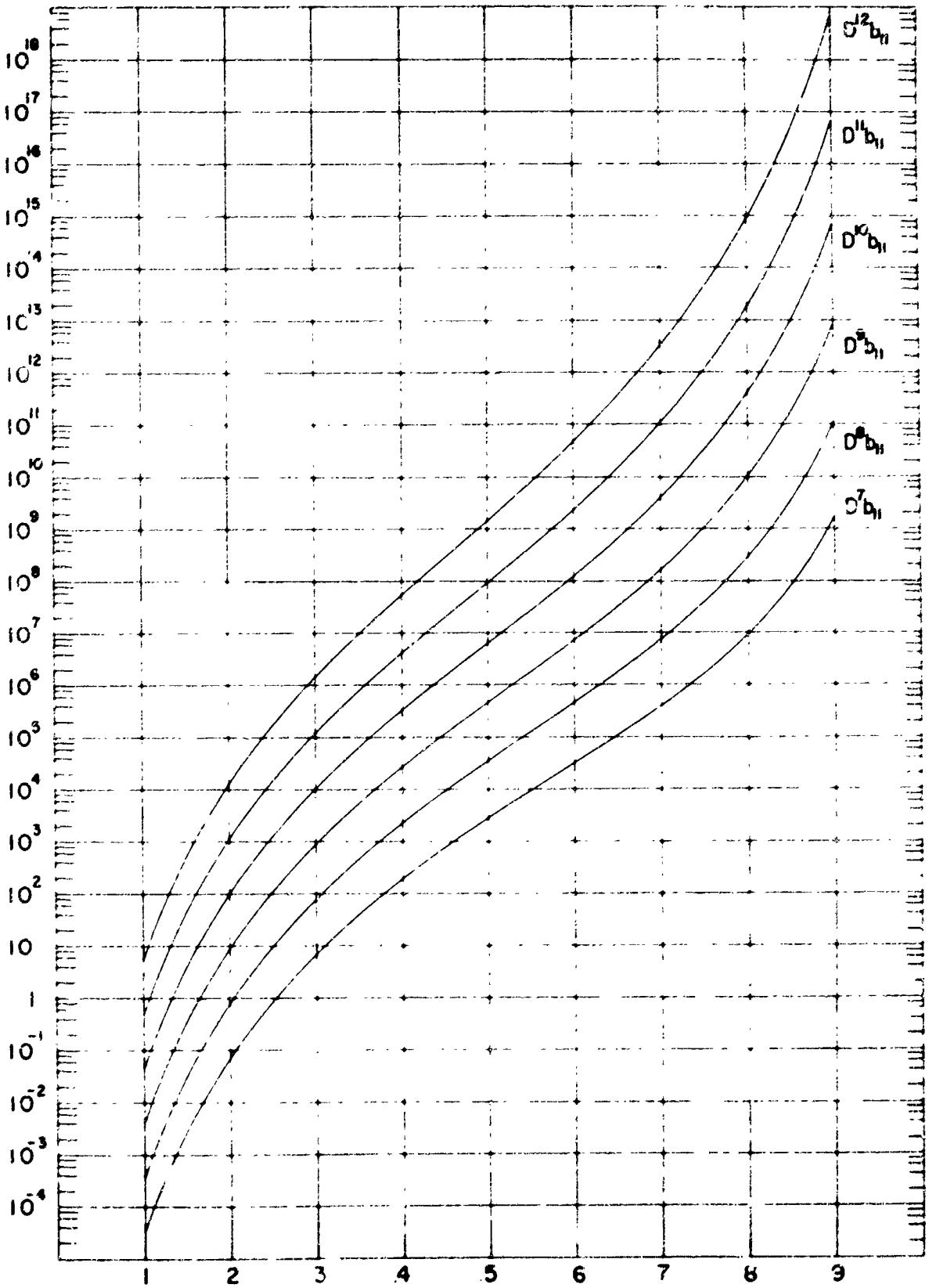
.8
.9

Figure 23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10^{12}
10^{11}
10^{10}
10^{9}
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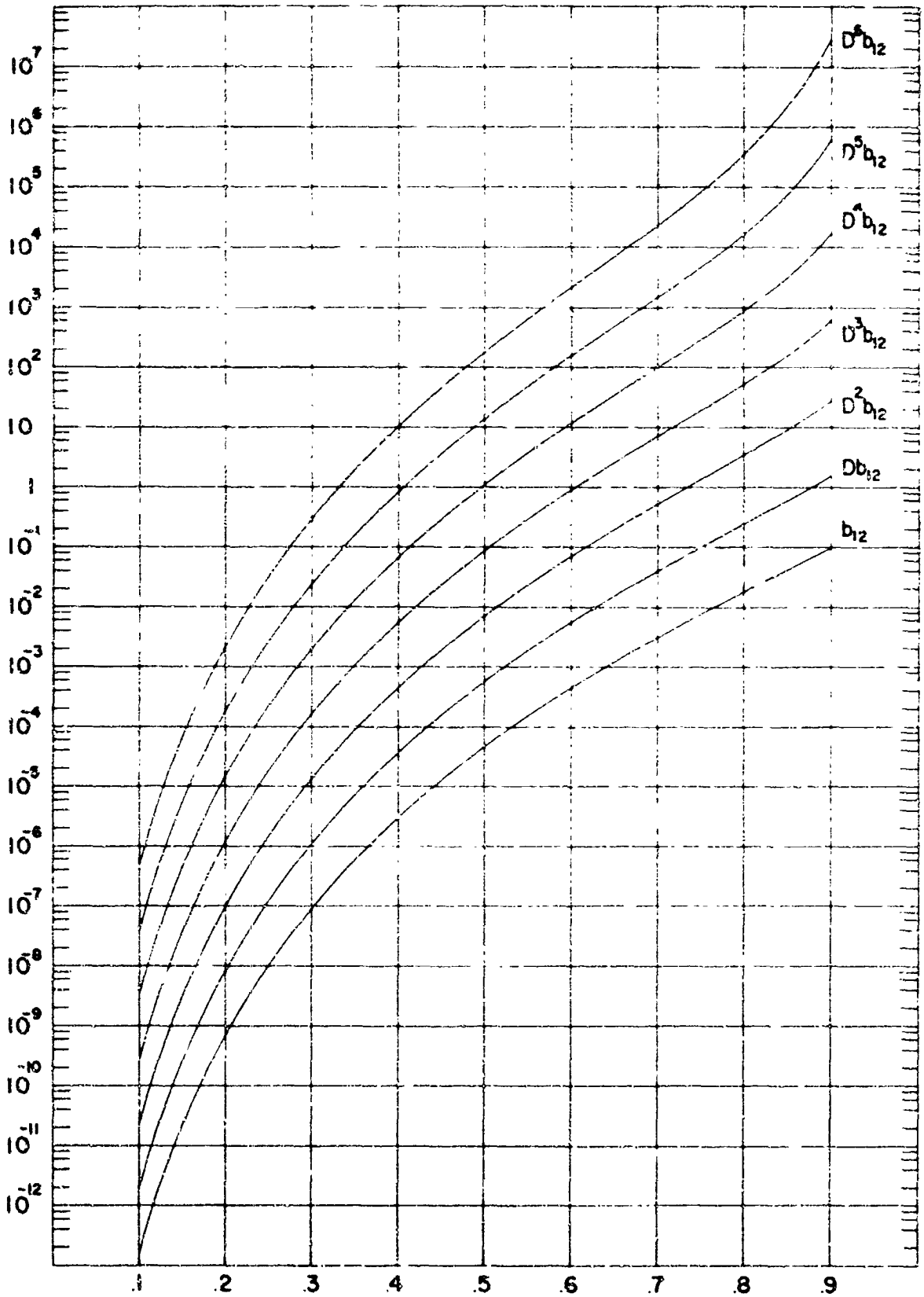
10^{5}
10^{4}
10^{3}
10^{2}
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
1
2
3
4
5
6
7
8
9
$D^{12}b_{11}$
$D^{11}b_{11}$
$D^{10}b_{11}$
$D^{9}b_{11}$
$D^{8}b_{11}$
$D^{7}b_{11}$

Figure 24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10^5
10^4
10^3
10^2
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1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
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10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
.1
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.3
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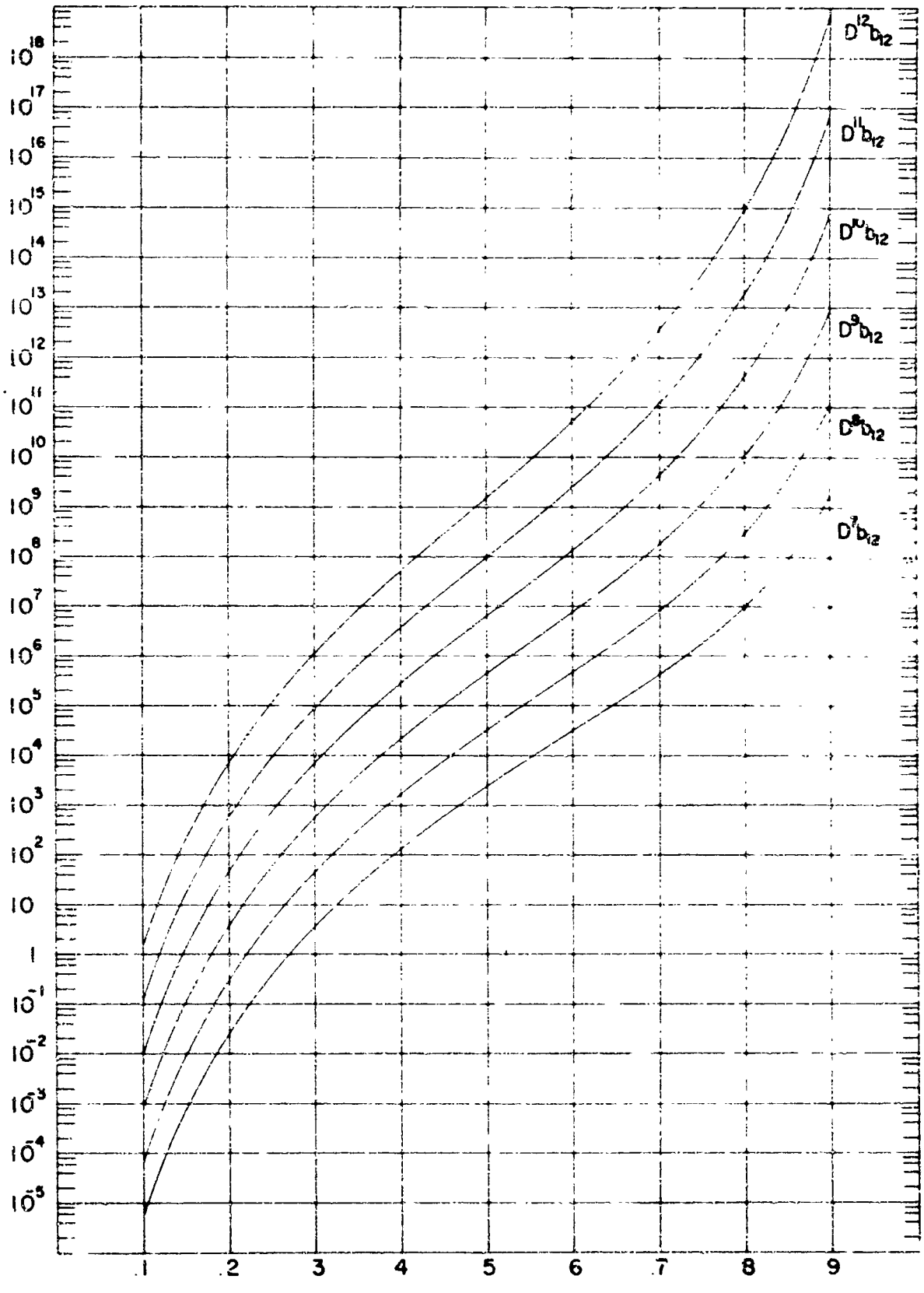
.5
.6
.7
.8
.9
D^6b_{12}
D^5b_{12}
D^4b_{12}
D^3b_{12}
D^2b_{12}
Db_{12}
b_{12}

Figure 25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10^{10}
10^{9}
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10^{2}
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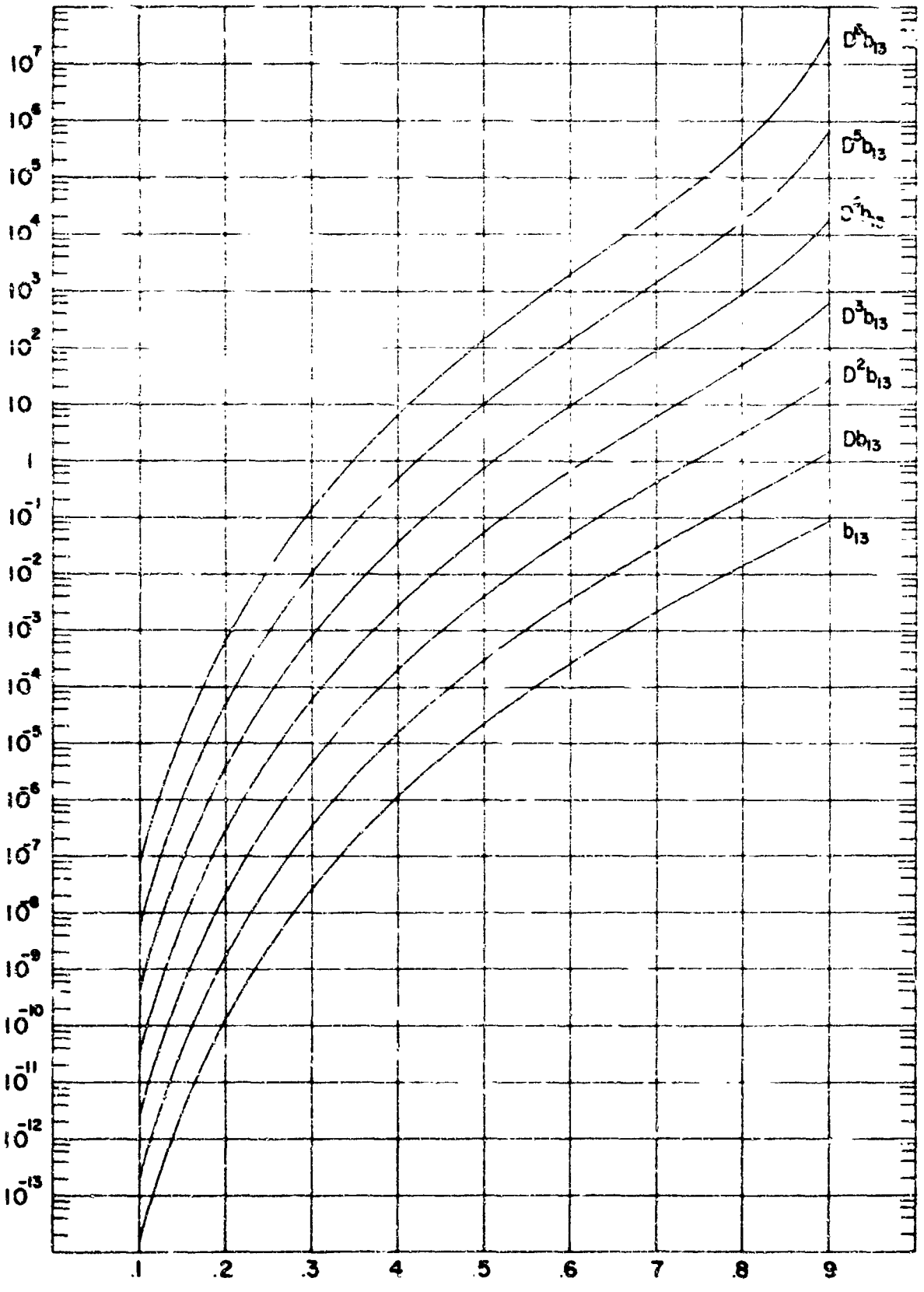
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
.1
.2
3
4
5
6
.7
8
9
$D^{12}b_{12}$
$D^{11}b_{12}$
$D^{10}b_{12}$
$D^{9}b_{12}$
$D^{8}b_{12}$
$D^{7}b_{12}$

Figure 26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10^6
10^5
10^4
10^3
10^2
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10^{-1}
10^{-2}
10^{-3}
10^{-4}
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10^{-7}
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10^{-10}
10^{-11}
10^{-12}
10^{-13}
.1
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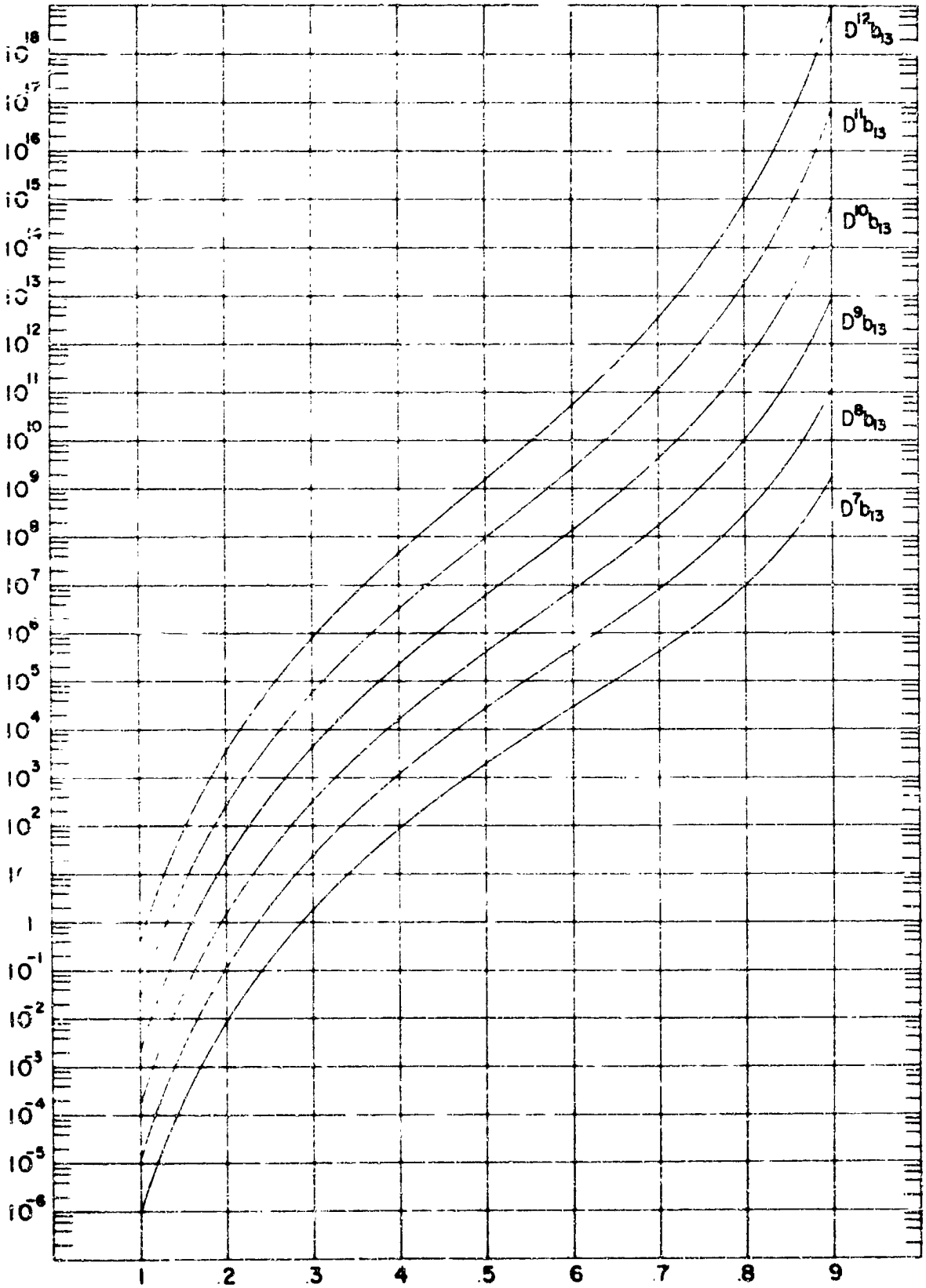
.4
.5
.6
.7
.8
.9
D^6b_{13}
D^5b_{13}
D^4b_{13}
D^3b_{13}
D^2b_{13}
Db_{13}
b_{13}

Figure 27


$D^{12}b_{13}$
$D^{11}b_{13}$
$D^{10}b_{13}$
$D^{9}b_{13}$
$D^{8}b_{13}$
$D^{7}b_{13}$
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10^{16}
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10^{7}
10^{6}
10^{5}
10^{4}
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10^{2}
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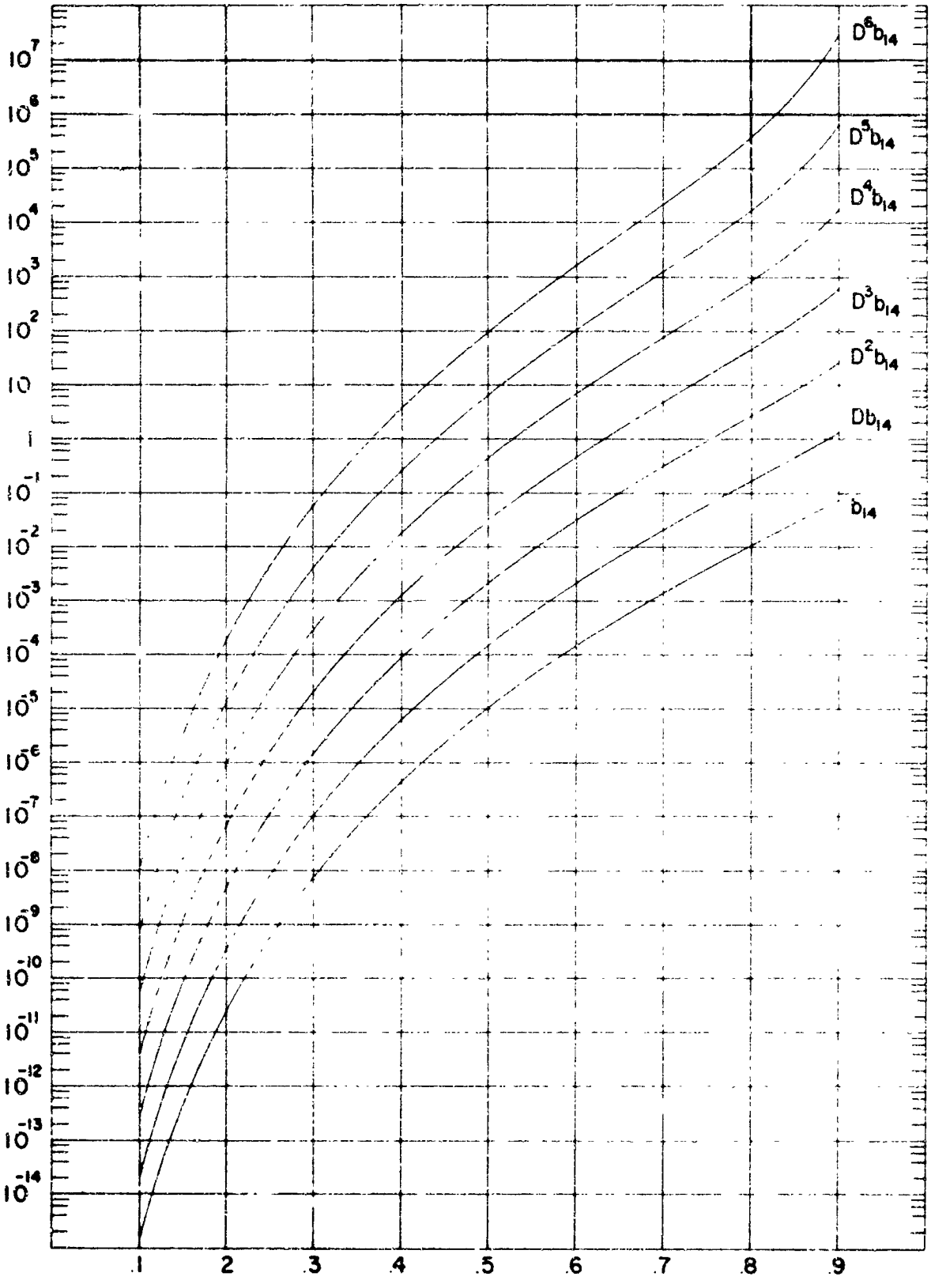
Figure 28


D^6b_{14}
D^5b_{14}
D^4b_{14}
D^3b_{14}
D^2b_{14}
Db_{14}
b_{14}
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10^6
10^5
10^4
10^3
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
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.8
.9

Figure 29

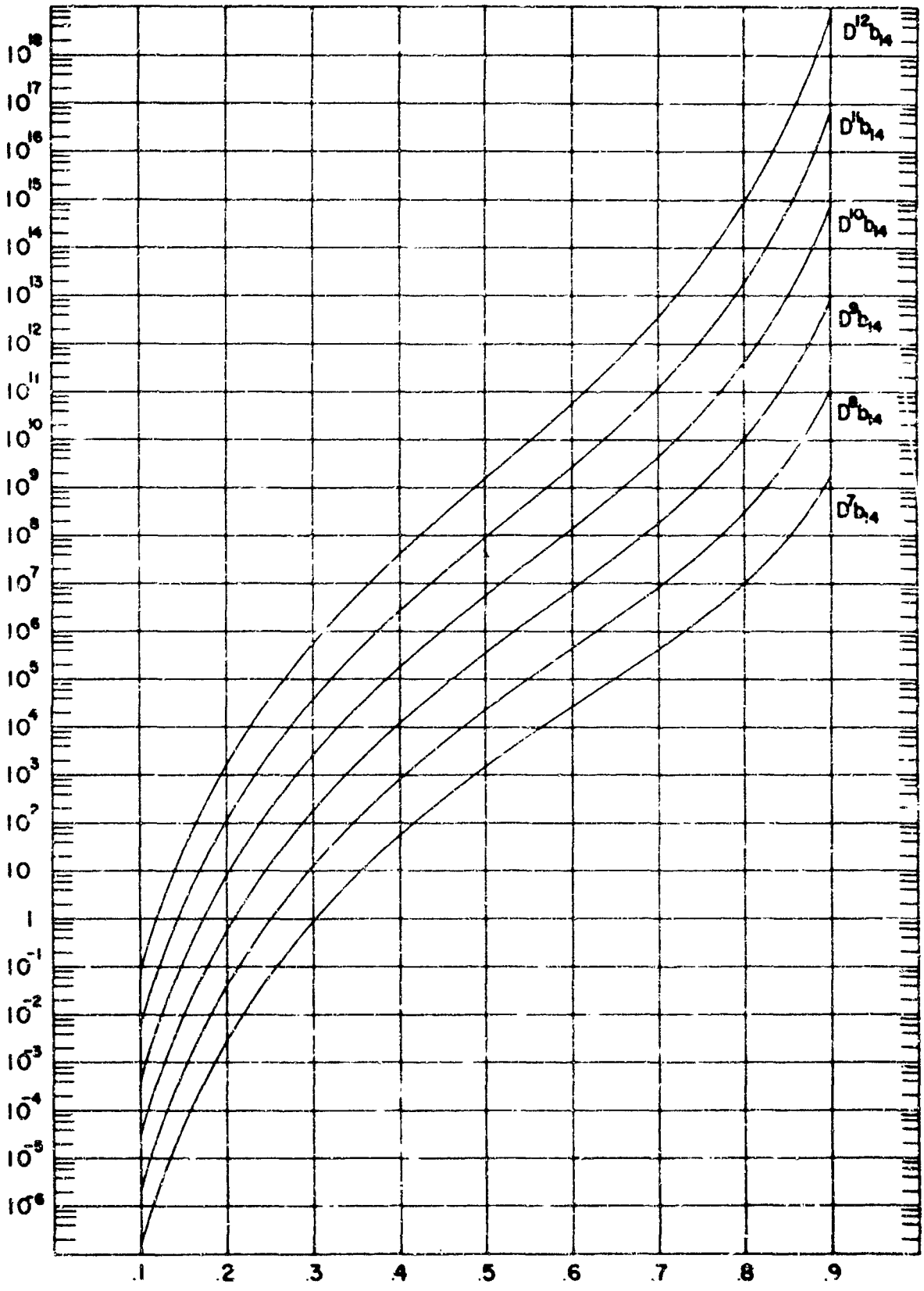
$D^{12}b_{14}$
$D^{11}b_{14}$
$D^{10}b_{14}$
$D^{9}b_{14}$
$D^{8}b_{14}$
$D^{7}b_{14}$
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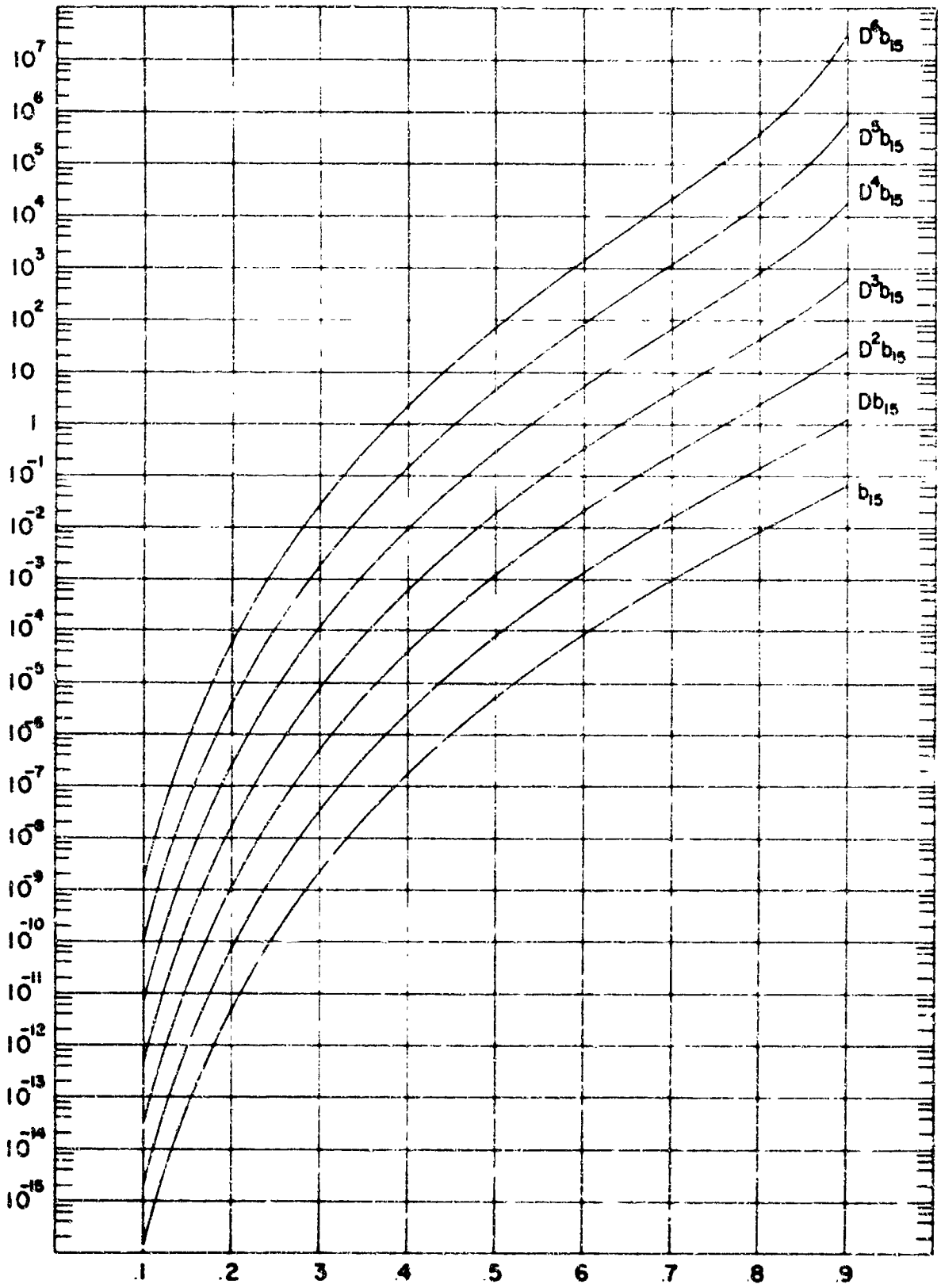
Figure 30


10^7
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10^5
10^4
10^3
10^2
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10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
10^{-15}
.1
.2
.3
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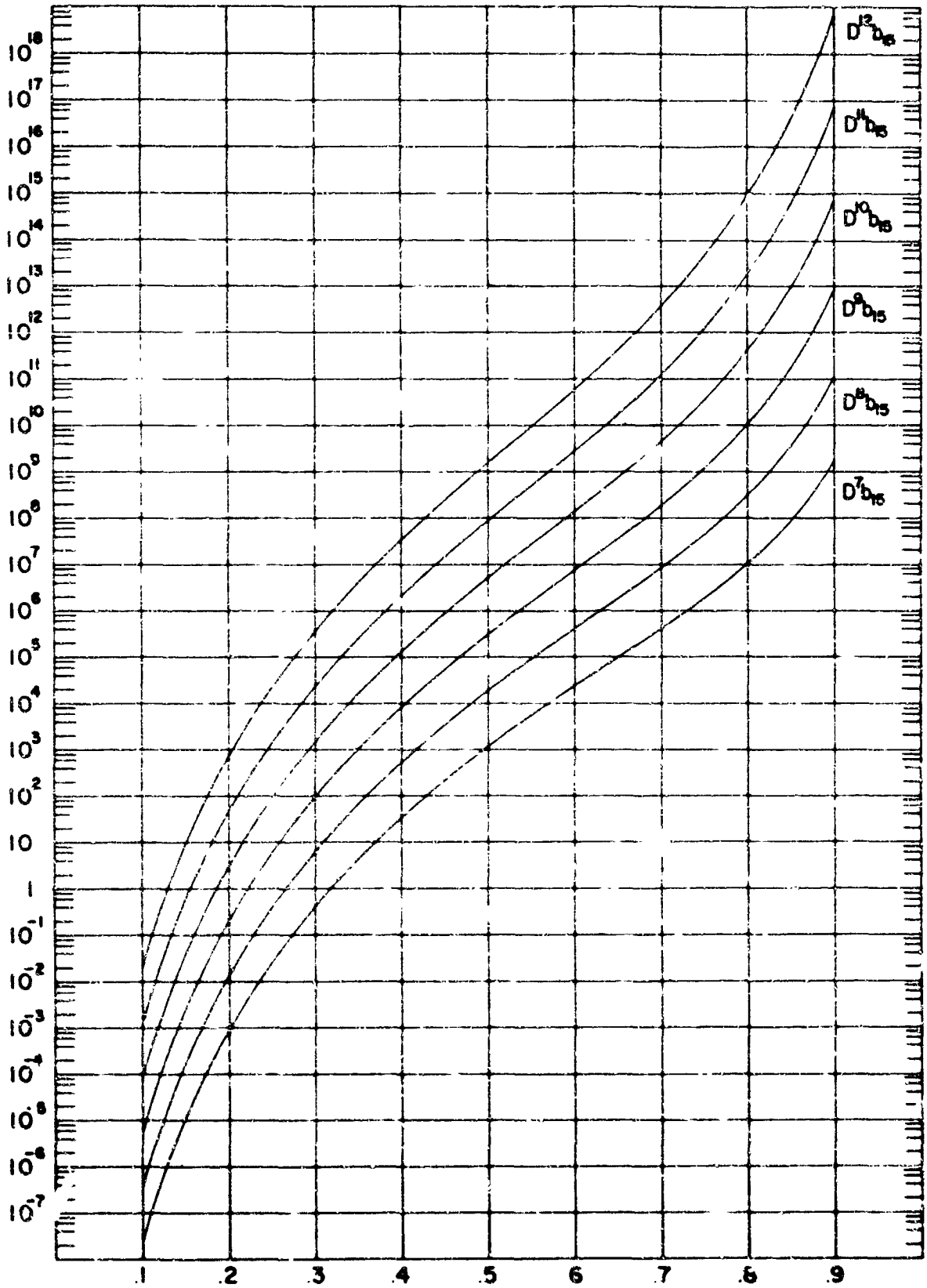
.5
.6
.7
.8
.9
D^6b_{15}
D^5b_{15}
D^4b_{15}
D^3b_{15}
D^2b_{15}
Db_{15}
b_{15}

Figure 31


$D^{13}b_{15}$
$D^{11}b_{15}$
$D^{10}b_{15}$
$D^{9}b_{15}$
$D^{8}b_{15}$
$D^{7}b_{15}$
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10^{5}
10^{4}
10^{3}
10^{2}
10
1
10^{-1}
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10^{-3}
10^{-4}
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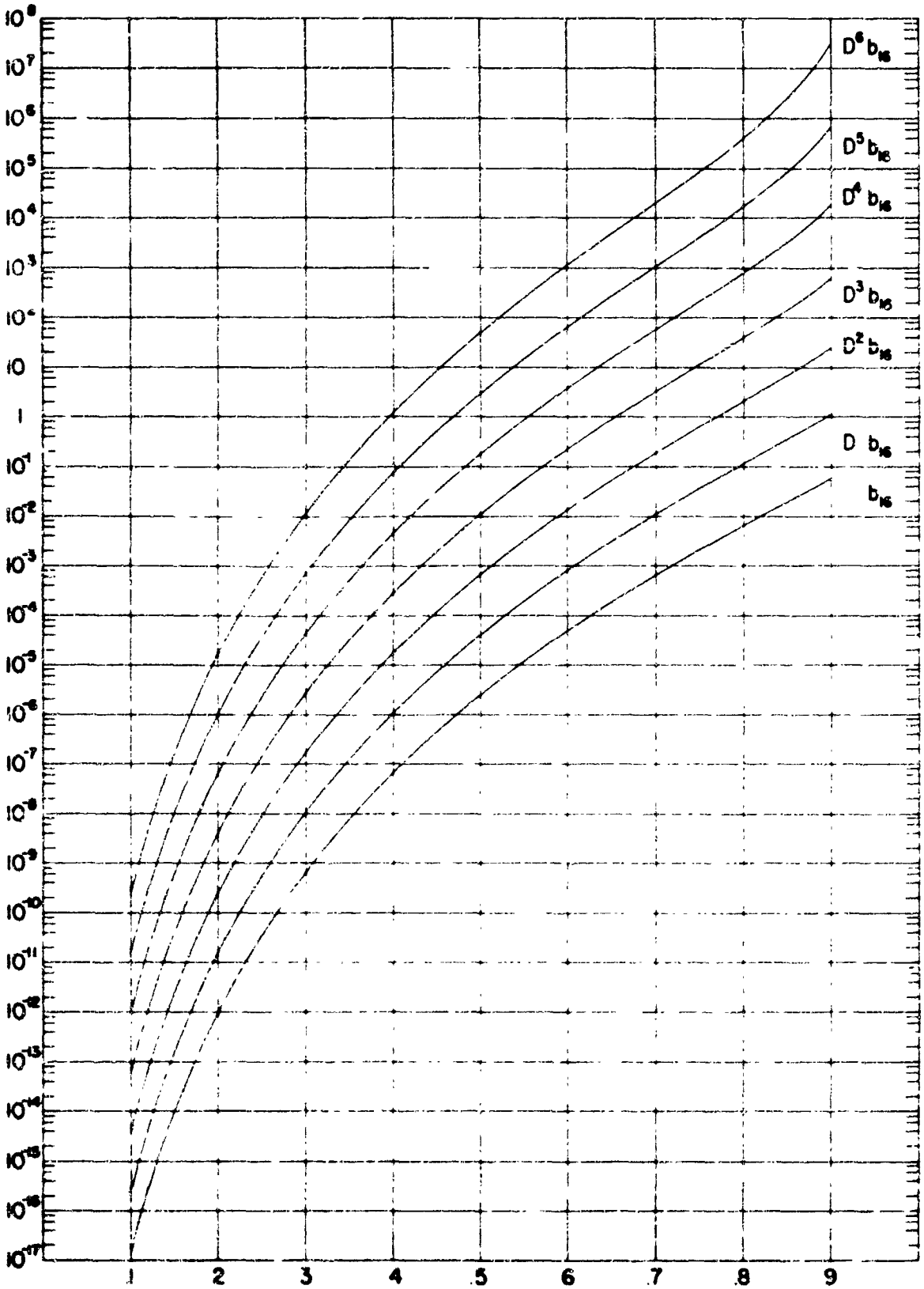
.8
.9

Figure 32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10^7
10^6
10^5
10^4
10^3
10^2
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10^{-1}
10^{-2}
10^{-3}
10^{-4}
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10^{-7}
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10^{-11}
10^{-12}
10^{-13}
10^{-14}
10^{-15}
10^{-16}
10^{-17}
.1
.2
.3
.4
.5
.6
.7
.8
.9
$D^6 b_{16}$
$D^5 b_{16}$
$D^4 b_{16}$
$D^3 b_{16}$
$D^2 b_{16}$
$D\ b_{16}$
b_{16}

Figure 33

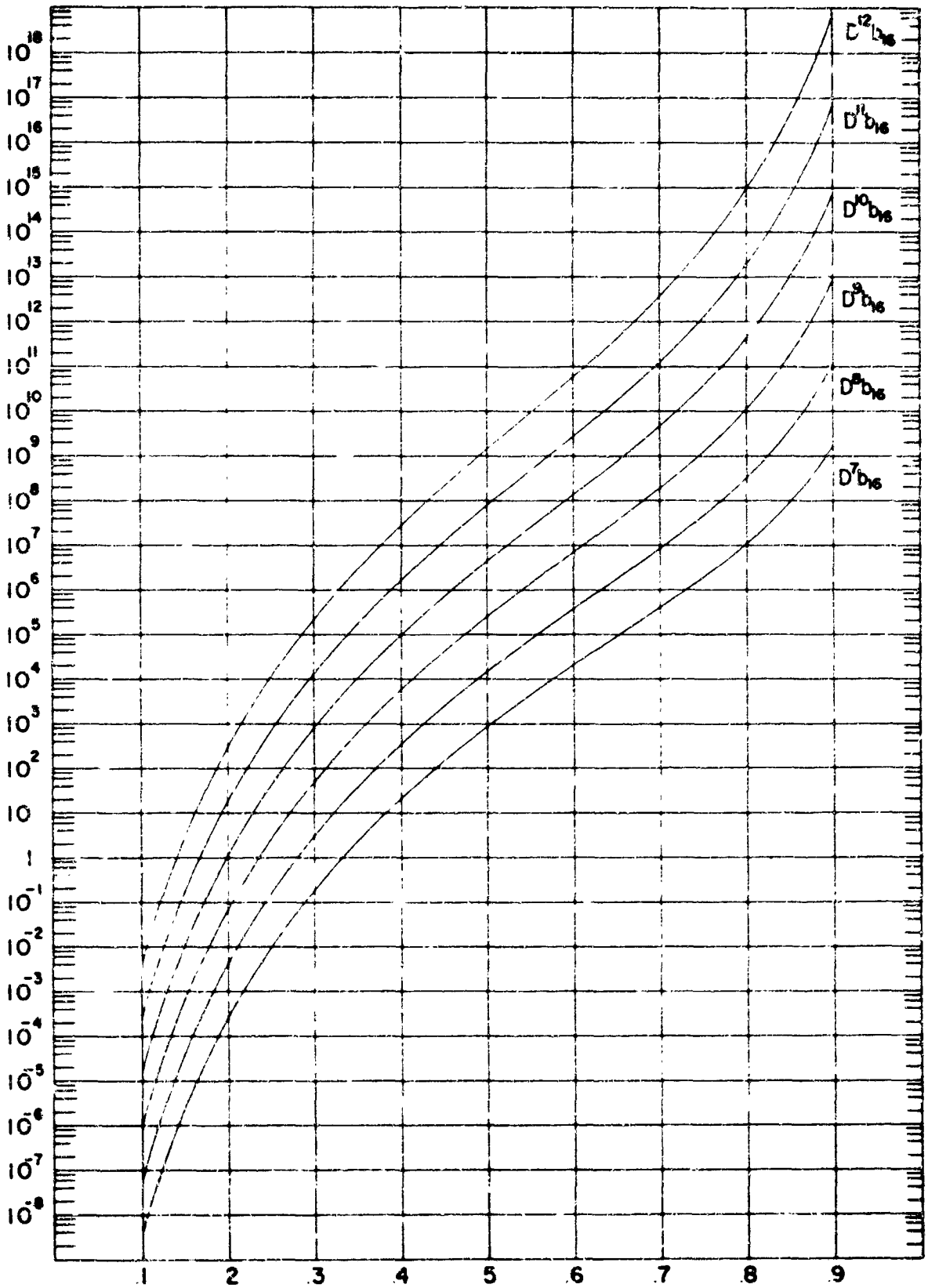
10^{18}
10^{17}
10^{16}
10^{15}
10^{14}
10^{13}
10^{12}
10^{11}
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{4}
10^{3}
10^{2}
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
.1
.2
.3
.4
.5
.6
.7
.8
.9
$D^{12}b_{16}$
$D^{11}b_{16}$
$D^{10}b_{16}$
$D^{9}b_{16}$
$D^{8}b_{16}$
$D^{7}b_{16}$

Figure 34

A COMPUTER PROGRAM FOR THE EVALUATION OF LAPLACE COEFFICIENTS AND THEIR NEWCOMB DERIVATIVES[1]

by

Benno Renima[2]

The program to compute the Laplace coefficients $b_k(\alpha)$ and their Newcomb derivatives $D^j b_k(\alpha)$ by the method described in the preceding paper of I. G. Izsak was written for an IBM-7090 digital computer in Fortran II using double precision arithmetic. The index s has been omitted since the program in its present form computes the Laplace coefficients only for $s = \frac{1}{2}$. It will calculate the array on the top of page 3 for any argument α in the interval $0 < \alpha < 1$ and up to any index $K \leq 32$ and any order $J \leq 12$ of the derivatives.

The program has the form of a subroutine and must be called by the user's program. The three input parameters, viz. the argument α, the maximum required index K and the maximum order of the derivatives J have to be supplied to the subroutine through the common area. An optional output routine is available that tabulates the functions for any given set of arguments. A sample of this output is printed in the Appendix. This output subroutine also is called from the user's program. The method according to which the program computes the array on page 3 is described in the following.

First the coefficients C_{kh} which are defined on p. 4 of the preceding paper are computed up to order max $\{K,J\}$ and stored. Their computational scheme is similar to that of a Pascal triangle. If the program is re-entered with a new argument, a check is made to see if the triangular array of the C_{kh} has been developed far enough for the new case. If so, the calculation of the C_{kh} is bypassed.

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Next we reduce the argument by the transformation (5) and compute the powers of α_1 until α_1^j becomes smaller than 10^{-16}. The reduced argument α_1 and its ascending powers are stored in the second column of a rectangular array of 33 rows and 13 columns in which all the intermediate results of the computation will be stored and which at the end of the program will contain the array of p. 3. The argument α_1 is again reduced by the same transformation to α_2, and α_2 and its powers are stored in the third column of the array. We go on in this fashion until the argument is so small that its column contains not more than four elements. Now we compute the first four Laplace coefficients for the final reduced argument α_I by taking the first one or two terms of the pertaining hypergeometric series, i.e.

$$b_0(\alpha_I) = 1 + \frac{\alpha_I^2}{4}$$

$$b_2(\alpha_I) = \frac{3}{8}\alpha_I^2$$

$$b_1(\alpha_I) = \frac{\alpha_I}{2}\left[1 + b_2(\alpha_I)\right]$$

$$b_3(\alpha_I) = \frac{5}{16}\alpha_I^3$$

and store these in the sixth column after the column in which α_I is stored. Back tracking now we compute the $b_k(\alpha_{I-1})$ from the $b_k(\alpha_I)$ with formula (6) and store the results each time one column to the left. Doing this (I-1)-times we end up with the $b_k(\alpha_1)$ in the eighth column. From the $b_k(\alpha_1)$ we now compute the $\beta_\ell(\alpha)$ with formula (9) and store the results in the first column. We form the alternating sum in formula (9) by subtracting from the product $C_{kh}b_h(\alpha_1)$ at each step the partial sum formed at the previous step and taking the absolute value of the final sum since the β_ℓ are always positive. Then the $D\beta_\ell$ are computed from the β_ℓ with formula (8) and stored in the second column. Now we go back to the first column and compute the Laplace coefficients $b_k(\alpha)$ from the $b_k(\alpha_1)$ which are still in the eighth column with formula (6) and store the results in the first column. Starting again from the second column we resume the computation of the triangular array of the $D^j\beta_\ell$ with formula (8) working to the right and upwards. This gives us the first row of our final array since $D^j b_0 = D^j\beta_0$. The second row of the final array, i.e. the $D^j b_1$ are now calculated with formula (15) from the first row. Actually, we use formula (15) in the form

$$D^j b_1 = \alpha^{-1} \sum_{m=0}^{j-1} (-1)^{j-(m+1)} \binom{j-1}{m} D^{m+1} b_0 .$$

Developing the binomial coefficients as we go along, we multiply at each step $D^{m+1}b_0$ with $\binom{j-1}{m}$ and subtract the partial sum formed at the previous step, setting the final sum positive as before. Division by α then gives $D^j b_1$. The elements of the second column, the Db_k, we find with formula (14) from the first column and the element of the second column that was computed in each previous step. Finally all remaining derivatives in the array are obtained column by column with the aid of formula (18).

The program was used to produce tables of the Laplace coefficients and their first twelve Newcomb derivatives for 81 arguments viz. $\alpha = 0.1(0.01)0.9$. The machine time for this run was 3.0 minutes. To achieve the best possible accuracy the decimal to binary conversion of the argument α should be done in double precision.

The results were also plotted on an EAI data-plotter, model 3033D. For each k two logarithmic graphs were made with the arguments as abscissae and the logarithms of the functional values as ordinates. One plot contains the Laplace coefficients and the first six derivatives and the other the last six derivatives. The 7090 program that produced the punched card input (ca. 20,000 cards) for the data-plotter, scaled the ordinates and adjusted the logarithmic scale for each plot separately. The graphs are given on pp. 12-45

To get an idea of the efficiency of the method and the accuracy of the results we have run the same program in single precision arithmetic. Furthermore we have written a program that computes the Laplace coefficients and their derivatives straightforward by the summation of hypergeometric series.

The single precision program took 1.9 minutes for the 81 arguments while the machine time to compute tables for nine arguments, viz. $\alpha = 0.1(0.1)0.9$ with the double precision hypergeometric series program was 3.5 minutes. The average computation time with hypergeometric series therefore is approximately ten times longer. Comparison of the results of these programs can be evaluated as follows.

In the program that is described above no loss of accuracy to speak of occurs in the first two columns and rows of the scheme on the top of p. 3. Beginning with the third column and third row that is with the use of the recurrence relation (18) one does loose accuracy in the sense that the last digits of the printed floating point numbers become gradually meaningless; the smaller the argument α the faster this happens. This is a perfectly natural and harmless phenomenon related to the fact that for small values of α the functions $D^j b_k$ decrease rapidly as k increases.

In the single precision version of the program the fourth digit of the printed floating point number becomes meaningless with the value of the index k for which $b_k < 2 \times 10^{-8}$. One will haruly ever go so far in the development of a planetary disturbing function. In the double precision version this limit is 2×10^{-16}. On the other hand, the double precision version gives the results with a maximum error of two units in the eighth place as long as $b_k > 2 \times 10^{-12}$. The value of the index j seems irrelevant to the accuracy of the computed values for the functions $D^j b_k$. For large values of the argument α the Newcomb derivatives of the Laplace coefficients increase very rapidly with the index j as shown by the figures on pp. 12-45. In a practical application, however, these functions are divided by large numbers and multiplied with at least the j-th power of the orbital eccentricity.

The program uses 5,356 memory locations plus an additional 268 locations for the output routine.

A print-out sample. The value $\alpha = 0.8250634$ pertains to the satellites Titan and Hyperion of Saturn. For a comparison see the table on p. 642 of the reference Innes (1909). In the present definition b_k is half as large as in that adopted by Innes.

THE LAPLACE COEFFICIENTS AND THEIR DERIVATIVES FOR S = 1/2 AND ALPHA = 0.8250634, ARE

	B(K)	D(1)B(K)	D(2)B(K)	D(3)B(K)	D(4)B(K)	D(5)B(K)	D(6)B(K)
K = 0	1.3037327E 00	1.1947925E 00	7.8746758E 00	8.5456506E 01	1.3531897E 03	2.8368253E 04	7.4113186E
K = 1	6.1331968E-01	1.4481220E 00	8.0962059E 00	8.5935150E 01	1.3565617E 03	2.8407428E 04	7.4176858E
K = 2	3.9834795E-01	1.4101518E 00	8.4560122E 00	8.7695472E 01	1.3678628E 03	2.8528970E 04	7.4370687E
K = 3	2.8118517E-01	1.3058961E 00	8.6989034E 00	9.0271585E 01	1.3882263E 03	2.8745522E 04	7.4704672E
K = 4	2.0643709E-01	1.1796811E 00	8.7736404E 00	9.3073107E 01	1.4170327E 03	2.9070105E 04	7.5196255E
K = 5	1.5510597E-01	1.0494003E 00	8.6832967E 00	9.5604445E 01	1.4521725E 03	2.9510406E 04	7.3866657E
K = 6	1.1833788E-01	9.2362804E-01	8.4510880E 00	9.7510879E 01	1.4906548E 03	3.0060747E 04	7.6735130E
K = 7	9.1278100E-02	8.0655875E-01	8.1066394E 00	9.8574561E 01	1.5291824E 03	3.0705394E 04	7.7813208E
K = 8	7.0986223E-02	7.0005406E-01	7.6796996E 00	9.8692918E 01	1.5645902E 03	3.1418158E 04	7.9100344E
K = 9	5.5559728E-02	6.0466466E-01	7.1972603E 00	9.7852504E 01	1.5941353E 03	3.2165446E 04	8.0581534E
K = 10	4.3709402E-02	5.2019562E-01	6.6823963E 00	9.5103933E 01	1.6156565E 03	3.2910556E 04	8.2226983E
K = 11	3.4551725E-02	4.4603871E-01	6.1540000E 00	9.3540204E 01	1.6276343E 03	3.3614844E 04	8.3993374E
K = 12	2.7377251E-02	3.8137356E-01	5.6269810E 00	9.0279140E 01	1.6291792E 03	3.4242047E 04	8.5826718E
K = 13	2.1765978E-02	3.2529124E-01	5.1126767E 00	8.6449851E 01	1.6199741E 03	3.4761374E 04	8.7665545E
K = 14	1.7356632E-02	2.7686961E-01	4.6193588E 00	8.2182757E 01	1.6001899E 03	3.5145352E 04	8.9444706E
K = 15	1.3867271E-02	2.3521836E-01	4.1527407E 00	7.7602637E 01	1.5703897E 03	3.5373639E 04	9.1098974E
K = 16	1.1101732E-02	1.9950440E-01	3.7164570E 00	7.2824096E 01	1.5314329E 03	3.5433313E 04	9.2566292E
K = 17	8.9031019E-03	1.6896480E-01	3.3124928E 00	6.7948905E 01	1.4843838E 03	3.5317115E 04	9.3790510E
K = 18	7.1509023E-03	1.4291166E-01	2.9415548E 00	6.3064768E 01	1.4304316E 03	3.5023980E 04	9.4723461E
K = 19	5.7514849E-03	1.2073197E-01	2.6033843E 00	5.8245119E 01	1.3708227E 03	3.4559142E 04	9.5326383E
K = 20	4.6316935E-03	1.0188445E-01	2.2970170E 00	5.3549672E 01	1.3068015E 03	3.3928311E 04	9.5570722E
K = 21	3.7341323E-03	8.5894612E-02	2.0209930E 00	4.9025098E 01	1.2395732E 03	3.3146787E 04	9.5435230E
K = 22	3.0136015E-03	7.2348977E-02	1.7735251E 00	4.4706861E 01	1.1702657E 03	3.2228580E 04	9.4921308E
K = 23	2.4343893E-03	6.0889023E-02	1.5526304E 00	4.0119995E 01	1.0999105E 03	3.1190572E 04	9.4021163E
K = 24	1.9681982E-03	5.1205158E-02	1.3562320E 00	3.6700819E 01	1.0294281E 03	3.0050758E 04	9.2748077E
K = 25	1.5925489E-03	4.3030987E-02	1.1822349E 00	3.3158242E 01	9.5962125E 02	2.8827569E 04	9.1119677E
K = 26	1.2895427E-03	3.6137982E-02	1.0285829E 00	2.9875061E 01	8.9117361E 02	2.7539310E 04	8.9159820E
K = 27	1.0448992E-03	3.0330617E-02	8.9329777E-01	2.6809132E 01	8.2465268E 02	[illegible].6203690E 04	8.6897303E
K = 28	8.4720410E-04	2.5441995E-02	7.7450624E-01	2.3994423E 01	7.6051574E 02	2.4837459E 04	8.4364607E
K = 29	6.8731836E-04	2.1329942E-02	6.7045711E-01	2.1421939E 01	6.9911795E 02	2.3456136E 04	8.1596692E
K = 30	5.5791404E-04	1.7873570E-02	5.7953015E-01	1.9080505E 01	6.4072182E 02	2.2073821E 04	7.8629891E
K = 31	4.5310708E-04	1.4970257E-02	5.0023941E-01	1.6957446E 01	5.8550740E 02	2.0703084E 04	7.5500925E
K = 32	3.6816697E-04	1.2533015E-02	4.3123228E-01	1.5039133E 01	5.3358258E 02	1.9354911E 04	7.2266045E

	D(7)B(K)	D(8)B(K)	D(9)B(K)	D(10)B(K)	D(11)B(K)	D(12)B(K)
K = 0	2.3198491E 07	8.4640592E 08	3.5273085E 10	1.[illegible]0888E 12	8.6057955E 13	4.9271359E
K = 1	2.3212029E 07	8.4676279E 08	3.5284324E 10	1.6535006E 12	8.6075177E 13	4.9279452E
K = 2	2.3252974E 07	8.4783885E 08	3.5318158E 10	1.6547394E 12	8.6126940E 13	4.9303764E
K = 3	2.3322409E 07	8.4965116E 08	3.5374946E 10	1.6568146E 12	8.6213543E 13	4.9344407E
K = 4	2.3422370E 07	8.5223075E 08	3.5455317E 10	1.6597420E 12	8.6335493E 13	4.9401563E
K = 5	2.3556051E 07	8.5562629E 08	3.5560223E 10	1.6635458E 12	8.6493517E 13	4.9475498E
K = 6	2.3727416E 07	8.5990702E 08	3.5691015E 10	1.6682588E 12	8.6688600E 13	4.9566562E
K = 7	2.3940900E 07	8.6516299E 08	3.5849522E 10	1.6739246E 12	8.6922013E 13	4.9675198E
K = 8	2.4200627E 07	8.7150124E 08	3.6038105E 10	1.6805997E 12	8.7195370E 13	4.9801961E
K = 9	2.4509641E 07	8.7903756E 08	3.6259652E 10	1.6883560E 12	8.7510703E 13	4.9947530E
K = 10	2.4869204E 07	8.8788468E 08	3.6517504E 10	1.6972819E 12	8.7870520E 13	5.0112731E
K = 11	2.5278247E 07	8.9813827E 08	3.6815290E 10	1.7074822E 12	8.8277889E 13	5.0298568E
K = 12	2.5733048E 07	9.0986245E 08	3.7156677E 10	1.7190772E 12	8.8736480E 13	5.0506244E
K = 13	2.6227140E 07	9.2307638E 08	3.7545058E 10	1.7321981E 12	8.9250584E 13	5.0737193E
K = 14	2.6751464E 07	9.3774372E 08	3.7983206E 10	1.7469815E 12	8.9825078E 13	5.0993106E
K = 15	2.7294700E 07	9.5376336E 08	3.8472916E 10	1.7635611E 12	9.0465332E 13	5.1275924E
K = 16	2.7843761E 07	9.7097664E 08	3.9014668E 10	1.7820[illegible] 12	9.1177049E 13	5.1587863E
K = 17	2.8384367E 07	9.8914835E 08	3.9607353E 10	1.8025713E 12	9.1966073E 13	5.1931364E
K = 18	2.8901666E 07	1.0079922E 09	4.0248060E 10	1.8251648E 12	9.2837858E 13	5.2309061E
K = 19	2.9380856E 07	1.0271691E 09	4.0931960E 10	1.8498592E 12	9.3797613E 13	5.2723699E
K = 20	2.9807751E 07	1.0463006E 09	4.1652327E 10	1.8766215E 12	9.4849449E 13	5.3178043E
K = 21	3.0169292E 07	1.0649815E 09	4.2400534E 10	1.9053586E 12	9.5998196E 13	5.3674754E
K = 22	3.0453958E 07	1.0827938E 09	4.3166307E 10	1.9359120E 12	9.7239016E 13	5.4216250E
K = 23	3.0652078E 07	1.0993206E 09	4.3937933E 10	1.9680559E 12	9.8577045E 13	5.4804552E
K = 24	3.0756047E 07	1.1141593E 09	4.4702576E 10	2.0014980E 12	1.0000710E 14	5.5441123E
K = 25	3.0760439E 07	1.1269335E 09	4.5446636E 10	2.0358824E 12	1.0152346E 14	5.6126713E
K = 26	3.0662027E 07	1.1373037E 09	4.6156121E 10	2.0707962E 12	1.0311774E 14	5.6861205E
K = 27	3.0459733E 07	1.1449754E 09	4.6817042E 10	2.1057779E 12	1.0477832E 14	5.7643485E
K = 28	3.0154503E 07	1.1497057E 09	4.7415780E 10	2.1403269E 12	1.0649294E 14	5.8471335E
K = 29	2.9749132E 07	1.1513075E 09	4.7939447E 10	2.1739158E 12	1.0824383E 14	5.9341350E
K = 30	2.9248097E 07	1.1496516E 09	4.8376188E 10	2.2060024E 12	1.1001293E 14	6.0248901E
K = 31	2.8657120E 07	1.1446669E 09	4.8715464E 10	2.2360426E 12	1.1177976E 14	6.1188118E
K = 32	2.7983314E 07	1.1363391E 09	4.8948254E 10	2.2635027E 12	1.1352224E 14	6.2151933E