N64-17224 *
CODE-1
(NASA CR-53219;

JPL-TR-
Technical Report No. 32-584) OTS:

A Geometrically Nonlinear Theory of Shells

Bayard Wilson 7 Nov. 1963 58 refs

OTS PRICE

XEROX $ 5.60 ph

MICROFILM $ 1.94 mf.

1304823

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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A GEOMETRICALLY NONLINEAR THEORY OF SHELLS

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Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

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PREFACE

The author, serving as a consultant at the Jet Propulsion Laboratory during the summer of 1963, presented the content of this Report in a series of seminars for the Engineering Research Section of the Engineering Mechanics Division. He is an associate professor of civil engineering at the University of Washington in Seattle, Washington.

ABSTRACT

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A general formulation of a geometrically nonlinear theory of shells is developed, utilizing basic concepts from vector and tensor analyses.

Large deflections and rotations are considered, consistent with the assumption of linear stress-strain relations. In addition, the fundamental Kirchhoff hypothesis is introduced to reduce the shell problem to one of two dimensions.

After the general theory governing the equilibrium configurations of a shell structure is established, the condition of stable or unstable equilibrium is considered.

Author

I. INTRODUCTION

The purpose of this Report is to develop a general formulation of a geometrically nonlinear theory of shells. Appropriate generality will be achieved through the use of basic concepts from vector and tensor analysis.

Physical or material nonlinearity will not be considered. Deformations are assumed to be sufficiently small to insure the validity of linear stress-strain relations, but this assumption still admits the possibility that a sufficiently-thin shell can sustain relatively-large deflections and rotations, such as occur in a snap-through problem. Even deflections that are the same order of magnitude as the shell thickness give rise to geometrical nonlinearities in the theory.

A shell may be considered as a three-dimensional elastic body, but the fundamental Kirchhoff hypothesis, which deals with the geometry of shell deformation, is introduced to reduce the problem to one of two dimensions. Although additional approximations are not necessary in the development of the theory, some special approximations, such as that used for shallow shells, will also be considered. In view of the modern, high-performance, computational equipment that is now readily available, these special approximations may not now be as important as they once were.

After the general theory governing the equilibrium configurations of a shell structure has been established, the conditions under which an equilibrium configuration is stable or unstable will be considered. The classical linear theory of elastic stability will be considered as a special case.

II. TENSOR ANALYSIS AND THEORY OF SURFACES

In order to formulate a geometrically nonlinear theory of shells with the desired degree of generality, basic concepts from vector and tensor analysis and the theory of surfaces will be applied. The following is an outline of results that are most directly applicable in the theory of shells. More detailed treatment of the subject may be found in treatises by Sokolnikoff, Synge and Schild, or Green and Zerna. (For precise references, see entries in the BIBLIOGRAPHY at the end of this Report.)

Modern vector-and-tensor analysis makes use of an indicial notation and summation convention. In this Report, the Einstein summation convention will be used throughout.

A. Coordinate Transformation, Definition of a Tensor

The concept of a tensor is intimately associated with the behavior of mathematical objects under a coordinate transformation or a change of independent variables. For our purposes, we shall be concerned with quantities in a three-dimensional Euclidean space and a two-dimensional Riemannian space, for which there are three or two independent variables, respectively.

Consider a set of three independent variables x^i, where the index $i = 1, 2, 3$. Now introduce another set of variables y^i that is related to the first set by the coordinate transformation

$$y^i = y^i(x^1, x^2, x^3) \tag{1}$$

This coordinate transformation is assumed to be single-valued and reversible, in which case the transformation can be written in the form

$$x^i = x^i(y^1, y^2, y^3) \tag{2}$$

By the chain rule of partial differentiation, with the understanding that repeated indices are summed, differentials can be expressed as

$$dx^i = \frac{\partial x^i}{\partial y^j} dy^j = \underset{\sim}{C}^i_j \, dy^j$$

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j = C^i_j \, dx^j \tag{3}$$

Evidently

$$C^i_k \underset{\sim}{C}^k_j = \underset{\sim}{C}^i_k C^k_j = \delta^i_j = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases} \tag{4}$$

The δ^i_j is known as the Kronecker delta.

We are now in a position to introduce some definitions that are fundamental in tensor analysis, as follows:

1. A *scalar invariant* is a function F for which

$$F(x) = F[x(y)] = \underset{\sim}{F}(y) \tag{5}$$

for values of x and y that are related by the coordinate transformation in Eq. (2).

2. A *covariant vector*, or covariant tensor of order one, is a set of three quantities $A_i(x)$ in the x coordinate system which is related to the quantities $\underset{\sim}{A}_i(y)$ in the y coordinate system by the equation

$$\underset{\sim}{A}_i = \underset{\sim}{C}^j_i A_j \tag{6}$$

By virtue of Eq. (4), this relation also takes the form

$$A_i = C^j_i \underset{\sim}{A}_j \tag{7}$$

3. A *contravariant vector* is a set of quantities $A^i(x)$ that are related to $\underset{\sim}{A}^i(y)$ by the equation

$$\begin{aligned} \underset{\sim}{A}^i &= C^i_j A^j \\ A^i &= \underset{\sim}{C}^i_j \underset{\sim}{A}^j \end{aligned} \tag{8}$$

From this definition, Eq. (3) indicates that dx^i is a contravariant vector.

4. A *covariant tensor of order two* is a set of nine quantities $\underset{\sim}{B}_{ij}(x)$ that are related to $\underset{\sim}{B}_{ij}(y)$ by the equation

$$\begin{aligned} \underset{\sim}{B}_{ij} &= \underset{\sim}{C}^m_i \underset{\sim}{C}^n_j B_{mn} \\ B_{ij} &= C^m_i C^n_j \underset{\sim}{B}_{mn} \end{aligned} \tag{9}$$

5. A *contravariant tensor of order two* is a set of quantities in the x and y coordinate systems that are related by the equation

$$\begin{aligned} B^{ij} &= \underset{\sim}{C}^i_m \underset{\sim}{C}^j_n \underset{\sim}{B}^{mn} \\ \underset{\sim}{B}^{ij} &= C^i_m C^j_n B^{mn} \end{aligned} \tag{10}$$

6. A *mixed tensor of order two* is a set of quantities in the x and y coordinate systems that are related by the equation

$$\begin{aligned} \underset{\sim}{B}^i_{\cdot j} &= C^i_m \underset{\sim}{C}^n_j B^m_{\cdot n} \\ \underset{\sim}{B}_i^{\cdot j} &= C^j_n \underset{\sim}{C}^m_i B_m^{\cdot n} \\ B_{i\cdot}^{\ j} &= C^j_n \underset{\sim}{C}^m_i B_{m\cdot}^{\ n} \end{aligned} \tag{11}$$

Higher-order tensors are defined in a manner similar to the above.

B. Curvilinear Coordinates and Base Vectors

Let z^i be rectangular cartesian coordinates with the origin 0 fixed in space as indicated in Fig. 1. The $\bar{i}_i$ are unit base vectors, and the position of a point P with coordinates z^i can be defined by the position vector $\bar{R}$, where

$$\bar{R} = z^i \, \bar{i}_i \tag{12}$$

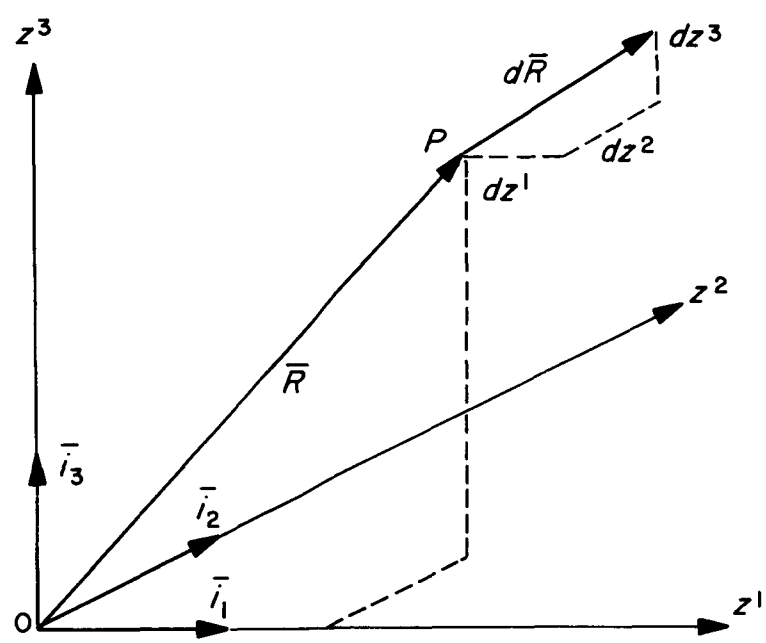

Fig. 1. Rectangular cartesian coordinates

The differential of the position vector is easily found to be

$$d\bar{R} = \frac{\partial \bar{R}}{\partial z^j} dz^j = \delta^i_j \bar{i}_i dz^j = \bar{i}_j dz^j \tag{13}$$

and the length of a line element is given by

$$ds^2 = d\bar{R} \cdot d\bar{R} = \bar{i}_i \cdot \bar{i}_j dz^i dz^j = \delta_{ij} dz^i dz^j \tag{14}$$

Let us now introduce a curvilinear coordinate system, x, defined by the coordinate transformation

$$x^i = x^i(z), \qquad \text{or} \qquad z^i = z^i(x) \tag{15}$$

If $\bar{R}$ is now regarded as a function of x^i and if we use the notation that a *comma* followed by a subscript i denotes partial differentiation with respect to x^i, then

$$d\bar{R} = \bar{R}_{,i} dx^i = \bar{g}_i dx^i \tag{16}$$

where

$$\bar{R}_{,i} \equiv \frac{\partial \bar{R}}{\partial x^i} \tag{17}$$

and

$$\bar{g}_i = \bar{R}_{,i} \tag{18}$$

Since z^i is analogous to y^i in Eq. (3), we write

$$dx^i = \frac{\partial x^i}{\partial z^j} dz^j = \underset{\sim}{C}^i_j dz^j \tag{19}$$

Substitution of this expression into Eq. (16) yields

$$d\bar{R} = \bar{g}_i \underset{\sim}{C}^i_j \, dz^j \tag{20}$$

which, by comparison with Eq. (13), leads to the result

$$\bar{i}_j = \underset{\sim}{C}^i_j \, \bar{g}_i, \qquad \text{or} \qquad \bar{g}_i = C^j_i \, \bar{i}_j \tag{21}$$

From the definition (Eq. 6), Eq. (21) is seen to be a covariant law of transformation. The $\bar{g}_i$ are therefore called *covariant base vectors* for the x coordinate system; their physical meaning is indicated in Fig. 2. Observe that, in general, the $\bar{g}_i$ are not orthogonal unit vectors, and the point P has no particular relation to the origin of the coordinates x^i. Although the base vectors $\bar{i}_j$ are constant, the $\bar{g}_i$ are functions of x^i.

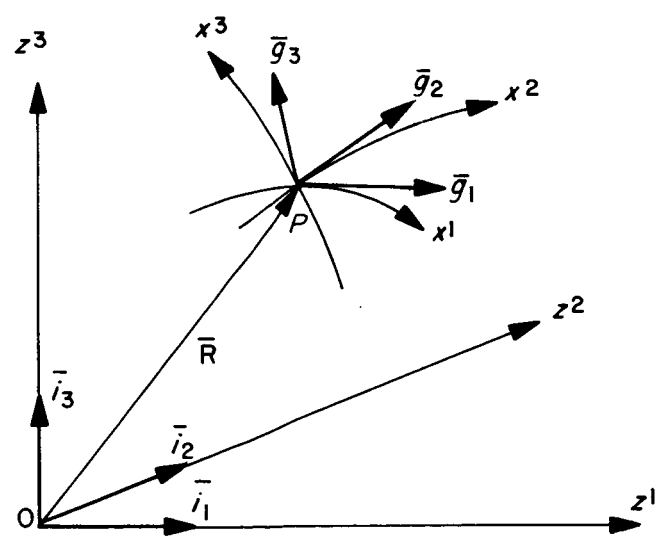

Fig. 2. Covariant base vectors

By using Eq. (16), the length of a line element is found to be

$$ds^2 = d\bar{R} \cdot d\bar{R} = \bar{g}_i \cdot \bar{g}_j \, dx^i \, dx^j = g_{ij} \, dx^i \, dx^j \tag{22}$$

where

$$g_{ij} = \bar{g}_i \cdot \bar{g}_j \tag{23}$$

is called the *covariant metric tensor* of the x coordinate system. The term "tensor" is justified since, by virtue of Eq. (22), (19), and (14),

$$\delta_{ij} = \underset{\sim}{C}^m_i \, \underset{\sim}{C}^n_j \, g_{mn} \tag{24}$$

which, according to the definition (Eq. 9), is a covariant tensor transformation. Observe that g_{ij} is a symmetric tensor. It should be clear that δ_{ij} is the metric tensor for the rectangular cartesian coordinate system z^i.

C. Reciprocal Base Vectors

There is no distinction between covariant and contravariant components of the base vectors $\bar{i}_i$ in a rectangular cartesian coordinate system. Thus we can write $\bar{i}^i = \bar{i}_i$ and define contravariant base vectors in the x coordinate system by the contravariant law of transformation

$$\bar{g}^i = \underset{\sim}{C}^i_j \, \bar{i}^j \tag{25}$$

In view of Eq. (24), it is easily shown that

$$\bar{g}^i \cdot \bar{g}_j = \delta^i_j \tag{26}$$

which indicates that, in general, the covariant and contravariant base vectors in the x coordinate system are orthogonal and reciprocal to each other. Analogous to Eq. (23), the contravariant metric tensor is

defined by

$$g^{ij} = \bar{g}^i \cdot \bar{g}^j \tag{27}$$

The following useful formulas are derived from Eq. (25), (21), and (4):

$$\begin{aligned} \bar{g}_i &= g_{ij}\, \bar{g}^j \\ \bar{g}^i &= g^{ij}\, \bar{g}_j \\ g^{ik}\, g_{jk} &= \delta^i_j \end{aligned} \tag{28}$$

Thus, the metric tensor can be used to raise or lower indices.

D. Tensor Components and Physical Components of a Vector

Any vector $\bar{V}$, such as a force vector in a three-dimensional Euclidean space, can be expressed in terms of components that are associated with a coordinate system. Thus

$$\bar{V} = V^i\, \bar{g}_i = V_i\, \bar{g}^i \tag{29}$$

In view of the formulas (Eq. 28)

$$V^i = g^{ij}\, V_j, \qquad \text{and} \qquad V_i = g_{ij}\, V^j \tag{30}$$

In order to show that V_i is a component of a covariant tensor, note that

$$\bar{V} = V_i\, \bar{g}^i = \underset{\sim}{V}_i\, \bar{i}^i \tag{31}$$

From Eq. (25), it follows that

$$\underset{\sim}{V}_i = \underset{\sim}{C}^j_i\, V_j \tag{32}$$

which is a covariant law of transformation.

Since $\bar{i}^i$ are unit vectors, $\underset{\sim}{V}_i$ are physical components of $\bar{V}$. The $\bar{g}^i$ are not unit vectors, but $\bar{V}$ can still be expressed in terms of physical components $V_{(i)}$ associated with $\bar{g}^i$ directions as follows:

$$\bar{V} = V_{(1)} \frac{\bar{g}^1}{|\bar{g}^1|} + V_{(2)} \frac{\bar{g}^2}{|\bar{g}^2|} + V_{(3)} \frac{\bar{g}^3}{|\bar{g}^3|} \tag{33}$$

Since $|\bar{g}^1| = \sqrt{\bar{g}^1 \cdot \bar{g}^1} = \sqrt{g^{11}}$ etc., the physical and tensor components of $\bar{V}$ are related by

$$V_{(1)} = V_1 \sqrt{g^{11}} \ , \ \text{etc.} \tag{34}$$

Similarly, the physical components in the $\bar{g}_i$ directions are found to be

$$V^{(1)} = V^1 \sqrt{g_{11}} \tag{35}$$

E. Derivative of a Vector--Covariant Derivatives

The covariant base vectors $\bar{g}_i$ are expressed in terms of the position vector $\bar{R}$ by Eq. (18). Since $\bar{R}$ is a continuous function of x^i, it follows that

$$\bar{g}_{i,j} = \bar{g}_{j,i} \tag{36}$$

From Eq. (21) and the definition of C^i_j, we have

$$\bar{g}_{i,j} = \frac{\partial}{\partial x^j} (C^m_i \bar{i}_m) = \frac{\partial^2 z^m}{\partial x^i \, \partial x^j} \bar{i}_m \tag{37}$$

We can also write

$$\bar{i}_m = \delta_{mn} \bar{i}^n = \delta_{mn} C^n_r \bar{g}^r = \delta_{mn} \frac{\partial z^n}{\partial x^r} \bar{g}^r \tag{38}$$

By combining Eq. (38) and (37), and introducing the definitions of the Christoffel symbols of the first and second kinds, respectively,

$$\Gamma_{ijr} = \delta_{mn} \frac{\partial^2 z^m}{\partial x^i \partial x^j} \frac{\partial z^n}{\partial x^r} \; ; \; \Gamma^m_{ij} = g^{mr} \Gamma_{ijr} \tag{39}$$

the derivative of the base vectors becomes

$$\bar{g}_{i,j} = \Gamma_{ijr} \bar{g}^r = \Gamma^m_{ij} \bar{g}_m \tag{40}$$

Now for any vector $\bar{V}$, which is expressed in terms of its components by Eq. (29), the derivative can be written as

$$\bar{V}_{,j} = (V^i \bar{g}_i)_{,j} = V^i_{,j} \bar{g}_i + V^i \bar{g}_{i,j} \tag{41}$$

By using the result (Eq. 40), the derivative (Eq. 41) becomes

$$\bar{V}_{,j} = V^m{}_{/j} \bar{g}_m \tag{42}$$

where

$$V^m{}_{/j} = V^m_{,j} + V^i \Gamma^m_{ij} \tag{43}$$

is the covariant derivative of the tensor V^m.

A similar line of reasoning leads to the result

$$\bar{V}_{,j} = V_{i/j} \bar{g}^i \tag{44}$$

where

$$V_{i,j} = V_{i,j} - V_m \Gamma^m_{ij} \tag{45}$$

It can be shown that $V^i{}_{/j}$ and $V_{i/j}$ are second order tensors.

It should be noted that from Eq. (24) and the definition (Eq. 39), there follows a formula that is usually more convenient for determining the Christoffel symbols:

$$\Gamma_{ijr} = \frac{1}{2} (g_{ir,j} + g_{jr,i} - g_{ij,r}) \tag{46}$$

F. Formulas from the Theory of Surfaces

Consider a two-dimensional surface that is imbedded in our three-dimensional Euclidean space. As indicated in Fig. 3, the position vector $\bar{r}$ of points on this surface from the fixed point 0 is a function of two variables

$$\bar{r} = \bar{r}(x^1, x^2) \tag{47}$$

The two vectors that are tangent to the surface in the direction of the coordinate lines are given by

$$\bar{a}_\alpha = \bar{r}_{,\alpha} \tag{48}$$

Here and in what follows, we use the convention that Greek indices take on values 1, 2, while we continue to let Latin indices take on values 1, 2, 3.

The length of a line element on the surface is given by

$$ds^2 = d\bar{r} \cdot d\bar{r} = a_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{49}$$

where

$$a_{\alpha\beta} = \bar{a}_\alpha \cdot \bar{a}_\beta \tag{50}$$

is the metric tensor of the surface. The components of this tensor are often called the *coefficients of the first fundamental form* of the surface.

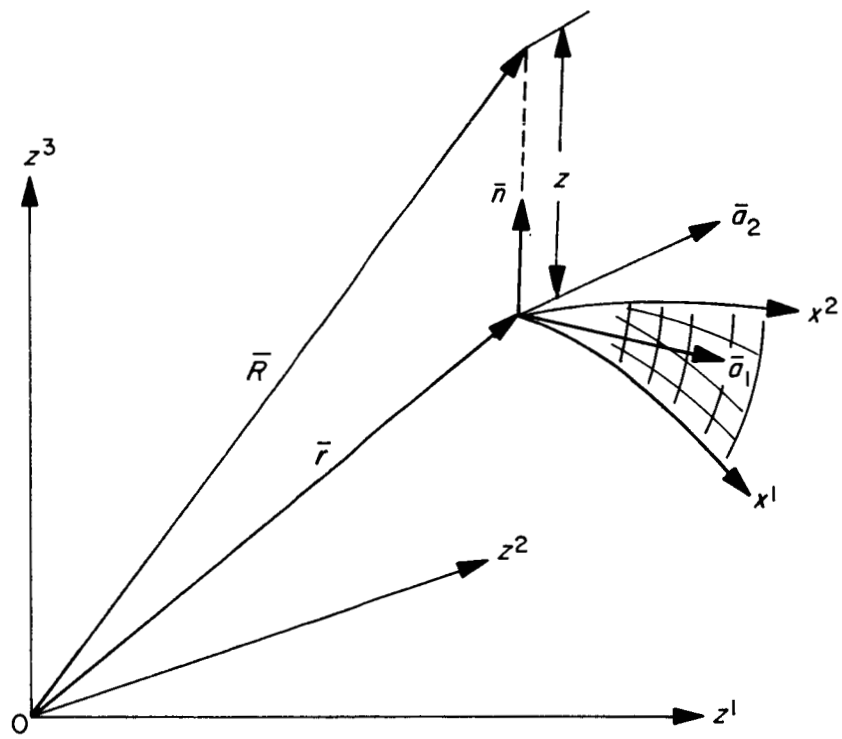

Fig. 3. Two-dimensional surface imbedded in three-dimensional Euclidean space

Let $\bar{n}$ be a unit vector that is normal to the surface. Now the second fundamental form of the surface is

$$d\bar{r} \cdot d\bar{n} = \bar{a}_\alpha \cdot \bar{n}_{,\beta}\, dx^\alpha dx^\beta = -\, b_{\alpha\beta}\, dx^\alpha dx^\beta \tag{51}$$

where

$$b_{\alpha\beta} = -\,\bar{a}_\alpha \cdot \bar{n}_{,\beta} = -\,\bar{a}_\beta \cdot \bar{n}_{,\alpha} \tag{52}$$

are *coefficients of the second fundamental form* of the surface.

Since the vectors $\bar{a}_\alpha$ and $\bar{n}$ are orthogonal, the coefficients of the second fundamental form can also be expressed as

$$b_{\alpha\beta} = \bar{a}_{\alpha,\beta} \cdot \bar{n} \tag{53}$$

Note that $b_{\alpha\beta}$ is a tensor with respect to coordinate transformations on the surface. Hence the $b_{\alpha\beta}$ and other such quantities are called *surface tensors.* Surface tensors play a fundamental role in the theory of shells.

In a manner similar to that followed in the three-dimensional case, contravariant base vectors $\bar{a}^\alpha$ and a contravariant metric tensor $a^{\alpha\beta}$ are defined. This metric tensor can be used to raise or lower indices of surface tensors.

Let us now construct a special three-dimensional coordinate system that refers to the two-dimensional surface. If z is the perpendicular distance from the surface of an arbitrary point in the three-dimensional space then, as indicated in Fig. 3, the position vector of this point is

$$\bar{R} = \bar{r} + z\bar{n} \tag{54}$$

If we set $x^3 = z$ in the formulas of Section II-B, then the base vectors and metric tensor of the three-dimensional space can be expressed in terms of surface vectors and tensors. Thus, by differentiating Eq. (54) and using Eq. (53), we find

$$\begin{aligned} \bar{g}_\alpha &= (\delta^\beta_\alpha - z b^\beta_\alpha)\, \bar{a}_\beta \\ \bar{g}_3 &= \bar{n} \end{aligned} \tag{55}$$

Hence from Eq. (23), remembering that $\bar{n}$ is a unit vector that is normal to $\bar{a}_\alpha$, the three-dimensional metric tensor becomes

$$\begin{aligned} g_{\alpha\beta} &= a_{\alpha\beta} - 2 b_{\alpha\beta} z + b^\gamma_\alpha b_{\gamma\beta} z^2 \\ g_{\alpha 3} &= 0 \\ g_{33} &= 1 \end{aligned} \tag{56}$$

The derivatives of the vectors $\bar{a}_\alpha$ are determined by setting $z = 0$ in Eq. (40) and observing that $b_{\alpha\beta} = \Gamma^{\beta}_{\alpha\beta}\Big|_{z=0}$. Thus,

$$\bar{a}_{\alpha,\beta} = \Gamma^{\gamma}_{\alpha\beta}\,\bar{a}_\gamma + b_{\alpha\beta}\,\bar{n} \tag{57}$$

Here and in the following developments, the Christoffel symbols $\Gamma^{\gamma}_{\alpha\beta}$ are based on the metric tensor $a_{\alpha\beta}$, unless otherwise specified.

Any vector $\bar{V}$ can be expressed in terms of components with respect to the three vectors $\bar{a}_\alpha$ and $\bar{n}$ as

$$\bar{V} = V^\alpha\,\bar{a}_\alpha + V\bar{n} \tag{58}$$

where V^α is a first order surface tensor and V is a scalar. From Eq. (57), (53), and (52), the derivative of $\bar{V}$ is expressed as

$$\bar{V}_{,\beta} = (V^\alpha_{//\beta} - b^\alpha_\beta V)\,\bar{a}_\alpha + (V_{,\beta} + b_{\alpha\beta}V^\alpha)\,\bar{n} \tag{59}$$

where the double stroke denotes covariant differentiation with respect to the surface, i.e.,

$$V^\alpha_{//\beta} = V^\alpha_{,\beta} + V^\gamma\,\Gamma^\alpha_{\gamma\beta} \tag{60}$$

Other useful formulas follow from the definition of the permutation symbol which is

$$\begin{aligned} \epsilon_{12} &= -\epsilon_{21} = \sqrt{a} \\ \epsilon^{12} &= -\epsilon^{21} = \frac{1}{\sqrt{a}} \\ \epsilon^{11} &= \epsilon^{22} = \epsilon_{11} = \epsilon_{22} = 0 \end{aligned} \tag{61}$$

where $a = |a_{\alpha\beta}|$ is the determinant of the metric tensor. From this definition, the following equations involving the cross products of vectors are derived:

$$\begin{aligned} \bar{a}_\alpha \times \bar{a}_\beta &= \epsilon_{\alpha\beta} \bar{n} \\ \bar{n} \times \bar{a}_\alpha &= \epsilon_{\alpha\beta} \bar{a}^\beta \\ \bar{n} &= \frac{1}{2} \epsilon^{\alpha\beta} \bar{a}_\alpha \times \bar{a}_\beta \end{aligned} \tag{62}$$

III. DEFORMATION OF THIN SHELLS

If a shell is sufficiently thin, it can be subjected to large displacements and rotations while the deformation or strain in the material remains relatively small. Our discussion of the deformation of thin shells will be based upon the Kirchhoff hypothesis which assumes that rectilinear material lines, which are normal to the undeformed middle surface, remain rectilinear and inextensible during the deformation and become normal to the deformed middle surface. Within the framework of this fundamental hypothesis, we shall account for the effects of large deflection and rotation in every detail.

The notation to be employed in the following will be similar to that used in Section II. To distinguish between the deformed and undeformed states of a body, geometrical quantities that refer to the deformed state will be designated by capital letters; the same quantities referring to the undeformed state will be designated by lower case letters.

A. Strain Tensor

Let $\underset{\sim}{\bar{r}}(x)$ be the position vector of points in an undeformed elastic body. When this body is subjected to external loads, it will deform into a new configuration. As indicated in Fig. 4, $\bar{U}(x)$ is the displacement vector of points in the body, and $\underset{\sim}{\bar{R}}(x)$ is the position vector of points in the deformed body. The x^i are material, or convected, coordinates which are associated with corresponding points in the deformed and undeformed body.

In the undeformed state, the length of a line element is given by

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{63}$$

and in the deformed state, the length of the same line element is

$$dS^2 = G_{ij}\, dx^i\, dx^j \tag{64}$$

The difference in the lengths of these line elements is a measure of the deformation experienced by the body as it moves from the undeformed to the deformed position. This deformation is described by a strain tensor

e_{ij} which is defined by

$$dS^2 - ds^2 = 2e_{ij}\, dx^i\, dx^j \tag{65}$$

from which it follows that

$$e_{ij} = \frac{1}{2}\,(G_{ij} - g_{ij}) \tag{66}$$

The strain tensor can be expressed in terms of the displacement vector $\bar{U}$ by noting that, as indicated in Fig. 4,

$$\bar{\underset{\sim}{R}} = \bar{\underset{\sim}{r}} + \bar{U} \tag{67}$$

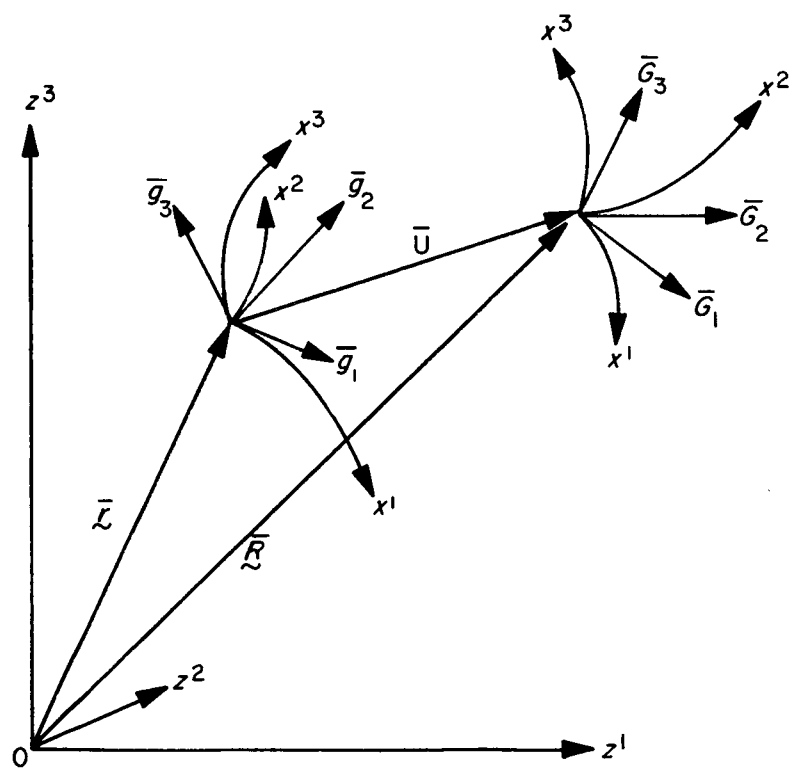

Fig. 4. Displacement vector and base vectors in deformed and undeformed states

Now, by using the formulas given in Section II-B, we write

$$
\begin{aligned}
\bar{g}_i &= \bar{\underset{\sim}{r}}_{,i} \\
g_{ij} &= \bar{g}_i \cdot \bar{g}_j \\
\bar{G}_i &= \bar{\underset{\sim}{R}}_{,i} = \bar{\underset{\sim}{r}}_{,i} + \bar{U}_{,i} = \bar{g}_i + \bar{U}_{,i} \\
G_{ij} &= \bar{G}_i \cdot \bar{G}_j = (\bar{g}_i + \bar{U}_{,i}) \cdot (\bar{g}_j + \bar{U}_{,j})
\end{aligned}
\tag{68}
$$

Then, substitute these expressions into Eq. (66) and obtain the result

$$
e_{ij} = \frac{1}{2} (\bar{g}_i \cdot \bar{U}_{,j} + \bar{g}_j \cdot \bar{U}_{,i} + \bar{U}_{,i} \cdot \bar{U}_{,j}) \tag{69}
$$

This result shows that e_{ij} is a symmetric tensor.

B. Deformation according to the Kirchhoff Hypothesis

Consider a thin shell for which $\bar{r}(x^1, x^2)$ is the position vector of points on its middle surface in the undeformed state and $\bar{R}(x^1, x^2)$ is the position vector of corresponding points in the deformed state. As indicated in Fig. 5, the point p on the undeformed middle surface moves with a displacement vector $\bar{u}(x^1, x^2)$ to a point P on the deformed middle surface. Thus,

$$
\bar{R} = \bar{r} + \bar{u} \tag{70}
$$

If $\bar{n}$ is a unit vector that is normal to the undeformed middle surface, then as in Section II-F, the position vector of any point $\underset{\sim}{p}$ at a distance z from the undeformed middle surface is given by

$$
\bar{\underset{\sim}{r}} = \bar{r} + z\bar{n} \tag{71}
$$

The essence of the Kirchhoff hypothesis is that upon deformation, the point $\underset{\sim}{p}$ moves with a displacement vector $\bar{U}(x^1, x^2, z)$ to the point $\underset{\sim}{P}$ which is on a normal line at the same distance from the deformed middle surface. Thus, if $\bar{N}$ is a unit vector that is normal to the deformed middle surface, then the position vector of $\underset{\sim}{P}$ is

$$\bar{\underset{\sim}{R}} = \bar{R} + z\bar{N} \tag{72}$$

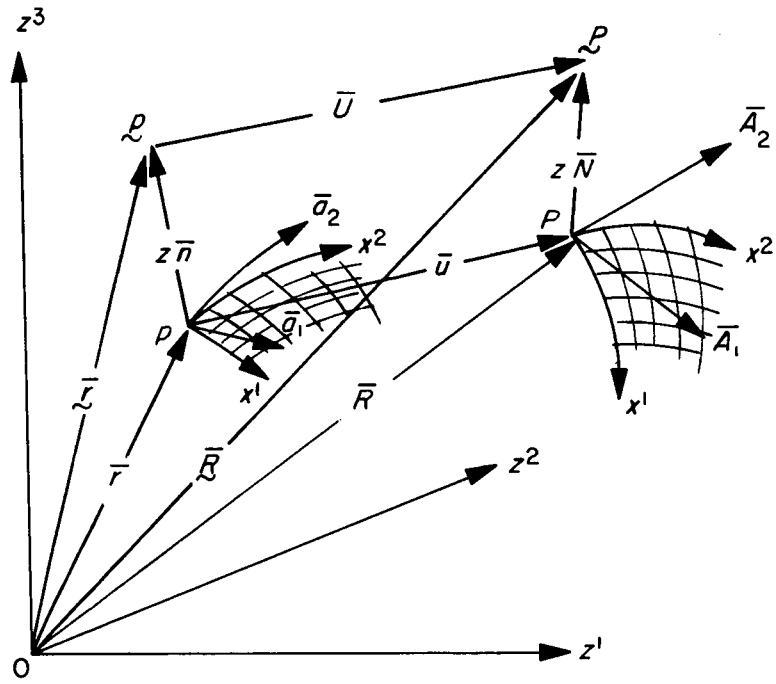

Fig. 5. Deformation according to the Kirchhoff hypothesis

Since

$$\bar{\underset{\sim}{R}} = \bar{\underset{\sim}{r}} + \bar{U} \tag{73}$$

it immediately follows that the displacement vector $\bar{U}$ of any point $\underset{\sim}{p}$ in the shell is given by

$$\bar{U} = \bar{u} + z\bar{m} \tag{74}$$

where

$$\bar{m} = \bar{N} - \bar{n} \tag{75}$$

is a measure of the rotation of the middle surface during the deformation. As shown by Eq. (74) and (75), the remarkable consequence of the Kirchhoff hypothesis is that dependence of the displacement vector $\bar{U}$ upon the variable z is explicitly determined.

On the basis of Eq. (74), the state of deformation at any point in a shell can be expressed in terms of the deformation of its middle surface and z. As in Section II-F, the vectors that are tangent to the deformed middle surface in the direction of the coordinate lines are

$$\bar{A}_\alpha = \bar{R}_{,\alpha} \tag{76}$$

and the coefficients of the first fundamental form of the deformed middle surface are

$$A_{\alpha\beta} = \bar{A}_\alpha \cdot \bar{A}_\beta \tag{77}$$

The coefficients of the second fundamental form of the deformed middle surface are

$$B_{\alpha\beta} = -\bar{A}_\alpha \cdot \bar{N}_{,\beta} = -\bar{A}_\beta \cdot \bar{N}_{,\alpha} \tag{78}$$

Hence from Eq. (72), the base vectors of the deformed shell are

$$\bar{G}_3 = \bar{N}; \qquad \bar{G}_\alpha = \bar{\underset{\sim}{R}}_{,\alpha} = \bar{A}_\alpha + z\bar{N}_{,\alpha} = (\delta^\beta_\alpha - zB^\beta_\alpha)\,\bar{A}_\beta \tag{79}$$

and the metric tensor is

$$\begin{aligned} G_{\alpha 3} &= 0 \\ G_{33} &= 1 \\ G_{\alpha\beta} &= A_{\alpha\beta} - 2B_{\alpha\beta}z + B^\gamma_\alpha B_{\gamma\beta}z^2 \end{aligned} \tag{80}$$

By using the expression (Eq. 56) for the metric tensor of the undeformed shell together with Eq. (80) in Eq. (66), the strain tensor in the Kirchhoffean theory of shells is found to be

$$e_{i\beta} = 0$$

$$e_{\alpha\beta} = \frac{1}{2}(A_{\alpha\beta} - a_{\alpha\beta}) - z(B_{\alpha\beta} - b_{\alpha\beta}) + \frac{z^2}{2}(B^{\gamma}_{\alpha} B_{\gamma\beta} - b^{\gamma}_{\alpha} b_{\gamma\beta}) \tag{81}$$

This result shows that, under the Kirchhoff hypothesis, the strain in the z-direction is zero.

Let us now define two surface tensors that characterize the deformation of the middle surface of a shell. The membrane strain tensor is defined as

$$\gamma_{\alpha\beta} = \frac{1}{2}(A_{\alpha\beta} - a_{\alpha\beta}) \tag{82}$$

and the change in curvature tensor is defined as

$$\chi_{\alpha\beta} = -(B_{\alpha\beta} - b_{\alpha\beta}) \tag{83}$$

If we use the assumption that strains are small, which implies that the $|a_{\alpha\beta}| \cong |A_{\alpha\beta}|$ to within a negligibly small strain, then the $a_{\alpha\beta}$ can be used to raise or lower indices on the $B_{\alpha\beta}$. Thus, when Eq. (82) and (83) are substituted into Eq. (81), the result is

$$e_{\alpha\beta} = \gamma_{\alpha\beta} + z\,\chi_{\alpha\beta} - \frac{z^2}{2}(b_{\alpha\gamma}\,\chi^{\gamma}_{\beta} + b_{\beta\gamma}\,\chi^{\gamma}_{\alpha} - \chi^{\gamma}_{\alpha}\,\chi_{\gamma\beta}) \tag{84}$$

Thus the deformation of a shell, under the Kirchhoff hypothesis, is completely determined in terms of the six quantities $\gamma_{\alpha\beta}$ and $\chi_{\alpha\beta}$, which characterize the deformation of the middle surface of the shell.

The membrane strain tensor can be expressed in terms of the displacement vector $\bar{u}$ of the middle surface by noting that from Eq. (76) and (70)

$$\bar{A}_{\alpha} = \bar{r}_{,\alpha} + \bar{u}_{,\alpha} = \bar{a}_{\alpha} + \bar{u}_{,\alpha} \tag{85}$$

Hence from Eq. (77),

$$A_{\alpha\beta} = (\bar{a}_{\alpha} + \bar{u}_{,\alpha}) \cdot (\bar{a}_{\beta} + \bar{u}_{,\beta}) \tag{86}$$

and, recalling Eq. (50), Eq. (82) becomes

$$\gamma_{\alpha\beta} = \frac{1}{2} (\bar{u}_{,\alpha} \cdot \bar{a}_\beta + \bar{u}_{,\beta} \cdot \bar{a}_\alpha + \bar{u}_{,\alpha} \cdot \bar{u}_{,\beta}) \tag{87}$$

If we express the displacement vector in terms of its tensor components as

$$\bar{u} = u^\alpha \bar{a}_\alpha + w\bar{n} \tag{88}$$

then from Eq. (59)

$$\bar{u}_{,\beta} = (u^\alpha{}_{//\beta} - b^\alpha_\beta w)\, \bar{a}_\alpha + (w_{,\beta} + b_{\beta\alpha} u^\alpha)\, \bar{n} \tag{89}$$

and the membrane strain tensor can be expressed in the alternative form

$$\gamma_{\alpha\beta} = \frac{1}{2} \left[u_{\alpha//\beta} + u_{\beta//\alpha} - 2b_{\alpha\beta} w + (u_{\gamma//\alpha} - b_{\gamma\alpha} w)(u^\gamma{}_{//\beta} - b^\gamma_\beta w) + (w_{,\alpha} + b_{\alpha\gamma} u^\gamma)(w_{,\beta} + b_{\beta\lambda} u^\lambda) \right] \tag{90}$$

The curvature change can be expressed in terms of middle surface displacement by using Eq. (78) and (52) in the definition (Eq. 83). Thus,

$$\chi_{\alpha\beta} = \bar{A}_\alpha \cdot \bar{N}_{,\beta} - \bar{a}_\alpha \cdot \bar{n}_{,\beta} \tag{91}$$

Hence, from Eq. (85) and (75)

$$\chi_{\alpha\beta} = (\bar{a}_\alpha + \bar{u}_{,\alpha}) \cdot \bar{m}_{,\beta} - b^\gamma_\beta \bar{a}_\gamma \cdot \bar{u}_{,\alpha} \tag{92}$$

To express $\bar{m}$ in terms of middle surface displacements, we proceed from the definition (Eq. 75) and the formulas (Eq. 62), again assuming that strains are small, which implies that the permutation symbols for the deformed and undeformed states are approximately equal, and obtain

$$\bar{m} = \frac{1}{2} \epsilon^{\alpha\beta} (\bar{A}_\alpha \times \bar{A}_\beta - \bar{a}_\alpha \times \bar{a}_\beta) \tag{93}$$

When Eq. (85) is substituted, this result becomes

$$\bar{m} = \frac{1}{2} \epsilon^{\alpha\beta} (2\,\bar{a}_\alpha + \bar{u}_{,\alpha}) \times \bar{u}_{,\beta} \tag{94}$$

Eq. (92) together with (94) completely determines the curvature change in terms of the middle surface displacement.

When $\bar{m}$ is expressed in terms of its tensor components as

$$\bar{m} = m^\alpha \bar{a}_\alpha + m\bar{n} \tag{95}$$

and use is made of the Codazzi equation

$$b_{\alpha\beta//\gamma} = b_{\alpha\gamma//\beta} \tag{96}$$

Eq. (92) and (94) become

$$\begin{aligned} \chi_{\alpha\beta} = {} & m_{\alpha//\beta} - b_{\alpha\beta} m - b^\gamma_\beta (u_{\gamma//\alpha} - b_{\gamma\alpha} w) \\ & + (u_{\gamma//\alpha} - b_{\gamma\alpha} w)(m^\gamma{}_{//\beta} - b^\gamma_\beta m) \\ & + (w_{,\alpha} + b_{\alpha\gamma} u^\gamma)(m_{,\beta} + b_{\beta\lambda} m^\lambda) \end{aligned} \tag{97}$$

$$m = u^\alpha{}_{//\alpha} - b^\alpha_\alpha w + \frac{1}{2} \epsilon^{\alpha\beta} \epsilon_{\gamma\lambda} (u^\gamma{}_{//\alpha} - b^\gamma_\alpha w)(u^\lambda{}_{//\beta} - b^\lambda_\beta w) \tag{98}$$

$$m_\alpha = -(w_{,\alpha} + b_{\alpha\beta} u^\beta) - \epsilon^{\lambda\beta} \epsilon_{\alpha\gamma} (u^\gamma{}_{//\lambda} - b^\gamma_\lambda w)(w_{,\beta} - b_{\beta\delta} u^\delta) \tag{99}$$

C. Special Approximations

In the geometrically nonlinear theory of shells, a widely-used approximation accounts for large deflections but assumes that rotations are small. It is further assumed that the terms $b^{\gamma}_{\alpha} u_{\gamma}$ are negligibly small compared to $w_{,\alpha}$. Thus, Eq. (89) becomes

$$\bar{u}_{,\beta} - (u^{\alpha}{}_{//\beta} - b^{\alpha}_{\beta}\, w)\, \bar{a}_{\alpha} + w_{,\beta}\, \bar{n} \tag{100}$$

As indicated in Fig. 6, a consequence of the assumption that rotations are small is that in Eq. (100), the first term is small compared to the last term. This implies that only nonlinear terms in $w_{,\alpha}$ need be retained in the membrane strain tensor.

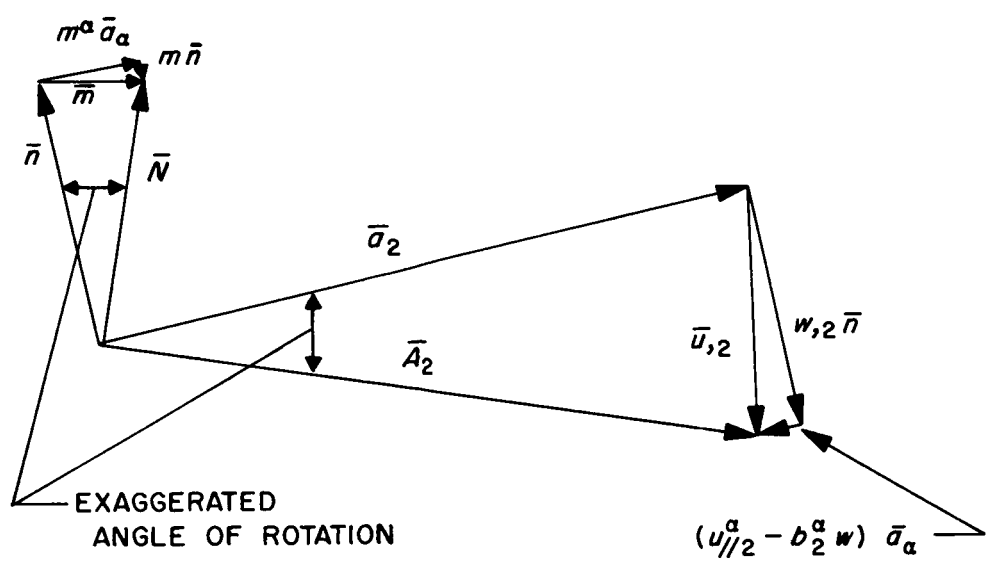

Fig. 6. Special approximations

Hence, using Eq. (100) in Eq. (87), the membrane strain tensor becomes

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha//\beta} + u_{\beta//\alpha} - 2b_{\alpha\beta}w + w_{,\alpha}w_{,\beta}\right) \tag{101}$$

To determine the change in curvature tensor, we need an expression for the vector $\bar{m}$ which has a relatively small component in the $\bar{n}$ direction as indicated in Fig. 6. For this purpose we take

$$\bar{u}_{,\alpha} = w_{,\alpha}\bar{n} \tag{102}$$

Hence, Eq. (94) takes the form

$$\bar{m} = \frac{1}{2}\epsilon^{\alpha\beta}\left(2\bar{a}_{\alpha} + w_{,\alpha}\bar{n}\right) \times w_{,\beta}\bar{n} = -w_{,\beta}\bar{a}^{\beta} \tag{103}$$

from which we see that

$$m = 0; \qquad m_{\alpha} = -w_{,\alpha} \tag{104}$$

Consequently,

$$\bar{m}_{,\beta} = -w_{//\alpha\beta}\bar{a}^{\alpha} - b^{\alpha}_{\beta}w_{,\alpha}\bar{n} \tag{105}$$

Now, with Eq. (105) and (102), Eq. (92) becomes

$$\chi_{\alpha\beta} = -\left(\bar{a}_{\alpha} + w_{,\alpha}\bar{n}\right)\cdot\left(w_{//\gamma\beta}\bar{a}^{\gamma} + b^{\gamma}_{\beta}w_{,\gamma}\bar{n}\right) \tag{106}$$

When the nonlinear terms in this expression are neglected, which is equivalent to neglecting the last term in Eq. (105), the change in curvature takes the well-known form

$$\chi_{\alpha\beta} = -w_{//\alpha\beta} \tag{107}$$

The geometrically nonlinear theory of shells that is based on Eq. (107) and (101) is associated with the names of Donnell and Vlasov. It is often called a "shallow-shell" theory because the effects of middle-surface rotation and tangential displacement are small compared to the effect of deflection normal to the middle surface of shallow shells; and assumptions regarding the relative size of these quantities were the basis of the derivation of Eq. (107) and (101).

When the nonlinear terms are neglected in Eq. (101), then this equation together with Eq. (107) forms the basis for Donnell's linear theory. The Flügge-Vlasov linear theory is obtained by neglecting the nonlinear terms in Eq. (90), (97), and (99), and, on the basis of assumed infinitesimal rotations, setting $m = 0$ in these equations.

IV. EQUILIBRIUM EQUATIONS

In this Section, the differential equations governing the equilibrium of a shell element in the deformed state will be derived. The initial undeformed state of the shell is assumed to be stress-free. Although strains are assumed to be small, displacements and rotations may be large; and the equilibrium must therefore be described in terms of the deformed state. Continuing with our two-dimensional treatment of a shell, the equilibrium conditions will deal with resultant forces and moments over the thickness of the shell. Let $\bar{T}^i/\sqrt{G^{ii}}$ (no sum) be the stress vector, which has the dimension of force per unit area, on the surface x^i = constant in the deformed state. It can be shown (see Green and Zerna) that $\bar{T}^i$ is a contravariant vector which leads to the definition

$$\bar{T}^i = T^{ij}\,\bar{G}_j \tag{108}$$

of the contravariant stress tensor T^{ij}. The stress vectors $\bar{T}^\alpha$ on a differential element of area in a shell are indicated in Fig. 7.

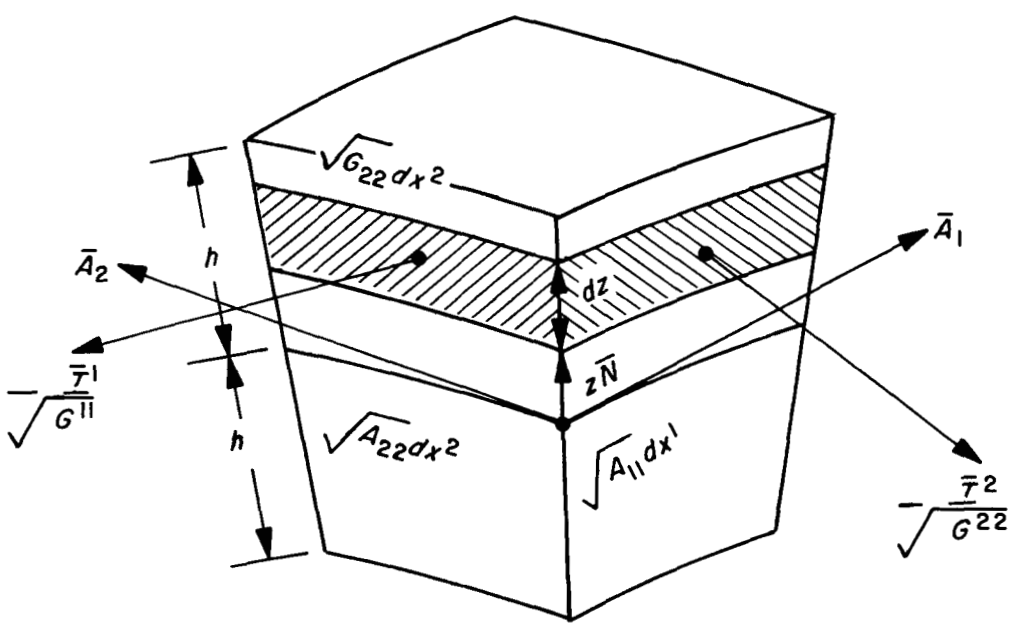

Fig. 7. Stress vectors on a differential element of area in a shell

Similarly, as indicated in Fig. 8, let $\overline{N}^{\alpha}/\sqrt{A^{\alpha\alpha}}$ (no sum) and $\overline{M}^{\alpha}/\sqrt{A^{\alpha\alpha}}$ (no sum) be the resultant force and moment vectors per unit length along the middle surface on $x^{\alpha}=$ constant. By integrating with respect to z over the thickness of the shell, the stress resultants can be expressed in terms of the stresses.

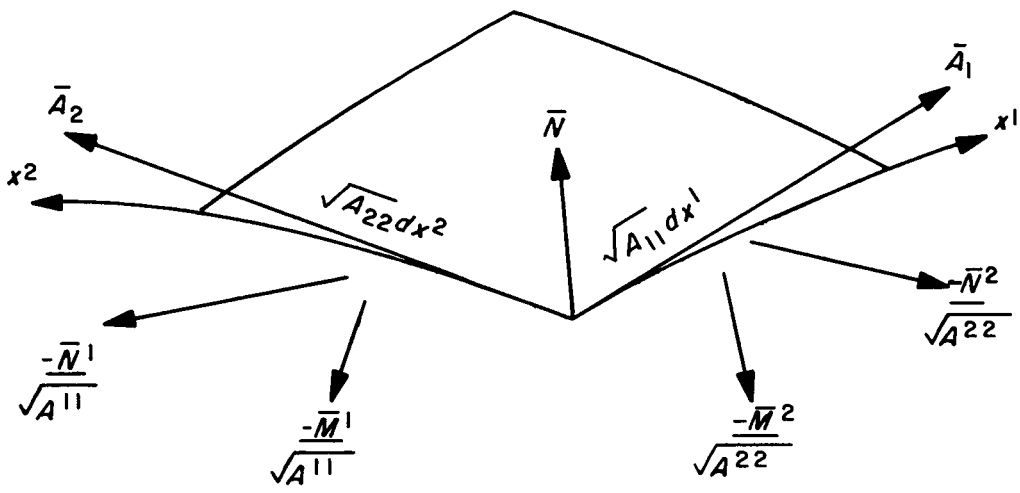

Fig. 8. Resultant force and moment vectors per unit length along middle surface

Thus, assuming the shell has a constant thickness $2h$, we have

$$\frac{\overline{N}^2}{\sqrt{A^{22}}}\sqrt{A_{11}}\;dx^1 = \int_{-h}^{h}\frac{\overline{T}^2}{\sqrt{G^{22}}}\sqrt{G_{11}}\;dx^1\,dz \tag{109}$$

When we use the result that

$$A_{11} = AA^{22}; \qquad G_{11} = GG^{22} \tag{110}$$

and cancel the dx^1, then Eq. (109) takes the form

$$\bar{N}^2 \sqrt{A} = \int_{-h}^{h} \bar{T}^2 \sqrt{G}\, dz \tag{111}$$

Now let us again use the small strain assumption, which enables us to write

$$A \cong a, \qquad G \cong g \tag{112}$$

Hence Eq. (111) can be generalized to

$$\bar{N}^\alpha = \int_{-h}^{h} \bar{T}^\alpha \sqrt{\frac{g}{a}}\, dz \tag{113}$$

Similar reasoning leads to the expression for the resultant moment:

$$\bar{M}^\alpha = \int_{-h}^{h} \bar{N} \times \bar{T}^\alpha \sqrt{\frac{g}{a}}\, z\, dz \tag{114}$$

The tensor components of the resultant force and moment vectors are defined by

$$\bar{N}^\alpha = N^{\alpha\beta} \bar{A}_\beta + Q^\alpha \bar{N} \tag{115}$$

$$\bar{M}^\alpha = M^{\alpha\beta} \bar{N} \times \bar{A}_\beta = M^{\alpha\beta} \epsilon_{\beta\gamma} \bar{A}^\gamma \tag{116}$$

When these definitions, together with Eq. (108) and (79), are used in Eq. (113) and (114), the resultant force and moment tensors become

$$N^{\alpha\beta} = \int_{-h}^{h} \mathrm{T}^{\alpha\gamma} (\delta^\beta_\gamma - z B^\beta_\gamma) \sqrt{\frac{g}{a}}\, dz \tag{117}$$

$$Q^\alpha = \int_{-h}^{h} \mathrm{T}^{\alpha 3} \sqrt{\frac{g}{a}}\, dz \tag{118}$$

$$M^{\alpha\beta} = \int_{-h}^{h} T^{\alpha\gamma} (\delta^{\beta}_{\gamma} - z B^{\beta}_{\gamma}) \sqrt{\frac{g}{a}} \; z \, dz \tag{119}$$

To develop the differential equations governing the stress resultants, let $\bar{P}$ be the externally-applied force vector per unit of middle surface area; and consider the forces and moments that act on an element of the middle surface as indicated in Fig. 9.

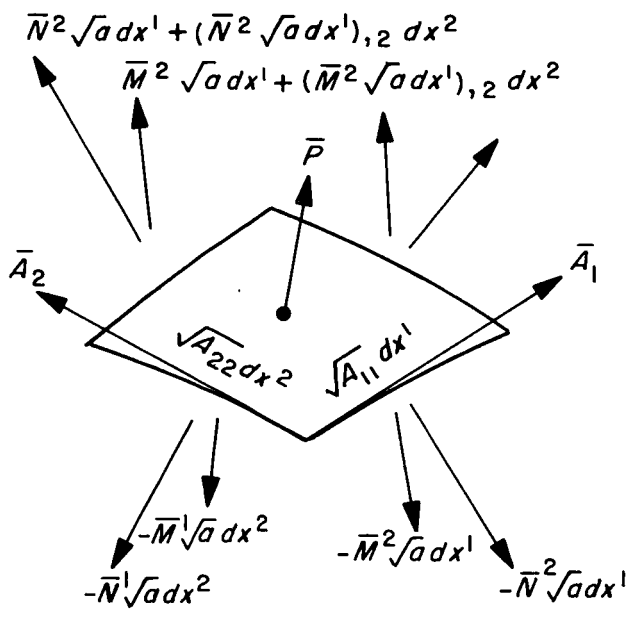

Fig. 9. Forces and moments acting on an element of the middle surface

In determining the total force or moment that acts on a side of the element, use is made of Eq. (110) and (112). Taking the vector sum of these forces and moments, the conditions of equilibrium become

$$(\bar{N}^{\alpha}\sqrt{a})_{,\alpha} + \bar{P}\sqrt{a} = 0$$

$$(\bar{M}^{\alpha}\sqrt{a})_{,\alpha} + \bar{A}_{\alpha} \times \bar{N}^{\alpha}\sqrt{a} = 0 \tag{120}$$

If we let

$$\bar{P} = P^{\alpha}\bar{A}_{\alpha} + P\bar{N} \tag{121}$$

and use Eq. (83), noting the identity

$$(\sqrt{a})_{,\alpha} = \sqrt{a}\,\Gamma^{\beta}_{\beta\alpha} \tag{122}$$

then Eq. (120) can be written in the form

$$\begin{aligned} N^{\alpha\beta}{}_{//\alpha} - (b^{\beta}_{\alpha} - \chi^{\beta}_{\alpha})\,Q^{\alpha} + P^{\beta} &= 0 \\ (b_{\alpha\beta} - \chi_{\alpha\beta})\,N^{\alpha\beta} + Q^{\alpha}{}_{//\alpha} + P &= 0 \\ M^{\alpha\beta}{}_{//\alpha} - Q^{\beta} &= 0 \\ \epsilon_{\alpha\beta}\,N^{\alpha\beta} - \epsilon_{\lambda\beta}(b^{\lambda}_{\alpha} - \chi^{\lambda}_{\alpha})\,M^{\alpha\beta} &= 0 \end{aligned} \tag{123}$$

In examining these equilibrium equations, it should be noted that, by Eq. (121), the components of the applied force vector $\bar{P}$ are referred to the deformed state. This is appropriate for a loading such as hydrostatic pressure. For other loadings, such as a gravity load, the components are more appropriately referred to the undeformed state in the form

$$\bar{P} = F^{\alpha}\,\bar{a}_{\alpha} + F\bar{n} \tag{124}$$

In this case, it follows from Eq. (85) and (75) that

$$\begin{aligned} P_{\alpha} &= F_{\alpha} + (u_{\beta//\alpha} - b_{\beta\alpha}w)\,F^{\beta} + (w_{,\alpha} + b_{\beta\alpha}u^{\beta})\,F \\ P &= (1 + m)\,F + m_{\alpha}F^{\alpha} \end{aligned} \tag{125}$$

For the special approximation for small rotations, that is discussed in Section III-C, the equilibrium equations (Eq. 123) take the simplified form

$$\begin{aligned} N^{\alpha\beta}{}_{//\alpha} + P^{\beta} &= 0 \\ b_{\alpha\beta}N^{\alpha\beta} + Q^{\alpha}{}_{//\alpha} + P &= 0 \\ M^{\alpha\beta}{}_{//\alpha} - Q^{\beta} &= 0 \end{aligned} \tag{126}$$

In this case, the last member of Eq. (123) is disregarded. Observe that for small rotations, the change in curvature $\chi_{\alpha\beta}$ is assumed to be negligibly small compared to the initial curvature $b_{\alpha\beta}$.

V. STRESS-STRAIN RELATIONS

To complete the formulation of a geometrically nonlinear theory of shells, we need equations that relate the stress resultants, $N^{\alpha\beta}$ and $M^{\alpha\beta}$, to quantities that characterize the deformation of a shell, namely, the membrane strain tensor $\gamma_{\alpha\beta}$ and the change in curvature tensor $\chi_{\alpha\beta}$. These relations are obtained through application of the classical, linear Hooke's law. Since strains are assumed to be small, the applicability of Hooke's law is certainly plausible, at least for metallic materials.

As indicated by the first member of Eq. (81), the Kirchhoff hypothesis is equivalent to the assumption of plane strain. This makes it possible to proceed directly to a two-dimensional Hooke's law relating the stress $T^{\alpha\beta}$ and the strain $e_{\alpha\beta}$. It is customary, however, in the development of shell theory, to use the Hooke's law that is based on plane stress, which has the form

$$T^{\alpha\beta} = \frac{E}{1-\nu^2}\left[\nu g^{\alpha\beta} g^{\lambda\gamma} + (1-\nu)\, g^{\alpha\lambda} g^{\beta\gamma}\right] e_{\lambda\gamma} \tag{127}$$

where E is Young's modulus and ν is Poisson's ratio. It should be emphasized that this plane stress assumption does not introduce any fundamental simplification into the theory of shells. It merely affects the manner in which Poisson's ratio enters into the theory.

When Eq. (127) and (84) are used in Eq. (117) and (119), the integration with respect to z can be carried out without further approximations in the theory. For this integration, it is necessary to note that, from Eq. (28) and (56),

$$g^{\alpha\beta} = a^{\alpha\beta} + 2b^{\alpha\beta} z + 3b^{\lambda\alpha} b_{\lambda}^{\beta} z^2 + \cdots \tag{128}$$

Furthermore,

$$\sqrt{\frac{g}{a}} = 1 - 2Hz + Kz^2 \tag{129}$$

where

$$H = \frac{1}{2} b^{\alpha}_{\alpha} \tag{130}$$

is the mean curvature and

$$K = \frac{1}{2} (b^{\alpha}_{\alpha} b^{\beta}_{\beta} - b^{\alpha\beta} b_{\alpha\beta}) \tag{131}$$

is the Gaussian curvature of the undeformed middle surface. Thus, the stress-strain relations take the form

$$N^{\alpha\beta} = \frac{2Eh}{1-\nu^2} [\nu a^{\alpha\beta} a^{\lambda\gamma} + (1-\nu) a^{\alpha\lambda} a^{\beta\gamma}] \gamma_{\lambda\gamma} + h^3 [A^{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma} + B^{\alpha\beta\lambda\gamma} \chi_{\lambda\gamma}] \tag{132}$$

$$M^{\alpha\beta} = \frac{2Eh^3}{3(1-\nu^2)} [\nu a^{\alpha\beta} a^{\lambda\gamma} + (1-\nu) a^{\alpha\lambda} a^{\beta\gamma}] \chi_{\lambda\gamma} + h^3 [C^{\alpha\beta\lambda\gamma} \gamma_{\lambda\gamma} + D^{\alpha\beta\lambda\gamma} \chi_{\lambda\gamma}] \tag{133}$$

where terms that are multiplied by powers of the thickness are retained through h^3 only.

In this form, the stress-strain relations (Eq. 132) and (Eq. 133) are nonlinear because the tensors $A^{\alpha\beta\lambda\gamma}$, $B^{\alpha\beta\lambda\gamma}$, $C^{\alpha\beta\lambda\gamma}$, and $D^{\alpha\beta\lambda\gamma}$ contain the change in curvature tensor $\chi_{\alpha\beta}$.

With Eq. (132) and (133), the last member of Eq. (123), which is often called the "sixth equilibrium equation," is satisfied identically. The third member of Eq. (123) determines the transverse shear forces Q^{α}; and, hence, with the results of Section III, all stress resultants are expressible in terms of the components u_{α} and w of middle-surface displacement. In this case, the first two members of Eq. (123) may be finally reduced to an eighth-order system of three nonlinear differential equations governing the three unknown u_{α} and w, respectively.

Another approximation that is frequently applied in the theory of thin shells is to neglect the second terms in Eq. (132) and (133). This is equivalent to setting $g^{\alpha\beta} \cong a^{\alpha\beta}$ in Eq. (127), neglecting the z^2 term in Eq. (84), and taking $\sqrt{g/a} \cong 1$ in Eq. (117) and (119). The basis for this approximation is the valid argument that for a thin shell, the ratio of shell thickness to minimum radius of curvature is small compared to unity. However, the approximation leads to a formal contradiction in the theory since, in this case, the

sixth equilibrium equation is not satisfied. For most practical problems, this contradiction is insignificant so that the simplified stress-strain relations

$$N^{\alpha\beta} = \frac{2Eh}{1-\nu^2} \left[\nu a^{\alpha\beta} a^{\lambda\gamma} + (1-\nu) a^{\alpha\lambda} a^{\beta\gamma}\right] \gamma_{\lambda\gamma} \tag{134}$$

$$M^{\alpha\beta} = \frac{2Eh^3}{3(1-\nu^2)} \left[\nu a^{\alpha\beta} a^{\lambda\gamma} + (1-\nu) a^{\alpha\lambda} a^{\beta\gamma}\right] \chi_{\lambda\gamma} \tag{135}$$

may be applied with a good degree of confidence. When the stress-strain relations, (Eq. 134), (Eq. 135), the equilibrium Eq. (123), and the strain-displacement relations, (Eq. 90), (Eq. 98) and (Eq. 99), are combined with appropriate boundary conditions, the formulation of a problem in the geometrically nonlinear theory of shells is complete.

VI. THEORY OF ELASTIC STABILITY

A. General Theory

The question of stability arises in the theory of elasticity because, for a given set of externally applied loads, the equilibrium configuration that is assumed by the elastic body, in general, is not unique. Furthermore, of the theoretically-possible equilibrium configurations associated with a certain load, some are less likely to occur than others. This means that a possible equilibrium configuration may be unstable. In the classical linear theory of elasticity, the equilibrium configuration for a given load is unique. The question of stability, which involves a non-unique equilibrium configuration, must therefore be examined from the point of view of a nonlinear theory of elasticity, which provides non-unique solutions of the governing differential equations.

To develop the theory of elastic stability, we need a suitable stability criterion. Such a criterion may be stated, in the words of Green and Adkins, as follows: "An equilibrium state is said to be stable if, in the motion following an arbitrary disturbance, the amplitude of the additional displacement is always vanishingly-small when the disturbance itself is sufficiently small." If this additional displacement is nonzero for a vanishingly-small disturbance, then the equilibrium is unstable. Our discussion will follow the line of reasoning set forth by Novozhilov, and by Green and Adkins, as it applies in the theory of shells.

Let $\overset{0}{\overline{u}}$ be the middle surface displacement vector, which we regard as known, of an equilibrium configuration whose stability we wish to investigate. Suppose this initial equilibrium configuration is associated with an applied surface load $\overset{0}{\overline{P}}$. Now, consider an adjacent equilibrium position which results from a vanishingly-small increase in the applied loads $\epsilon \overset{1}{\overline{P}}$, where ϵ is a small constant and $\overset{1}{\overline{P}}$ is a vanishingly-small vector. Thus, the load vector for this new equilibrium configuration is

$$\overline{P} = \overset{0}{\overline{P}} + \epsilon \overset{1}{\overline{P}} \tag{136}$$

and the new position vector can be expressed in a power series in the small parameter ϵ:

$$\overline{u} = \overset{0}{\overline{u}} + \epsilon \overset{1}{\overline{u}} + \epsilon^2 \overset{2}{\overline{u}} + \cdots \tag{137}$$

Since ϵ is assumed to be vanishingly small, we may neglect the higher-order terms in this expression and regard $\epsilon \overset{1}{u}$ as the additional displacement due to the additional load.

By virtue of the relations between deformation and displacement, Eq. (137) can be used to determine the membrane-strain and curvature-change tensors for the new equilibrium configuration in the form

$$\gamma_{\alpha\beta} = \overset{0}{\gamma}_{\alpha\beta} + \epsilon \overset{1}{\gamma}_{\alpha\beta} + \cdots$$

$$\chi_{\alpha\beta} = \overset{0}{\chi}_{\alpha\beta} + \epsilon \overset{1}{\chi}_{\alpha\beta} + \cdots \tag{138}$$

Hence from stress-strain relations,

$$N^{\alpha\beta} = \overset{0}{N}{}^{\alpha\beta} + \epsilon \overset{1}{N}{}^{\alpha\beta} + \cdots$$

$$M^{\alpha\beta} = \overset{0}{M}{}^{\alpha\beta} + \epsilon \overset{1}{M}{}^{\alpha\beta} + \cdots \tag{139}$$

Since the new position is an equilibrium configuration, the equilibrium Eq. (123) must be satisfied. When Q^{α} is eliminated, this equation takes the form

$$N^{\alpha\beta}{}_{//\alpha} - (b^{\beta}_{\alpha} - \chi^{\beta}_{\alpha})\, M^{\gamma\alpha}{}_{//\gamma} + P^{\beta} = 0$$

$$(b_{\alpha\beta} - \chi_{\alpha\beta})\, N^{\alpha\beta} + M^{\alpha\beta}{}_{//\alpha\beta} + P = 0 \tag{140}$$

Moreover, since the initial position is an equilibrium configuration, the following equilibrium equation must also be satisfied:

$$\overset{0}{N}{}^{\alpha\beta}{}_{//\alpha} - (b^{\beta}_{\alpha} - \overset{0}{\chi}{}^{\beta}_{\alpha})\, \overset{0}{M}{}^{\gamma\alpha}{}_{//\gamma} + \overset{0}{P}{}^{\beta} = 0$$

$$(b_{\alpha\beta} - \overset{0}{\chi}_{\alpha\beta})\, \overset{0}{N}{}^{\alpha\beta} + \overset{0}{M}{}^{\alpha\beta}{}_{//\alpha\beta} + \overset{0}{P} = 0 \tag{141}$$

where $\overset{0}{P}{}^{\alpha}$ and $\overset{0}{P}$ are the initial load tensors.

If we now let

$$P^{\alpha} = \overset{0}{P}{}^{\alpha} + \epsilon \overset{1}{P}{}^{\alpha} + \cdots$$

$$P = \overset{0}{P} + \epsilon \overset{1}{P} + \cdots \tag{142}$$

and substitute the ϵ series expressions into Eq. (140), and then subtract Eq. (141) from the result, the following equation is obtained:

$$\overset{1}{N}{}^{\alpha\beta}{}_{//\alpha} - (b_{\alpha}^{\beta} - \overset{0}{\chi}{}_{\alpha}^{\beta})\, \overset{1}{M}{}^{\gamma\alpha}{}_{//\gamma} + \overset{1}{\chi}{}_{\alpha}^{\beta}\, \overset{0}{M}{}^{\gamma\alpha}{}_{//\gamma} + \overset{1}{P}{}^{\beta} = 0$$

$$(b_{\alpha\beta} - \overset{0}{\chi}_{\alpha\beta})\, \overset{1}{N}{}^{\alpha\beta} - \overset{1}{\chi}_{\alpha\beta}\, \overset{0}{N}{}^{\alpha\beta} + \overset{1}{M}{}^{\alpha\beta}{}_{//\alpha\beta} + \overset{1}{P} = 0 \tag{143}$$

In these equations, an ϵ has been cancelled; and higher-order terms in ϵ have been neglected.

Equation (143) is fundamental in this theory of elastic stability. By making appropriate substitutions, this equation can be shown to be expressible in a form that is linear in the unknown displacement components $\overset{1}{u}_{\alpha}$ and $\overset{1}{w}$. Before we examine this point, however, let us clarify the meaning of the terms $\overset{1}{P}{}^{\alpha}$ and $\overset{1}{P}$ that appear in Eq. (143).

We shall distinguish two types of loading on the shell surface: (1) dead weight (inertia), and (2) hydrostatic pressure. For a dead weight loading, the direction of the load vector $\bar{P}$ is independent of the displacement; while for a hydrostatic pressure, this vector is always normal to the surface of the shell. It will be convenient to use Eq. (124) and refer the load vector to the undeformed state. Thus, using an ϵ series, we write

$$\bar{P} = (\overset{0}{F}{}^{\alpha} + \epsilon \overset{1}{F}{}^{\alpha})\, \bar{a}_{\alpha} + (\overset{0}{F} + \epsilon \overset{1}{F})\, \bar{n} \tag{144}$$

By comparing this with Eq. (136) and recalling that $\overset{1}{\bar{P}}$ is vanishingly-small, we conclude that $\overset{1}{F}{}^{\alpha}$ and $\overset{1}{F}$ are also vanishingly-small. Hence, by making appropriate substitutions into Eq. (125), and setting $\overset{1}{F}{}^{\alpha} = \overset{1}{F} = 0$, we obtain

$$\overset{1}{P}_{\alpha} = (\overset{1}{u}_{\beta//\alpha} - b_{\beta\alpha}\overset{1}{w})\,\overset{0}{F}{}^{\beta} + (\overset{1}{w}_{,\alpha} + b_{\beta\alpha}\overset{1}{u}{}^{\beta})\,\overset{0}{F}$$

$$\overset{1}{P} = \overset{1}{m}\,\overset{0}{F} + \overset{1}{m}_{\alpha}\,\overset{0}{F}{}^{\alpha} \tag{145}$$

For the case of dead weight or inertia loading, $\overset{0}{F}{}^{\beta}$ and $\overset{0}{F}$ are merely the load components referred to the undeformed state.

For the case of a hydrostatic pressure load, p, we write

$$\overset{0}{\overline{P}} = \overset{0}{F}{}^{\alpha}\,\overline{a}_{\alpha} + \overset{0}{F}\,\overline{n} = -\,p\,\overset{0}{\overline{N}} \tag{146}$$

Hence, using Eq. (75) and (95), we find

$$\overset{0}{F}{}^{\alpha} = -\,p\,\overset{0}{m}{}^{\alpha}$$

$$\overset{0}{F} = -\,p\,(\overset{0}{m} + 1) \tag{147}$$

which are to be used in the load expressions (Eq. 145).

As we see from the form of the load terms (Eq. 145), the basic Eq. (143) can be put in a form that is homogeneous in the unknown displacement components $\overset{1}{u}_{\alpha}$ and $\overset{1}{w}$.

B. Classical Theory of Elastic Stability

An important application of the theory of elastic stability is the case for which the equilibrium configuration whose stability is being examined is governed by the classical linear theory of elasticity. In this case, the initial stress resultants $\overset{0}{N}{}^{\alpha\beta}$ and $\overset{0}{M}{}^{\alpha\beta}$ are determined from a linear theory of shells. Since the displacements and rotations are infinitesimal, the $\overset{0}{\chi}_{\alpha\beta}$ are neglected in comparison with $b_{\alpha\beta}$ in Eq. (143). In the load terms (Eq. 145), the products of the vectors $\overset{0}{\overline{u}}$ and $\overset{1}{\overline{u}}$ are negligible, and the component $\overset{0}{m}$ of the vector $\overset{0}{\overline{m}}$ is taken as zero. Thus, for the case of hydrostatic pressure, we have

$$\overset{1}{P}_{\alpha} = -\,(\overset{1}{w}_{,\alpha} + b_{\alpha\beta}\overset{1}{u}{}^{\beta})\,p$$

$$\overset{1}{P} = -\,\overset{1}{m}\,p \tag{148}$$

In practical applications, the $\overset{1}{P}_{\alpha}$ in Eq. (148) is usually taken to be zero.

In considering the stability of shell structures, by far the most widely-used approach is to examine the stability of the equilibrium configuration that is governed by the linear membrane theory of shells. In this case, the $\overset{0}{M}{}^{\alpha\beta}$ are zero; and Eq. (143) reduces to

$$\overset{1}{N}{}^{\alpha\beta}{}_{//\alpha} - b^{\beta}_{\alpha}\overset{1}{M}{}^{\gamma\alpha}{}_{//\gamma} + \overset{1}{P}{}^{\beta} = 0$$

$$b_{\alpha\beta}\overset{1}{N}{}^{\alpha\beta} - \overset{1}{\chi}_{\alpha\beta}\overset{0}{N}{}^{\alpha\beta} + \overset{1}{M}{}^{\alpha\beta}{}_{//\alpha\beta} + \overset{1}{P} = 0 \tag{149}$$

By making appropriate substitutions, this set of equations can be reduced to a set of homogeneous differential equations in the $\overset{1}{u}_{\alpha}$ and $\overset{1}{w}$. Since the appropriate boundary conditions are also homogeneous, the problem is a linear *eigenvalue* problem.

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