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PREFACE

This Memorandum is part of a continuing theoretical study of the geomagnetic field. It develops a new and powerful integral method for extrapolating a field. The results should aid the development of representations of the geomagnetic field in space, and are directly applicable to both geophysical problems involving scalar and vector potential fields, and to the design of conjugate point experiments.

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ABSTRACT

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The method of field analysis by integrals is developed and applied, with an electronic computer, to various test fields prescribed over the surface of a sphere. The results are compared with the true values, and a scheme of numerical integration is devised to ensure suitable accuracy. Finally, this method is successfully applied to the geographical distribution of the daily mean field of disturbance. The part of the observed surface field that is of external origin is isolated, and atmospheric current functions capable of yielding it are derived.

Author

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It is a pleasure to record our indebtedness to W. L. Sibley's helpful discussion of various numerical procedures used in this paper.

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I. INTRODUCTION

The present paper extends and applies a method considered in a previous paper (Vestine, 1941) for analyzing magnetic fields by integrals. It also makes use of later studies (Taylor, 1944; Benkova, 1953). For the earth, Gaussian spherical-harmonic analysis is usually the most convenient and satisfactory method (Chapman and Bartels, 1940). For terrestrial fields too complex for spherical-harmonic series, integral analysis may be preferable. The latter tends, theoretically, to be more accurate, since figures are less likely to be lost than in spherical harmonics, which may involve the differencing of numerical quantities. Moreover, a high-speed computer can make the numerical calculation less burdensome. Even so, when data are incomplete or known only pointwise, it may prove best to extrapolate or interpolate surface field data by an analytical method such as spherical harmonics, based on fitting the observed data with polynomials.

The present paper develops methods mainly in order to analyze certain average aspects of the magnetic storm field — particularly the daily means of disturbance.

II. SEPARATION OF SURFACE MAGNETIC FIELD INTO PARTS OF EXTERNAL AND INTERNAL ORIGIN

It is convenient to begin by noting some results based on Green's theorem (Vestine, 1941). Consider a closed surface s enclosing a volume v. Let U and $V = V_i + V_e$ satisfy Laplace's equation, where V_i is due to sources inside s, V_e to sources outside, so that

$$\int_v (U\nabla^2 V - V\nabla^2 U)dv = \int_s \left(U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n}\right) ds. \tag{1}$$

On writing $U = 1/r$, where r is the distance from a variable point in v, to the fixed point P, there results

$$\int_v \left(\frac{1}{r}\nabla^2 V - V\nabla^2 \frac{1}{r}\right) dv = \int_s \left(\frac{1}{r}\frac{\partial V}{\partial n} - V \frac{\partial}{\partial n}\frac{1}{r}\right) ds . \tag{2}$$

If we consider only the sources inside v, then from an external point P_e,

$$\nabla^2 \frac{1}{r} = 0, \ \nabla^2 V_e = 0, \ \nabla^2 V_i = -4\pi\mu_i ,$$

where μ_i is the volume density of magnetic poles. Consequently,

$$\int_v \frac{1}{r}\nabla^2 V \, dv = -4\pi \int_v \frac{\mu_i}{r} dv = -4\pi V_i ,$$

so that

$$V_i = -\frac{1}{4\pi} \int_s \left(\frac{1}{r}\frac{\partial V}{\partial n} - V \frac{\partial}{\partial n}\frac{1}{r}\right) ds \tag{3}$$

at the external point P_e.

Similarly, at an internal point P_i, for external sources

$$V_e = \frac{1}{4\pi} \int_s \left(\frac{1}{r}\frac{\partial V}{\partial n} - V \frac{\partial}{\partial n}\frac{1}{r}\right) ds . \tag{4}$$

Consider the value approached by $V_e - V_i$ as P_e, P_i approach a point P on s. This value can be inferred by noting that

$$-\frac{1}{4\pi}\int_s \frac{1}{r}\frac{\partial V}{\partial n}\,ds$$

may be regarded as the potential of a single layer of pole density

$$\sigma = -\frac{1}{4\pi}\frac{\partial V}{\partial n} ,$$

and

$$\frac{1}{4\pi}\int_s V\frac{\partial}{\partial n}\left(\frac{1}{r}\right) ds$$

as the potential of a double layer of strength, or magnetic moment per unit area $\mu = \frac{1}{4\pi} V$. Thus,

$$V_e \to \frac{1}{4\pi}\int_s \left(\frac{1}{r}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\frac{1}{r}\right) ds + \frac{V(P)}{2} \tag{5}$$

as $P_i \to P$, and

$$V_i \to -\frac{1}{4\pi}\int_s \left(\frac{1}{r}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\frac{1}{r}\right) ds + \frac{V(P)}{2} \tag{6}$$

when $P_e \to P$. Consequently,

$$V_e - V_i = \frac{1}{2\pi}\int_s \left(\frac{1}{r}\frac{\partial V}{\partial n} - V\frac{\partial}{\partial n}\frac{1}{r}\right) ds . \tag{7}$$

If a uniform double layer of strength μ_o is placed on s, its potential V_o is zero outside s and $-4\pi\,\mu_o$ inside, so that if $\mu_o = -V_o/4\pi$ the potential is constant inside and equal to V_o. Thus for Eq. (3), we write what is the same thing

$$V_i = -\frac{1}{4\pi}\int_s \left[\frac{1}{r}\frac{\partial V}{\partial n} - (V - V_o)\frac{\partial}{\partial n}\frac{1}{r}\right] ds ,$$

and as Vestine (1941) has shown

$$V_e = \frac{1}{4\pi} \int_s \left[\frac{1}{r} \frac{\partial V}{\partial n} - (V - V_o) \frac{\partial}{\partial n} \frac{1}{r} \right] ds + V_o \; . \tag{8}$$

Also as $P_e \to P$, $P_i \to P$

$$V_e \to \frac{1}{4\pi} \int_s \left[\frac{1}{r} \frac{\partial V}{\partial n} - (V - V_o) \frac{\partial}{\partial n} \frac{1}{r} \right] ds + \frac{V - V_o}{2} + V_o \; , \tag{9}$$

$$V_i \to - \frac{1}{4\pi} \int_s \left[\frac{1}{r} \frac{\partial V}{\partial n} - (V - V_o) \frac{\partial}{\partial n} \frac{1}{r} \right] ds + \frac{V - V_o}{2} \; , \tag{10}$$

and

$$V_e - V_i = \frac{1}{2\pi} \left[\frac{1}{r} \frac{\partial V}{\partial n} - (V - V_o) \frac{\partial}{\partial n} \frac{1}{r} \right] ds + V_o \; . \tag{11}$$

Thus, if $V = V_e + V_i$ on s is known only apart from a constant, $V_e - V_i$ will be affected, though actually only in V_e.

A corresponding result is found in Taylor (1944) for a sphere of radius a:

$$Z_e - Z_i = - \frac{1}{2\pi} \int_s \frac{V(Q) - V(P)}{r^3} ds + \frac{V_i(P)}{a} \; , \tag{12}$$

where P is the point at which $Z_e - Z_i$ is being evaluated and Q is the variable point on s.

III. A SPHERE

For a sphere of radius a, if $Z = \partial V/\partial n$, then using Eq. (7) at a surface point $P(a, \theta, \varphi)$,

$$V_e - V_i = \frac{1}{2\pi} \int_s \left[\frac{Z}{r} - V \frac{\partial}{\partial a}\left(\frac{1}{r}\right) \right] ds \quad .$$

Here r is the distance from P to the element of area ds at $Q(a, \theta', \varphi')$, so

$$r^2 = 2a^2 (1 - \cos\gamma) \quad ,$$

where γ is the angular distance between P and Q, subtended at the center of the sphere. Thus,

$$\frac{\partial}{\partial a}\left(\frac{1}{r}\right) = - \frac{1}{2ar}$$

and

$$V_e - V_i = \frac{1}{4\pi a} \int_s (V + 2aZ) \frac{ds}{r} \quad .$$

For convenience in numerical evaluation, we can write

$$\begin{aligned} V_e - V_i &= \frac{1}{4\sqrt{2}\,\pi a} \int_s \frac{(V + 2aZ)}{(1 - \cos\gamma)^{\frac{1}{2}}} ds \\ Z_e - Z_i &= - \frac{1}{4\sqrt{2}\,\pi a} \int_s \frac{[V(Q) - V(P)]}{(1-\cos\gamma)^{3/2}} ds \end{aligned} \tag{13}$$

where $\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi' - \varphi)$.

IV. EXTRAPOLATION OF A PRESCRIBED SURFACE FIELD, USING POISSON'S INTEGRAL

When V_i is known and equal to f_i at all points Q over a sphere of radius a, values of V_i for $R > a$, are given by Poisson's integral (MacRobert, 1928). Thus,

$$V_i = \frac{(R^2-a^2)}{4\pi a} \int_s \frac{f_i(Q)}{r^3} \, ds \tag{14}$$

where $r^2 = (a^2 + R^2 - 2aR \cos \gamma)^{1/2}$.

In the same way, for $R < a$,

$$V_e = \frac{(a^2 - R^2)}{4\pi a} \int_s \frac{f_e(Q)}{r^3} \, ds \ . \tag{15}$$

These integrals apply to potential functions V, but may also be applied to magnetic field components separately, if these are directed parallel to a fixed Cartesian or skew reference frame (Vestine, et al., 1962).

To extrapolate prescribed surface fields over a sphere towards the source of field, one can use spherical harmonics (Chapman and Bartels, 1940). There are advantages in using integrals, however, if fitting of the field requires too many spherical-harmonic terms. Either method may give uncertain results, because small errors in the observational data tend to be accentuated as a singular source of field is approached.

If a potential V is known at points $Q(\theta, \varphi)$ over a spherical surface s_1 of radius a, then for a point P on a concentric spherical surface s of radius $R < a$ we may write

$$V(P) = \frac{a^2 - R^2}{4\pi a} \int_{s_1} \frac{f(Q)}{r^3} \, ds_1 \equiv pf \ , \tag{16}$$

where f(Q) originates outside s_1, and p indicates the application of Poisson's integral as an operator to all values V(Q) = f(Q).

It may be possible to estimate f(Q) on s_1, knowing V(P) over s. If there are no sources of magnetic field between s and s_1, V(P) may be taken as a first approximation to V(Q) on s_1. Then $V_1 = pV$ is an estimate of V on s_1. The value V_1 will fail to agree with V(P) by the amount V(P) - p V(P) = (1 - p)V.

The second approximation for the estimated value of V on s_1 becomes V + (1 - p) V. The field calculated at s is

$$p[V + (1 - p)V] = p[1 + (1 - p)] \cdot V \quad ,$$

which fails to agree with the observed value V by the amount

$$V - p[1 + (1 - p)] \cdot V = (1 - p)^2 \cdot V \quad ,$$

and so on. We thus estimate f(Q) from observed values V(P). It is apparent that the various approximations to f(Q) in sequence become V; $[1 + (1 - p)] \cdot V$; $[1 + (1 - p) + (1 - p)^2] \cdot V$; and the j-th approximation to V on s, becomes

$$f_j(Q) = \sum_{o}^{j} (1 - p)^j \cdot V(P) = \frac{1 - (1 - p)^j}{p} \cdot V(P) \qquad (17)$$

for R < a. When R > a, the operator p is given by that for V_e in Eq. (15). If we consider the series summable, the approached limit is 1/p, since the result pV is less than V, if averaged over s. This would imply that V(Q) = 1/p • V(P), or that p • V(Q) = V(P), as it should if the operator p is intended to imply the use of Poisson's integral.

Caution is needed in the application of Eq. (17). Extrapolation should not extend into the space occupied by the field's sources. The success achieved will also depend upon the accuracy of the data provided, since errors may be amplified, especially as the distance of the extrapolation increases.

V. APPLICATIONS OF INTEGRAL METHODS TO SIMPLE FIELDS

The potential on the surface of a sphere of radius R can be expressed in spherical harmonics as

$$V = a \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[\left(\frac{R}{a}\right)^{n} E_n^m + \left(\frac{a}{R}\right)^{n+1} I_n^m \right] \tag{18}$$

$= V_e + V_i$, say, where a is the radius of a sphere. The potential of external components is $E_n^m = (a_n^m \cos m\varphi + b_n^m \sin m\varphi)\, P_n^m (\cos\theta)$, and the internal component $I_n^m = (c_n^m \cos m\varphi + d_n^m \sin m\varphi)\, P_n^m (\cos\theta)$. Here θ and φ are colatitude and longitude, respectively; a_n^m, b_n^m, c_n^m, d_n^m are constants; and $P_n^m (\cos\theta)$ is the associated Legendre polynomial of Schmidt, order m and degree n (Chapman and Bartels, 1940). The vertical intensity $\partial V/\partial R$, positive when directed downwards, is

$$Z = a \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[n\left(\frac{R}{a}\right)^{n-1} E_n^m - (n+1) \left(\frac{a}{R}\right)^{n+2} I_n^m \right] . \tag{19}$$

To test the integral method of separating an observed field into external and internal parts, we chose the field obtained for a single term m = 0, n = 2, of Eq. (18) with a = R = 1, $a_2^0 = -200$, $b_2^0 = 0$, $c_2^0 = 100$, and $d_2^0 = 0$. Then $V = -50\,(3\cos^2\theta - 1)$. Here, $V_e = -100\,(3\cos^2\theta - 1)$, and $V_i = 50\,(3\cos^2\theta - 1)$. Also the vertical force is $Z = -350\,(3\cos^2\theta - 1)$ with $Z_e = -200\,(3\cos^2\theta - 1)$ and $Z_1 = -150\,(3\cos^2\theta - 1)$.

On putting a = 1 in the first integral of Eq. (13), we see that

$$V_e - V_i = \frac{1}{4\sqrt{2}\,\pi} \int_0^{\pi} (V + 2Z) \sin\theta' \left[\int_0^{2\pi} \frac{d\varphi'}{(1 - \cos\gamma)^{\frac{1}{2}}} \right] d\theta', \tag{20}$$

where $\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi' - \varphi)$ evaluated at the point $P(a,\theta,\varphi)$ using "observed" values at the variable point $Q(a, \theta', \varphi')$ on s.

A problem arises at the point where P = Q, however, since the denominator of the integral becomes zero. The integral has a definite limit (Taylor, 1944). In the application to the simple field described above, the values of $V_e - V_i$ (at the point P) were computed at odd degrees of latitude, while the values of V + 2Z at Q were taken at even degrees of latitude. Using an IBM 7090 computer, we evaluated the integrand for every four degrees of longitude and two degrees of latitude. The integration was performed as indicated in Eq. (20), first by summing around each circle of latitude, and then by summing these results. The quadrature formula used for the integration was

$$\int_{x_o}^{x_n} y(x)\, dx = h \sum_{i=o}^{n} \delta_i\, y_i - E \quad , \tag{21}$$

where E is the truncation error

$$E = \frac{3}{80}\, y^{(4)}\, h^5 + (n-4)\left(\frac{2}{945}\right) y^{(6)}\, h^7 \tag{22}$$

and

$$\delta_o = 0.348611111$$

$$\delta_1 = 1.24583333$$

$$\delta_2 = 0.879166667$$

$$\delta_3 = 1.02638889$$

$$\delta_i = 1, \quad 4 \le i \le n - 4$$

$$\delta_{n-i} = \delta_i, \quad n - 4 < i \le n \ .$$

The test potential and its calculated parts of internal and external origin are shown in Fig. 1. The error of one to two per cent is probably due to the approximate method of treating the singular point.

For the vertical intensity, the second integral of Eq. (13) gives us

$$Z_e - Z_i = -\frac{1}{4\sqrt{2}\,\pi} \int_0^{\pi} [V(Q)-V(P)]\sin\theta' \left[\int_0^{2\pi} \frac{d\varphi'}{(1-\cos\gamma)^{3/2}}\right] + V_i \,. \tag{23}$$

The computed values of $Z_e - Z_i$ corresponding to $V_e - V_i$ are also shown in Fig. 1. Here the calculation is more precise than for the potential, since the numerator and denominator of the integrand approach zero together.

No special difficulty arises, therefore, in obtaining $V_e - V_i$ and $Z_e - Z_i$, when $V_e + V_i = V$ and when $Z_e + Z_i = Z$ are known. Hence, V_i, V_e, Z_i, and Z_e can be obtained separately (Vestine, 1941). The separation can also be shown to be unique (Taylor, 1944).

Equation (17) can be used to extrapolate surface values of fields to interior and exterior points. It is instructive to see how the integral method works with fields represented by harmonics of varying degrees n. The present paper examines the procedure for extrapolating V_e from the surface s (of radius a) to an exterior surface s_1 for which $R > a$. Now for points interior to a given surface, V_e is defined by Eq. (15). This equation also defines an integral operator p in the same manner as does Eq. (16). It follows that Eq. (17) can be applied directly to the extrapolation of V_e. The problem therefore is that of

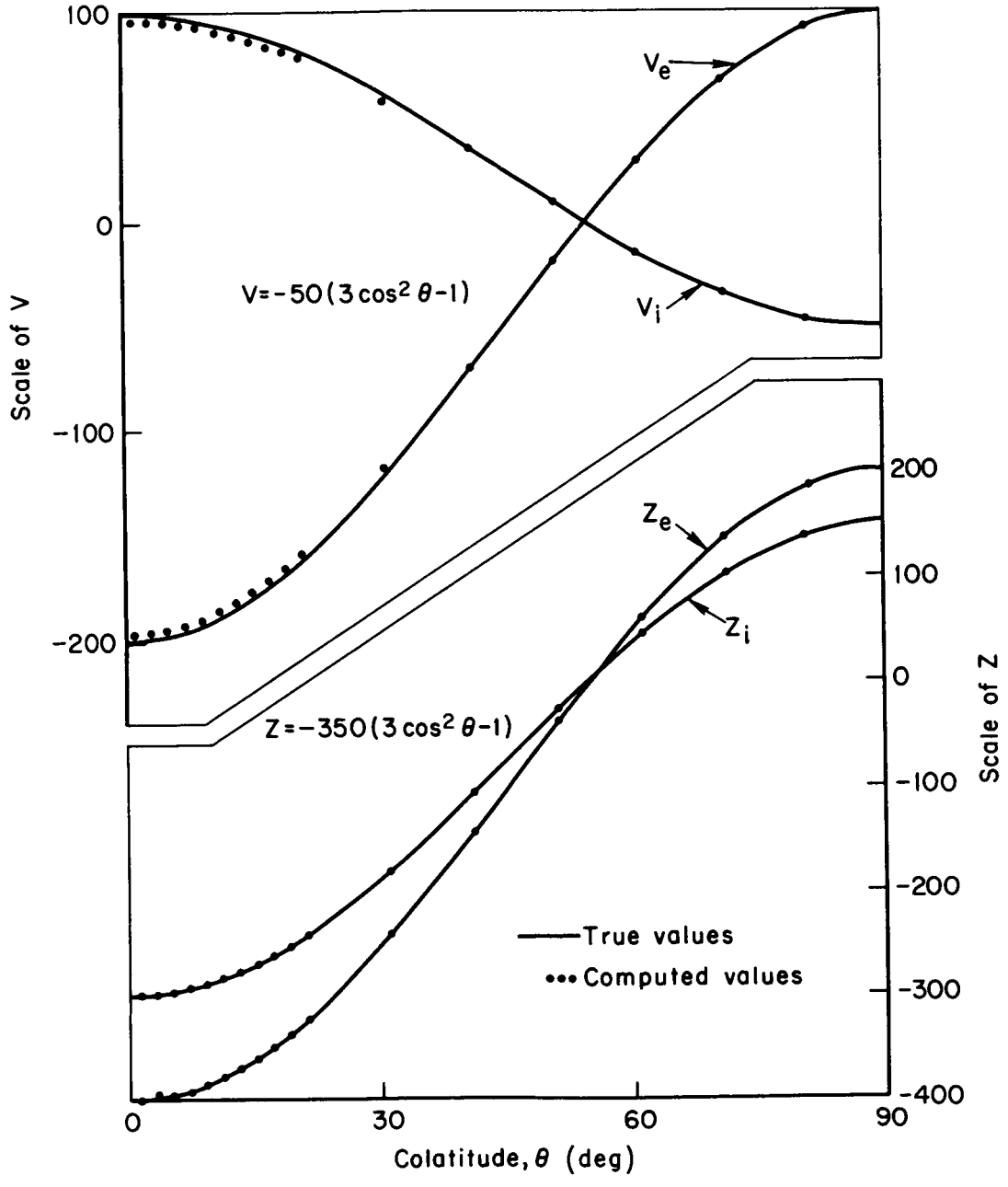

Fig. 1 Computed separation of magnetic field into external and internal parts by integrals, compared with true values

estimating $f(Q) = V_e$ on s_1 from the observed potential on s, when there are no sources of field between s and s_1. This is very nearly the situation between the earth's surface and the lower E-region of the ionosphere, where the sources of V_e are located at higher levels above the ground. Since the convergence of Eq. (17) may depend both on r and on the nature of the field extrapolated, it is convenient to effect the extrapolations for various values of r and degrees n of spherical-harmonic field terms. Though for convenience only zonal harmonics will be used here, no loss in generality ensues.

Equation (18) indicates that more complex surface fields, which give rise to harmonics of higher degree n, change rapidly in magnitude as R increases, since the dependence is upon $(R/a)^n$, where $R > a$. Therefore, by writing $\alpha = a/R$ and for a potential V(P) expressible by a spherical harmonic of degree n, and by using Eqs. (17) and (18), we derive for the j-th approximation to V at the point Q on s_1

$$V_j(Q) = \frac{1 - (1-\alpha^n)^j}{\alpha^n} \cdot V(P) \quad . \tag{24}$$

Also note from Eq. (18) that $V_e(Q) = (R/a)^n V_e(P)$, or $V_e(P) = \alpha^n V_e(Q)$. Consequently, the error resulting from use of Eq. (24) can be estimated. The approximate result is $V_e(P) = [1-(1-\alpha^n)^j] \cdot V_e(P)$, whereas the correct result is $V_e(P) = V_e(P)$. Hence, the convergence of Eq. (24) depends on the rate at which $(1 - \alpha^n)^j$ tends toward zero as j, the number of approximations, increases. The table gives values of this function for various values of $\alpha = a/R$, n and j. At heights of 100 to 200 km, the convergence is good to about one per cent or better, if $n < 10$ and $j = 4$ at most. For complex fields near the auroral zone (supposing

Table

CONVERGENCE OF $(1-\alpha^n)^j$

h = 100 km α = 0.984544

j/n	0	1	2	3	4	5	6	7	8	9	10	25	50	100
1	0	0.0155	0.0307	0.0457	0.0604	0.0749	0.0892	0.1033	0.1172	0.1308	0.1442	0.3224	0.5408	0.7891
2		0.0002	0.0009	0.0021	0.0036	0.0056	0.0080	0.0107	0.0137	0.0171	0.0208	0.1039	0.2925	0.6227
3		0	0	0.0001	0.0002	0.0004	0.0007	0.0011	0.0016	0.0022	0.0030	0.0335	0.1582	0.4914
4				0	0	0	0.0001	0.0001	0.0002	0.0003	0.0004	0.0108	0.0855	0.3877
5							0	0	0	0	0.0001	0.0035	0.0463	0.3060
6											0	0.0011	0.0250	0.2414
9												0.00004	0.0040	0.1186
17													0.00003	0.0178
42														0.00004

h = 200 km α = 0.969559

j/n	0	1	2	3	4	5	6	7	8	9	10	25	50	100
1	0	0.0304	0.0600	0.0886	0.1163	0.1432	0.1693	0.1946	0.2191	0.2429	0.2659	0.5384	0.7869	0.9546
2		0.0009	0.0036	0.0078	0.0135	0.0205	0.0287	0.0379	0.0480	0.0590	0.0707	0.2899	0.6192	0.9113
3		0	0.0002	0.0007	0.0016	0.0029	0.0049	0.0074	0.0105	0.0143	0.0188	0.1561	0.4873	0.8699
4			0	0.0001	0.0002	0.0004	0.0008	0.0014	0.0023	0.0035	0.0050	0.0840	0.3834	0.8304
5				0	0	0.0001	0.0001	0.0003	0.0005	0.0008	0.0013	0.0452	0.3017	0.7927
6						0	0	0.0001	0.0001	0.0002	0.0004	0.0244	0.2374	0.7567
7								0	0	0	0.0001	0.0131	0.1868	0.7223
8											0	0.0071	0.1470	0.68955
16												0.000049	0.0216	0.4755
42													0.00004	0.1421
214														0.000048

that a ribbon current about 400 km wide flows there), we may need harmonics of degree $n = 100$ to fit the field at ground level. Though a value for j of about 100 might be necessary, a modern computer would still require only a moderate interval of time to extrapolate field values to the base of the E-region.

Now that the integral method has been successfully applied to various model fields, let us next consider a geophysical application.

VI. APPLICATION TO THE DISTURBED GEOMAGNETIC FIELD

Integral analysis may be applied to those complex features of the magnetic disturbance field that ordinarily require too many spherical-harmonic terms for practical representation. A simple example is the international disturbed-day means minus the quiet-day means of disturbance (D_{mi}) averaged for the years 1922-1933; in high latitudes, data for the polar year of 1932-1933 were adjusted to the level of disturbance for 1932-1933 (Vestine, et al., 1947).

The solid curves in Figs. 2(A) and (B) represent these data in geomagnetic latitude in terms of the potential V and the vertical gradient $Z = \partial V/\partial n$. The values of V were derived apart from a constant by integrating a graph of the observational data in the geomagnetic-north component X', using the relationship $X' = 1/\alpha\ \partial V/\partial\theta = \partial V/\partial\theta$, when $a = 1$.

The potential and vertical force of this field were separated into components of internal and external origin, using Eqs. (20) and (23). Figures 2(A) and (B) show the results. For the potential, we analyzed two sets of values that differ by an additive constant. Figure 3(A) compares the results of separation for both sets. As we indicated in Section II, both sets give the same value of V_i, while the values of V_e differed by a constant. The potential that is symmetrical around the equator (a) is considered to be the correct potential, since V arises from electric currents or magnetic material. If we use this potential, the internal part will be almost directly proportional to the external part; this is to be expected if the cause of V_i is electromagnetic induction by V_e. These results are consistent with

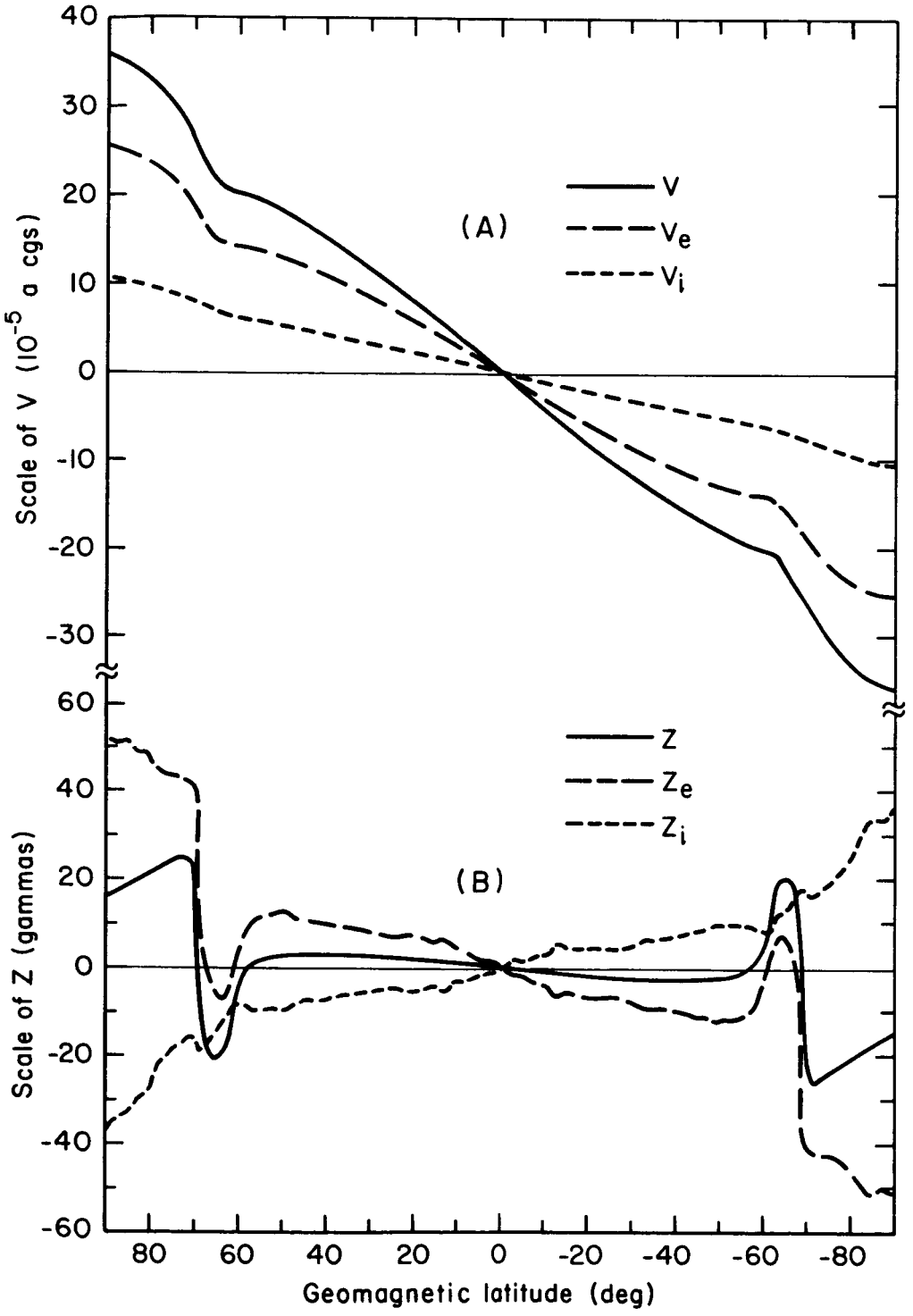

Fig. 2 Separation into internal and external parts (A) potential, (B) vertical force

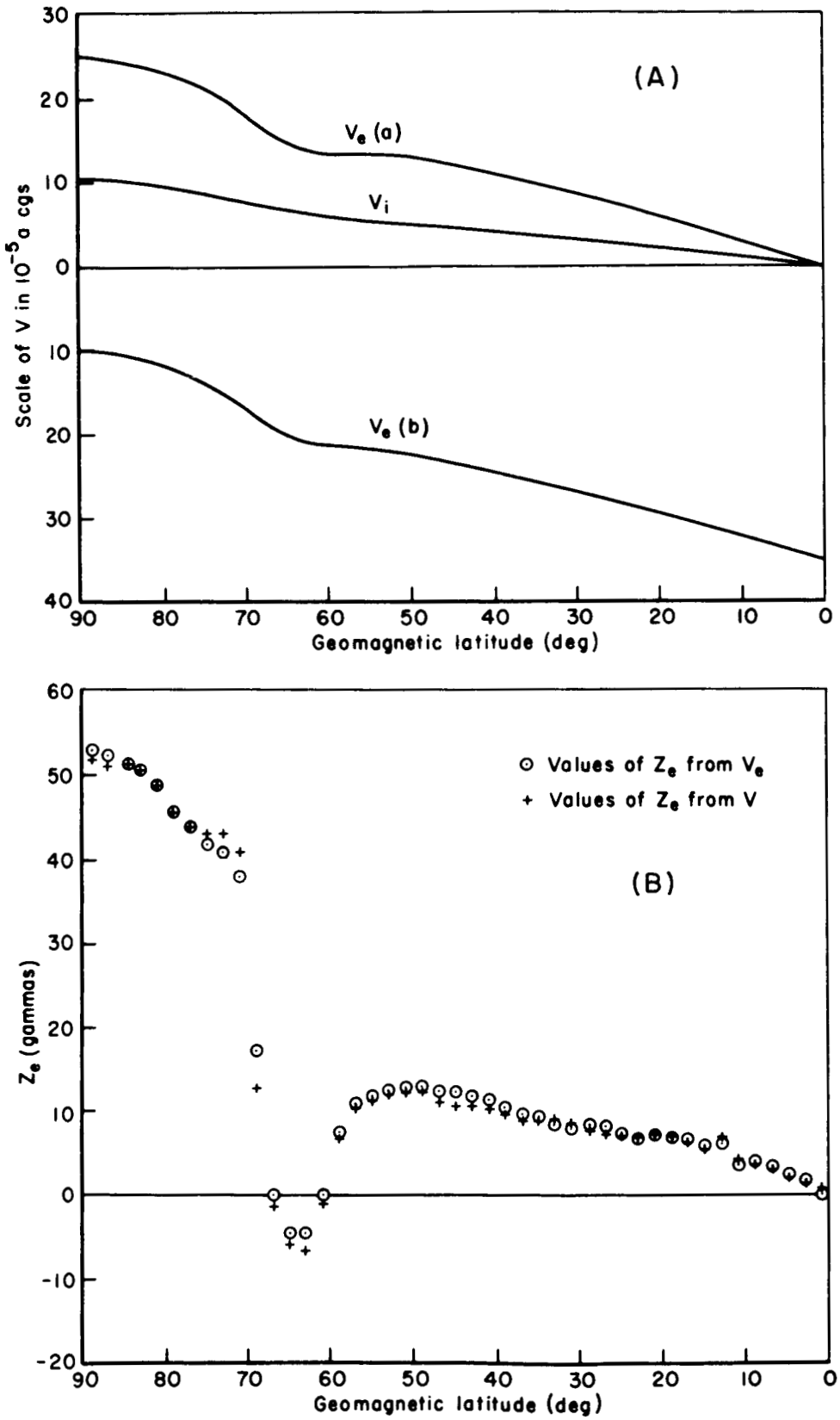

Fig. 3 (A) Separation of (a) potential, and (b) potential plus constant into internal and external parts; (B) estimates of external part of vertical intensity

previous separations of the field made by spherical harmonics (Chapman and Bartels, 1940).

Figure 3(B) shows results for Z_e computed directly from V_e using the relationship, found in Taylor (1944)

$$Z_e(P) = -\frac{1}{2\pi} \int_s \frac{V_e(Q) - V_e(P)}{r^3} \, ds$$

together with similar values obtained by using Eq. (23). The irregularities in both sets of values for Z_e (and the differences between them) are ascribed to the lack of significant figures in the original data. In the integrand, the term $[V(Q) - V(P)]/r^3$ is most important when P and Q are close together, where both the numerator and denominator become quite small. Significant figures in the numerator are lost in the differencing process under this condition.

VII. CALCULATION OF EQUIVALENT OVERHEAD CURRENT-FUNCTION FOR D_{mi}

The potential of external origin most likely derives from currents flowing at various levels above the earth. No one can determine the location, strength, and direction of these currents solely from a knowledge of the surface potential. A given potential distribution could be attributed, for instance, to an infinite number of current shell distributions (Chapman, 1935).

The current function J(Q) at a point Q can be regarded as given by the magnetic moment per unit area, $\mu(Q)$ of such a current shell. Thus,

$$V(P) = \int_s \mu(Q) \frac{\partial}{\partial a}\left(\frac{1}{r}\right) ds$$

$$V(P) = \int_s J(Q) \frac{\partial}{\partial a}\left(\frac{1}{r}\right) ds \quad . \tag{25}$$

By using spherical harmonics, one can also express this result as

$$V(P) = -4\pi \sum_{n=0}^{\infty} \frac{n+1}{2n+1} J_n(Q) \left(\frac{R}{a}\right)^n \tag{26}$$

where

$$J = \sum_{n=0}^{\infty} J_n$$

(Chapman and Bartels, 1940).

Since V is known, rather than J, a method of successive approximations similar to that used for extrapolating the potential can be used. What is formally required is to replace the single

plus double distribution in Eq. (4) with a double distribution of strength μ. It has been noted that $V_e - V_i$ on s can be expressed in terms of a single distribution $V = 2aZ$ on s. Treatment of this problem in terms of integral equations has been discussed in Benkova (1953), where it was also remarked that Eqs. (13) and (25) are equivalent. So

$$V(P) = \int_s \mu(Q) \frac{\partial}{\partial a}\left(\frac{1}{r}\right) ds$$

$$= \frac{a^2-R^2}{4\pi a} \int_s \frac{V(Q)ds}{r^3} \quad .$$

Thus, if $r^2 = a^2 + R^2 - 2aR \cos \gamma$, with Q a point on an outer sphere of radius a,

$$\int_s \mu(Q) \frac{\partial}{\partial a}\left(\frac{1}{r}\right) ds = - \int_s \frac{\mu(Q)(a-R\cos \gamma)\, ds}{r^3}$$

$$= \frac{a^2-R^2}{4\pi a} \int_s \frac{V(Q)}{r^3} ds$$

suggesting that $-V(Q)/4\pi$ might afford a useful first approximation $\mu_1(Q)$ to $\mu(Q)$ and hence to $J(Q)$. This result, substituting $\mu_1(Q)$ for V_1 in Eq. (17) would then provide an estimate of $\mu(Q)$ under favorable circumstances from known values of V_e on the surface of an interior sphere of radius R. The convergence of Eq. (17) would be rapid when $a \sim R$, but would be poor when $a \gg R$.

In practice, we found Eq. (26) to be more useful, taking the first approximation to $\mu(Q)$ from a rough spherical-harmonic analysis of the present data to $n = 3$. In the numerical integration grid

previously used for the field separation, the integrand was evaluated at every two degrees of latitude and four degrees of longitude. This proved to be unsatisfactory for the current density integrations, a finer grid being necessary when Q was in the vicinity of P. Consequently, within two degrees of latitude and eight degrees of longitude of P, the integrand of Eq. (25) was evaluated at every $1/4^{\circ}$ of latitude and $1/2^{\circ}$ of longitude. We used a different grid when P was within 14° of one of the poles, evaluating the integrand for every $1/2^{\circ}$ of latitude and 2° of longitude over the entire polar cap down to colatitude 16°. With these modifications, then, for all current density evaluations attempted, the calculated potential converged to within 0.1% of the true potential after three to five approximations.

Using Eqs. (17) and (25), we thus determined the current density at a height of 100 km, the assumed height of the auroral zone current. The current intensity found is shown in Fig. 4(A). The potential at the ground could be due to this or to equivalent currents at other heights, or to some combination of currents. We here took the large current in the high latitudes to be the auroral zone current, while assuming that the remainder flowed higher above the earth. The small eastward-flowing current at 60° latitude was left for later study. The current flow between the auroral zones, though calculated for a height of 100 km, we regarded as more likely to flow as a ring current at a level yet to be defined. Figure 4(B) shows the potential V_e and the two components of V_e at ground level that yield the calculated total current flow in Fig. 4(A).

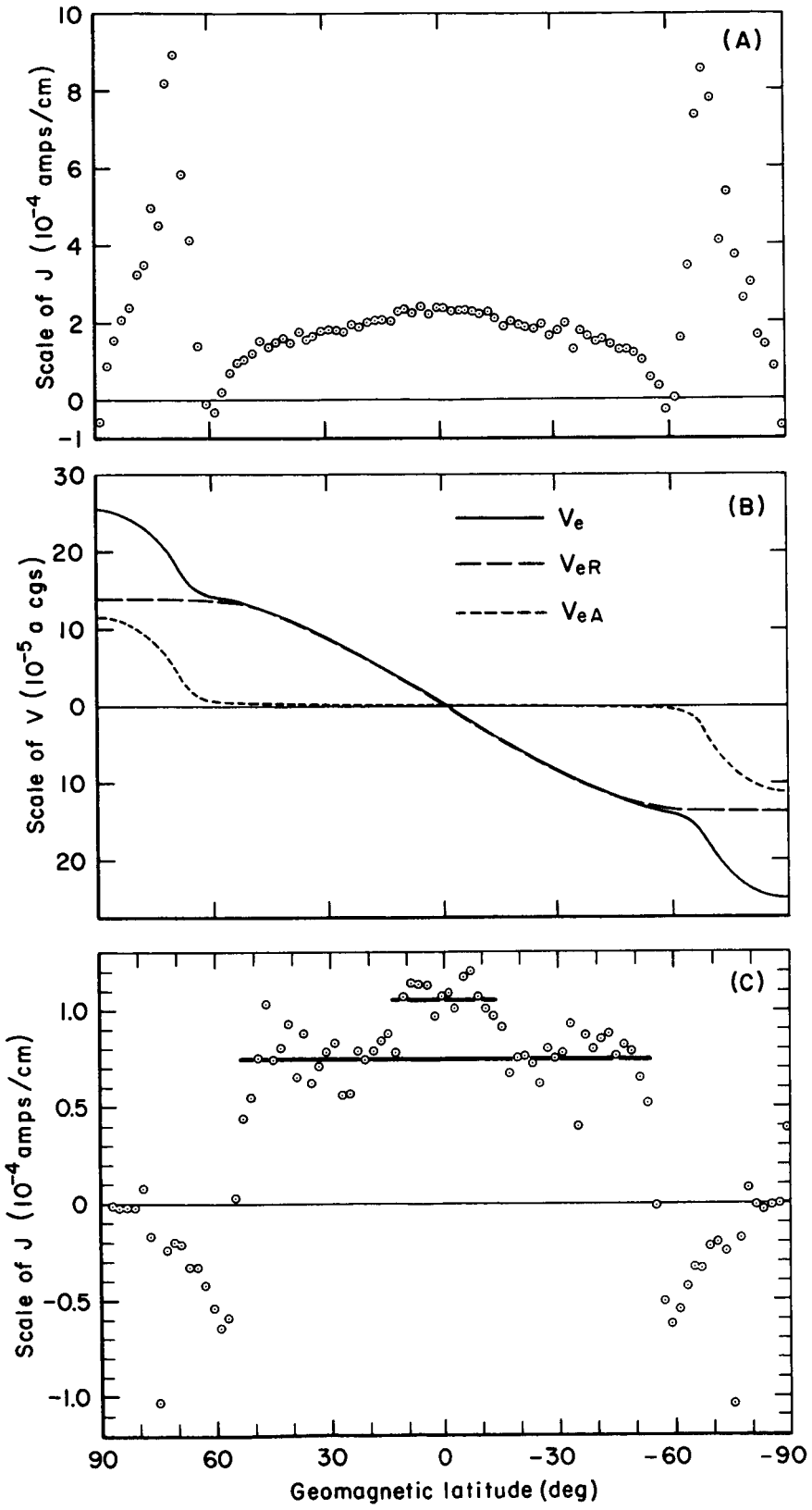

Fig. 4 (A) Westward current intensity at height 100 km from total external potential V_e; (B) division of external potential V_e into auroral component V_{eA} and ring-current component V_{eR}; (C) westward current intensity at 2000 km from ring-current potential V_{eR}

Figure 4(C) illustrates another possible model current system that could produce the potential V_{eR} between the auroral zones shown in Fig. 4(B), if the current flowed at a height of 2000 km; the calculated eastward-flowing current just south of the auroral zone is accentuated, and may be unreal. It could, however, represent an effect in the daily mean that is due to the eastward-directed electrojet of bays, if some of these flow at a latitude a little south of the westward-directed electrojet (Chapman and Bartels, 1940). If so, it can be regarded as an unwanted signal from a lower level, irrelevant to a ring-current flow. In contrast, the short horizontal line in Fig. 4(C) indicates a fairly pronounced increase in current from about 20°S to 20°N. This may indicate a significant current flow from east to west near this level, or in the ionosphere below. A large part of the current, indicated by the longer line in Fig. 4(C), is probably in the form of a broadly distributed ring current at greater heights. For instance, the currents indicated (for flow in thin shells) in Fig. 5 could reproduce the potential V_e derived from the twelve-year means of daily means D_{mi}. The method of analysis by integrals therefore permits a formal analysis of data which would be very difficult practically to represent by spherical harmonics.

One can then interpret the equatorial current at a height of 2000 km as being due to energetic particles — principally protons having mirror points extending mainly 20°N or S and equatorwards, augmented in energy flux during storm days. One of us has previously mentioned this, without estimating the possible height of the current flow (Vestine, 1961). On the other hand, our analysis here does not

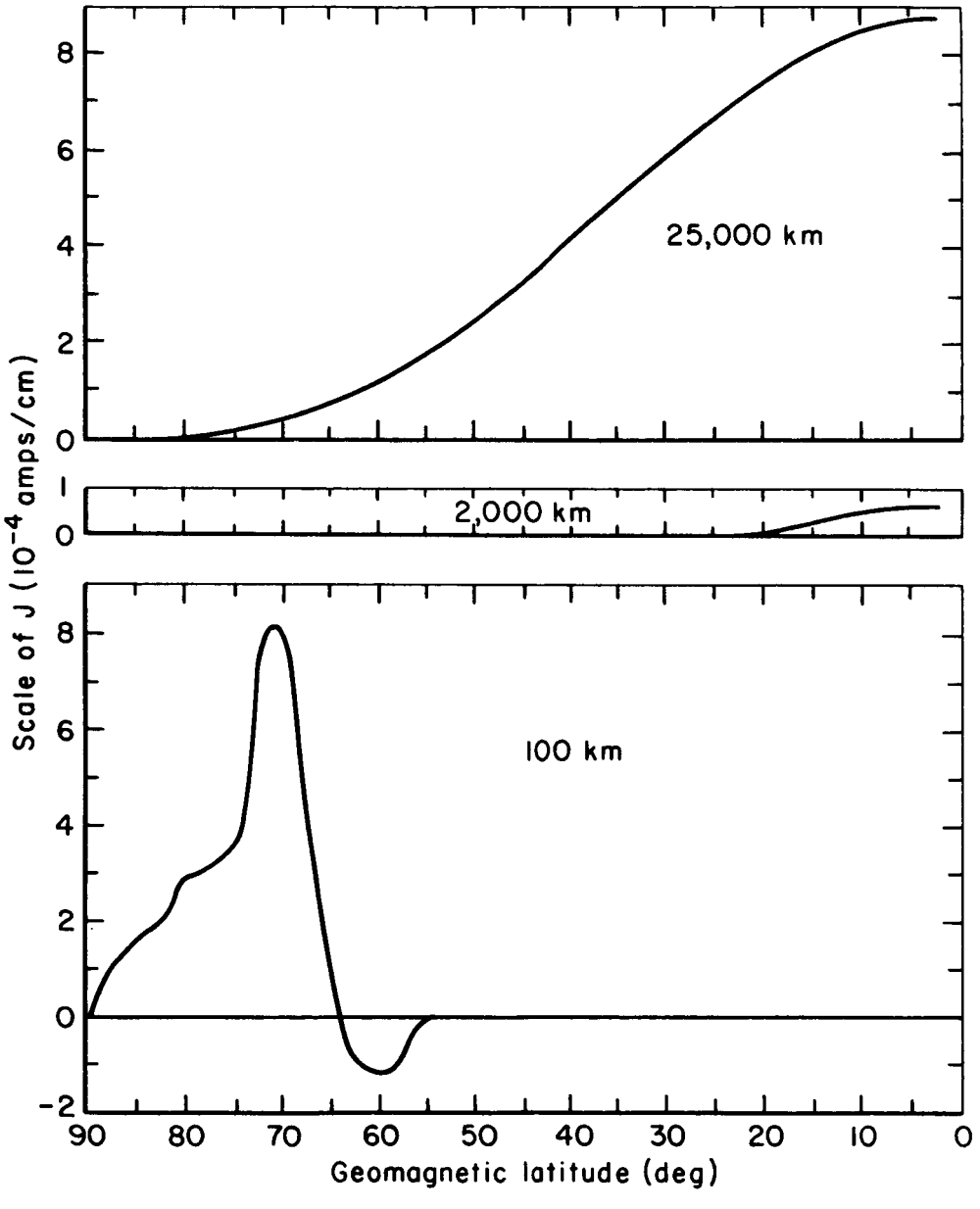

Fig. 5 Current intensities at the various altitudes

sufficiently indicate the need for postulating current at so great a height. Even so, since the character of the field change is such that the signal could scarcely arise at twice this height, we have inferred that the responsible current may flow in lower reaches of the outer Van Allen belt. Because there is also the possibility that the flow occurs within the ionosphere, a later study should consider this possibility.

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