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A PRELIMINARY ANALYSIS OF A REACTING BOUNDARY LAYER WITH COUPLED PRESSURE INTERACTION

TECHNICAL REPORT NO. 377

By

Raymond Edelman
Paul A. Libby

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ABSTRACT

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An analysis is developed for a high speed two-dimensional reacting laminar boundary layer with coupled pressure interaction. The external pressure is assumed to depend linearly on the slope of the displacement thickness in accordance with linearized supersonic theory. Simple transport properties are assumed with all Lewis and Prandlt numbers taken to be equal to unity. A moment method of solution is employed which is an extension of the Karman-Pohlhausen integral technique and involves the use of the normal coordinate as a weighting function. Although a general analysis is presented the treatment of the hydrogen-air system and of the zero and first moments of the conservation equations is carried forward in more detail. As might be expected a large system of first-order, non-linear ordinary differential equations results from this treatment.

Author

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A PRELIMINARY ANALYSIS OF A REACTING BOUNDARY LAYER WITH COUPLED PRESSURE INTERACTION

I. INTRODUCTION

The injection of combustibles such as hydrogen into a high speed air stream arises in several practical flight vehicles. In some applications combustion with the air stream is undesirable such as in the dumping of fuel from the upper stages of a launch vehicle. Other applications, such as with air-breathing propulsion units combustion is indeed desirable. Accordingly, it is necessary in the study of such flows to understand the chemical kinetics of the system. Furthermore, in many applications the mode of injection is such that the effects of viscosity, diffusion and viscous heating play an important role in determining the flow. Finally, in some applications which can be studied within the framework of boundary layer theory the occurrence of combustion within the boundary layer can deflect the external flow altering the pressure distribution, i.e., can lead to a boundary layer interaction problem. Indeed it is expected that in some cases the boundary layer can act as a flame holder, initiating combustion close to the wall but causing a compression in the external flow so that reaction is initiated there. This process can take place in the flow over flat surfaces and indeed over flared surfaces, i.e., over vehicle surfaces with discontinuous changes in cross section.

It is the purpose of the present study to present an analysis which can consider at least in an approximate manner some of the aspects of the injection of fuel cited above. For orientation it is convenient to consider two flow configurations; in one, a fuel, such as hydrogen is considered to be injected through a slot adjacent to a wall, such as a vehicle surface over which a supersonic airstream is flowing. In this case the fuel is confined to the boundary layer over the surface; if combustion occurs, the external airstream may be deflected significantly and the pressure distribution over the surface altered. In a second flow configuration, a uniform mixture of fuel and oxidizer at a temperature sufficiently low so that combustion does not occur is considered to flow over a surface. Either the surface temperature or the temperature rise due to aerodynamic heating is sufficiently high so that combustion is initiated within the boundary layer. The resulting compression of the external stream can initiate combustion there as well.

The method of approach is that of Reference 1 which suggests an integral method of solution to homogeneous boundary layers. In contrast to the usual integral methods based on the Karman-Pohlhausen approach, a method of moments is advanced in Reference 1, so that additional accuracy is achieved and so that in principle systematically improved. An integral procedure, even though of a sophisticated type, will be recognized as leading to approximate solutions in contrast to a finite difference procedure which is in principle exact. However, the latter procedure, when applied to flows

with finite rate chemistry have in the past led to great computational difficulty even without boundary layer interaction. Accordingly, the present approach is considered to have a provisional value.

In the integral method of Reference 1 as employed here the transformed normal coordinate is used as a weighting function. The transformation renders the equations practically incompressible and the moment method permits the introduction of an arbitrary number of free parameters which account for the non-similar behavior of the flow field, and which are determined by a corresponding number of moment equations. Thus, the free parameters, or coefficients of the assumed profiles, are determined by additional ordinary differential equations and not by merely satisfying additional slope, curvature, etc. requirements at the wall and "outer edge" of the boundary layer characteristic of the basic Karman-Pohlhausen method. The nearly incompressible behavior in the transformed plane plus the flexibility offered by the moment method should improve the integral method approximation in the physical plane. Other modified integral methods similar to the present approach have been employed in the past (2, 3, 4, 5).

II. ANALYSIS

The conservation equations for two-dimensional boundary layer flow with all Lewis numbers and with the number Prandtl equal to unity are given below:

Global Continuity of Mass:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

Species Continuity:

$$\frac{\partial \rho u Y_i}{\partial x} + \frac{\partial \rho v Y_i}{\partial y} = \rho \dot{w}_i + \frac{\partial}{\partial y}\left(\mu \frac{\partial Y_i}{\partial y}\right)$$

Momentum:

$$\frac{\partial \rho u u}{\partial x} + \frac{\partial \rho u v}{\partial y} = \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right)$$

Energy:

$$\frac{\partial \rho u H}{\partial x} + \frac{\partial \rho v H}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial H}{\partial y}\right)$$

Equations 1-4 together with equations of state, a description of the production term, $\dot{w}_i$, and boundary and initial conditions, determine the behavior of the flow field, provided that the pressure dependence with x is specified. In the present analysis the pressure is assumed linearly dependent on the slope of the displacement thickness, δ^*, as in linearized supersonic flow theory.

In view of the previous discussion, transform the independent variables x, y to the Levy-Lees variables, $\bar{s}$, η and introduce a modified stream function, f, where

$$u = u_e(\bar{s})\left[f(\bar{s},\eta)\right]_\eta$$

with

$$\psi = \frac{(2\bar{s})^{1/2}}{P_e} f(\bar{s},\eta)$$

5 a, b, c, d

and

$$\bar{s} = \int_0^x \rho_e u_e \mu_e \, dx$$

$$\eta = \frac{\rho_e u_e}{(2\bar{s})^{1/2}} \int_0^y \frac{\rho}{\rho_e} \, dy$$

Introducing the new variables results in the following set of governing equations for the flow in the transformed plane:

Momentum:

$$(C f_{\eta\eta})_\eta + f f_{\eta\eta} + \frac{2\bar{s}}{u_e}\frac{du_e}{d\bar{s}}\left[\frac{\rho_e}{\rho} - (f_\eta)^2\right] = 2\bar{s}\left[f_\eta f_{\eta\bar{s}} - f_{\bar{s}} f_{\eta\eta}\right] \quad 6$$

Species Continuity.

$$(C Y_{i\eta})_\eta + f Y_{i\eta} = 2\bar{s}\left[f_\eta Y_{i\bar{s}} - f_{\bar{s}} Y_{i\eta}\right] - 2\bar{s}\frac{\dot{W}_i}{\rho_e u_e^2 \mu_e} \quad 7$$

Where the element mass fractions are governed by:

$$(C \tilde{Y}_{i\eta})_\eta + f \tilde{Y}_{i\eta} = 2\bar{s}\left[f_\eta \tilde{Y}_{i\eta} - f_{\bar{s}} \tilde{Y}_{i\eta}\right] \quad 8$$

Energy:

[illegible] 9

where,

$$C = \frac{\rho\mu}{\rho_e \mu_e}$$

The boundary and initial conditions will be discussed in conjunction with the integral formulation.

INTEGRAL FORMULATION:

To obtain an approximate solution by the method of moments take a weighting function given by η^m and anticipate assumed profiles for the variables governed by Equations 6 through 9 in terms of polynomials in $\tilde{\eta} = \eta / \eta_e$. To complete a simplified treatment of the transport properties take $C = \frac{\rho\mu}{\rho_e \mu_e} = 1$.

Multiplying Equations 6 through 9 by $\eta^m d_\eta$ and carrying out the integrations from 0 to η_e* results for m = 0, 1** in the following set of integral differential equations involving $f = \eta / \eta_e$.

Momentum:

$$2\bar{s}\left\{1c\frac{d}{d\bar{s}}\left(\int_0^1 [f_\eta^2 - f_\eta]\,d\tilde{\eta} + \int_0^1 [f_\eta^2 - f_\eta]\,d\tilde{\eta}\,\frac{d\eta_c}{d\bar{s}}\right\} = \frac{2\bar{s}}{u_e}\frac{du_e}{d\bar{s}}\,\eta_c\left[\int_0^1 \frac{\rho_e}{\rho}\,d\tilde{\eta} - \int_0^1 f_\eta\,d\tilde{\eta}\right] - \frac{\left[(f_\eta)\tilde{\eta}\right]_w}{\eta_c} + \eta_c\int_0^1 (f_\eta - f_\eta^2)\,d\tilde{\eta}\ , \quad \text{for } m = 0 \tag{10a}$$

and

$$2\bar{s}\,\eta_c\left\{\eta_e\frac{d}{d\bar{s}}\int_0^1 [\tilde{\eta} f_\eta^2 - f_\eta]\,d\tilde{\eta} + \int_0^1 [2\tilde{\eta} f_\eta^2 - f + \eta_c\int_0^1 f_\eta\left(\int_0^{\tilde{\eta}} f_{\eta\bar{s}}\,d\tilde{\eta}'\right)d\tilde{\eta}\right\} = \eta_e^2\int_0^1 [f_\eta - f_\eta^2\tilde{\eta}]\,d\tilde{\eta} - 1 + \eta_e^2\frac{2\bar{s}}{u_e}\frac{du_e}{d\bar{s}}\int_0^1 \left[\frac{\rho_e}{\rho} - f_\eta^2\right]\tilde{\eta}\,d\tilde{\eta} - \eta_c^2\int_0^1 f_\eta\left(\int_0^{\tilde{\eta}} f_\eta\,d\eta^{-1}\right)d\tilde{\eta}, \quad \text{for } m = 1 \tag{10b}$$

---

* Note that a single thickness is assumed here; this is consistent with the assumption for the transport properties.

** The general set of equations in terms of arbitrary m are given in Appendix I.

Species:

$$2\bar{\xi}\left[Y_{ie}\left\{\eta_e\frac{d}{d\bar{\xi}}\int_0^1 f_\eta\, d\tilde{\eta}+\int_0^1 f_\eta\, d\tilde{\eta}\,\frac{d\eta_e}{d\bar{\xi}}\right\}-\eta_e\frac{d}{d\bar{\xi}}\int_0^1 Y_i f_\eta\, d\tilde{\eta}-\int_0^1 Y_i f_\eta\, d\tilde{\eta}\,\frac{d\eta_e}{d\bar{\xi}}\right]+$$

$$+\eta_e\left[Y_{ie}\int_0^1 f_\eta\, d\tilde{\eta}-\int_0^1 f_\eta Y_i\, d\tilde{\eta}\right]=-2\bar{\xi}\eta_e\int_0^1\frac{\dot{w}_i}{\rho_e u_e^2\mu_e}\,d\tilde{\eta},\ \text{for } m=0 \qquad \text{11a}$$

and,

$$2\bar{\xi}\eta_e\left\{\eta_e\left[Y_{ie}\frac{d}{d\bar{\xi}}\left(\int_0^1 f_\eta\, d\tilde{\eta}-\frac{d}{d\bar{\xi}}\left(\tilde{\eta}f_\eta Y_i\, d\tilde{\eta}-\left(\int_0^1 Y_i\left(\int_0^{\tilde{\eta}} f_{\bar{\xi}\eta}\, d\tilde{\eta}'\right)d\tilde{\eta}\right]+\right.\right.\right. \qquad \text{11b}$$

$$+\left[Y_{ie}\int_0^1 f_\eta\, d\tilde{\eta}-2\int_0^1\tilde{\eta}f_\eta Y_i\, d\tilde{\eta}\right]\frac{d\eta_e}{d\bar{\xi}}\Bigg\}+\eta_e^2\left[Y_{ie}\int_0^1 f_\eta\, d\tilde{\eta}-\int_0^1\tilde{\eta}f_\eta Y_i\, d\tilde{\eta}\right]-$$

$$-\eta_e^2\left(\int_0^1 Y_i\left(\int_0^{\tilde{\eta}} f_\eta\, d\tilde{\eta}'\right)d\tilde{\eta}-\left[Y_{ie}-Y_{iw}\right]=-2\bar{\xi}\eta_e^2\int_0^1\frac{\dot{w}_i\tilde{\eta}}{\rho_e u_e^2\mu_e}\,d\tilde{\eta},$$

for $m=1$

<u>Element Mass Fractions:</u>

In Equations 11a and 11b replace:

$$\left.\begin{array}{l}\dot{w}_i \text{ by } 0\\ Y_i \text{ by } \tilde{Y}_i\\ \tilde{Y}_{ie}\equiv \text{const.}\end{array}\right\} \qquad \begin{array}{r}\text{12a}\\ \\ \text{12b}\end{array}$$

Energy:

$$2\bar{\xi}\left\{\eta_e \frac{d}{d\bar{\xi}}\int_0^1 f_\eta(g-1)\,d\tilde{\eta} + \int_0^1 f_\eta(g-1)\,d\tilde{\eta}\,\frac{d\eta_e}{d\bar{\xi}}\right\} + \qquad 13a$$

$$+\eta_e\int_0^1 f_\eta(g-1)\,d\tilde{\eta} + \frac{(g_{\tilde{\eta}})_w}{\eta_e} = 0, \quad \text{for } m=0$$

and

$$2\bar{\xi}\left\{\eta_e\int_0^1 g\left(\int_0^{\tilde{\eta}} f_{\bar{\xi}\eta}\,d\tilde{\eta}'\right)d\tilde{\eta} + \eta_e\frac{d}{d\bar{\xi}}\int_0^1 f_\eta(g\tilde{\eta}-1)\,d\tilde{\eta} + \right. \qquad 13b$$

$$\left. + \left[2\int_0^1 g f_\eta\,\tilde{\eta}\,d\tilde{\eta} - \int_0^1 f_\eta\,d\tilde{\eta}\right]\frac{d\eta_e}{d\bar{\xi}}\right\} =$$

$$= (g_{\tilde{w}}-1) + \eta_e^2\left[\int_0^1\left[f_\eta(1-\tilde{\eta}g)\,d\tilde{\eta} - \int_0^1 g\left(\int_0^{\tilde{\eta}} f_\eta\,d\tilde{\eta}'\right)d\tilde{\eta}\right]\right], \quad \text{for } m=1$$

Velocity Profile, f_η

One of the integral momentum equations can be considered on an equation for thickness parameter, η_e. Thus, introduce a second parameter, a_2 in the velocity profile governed by the second integral momentum equation*, and take a fourth degree polynomial given by:

$$f_\eta = (2\tilde{\eta} - 2\tilde{\eta}^3 + \tilde{\eta}^4) - \frac{a_2}{3}(\tilde{\eta} - 3\tilde{\eta}^2 + 3\tilde{\eta}^3 - \tilde{\eta}^4) \qquad 14$$

which satisfies the conditions:

$$\begin{aligned}
&(f_\eta(\bar{\xi},1))_{\tilde{\eta}} = 0\\
&(f_\eta(\bar{\xi},1)_{\tilde{\eta}\tilde{\eta}} = 0\\
&f_\eta(\bar{\xi},0) = 0\\
&f_\eta(\bar{\xi},1) = 1
\end{aligned} \qquad 15$$

*Note that as in Reference 1 the point of view promulgated by Tani (Reference 5) is employed; namely, the compatability conditions which apply at the wall and which could be used to determine a_2 are not employed.

Stagnation Enthalpy Profile, g:

Since all thicknesses are taken equal, the energy integrals provide two equations for two additional $\bar{s}$ dependent parameters. Here, take a fifth degree polynomial given by:

$$g = g_w + (1-g_w)(10\tilde{\eta}^3 - 15\tilde{\eta}^4 + 6\tilde{\eta}^5) + b_1(\bar{s})(\tilde{\eta} - 6\tilde{\eta}^3 + 8\tilde{\eta}^4 - 3\tilde{\eta}^5) + b_2(\bar{s})(\tilde{\eta}^2 - 3\tilde{\eta}^3 + 3\tilde{\eta}^4 - \tilde{\eta}^5) \qquad 16$$

which satisfies the conditions:

$$g(\bar{s},0) = g_w, \ (T_w \text{ given})$$
$$g(\bar{s},1) = 1 \qquad 17$$
$$(g(\bar{s},1))_{\tilde{\eta}} = (g(\bar{s},1))_{\tilde{\eta}\tilde{\eta}} = 0$$

Species, or Element Mass Fractions $\tilde{Y}_i$ or Y_j:

Here, as in the profile for the stagnation enthalpy, two $\bar{s}$-dependent parameters are required. One parameter is the mass fraction at the wall, $Y_i w$. Thus, take a fifth degree polynomial given by:

$$Y_i = Y_{iw} + (Y_{i,e} - Y_{i,w})\left[10\tilde{\eta}^3 - 15\tilde{\eta}^4 + 6\tilde{\eta}^5\right] + C_{i,2}\left[\tilde{\eta}^2 - 3\tilde{\eta}^3 + 3\tilde{\eta}^4 - \tilde{\eta}^5\right] \qquad 18$$

which satisfies the conditions:

$$Y_i(\bar{s},0) = Y_{iw}$$
$$(Y_i(\bar{s},0))_{\tilde{\eta}} = 0^{*}$$
$$(Y_i(\bar{s},1))_{\tilde{\eta}} = (Y_i(\bar{s},1))_{\tilde{\eta}\tilde{\eta}} = 0$$

$$Y_i(\bar{s},1) = Y_{ie}$$

1.) Constant for element mass fractions and for species which are non-reactive in the external stream.
2.) $Y_{ie}(\bar{s})$ for species mass fractions in an external chemically reacting flow.

* Note that no surface reactions are admitted here.

Pressure Interaction:

Assume that the pressure interaction can be adequately described by the linearized supersonic flow theory. In this approximation the local streamwise pressure is linearly related to the gradient of the boundary layer displacement thickness given by:

$$\frac{p_e - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{1}{\sqrt{M_\infty^2 - 1}} \frac{d\delta^*}{d\bar{s}} \frac{d\bar{s}}{dx} \qquad 20$$

where, $$\delta^* = \frac{(2\bar{s})^{1/2}}{\rho_e U_e} \eta_k \int_0^1 \left(\frac{\rho_e}{\rho} - f_\eta \right) d\eta^2 \qquad 21$$

Now, $\frac{d\delta^*}{d\bar{s}}$ may be eliminated from Equation 20 by evaluating the derivative of Equation 21. Note here that δ^* may be written formally as a function of the following variables:

$$\delta^* = \delta^*(\rho_e, U_e, \bar{s}, \eta_e, S_j) \qquad 22$$

where Sj are the $\bar{s}$-dependent parameters appearing in the assumed profiles. Thus:

$$\frac{d\delta^*}{d\bar{s}} = \frac{\partial\delta^*}{\partial\rho_e}\frac{d\rho_e}{d\bar{s}} + \frac{\partial\delta^*}{\partial U_e}\frac{dU_e}{d\bar{s}} + \frac{\partial\delta^*}{\partial\bar{s}} + \frac{\partial\delta^*}{\partial\eta_e}\frac{d\eta_e}{d\bar{s}} + \sum_j \frac{\partial\delta^*}{\partial S_j}\frac{dS_j}{d\bar{s}} \qquad 23$$

In order to evaluate $\frac{d\rho_e}{d\bar{s}}$ and $\frac{dU_e}{d\bar{s}}$ the following governing relations for the external flow must be considered:

Momentum: 24

$$\rho_e U_e \frac{dU_e}{d\bar{s}} = -\frac{dp_e}{d\bar{s}}$$

Species Continuity.

$$\rho_e u_e \frac{dY_{ie}}{d\bar{s}} = \rho_e \dot{W}_{ie} \left(\frac{d\bar{s}}{dx}\right)^{-1} \qquad 25$$

Energy:

$$H_e = \text{CONSTANT} \qquad 26$$

State:

$$P_e = \frac{\rho_e R T_e}{W_e} \qquad 27$$

Equations 24 through 27 together with Equation 23 lead to the following differential equation for $\frac{d\delta^x}{d\bar{s}}$: 28

$$\frac{d\delta^x}{d\bar{s}} = \frac{\rho_e}{P_e}\frac{\partial\delta^x}{\partial\rho_e}\left\{\frac{dP_e}{d\bar{s}}\left[1-\frac{P_e}{\rho_e T_e \sum_i C_{pi,e}Y_{ie}}\right]+\sum\frac{\rho_e \dot{W}_{i,e}}{(\rho_e u_e)^2 u_e}\left[\frac{P_e h_i}{T_e \sum Y_{ie} C_{pi,e}}-\frac{\rho_e R T_e}{W_i}\right]\right\}$$
$$-\frac{1}{\rho_e u_e}\frac{\partial\delta^x}{\partial u_e}\frac{dP_e}{d\bar{s}}+\frac{\partial\delta^x}{\partial\bar{s}}+\frac{\partial\delta^x}{\partial\eta_e}\frac{d\eta_e}{d\bar{s}}+\sum_j\frac{\partial\delta^x}{\partial s_j}\frac{ds_j}{d\bar{s}}$$

where h_e is given by:

$$h_e = \sum_i h_{ie} Y_{ie} \qquad 29$$

It is convenient and usually sufficiently accurate to assume:

$$h_{ie} = C_{pi,e} T_e + \Delta_i \qquad 30$$

where Δ_i is a constant associated with the heat of formation of the molecular species and the particular base static chosen for the calculation of thermodynamic quantities. Thus, Equation 28 together with Equation 20 gives a differential equation for the streamwise pressure variation.

Chemical System:

The analysis at this point in its development is general with respect to the chemical system under consideration. It is now convenient to discuss the system of particular interest in this study; namely, hydrogen-

air. It will here be idealized to consist of the following species:

1. OH 2. H_2 3. H_2O 4. N_2 5. O 6. H 7. OH

where the numbers will be used for subscript identification and where it will be noted nitrogen is treated as inert. The elements in this system are oxygen, hydrogen, and nitrogen. Thus, the element mass fractions are given by:

$$\tilde{Y}_i = Y_1 + \tfrac{1}{2}\frac{W_1}{W_3} Y_3 + Y_5 + \tfrac{1}{2}\frac{W_1}{W_7} Y_7 \qquad 31$$

$$\tilde{Y}_2 = Y_2 + \frac{W_2}{W_3} Y_3 + Y_6 + \tfrac{1}{2}\frac{W_2}{W_7} Y_7 \qquad 32$$

$$\tilde{Y}_4 = Y_4 \qquad 33$$

where,

$$\sum_k Y_i = 1 \qquad 34$$

The four equations, 31 through 34, may be considered as four independent relations among the species. Thus, the seven species mass fractions and the three element mass fractions lead to ten minus four, or six unknown species and element mass fractions for which conservation equations must be supplied. The six unknown mass concentrations considered herein are:

$\tilde{Y}_1$ Y_2 $\tilde{Y}_2$ Y_5 Y_6 Y_7

In the method of moments with the profiles as assumed here wall and edge ($\tilde{\eta}=1$) values of these six dependent variables appear in the describing integral equations along with the six $\bar{s}$ dependent coefficients C_i in the profiles. The element mass fractions in any case will be constant in the external stream; for flows wherein the external stream is chemically reactive the values of the four species mass fractions, $Y_{i,e}$, $i=2,5,6,7$, are obtained from their own conservation equations corresponding to inviscid, non-diffusive flow. On the contrary, for flows, which are non-reactive external to the boundary layer, not only $\tilde{Y}_{i,e}$ $i=1,2$ but also $Y_{i,e}$ $i=2,5,6,7$ are constants. Thus the chemistry of the hydrogen-air system according to the present description can lead to either 16 equations or 12 equations depending on whether or not the external stream is reactive or frozen.

The Governing Differential Equations:

A system of ordinary differential equations is produced by substitution of Equations 14, 16 and 18 into the moment equations 10a through 13b. These are supplemented by the governing equations for the external flow in case it is reactive. The general form of these resulting differential equations is given by

$$\sum_j A_{ij} \frac{dQ_j}{d\bar{s}} = F_i \qquad 35$$

where the Q_j's are the dependent variables:

$Q_1=\eta_e$	$Q_5=Y_{5e}$	$Q_9=\tilde{Y}_{2w}$
$Q_2=a_2$	$Q_6=Y_{6e}$	$Q_{10}=Y_{2w}$
$Q_3=U_e$	$Q_7=Y_{7e}$	$Q_{11}=Y_{5w}$
$Q_4=Y_{2e}$	$Q_8=\tilde{Y}_{iw}$	$Q_{12}=Y_{6w}$

$Q_{13} = Y_{7w}$ $\quad$ $Q_{17} = C_{5,2}$ $\quad$ $Q_{21} = b_1$

$Q_{14} = \tilde{C}_{1,2}$ $\quad$ $Q_{18} = C_{6,2}$ $\quad$ $Q_{22} = b_2$

$Q_{15} = \tilde{C}_{2,2}$ $\quad$ $Q_{19} = C_{7,2}$ $\quad$ $Q_{23} = P_e$

$Q_{16} = C_{2,2}$ $\quad$ $Q_{20} = g_w$ $\quad$ $Q_{24} = \rho_e$

$Q_{25} = T_e$

The A_{ij}'s and the F_i's are functions of the Q_j's and are listed in Appendix II.

Note on the Equations and Initial Conditions

Before discussing the two cases of present interest there is a general comment regarding the behavior of Equations 35. In this regard, the system of Equations 35 must be singular at $\bar{s} = 0$ and requires the regularity condition to be imposed in the neighborhood of this point if the integration is to initiate there. This is more clearly illustrated by successive elimination of the derivatives of the external flow variables from System 35. The procedure leads to two coupled systems of ordinary differential equations where one system exhibits the variation of the external flow variables. Note that in the case of flow over a flat surface, the pressure interaction is the only driving force on the external flow which otherwise remains unaltered.

The resulting two systems of equations have the following forms:

$$\sum_{\mu} A_{\tau\mu} \frac{dQ_\mu}{d\bar{s}} = F_\tau \qquad 36a$$

where $\mu = \rho_e, P_e, U_e, T_e$ and Y_{ie}, i = 2, 5, 6, and 7. The second system, however, has the form:

$$\bar{s} \sum_{\beta} A_{\gamma\beta} \frac{dQ_\beta}{d\bar{s}} = F_\gamma \qquad 36b$$

where $\beta = Q_j$, $j \neq \mu$. It is observed in Equation 36b that at $\bar{s} = 0$ finite derivatives can exist only if $F_\gamma = 0$. This is the regularity condition which must be imposed if the integration is to initiate at $\bar{s} = 0$. The consequences of this are pertinent to the determination of initial conditions for configurations when integration is to start at $\bar{s} = 0$. Therefore, in the case of the uniform mixture of non-reacting fuel and air flowing over the plate the initial conditions can be determined as follows: Impose the regularity condition $F_\gamma = 0$. This provides a system of algebraic relations for the initial Q_β's in terms of the initial Q_μ's. It is noted, however, that the task of solving the complex algebraic system may be avoided by observing the condition that $F_\gamma = 0$ is the required condition for the existence of a similar flow. Similarity requires that η_e, a_2, b_1, b_2 and the pertinent C_i'2s must be independent of $\bar{s}$, their values depending only on the specified initial flow conditions. Inspection of the F_γ for the present flat plate configuration shows that this requirement is satisfied for a uniform external flow at the edge of the boundary layer as well as constant wall conditions and frozen chemistry in the external flow i.e., $U_e = U_e(0)$, $P_e = P_e(0)$, $\rho_e = \rho_e(0)$, $T_e = T_e(0)$, and $Y_{iw} = Y_{iw}(0)$, $g_w = g_w(0)$, respectively and $\dot{w}_{ie} = 0$ in general. Now, the $C_i 2s(0) = 0$ in virtue of the fact that at $\bar{s} = 0$ the flow is uniform. Thus, the problem reduces to finding the values of η_e, a_1, b_1, and b_e corresponding to the specified initial flow conditions. Therefore, the initial conditions appropriate to Equation 35 can be obtained from the same program used in the integration of this set

of equations. The requirements for similarity discussed above are inserted into Equation 35. The indeterminancy is ~~revived~~ removed by starting the integration at an arbitrarily small $\bar{s} = \bar{s}_i > 0$ with estimates made for η_e, a_2, b_1, and b_2. The equations are integrated until η'_e, a'_2, b'_1, and b'_2 are sensibly zero. There are the initial conditions used to integrate Equation 35 with the similarity requirements removed.

In the case of slot injection, the integration of Equation 35 is to initiate downstream from the point of injection. Indeed, the flow in the vicinity of the slot is not simply a boundary layer flow but consists of a wake like mixing region originating from the trailing edge of the splitter plate and a boundary layer adjacent to the lower surface (6). The present analysis is applicable from the point, $\bar{s} = \bar{s}_i$, where the splitter plate wake and the boundary layer merge. Thus, at $\bar{s} = \bar{s}_i$ the distributions of the flow variables are considered known which supply the initial values of a_2, b_1, b_2 and the C_i,2's by fitting the assumed polynomial distributions. The value of η_e at $\bar{s} = \bar{s}_i$ is calculated through Equation 5d. The incorporation of the extra parameters makes the present analysis superior to the conventional integral method but may not be sufficiently accurate for some problems.

III. DISCUSSION

A preliminary analysis for the two-dimensional, reacting laminar boundary layer with coupled pressure interaction is developed. The pressure is assumed to depend linearly on the slope of the displacement thickness in accordance with linearized supersonic theory.

A method of moments for chemically reacting flows is presented; it affords a means of obtaining any desired degree of accuracy. Increasing the number of moments provides additional free parameters which better describe the local flow behavior within the boundary layer. In the limit as $m \rightarrow \infty$, the method approaches a finite differencing scheme in the transverse direction.

In the present analysis, two moments are used which will provide a better approximation than that of the basic Karman-Pohlhausen method. In order to assess the accuracy of the present two-moment method, additional moments will have to be calculated.

Two applications of current interest are considered; in one a uniform mixture of fuel and oxidizer under conditions wherein no reaction occurs is considered to flow at hypersonic velocity on a flat plate. Because of either the temperature of the plate or aerodynamic heating, combustion within the boundary layer is assumed to occur and to lead to compression of the external stream so that reaction there is initiated. This application will be seen to be related to the use of boundary layers as flame holders. The second pertains to the slot injection of a reactive gas into an oxidizing

supersonic stream, in this case combustion within the boundary layer can lead to flow deflection of the external stream and thus to pressure distributions on the body.

Although the analysis as presented is rather general, the hydrogen-air system has been considered in more detail.

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LIST OF SYMBOLS

A_{ij} coefficient matrix

a_2 = velocity profile parameter

b_1, b_2 = stagnation enthalpy profile parameters

$C = \rho\mu / \rho_e \mu_e$

C_{pi} = energy constant appearing in Equation 38

C_i, 2 = species profile parameter

F_i = i^{th} forcing function

f = modified stream function defined by Equation 5a

$g = \frac{H}{H_e}$ = stagnation enthalpy ratio

h = static enthalpy

m = moment

p = static pressure

Q_j j^{th} dependent variable

R = universal gas constant

S = transformed streamwise coordinate defined by Equation 5c

T = static temperature

u = streamwise velocity component

v = normal velocity component

w_i = molecular weight of the ith component

$\dot{w}_i$ = production rate of the ith component

x = physical streamwise coordinate

y = physical normal coordinate

Y_i = i^{th} species mass fraction

$\tilde{Y}_i$ = i^{th} element mass fraction

Greek

Δ_i = energy constant appearing in Equation 38

δ^* = boundary layer displacement thickness

η = transformed normal coordinate

μ = absolute viscosity

ρ = density

ψ = stream function defined by Equation 5b

Subscripts

e = external

i = i^{th} chemical component

$\bar{s}$ = derivative with respect to $\bar{s}$

η = derivative with respect to η

∞ = undisturbed free stream

Superscript

prime = derivative with respect to $\bar{s}$

APPENDIX I

The general set of governing equations for m>1 are given below where the governing set for m=0 and 1 are given by Equations 10a through 13b in the text.

Momentum: (m> 1)

$$2\bar{\xi}\left\{\int_0^1\left[(m+1)\tilde{\eta}^m f_\eta^2 - f_\eta\right]d\tilde{\eta}\,\frac{d\eta_e}{d\bar{\xi}} + \eta_e\frac{d}{d\bar{\xi}}\int_0^1\left[\tilde{\eta}^m f_\eta^2 - f_\eta\right]d\tilde{\eta} + m\eta_e\int_0^1\tilde{\eta}^{m-1} f_\eta\left(\int_0^{\eta^2} f_{\eta\bar{\xi}}\,d\tilde{\eta}'\right)d\tilde{\eta}\right\} = \eta_e\int_0^1\left[f_\eta - f_\eta^2\tilde{\eta}^m\right]d\tilde{\eta} -$$

$$-\frac{m}{\eta_e}\left[1-(m-1)\int_0^1 f_\eta\tilde{\eta}^{m-2}d\tilde{\eta}\right] +$$

$$+\eta_e\frac{2\bar{\xi}}{u_e}\frac{du_e}{d\bar{\xi}}\int_0^1\left[\frac{\rho_e}{\rho} - f_\eta^2\right]\tilde{\eta}^m d\tilde{\eta} -$$

$$-m\eta_e\int_0^1 f_\eta\tilde{\eta}^{m-1}\left(\int_0^{\tilde{\eta}} f_\eta\,d\tilde{\eta}'\right)d\eta^2 \qquad \text{I}$$

Species: (m > 1)

$$2\bar{\xi}\left\{\eta_e\left[Y_{ie}\frac{d}{d\bar{\xi}}\int_0^1 f_\eta\, d\tilde{\eta} - \frac{d}{d\bar{\xi}}\int_0^1 \tilde{\eta}^m f_\eta Y_i\, d\tilde{\eta}\right] + \right.$$

$$+\left[Y_{ie}\int_0^1 f_\eta\, d\tilde{\eta} - (m+1)\int_0^1 \tilde{\eta}^m f_\eta Y_i\, d\tilde{\eta}\right]\frac{d\eta_e}{d\bar{\xi}} -$$

$$\left. - m\eta_e\int_0^1 Y_i\tilde{\eta}^{m-1}\left(\int_0^{\tilde{\eta}} f_{\bar{\xi}\eta}\, d\tilde{\eta}\right)\right\} + \eta_e\left[Y_{ie}\int_0^1 f_\eta\, d\tilde{\eta} - \int_0^1 \tilde{\eta}^m Y_i f_\eta\, d\tilde{\eta}\right] -$$

$$- m\eta_e\int_0^1 Y_i\tilde{\eta}^{m-1}\left(\int_0^{\tilde{\eta}} f_\eta\, d\tilde{\eta}'\right) d\tilde{\eta} - \frac{m}{\eta_e}\left[Y_{ie} - (m-1)\int_0^1 Y_i\tilde{\eta}^{m-2}\, d\tilde{\eta}\right] =$$

$$= -2\bar{\xi}\eta_e\int_0^1 \frac{W_i\tilde{\eta}^m}{\rho_e u_e^2 \mu_e}\, d\tilde{\eta} \qquad \text{II}$$

Elements: (m > 1)

The conservation of element mass fractions is given by Equation II with:

$$\dot{W}_i = 0$$
$$Y_i = \tilde{Y}_i \qquad \text{III}$$
$$\text{and } Y_{ie} = \text{Const.}$$

Energy: ($m > 1$)

$$2\bar{s}\left\{m\eta_e\int_0^1 g\tilde{\eta}^{m-1}\left(\int_0^{\tilde{\eta}} f_{\bar{s}\eta}\,d\tilde{\eta}'\right)d\tilde{\eta}+\eta_e\frac{d}{d\bar{s}}\int_0^1 f_\eta\left(g\tilde{\eta}^m-1\right)d\tilde{\eta}+\right.$$

$$\left.+\left(\frac{d\eta_e}{d\bar{s}}\right)\left[(m+1)\int_0^1 g f_\eta\tilde{\eta}^m d\tilde{\eta}-\int_0^1 f_\eta\,d\tilde{\eta}\right]\right\}=-\frac{m}{\eta_e}\left[1-(m-1)\int_0^1 g\tilde{\eta}^{m-2}d\tilde{\eta}\right]+$$

$$+\eta_e\left[\int_0^1\left(f_\eta-\tilde{\eta}^m g\right)d\tilde{\eta}-m\int_0^1 g\tilde{\eta}^{m-1}\left(\int_0^{\tilde{\eta}} f_\eta\,d\tilde{\eta}'\right)d\tilde{\eta}\right]$$

IV

APPENDIX II

The boundary layer momentum equation leads to the following coefficients for m= 0 and 1, respectively

m = 0

$$A_{11} = 2\bar{\xi}\left[\frac{a_2^2}{2268} - \frac{2a_2}{945} - \frac{37}{315}\right]$$

$$A_{12} = 4\bar{\xi}\eta_e\left[\frac{a_2}{2268} - \frac{1}{945}\right]$$

$$A_{13} = 2\bar{\xi}\frac{\eta_e}{u_e}\left[\left(\frac{a_2^2}{2268} - \frac{71a_2}{3780} \quad \frac{367}{630}\right) - \int_0^1 \frac{\rho_e}{\rho}\, d\tilde{\eta}\right]$$

$$A_{14} = A_{1,25} = 0$$

$$F_1 = \frac{1}{3\eta_e}\left[a_2 - 6\right] + \eta_e\left[\frac{37}{315} + \frac{2\,a_2}{945} - \frac{1}{2268}a_2^2\right]$$

m = 1

$$A_{21} = 2\bar{\xi}\left[\frac{41\,a_2^2}{151200} - \frac{101\,a_2}{37800} + \frac{773}{12600}\right]$$

$$A_{22} = 2\bar{\xi}\eta_e\left[\frac{61\,a_2}{151200} + \frac{1157}{241200}\right]$$

$$A_{23} = \frac{2\bar{\xi}\eta_e}{u_e}\left[\frac{a_2^2}{7560} - \frac{29}{3780}a_2 + \frac{124}{315} - \int_0^1 \frac{\rho_e}{\rho}\,\tilde{\eta}\, d\tilde{\eta}\right]$$

$$A_{24} = a_2, 25 = 0$$

$$F_2 = \eta_e\left[\frac{773}{12600} + \frac{101}{37800}a_2 - \frac{41}{151200}a_2^2\right] - \frac{1}{\eta_e}$$

The boundary layer species conservation equations give the following coefficients for m=0 and 1 respectively.

m = 0

$$A_{31} = 2\bar{\xi}\left[Y_{2e}\left(\frac{17}{70} - \frac{31}{2520}a_2\right) - Y_{2w}\left(\frac{17}{70} - \frac{31}{2520}a_2\right) - C_{2,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right)\right]$$

$$A_{32} = 2\bar{\xi}\eta_e\left[\frac{31}{2520}\left(Y_{2e} - Y_{2w}\right) - \frac{19}{7560}C_{2,2}\right]$$

$$A_{33} = 0$$

$$A_{34} = 2\bar{\xi}\eta_e\left[-\frac{16}{35} - \frac{11}{2520}a_2\right]$$

$$A_{35} = A_{39} = 0$$

$$A_{3,10} = 2\bar{\xi}\eta_e\left[-\frac{17}{70} - \frac{31}{2520}a_2\right]$$

$$A_{3,11} = A_{3,15} = 0$$

$$A_{3,16} = 2\bar{\xi}\eta_e\left[\frac{79}{1260} - \frac{19}{7560}a_2\right]$$

$$A_{3,17} = A_{3,25} = 0$$

$$F_3 = \eta_e\left[C_{2,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right) - \left(Y_{2e} - Y_{2w}\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right)\right] - 2\bar{\xi}\eta_e\int_0^1 \frac{\dot{W}_2}{\rho_e U_e^2 \Lambda_e}\,d\tilde{\eta}$$

$$A_{41} = 2\bar{\xi}\left[\left(Y_{5e} - Y_{5w}\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right) - C_{5,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right)\right]$$

$$A_{42} = 2\bar{\xi}\eta_e\left[\frac{31}{2520}\left(Y_{5e} - Y_{5w}\right) - \frac{19}{7560}C_{5,2}\right]$$

$$A_{43} = A_{44} = 0$$

$$A_{45} = 2\bar{\xi}\eta_e\left[\frac{16}{35} - \frac{11}{2520}a_2\right]$$

$$A_{46} = A_{4,16} = 0$$

$$A_{4,17} = 2\bar{\xi}\eta_e\left[\frac{79}{1260} - \frac{19}{7560}a_2\right]$$

$$A_{4,18} = A_{4,25} = 0$$

$$F_4 = \eta_e\left[C_{5,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right) - \left(Y_{5e} - Y_{5w}\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right)\right] - 2\bar{\xi}\eta_e\int_0^1 \frac{\dot{W}_5}{\rho_e u_e^2 M_e}\,d\tilde{\eta}$$

$$A_{51} = 2\bar{\xi}\left[\left(Y_{6e} - Y_{6w}\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right) - C_{6,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right)\right]$$

$$A_{52} = 2\bar{\xi}\eta_e\left[\frac{31}{2520}\left(Y_{6e} - Y_{6w}\right) - \frac{19}{7560}C_{6,2}\right]$$

$$A_{53} = A_{54} = A_{55} = 0$$

$$A_{56} = 2\bar{\xi}\eta_e\left[\frac{16}{35} - \frac{11}{2520}a_2\right]$$

$$A_{57} = A_{5,11} = 0$$

$$A_{5,12} = 2\bar{\xi}\eta_e\left[\frac{17}{70} - \frac{31}{2520}a_2\right]$$

$$A_{5,13} = A_{5,17} = 0$$

$$A_{5,18} = 2\bar{\xi}\eta_e\left[\frac{79}{1260} - \frac{19}{7560}a_2\right]$$

$$A_{5,19} = A_{5,25} = 0$$

$$F_5 = \eta_e C_{6,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right)\left(Y_{6e} - Y_{6w}\right)\left(\frac{17}{70} - \frac{31}{2520}a_2 - 2\bar{\xi}\eta_e\right)\int_0^1 \frac{\dot{W}_{6}}{\rho_e l_e u_e^2 \mu_e}\, d\tilde{\eta}$$

$$A_{61} = 2\bar{\zeta}\left[\left(Y_{7e} - Y_{7w}\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right) - C_{7,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right)\right]$$

$$A_{62} = 2\bar{\zeta}\eta_e\left[\frac{31}{2520}\left(Y_{7e} - Y_{7w}\right) - \frac{19}{7560}C_{7,2}\right]$$

$$A_{63} = A_{66} = 0$$

$$A_{67} = 2\bar{\zeta}\eta_e\left[\frac{16}{35} - \frac{11}{2520}a_2\right]$$

$$A_{68} = A_{6,12} = 0$$

$$A_{6,13} = 2\bar{\zeta}\eta_e\left[\frac{17}{70} - \frac{31}{2520}a_2\right]$$

$$A_{6,14} = A_{6,18} = 0$$

$$A_{6,19} = 2\bar{\zeta}\eta_e\left[\frac{79}{1260} - \frac{19}{7560}a_2\right]$$

$$A_{6,20} = A_{6,25} = 0$$

$$F_6 = \eta_e\left[C_{7,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right) - \left(Y_{7e} - Y_{7w}\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right) - 2\bar{\zeta}\eta_e\right]\int_0^1 \frac{\dot{W}_7}{\rho_e l_e^2 \mu_e}\, d\tilde{\eta}$$

$$A_{71} = 2\bar{\eta}\left[(\tilde{Y}_{1e} - \tilde{Y}_{1w})\left(\frac{17}{70} - \frac{31}{2520}a_2\right) - \tilde{C}_{1,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right)\right]$$

$$A_{72} = 2\bar{\eta}\eta_e\left[\frac{31}{2520}(\tilde{Y}_{1e} - Y_{1w}) - \frac{19}{7560}\tilde{C}_{1,2}\right]$$

$$A_{73} = A_{77} = 0$$

$$A_{78} = 2\bar{\eta}\eta_e\left[\frac{17}{70} - \frac{31}{2520}a_2\right]$$

$$A_{79} = A_{7,13} = 0$$

$$A_{7,14} = 2\bar{\eta}\eta_e\left[\frac{79}{1260} - \frac{19}{7560}a_2\right.$$

$$A_{7,15} = A_{7,25} = 0$$

$$F_7 = \eta_e\left[\tilde{C}_{12}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right) - (\tilde{Y}_{1e} - \tilde{Y}_{1w})\left(\frac{17}{70} - \frac{31}{2520}a_2\right)\right]$$

$$A_{81} = 2\bar{\xi}\left[\left(\tilde{Y}_{2e} - \tilde{Y}_{2w}\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right) - \tilde{C}_{2,2}\left(\frac{79}{1260} - \frac{19}{7560}a_2\right)\right]$$

$$A_{82} = 2\bar{\xi}\eta_e\left[\frac{31}{2520}\left(\tilde{Y}_{2e} - \tilde{Y}_{2w}\right) - \frac{19}{7560}\tilde{C}_{2,2}\right]$$

$$A_{83} = A_{88} = 0$$

$$A_{89} = 2\bar{\xi}\eta_e\left[\frac{17}{70} - \frac{31}{2520}a_2\right]$$

$$A_{8,10} = A_{8,14} = 0$$

$$A_{8,15} = 2\bar{\xi}\eta_e\left[\frac{79}{1260} - \frac{19}{7560}a_2\right]$$

$$A_{8,16} = A_{8,25} = 0$$

$$F_8 = \eta_e\left[\tilde{C}_{2,2}\left(\frac{79}{1260}\ \frac{19}{7560}a_2\right) - \left(\tilde{Y}_{2e} - \tilde{Y}_{2w}\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right)\right]$$

In the following coefficients $A_{9,\ j}$ through $A_{14,\ j}$ the numerical coefficients are given by:

(a) = $\frac{2862783}{19209960}$

(b) = $\frac{10287}{1496880}$

(c) = $\frac{350889}{8004150}$

(d) = $\frac{98208}{54885600}$

(e) = $\frac{12220956}{1778293440}$

(f) = $\frac{98208}{5488560}$

(g) = $\frac{35085}{103950}$

(h) = $\frac{1629}{748440}$

(i) = $\frac{664}{3465}$

(j) = $\frac{281}{41580}$

(k) = $\frac{389}{13860}$

(l) = $\frac{23}{27720}$

(m) = $\frac{350889}{54885600}$

(n) = $\frac{98208}{54885600}$

(o) = $\frac{2862783}{19209960}$

(p) = $\frac{10287}{1496880}$

$$A_{91} = 2\eta_e \bar{s}\left[\left(Y_{2e} - Y_{2w}\right)\left((a) - (b)a_2\right) - C_{2,2}\left((c) - (d)a_2\right)\right]$$

$$A_{92} = 2\eta_e^2 \bar{s}\left[\left(Y_{2e} - Y_{2w}\right)\left(-(e) + (f)C_{2,2}\right)\right]$$

$$A_{93} = 0$$

$$A_{94} = 2\eta_e^2 \bar{s}\left[(g) - (h)a_2\right]$$

$$A_{95} = A_{99} = 0$$

$$A_{9,10} = \eta_e^2 \bar{s}\left[(i) - (j)a_2\right]$$

$$A_{9,11} = A_{9,15} = 0$$

$$A_{9,16} = 2\eta_e^2 \bar{s}\left[(k) - (l)a_2\right]$$

$$A_{9,17} = A_{9,25} = 0$$

$$F_9 = \eta_e^2 C_{2,2}\left[(m) - (n)a_2\right] + \left(Y_{2e} - Y_{2w}\right)\left[1 - \eta_e^2\left((o) - (p)\right)\right]$$
$$- 2\bar{s}\eta_e^2 \int_0^1 \frac{W_2\tilde{\eta}}{\xi_e \Pi_e \Lambda_e}\, d\tilde{\eta}$$

$$A_{10,1} = 2\eta_e \bar{\xi}\left[\left(Y_{5e} - Y_{5w}\right)(a) - (b)a_2\right] - C_{5,2}\left[(c) - (d)a_2\right]$$

$$A_{10,2} = 2\eta_e^2 \bar{\xi}\left[\left(Y_{5e} - Y_{5w}\right)(e) + (f)C_{5,2}\right]$$

$$A_{10,3} = A_{10,4} = 0$$

$$A_{10,5} = -2\eta_e^2 \bar{\xi}\left[(g) - (h)a_2\right]$$

$$A_{10,6} = A_{10,10} = 0$$

$$A_{10,11} = \eta_e^2 \bar{\xi}\left[\frac{664}{3465} - \frac{281}{41580}a_2\right]$$

$$A_{10,12} = A_{10,16} = 0$$

$$A_{10,17} = 2\bar{\xi}\eta_e^2\left[(i) - \frac{23}{13860}a_2\right]$$

$$A_{10,18} = A_{10,25} = 0$$

$$F_{10} = \eta_e^2 C_{5,2}\left[(k) - (l)a_2\right] + \left(Y_{5e} - Y_{5w}\right)\left[1 - \eta_e^2\left[(m) - (n)a_2\right]\right]$$
$$- 2\bar{\xi}\eta_e^2 \int_0^1 \frac{W_5\tilde{\eta}}{\rho_e u_e \lambda_e} d\tilde{\eta}$$

$$A_{11,1} = 2\eta_e \bar{s}\left[\left(Y_{6e} - Y_{6w}\right)\left((a) - (b)a_2\right) - C_{6,2}\left((c) - (d)a_2\right)\right]$$

$$A_{11,2} = 2\eta_e^2 \bar{s}\left[\left(Y_{6e} - Y_{6w}\right)\left((e) + (f)C_{6,2}\right)\right]$$

$$A_{11,3} = A_{11,5} = 0$$

$$A_{11,6} = 2\eta_e^2 \bar{s}\left[(g) - (h)a_2\right]$$

$$A_{11,7} = A_{11,11} = 0$$

$$A_{11,12} = -\eta_e^2 \bar{s}\left[\frac{6669}{3465} - \frac{281}{41580}a_2\right]$$

$$A_{11,13} = A_{11,17} = 0$$

$$A_{11,18} = 2\bar{s}\eta_e^2\left[(i) - \frac{23}{13860}a_2\right]$$

$$A_{11,19} = A_{11,25} = 0$$

$$F_{11} = \eta_e^2 C_{6,2}\left[(k) - (l)a_2\right] + Y_{6e} - Y_{6w}\left[1 - \eta_e^2\left((m) - (n)a_2\right)\right]$$
$$- 2\bar{s}\eta_e^2 \int_0^1 \frac{\dot{w}_i \tilde{\eta}}{\rho_e U_e \lambda_e} d\tilde{\eta}$$

$$A_{12,1} = 2\eta_e \bar{\xi}\left[\left(Y_{7e} - Y_{7w}\right)\left((a) - (b)a_2\right) - C_{7,2}\left((c) - (d)a_2\right)\right]$$

$$A_{12,2} = 2\eta_e^2 \bar{\xi}\left[\left(Y_{7e} - Y_{7w}\right)(e) + (f)C_{7,2}\right]$$

$$A_{12,3} = A_{12,6} = 0$$

$$A_{12,7} = -2\eta_e^2 \bar{\xi}\left[(g) - (h)a_2\right]$$

$$A_{12,8} = A_{12,12} = 0$$

$$A_{12,13} = -\eta_e^2 \bar{\xi}\left[\frac{664}{3465} - \frac{281}{41580}a_2\right]$$

$$A_{12,14} = A_{12,18} = 0$$

$$A_{12,19} = 2\bar{\xi}\eta_e^2\left[(i) - \frac{23}{13860}a_2\right]$$

$$A_{12,20} = A_{12,25} = 0$$

$$F_{12} = \eta_e^2 C_{7,2}\left[(k) - (l)a_2\right] + \left(Y_{7e} - Y_{7w}\right)\left[1 - \eta_e^2\left((m) - (n)a_2\right)\right]$$
$$- 2\bar{\xi}\eta_e^2 \int_0^1 \frac{W_7 \tilde{\eta}}{\rho_e U_e \mu_e} d\tilde{\eta}$$

$$A_{13,1} = 2\eta_e \bar{3}\left[\left(\tilde{\Psi}_{1e} - \tilde{\Psi}_{1w}\right)(a) - (b)a_2 - \tilde{C}_{1,2}\left((c) - (d)a_2\right)\right]$$

$$A_{13,2} = 2\eta_e^2 \bar{3}\left[\left(\tilde{\Psi}_{1e} - \tilde{\Psi}_{1w}\right)(e) + (f)\tilde{C}_{1,2}\right]$$

$$A_{13,3} = A_{13,7} = 0$$

$$A_{13,8} = -\eta_e^2 \bar{3}\left[\frac{6664}{3465} - \frac{281}{41580}a_2\right]$$

$$A_{13,9} = A_{13,13} = 0$$

$$A_{13,14} = 2\bar{3}\eta_e^2 (i) - \frac{23}{13860}a_2$$

$$A_{13,15} = A_{13,25} = 0$$

$$F_{13} = \eta_e^2 \tilde{C}_{1,2}\left((k) - (l)a_2\right) + \left(\tilde{\Psi}_{1e} - \tilde{\Psi}_{1w}\right)\left[1 - \eta_e^2 (m) - (n)a_2\right]$$

$$A_{14,1} = 2\eta_e \bar{\xi} \left[\left(\tilde{\Psi}_{2e} - \Psi_{2w} \right) (a) - (b) a_2 - \tilde{C}_{2,2} \left((c) - (d) a_2 \right) \right]$$

$$A_{14,2} = 2\eta_e^2 \bar{\xi} \left[\left(\tilde{\Psi}_{2e} - \Psi_{2w} \right) (e) + (f) \tilde{C}_{2,2} \right]$$

$$A_{14,3} = A_{14,8} = 0$$

$$A_{14,9} = -\eta_e^2 \bar{\xi} \left[(g) - (h) a_2 \right]$$

$$A_{14,10} = A_{14,14} = 0$$

$$A_{14,15} = 2\bar{\xi}\eta_e^2 \left[(i) - \frac{23}{13860} a_2 \right]$$

$$A_{14,16} = A_{14,25} = 0$$

$$F_{14} = \eta_e^2 \tilde{C}_{2,2} \left[(k) - (l) a_2 + \left(\tilde{\Psi}_{2e} - \tilde{\Psi}_{2w} \right) \left[1 - \eta_e^2 (m) - (n) a_2 \right] \right]$$

The boundary layer energy equation has the following coefficients for m = 0 and 1, respectively:

m = 0

$$A_{15,1} = 2\bar{\xi}\left[\left(g_w - 1\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right) + b_1\left(\frac{79}{1260} - \frac{19}{7560}a_2\right) + b_2\left(\frac{29}{2520} - \frac{a_2}{2520}\right)\right]$$

$$A_{15,2} = 2\bar{\xi}\eta_e\left[\frac{31}{2520}\left(1 - g_w\right) - \frac{19}{7560}b_1 - \frac{b_2}{2520}\right]$$

$$A_{15,3} = A_{15,9} = 0$$

$$A_{15,20} = 2\bar{\xi}\eta_e\left[\frac{17}{70} - \frac{31}{2520}a_2\right]$$

$$A_{15,21} = 2\bar{\xi}\eta_e\left[\frac{79}{1260} - \frac{19}{7560}a_2\right]$$

$$A_{15,22} = 2\bar{\xi}\eta_e\left[\frac{29}{2520} - \frac{a_2}{2520}\right]$$

$$A_{15,23} = A_{15,25} = 0$$

$$F_{15} = \eta_e\left(1 - g_w\right)\left(\frac{17}{70} - \frac{31}{2520}a_2\right) - \eta_e b_1\left(\frac{79}{1260} - \frac{19}{7560}a_2\right)$$

$$- \eta_e b_2\left(\frac{29}{2520} - \frac{a_2}{2520}\right) - \frac{b_1}{\eta_e}$$

m=1

$$A_{16,1} = 2\bar{\eta}\eta_e^2\left[\frac{1-g_w}{\eta_e} - \left(\frac{2862783}{19209960} + \frac{10287}{1496880}a_2\right) + \frac{b_1}{\eta_e} + \left(\frac{350889}{8004150} - \frac{98208}{54885600}a_2\right)\right.$$
$$\left. + \frac{b_2}{\eta_e}\left(\frac{246092}{28459200} - \left(\frac{53559}{164656800}a_2\right)\right)\right]$$

$$A_{16,2} = 2\bar{\eta}\eta_e^2\left[\left(1-g_w\right)\frac{12220956}{1778293440} - b_1\frac{98208}{54885600} - b_2\frac{53559}{164656800}\right]$$

$$A_{16,3} = A_{16,19} = 0$$

$$A_{16,20} = 2\bar{\eta}\eta_e^2\left[\frac{332}{3465} - \frac{281}{83160}a_2\right]$$

$$A_{16,21} = 2\bar{\eta}\eta_e^2\left[\frac{389}{13860} - \frac{23}{27720}a_2\right]$$

$$A_{16,22} = 2\bar{\eta}\eta_e^2\left[\frac{17}{3080} - \frac{a_2}{6930}\right]$$

$$A_{16,23} = A_{16,23} = 0$$

$$F_{16} = \left(g_w - 1\right) + \eta_2^e\left[\left(1-g_w\right)\frac{2862783}{19209960} - \frac{10287}{1496880}a_2 - \right.$$
$$-b_1\left(\frac{350889}{8004150} - \frac{98208}{54885600}a_2\right)$$
$$\left. -b_2\left(\frac{246092}{28459200} - \frac{53559}{164656800}a_2\right)\right]$$

The reacting external flow field is coupled to the boundary layer flow through the pressure interaction. The coefficients of the derivatives of the governing equations for the external flow are given below:

Momentum Equation

$$A_{17,1} = A_{17,2} = 0$$

$$A_{17,3} = \rho_e U_e$$

$$A_{17,4} = A_{17,22} = 0$$

$$A_{17,23} = 1$$

$$A_{17,24} = A_{17,25} = 0$$

$$F_{17} = 0$$

Species Continuity

$A_{18,1} = A_{18,3} = 0$

$A_{18,4} = \rho_e u_e$

$A_{18,5} = A_{18,25} = 0$

$F_{18} = \rho_e \dot{w}_{2e} \left| \frac{d\xi}{dx} \right|^{-1}$

$A_{19,1} = A_{19,4} = 0$

$A_{19,5} = \rho_e u_e$

$A_{19,6} = A_{19,25} = 0$

$F_{19} = \rho_e \dot{w}_{5e} \left| \frac{d\xi}{dx} \right|^{-1}$

$A_{20,1} = A_{20,5} = 0$

$A_{20,6} = \rho_e u_e$

$A_{20,7} = A_{20,25} = 0$

$F_{20} = \rho_e \dot{w}_{6e} \left| \frac{d\xi}{dx} \right|^{-1}$

$A_{21,1} = A_{21,6} = 0$

$A_{21,7} = \rho_e U_e$

$A_{21,8} = A_{21,25} = 0$

$F_{21} = \rho_e \dot{W}_{7e} \frac{d\bar{s}}{dx}^{-1}$

Energy Equation

$A_{22,1} = A_{22,2} = 0$

$A_{23,3} = U_e$

$A_{22,4} = h_{2e} + \frac{1}{2} \frac{W_1}{W_2} h_{1e} - \frac{W_3}{W_2} h_{3e}$

$A_{22,5} = h_{5e} - h_{1e}$

$A_{22,6} = h_{6e} + \frac{1}{2} \frac{W_1}{W_2} h_{1e} - \frac{W_3}{W_2} h_{3e}$

$A_{22,7} = h_{7e} - \frac{1}{2} \frac{W_3}{W_7} h_{3e}$

$A_{22,8} = A_{22,24} = 0$

$A_{22,25} = Y_{2e} h_{2e} + Y_{5e} h_{5e} + Y_{6e} h_{6e} + Y_{7e} h_{7e} + Y_{1e} h_{1e} + Y_{3e} h_{3e} + Y_{4e} h_{4e}$

$F_{22} = 0$

State

$$A_{23,1} = A_{23,3} = 0$$

$$A_{23,4} = \frac{1}{2} \frac{\rho_e W_e}{W_2}$$

$$A_{23,5} = \frac{\rho_e W_e}{W_1}$$

$$A_{23,6} = \frac{3}{2} \frac{\rho_e W_c}{W_2}$$

$$A_{23,7} = \frac{1}{2} \frac{\rho_e W_e}{W_7}$$

$$A_{23,8} = A_{23,22} = 0$$

$$A_{23,23} = \frac{\rho_e}{P_e}$$

$$A_{23,24} = 1$$

$$A_{23,25} = \frac{\rho_e}{T_e}$$

$$F_{23} = 0$$

The relation expressing the pressure dependence on the displacement thickness is given by:

$$A_{24,i} = \bar{\zeta}\,\frac{\partial \rho^{x}}{\partial Q_i}, \quad i = 23 \text{ or } 25$$

$$A_{24,23} = A_{24,25} = 0$$

$$F_{24} = \bar{\zeta}\,\frac{P_e - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}\sqrt{M_\infty^2 - 1}\;\frac{d\bar{\zeta}}{dx}^{-1} - \bar{\zeta}\,\frac{\partial \rho^{x}}{\partial \bar{\zeta}}$$

Wall Stagnation Enthalpy

$$A_{25,1} = A_{25,7} = 0$$

$$A_{25,8} = \frac{1}{He}\left[\Delta_1 + C_{p1}T_w\right]$$

$$A_{25,9} = \frac{1}{He}\left[\frac{W_3}{W_2}\Delta_3 + C_{p3}T_w - \frac{1}{2}\frac{W_3}{W_2}\Delta_3 + C_{p3}T_w\right]$$

$$A_{25,10} = \frac{1}{He}\left[\frac{1}{2}\frac{W_1}{W_2}\right]$$

$$A_{25,11} = \frac{1}{He}\left[\Delta_5 + C_{p5}T_w - \Delta_1 + C_{p1}T_w\right]$$

$$A_{25,12} = \frac{1}{He}\left[\Delta_1 + C_{p1}T_w\,\frac{1}{2}\frac{W_1}{W_2} - \Delta_3 + C_{p3}T_w\,\frac{W_3}{W_2} + \Delta_6 + C_{p6}T_w\right]$$

$$A_{25,13} = \frac{1}{He}\left[\Delta_7 + C_{p7}T_w - \frac{1}{2}\frac{W_3}{W_7}\Delta_3 + C_{p3}T_w\right]$$

$$A_{25,14} = A_{25,19} = 0$$

$A_{25,20} = -1$

$A_{25,21} = A_{25,25} = 0$

$F_{25} = 0$

Note that the wall stagnation enthalpy is given by:

$$g_w = \frac{h_w}{h_e} \qquad 36$$

where

$$h_{w2} = h_w = \sum_i Y_{iw} h_{iw} \qquad 37$$

and, in general, we assume an analytical expression for h_{iw} given by:

$$h_i = \Delta_i + C_{pi} T \qquad 38$$

where Δ_i and C_{pi} are constants giving the proper behavior of h_i in the existing temperature range.