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DETERMINATION OF MEAN ELEMENTS FOR VINTI'S SATELLITE THEORY

by N. L. Bonavito

Goddard Space Flight Center
Greenbelt, Maryland

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SUMMARY

This report presents the orbit improvement method by which mean or Izsak elements that exactly factor Vinti's two quartic polynominals are determined by comparison with observations. Also included are analytic partial derivatives for the first order Taylor expansion of the conditional equations.

CONTENTS

Summary i

INTRODUCTION 1

STATEMENT OF THE PROBLEM 2

PARTIAL DERIVATIVES 3

REMARKS 52

ACKNOWLEDGMENTS 52

References 52

Appendix A – Computation with Range and Range Rate Data 53

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INTRODUCTION

The classical problem of the determination of an orbit of an asteroid from three observations is a rather difficult one. In determining the orbit of an artificial earth satellite, most of this difficulty can be avoided through modern tracking techniques. These techniques provide a reasonably good set of initial conditions and therefore orbital elements; this gives an orbit. Comparing this orbit with one obtained, for example, by radio methods (which gives a good determination of directions) we can reduce the problem of improving initial orbital elements to a differential one.

Assuming the orbital elements are to be determined by an iterated least-square fitting of the solution to many revolutions in the orbit results in a set of elements introduced by Izsak; namely, a, e, η_0, β_1, β_2, and β_3. This provides the immediate factoring of the quartics $F(\rho)$ and $G(\eta)$ (Reference 1). This is in contrast to the numerical factorization of these quartics to the second order in the oblateness parameter k_0, as shown in the section on prime constants. Also, inaccuracies of the initial observations are taken out, and the effects of forces not considered in the theory are accounted for in producing this mean set of elements.

First partials for the normal equations are derived for use in the method of direction cosines. Since the quantities $\partial E/\partial q_i$, $\partial\psi/\partial q_i$, and $\partial\phi/\partial q_i$ (where q_i for $i = 1, 2, \cdots, 6$, are a, e, η_0, β_1, β_2, and β_3) are completely general, and functions only of the theory, then first partials for use with any method of tracking whatsoever, can be written down immediately. The procedure for obtaining $\partial E/\partial q_i$, $\partial\psi/\partial q_i$, and $\partial\phi/\partial q_i$ is given and applied in the section on the partial derivatives. By inserting coordinates obtained from initial conditions for each specified time of observation (Reference 2) into the normal equations, the results of the iterated least-square procedure will produce the Izsak elements and, consequently, coordinate predictions. Classically, the initial coordinates of the system point have been shifted in p, q phase space, such that the continuous evolution of a canonical transformation with this new value of the Hamiltonian produces a motion of a mechanical system consistent with observation, in this case the satellite orbit.

STATEMENT OF THE PROBLEM

If L_0 and M_0 denote the values of the observed direction cosines of a satellite for a given time of observation, then the corresponding computed values of these functions are

$$L_c = \frac{x_m}{\sqrt{x_m^2 + y_m^2 + z_m^2}},$$

$$M_c = \frac{y_m}{\sqrt{x_m^2 + y_m^2 + z_m^2}}$$

where x_m, y_m, and z_m are the corresponding local coordinates:

$$\begin{pmatrix} x_m \\ y_m \\ z_m \end{pmatrix} = (A_I)^{-1} \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_T \\ y_T \\ z_T \end{pmatrix} \right\}.$$

The values x, y, and z define the satellite's coordinates with respect to an earth-centered, right-hand, orthogonal inertial system; x_T, y_T, and z_T are the coordinates of the observation point; and the matrix A_I rotates the local x_m, y_m, z_m frame parallel to the inertial x, y, z. The functions are then

$$L_c = \frac{\left[(A_I)^{-1} (\xi - T)\right]_1}{\left\{ \left[(A_I)^{-1} (\xi - T)\right]_1^2 + \left[(A_I)^{-1} (\xi - T)\right]_2^2 + \left[(A_I)^{-1} (\xi - T)\right]_3^2 \right\}^{1/2}},$$

$$M_c = \frac{\left[(A_I)^{-1} (\xi - T)\right]_2}{\left\{ \left[(A_I)^{-1} (\xi - T)\right]_1^2 + \left[(A_I)^{-1} (\xi - T)\right]_2^2 + \left[(A_I)^{-1} (\xi - T)\right]_3^2 \right\}^{1/2}},$$

where

$$\xi = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} x_T \\ y_T \\ z_T \end{pmatrix}.$$

The subscripts 1, 2, and 3 denote the rows of the product matric $(A_I)^{-1} (\xi - T)$. Thus, the functions are obtained for a specified time of observation by inserting into the matrix ξ those values of the earth satellite coordinates computed with the Vinti orbit generator (Reference 2).

THE PARTIAL DERIVATIVES

If the following substitutions are made in the ξ matrix:

$$x = \sqrt{(\rho^2 + c^2)(1-\eta^2)} \cos \phi,$$

$$y = \sqrt{(\rho^2 + c^2)(1-\eta^2)} \sin \phi,$$

$$z = \rho\eta,$$

where

$$\rho = a(1 - e \cos E),$$

$$\eta = \eta_0 \sin \psi,$$

then the computed direction cosines L_c and M_c can be expressed as functions of the six parameters a, e, η_0, E, ψ, and ϕ: Here a and e are the semimajor axis and eccentricity and η_0 is related to the orbit inclination by the expression $\eta_0 = \sin I$. The parameters E and ψ are uniformizing variables defined as the eccentric anomaly and a variable analogous to the argument of latitude, respectively. The element ϕ is the geocentric right ascension.

In the expression

$$L_c = \frac{x_m}{\sqrt{x_m^2 + y_m^2 + z_m^2}},$$

x_m, y_m, and z_m are functions of the quantities a, e, η_0, E, ψ, and ϕ, where E, ψ, and ϕ are in turn functions of the elements a, e, η_0, β_1, β_2, and β_3. Since the local coordinates are themselves functions of the computed interial coordinates, if we denote any one of the elements by q_i then

$$\frac{\partial L_c}{\partial q_i} = \frac{\partial L_c}{\partial x_m}\frac{\partial x_m}{\partial q_i} + \frac{\partial L_c}{\partial y_m}\frac{\partial y_m}{\partial q_i} + \frac{\partial L_c}{\partial z_m}\frac{\partial z_m}{\partial q_i},$$

with

$$\frac{\partial L_c}{\partial x_m} = \frac{1}{\sqrt{x_m^2 + y_m^2 + z_m^2}} - \frac{x_m^2}{\left(x_m^2 + y_m^2 + z_m^2\right)^{3/2}} = K_{00},$$

$$\frac{\partial L_c}{\partial y_m} = \frac{-x_m\, y_m}{\left(x_m^2 + y_m^2 + z_m^2\right)^{3/2}} = K_{01},$$

$$\frac{\partial L_c}{\partial z_m} = \frac{-x_m\, z_m}{\left(x_m^2 + y_m^2 + z_m^2\right)^{3/2}} = K_{02}.$$

Since

$$\begin{pmatrix} x_m \\ y_m \\ z_m \end{pmatrix} = (A_I)^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - (A_I)^{-1} \begin{pmatrix} x_T \\ y_T \\ z_T \end{pmatrix},$$

then

$$\begin{pmatrix} \frac{\partial x_m}{\partial q_i} \\ \frac{\partial y_m}{\partial q_i} \\ \frac{\partial z_m}{\partial q_i} \end{pmatrix} = (A_I)^{-1} \begin{pmatrix} \frac{\partial x}{\partial q_i} \\ \frac{\partial y}{\partial q_i} \\ \frac{\partial z}{\partial q_i} \end{pmatrix},$$

since the x_T, y_T, and z_T are independent of q_i. Substituting $\rho = a(1 - e \cos E)$ and $\eta = \eta_0 \sin \psi$ into

$$x = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \cos \phi,$$

$$y = \sqrt{(\rho^2 + c^2)(1 - \eta^2)} \sin \phi,$$

gives

$$x = \sqrt{\left[a^2 (1 - e \cos E)^2 + c^2\right]} \left(1 - \eta_0^2 \sin^2 \psi\right) \cos \phi,$$

$$y = \sqrt{\left[a^2(1-e\cos E)^2 + c^2\right]\left(1-\eta_0^2\sin^2\psi\right)}\,\sin\phi,$$

$$z = a(1-e\cos E)\,\eta_0\sin\psi.$$

Differentiating x, y, and z with respect to a, e, η_0, β_1, β_2, and β_3 and placing this in the partial matrix yields $\partial x_m/\partial q_i$, $\partial y/\partial q_i$, and $\partial z_m/\partial q_i$.

The following procedure is used to determine $\partial E/\partial q_i$, $\partial\psi/\partial q_i$, and $\partial\phi/\partial q_i$, where $q_i = a, e, \eta_0, \beta_1, \beta_2$, or β_3 :

With the expressions on page 199 of Reference 1, we compute:

1. $\dfrac{\partial M_s}{\partial q_i}$ and $\dfrac{\partial \psi_s}{\partial q_i}$,

2. $\dfrac{\partial E_0}{\partial q_i}$ from the Kepler equation,

3. $\dfrac{\partial v_0}{\partial q_i}$, by using $v = M_s + v_0$, $E = M_s + E_0 = \mathcal{E}$, and the anomaly connections,

4. $\dfrac{\partial \psi_0}{\partial q_i}$,

5. $\dfrac{\partial M_1}{\partial q_i}$,

6. $\dfrac{\partial E_1}{\partial q_i}$,

7. $\dfrac{\partial v_1}{\partial q_i}$, by using $v = M_s + v_0 + v_1$, $E = M_s + E_0 + E_1 = \mathcal{E} + E_1$, and the anomaly connections,

8. $\dfrac{\partial \psi_1}{\partial q_i}$,

9. $\dfrac{\partial M_2}{\partial q_i}$,

10. $\dfrac{\partial E_2}{\partial q_i}$, which gives $\dfrac{\partial E}{\partial q_i} = \dfrac{\partial \mathcal{E}}{\partial q_i} + \dfrac{\partial E_1}{\partial q_1} + \dfrac{\partial E_2}{\partial q_1}$,

11. $\frac{\partial v_2}{\partial q_i}$, by using $v = M_s + v_0 + v_1 + v_2$, $E = M_s + E_0 + E_1 + E_2 = \mathcal{E} + E_1 + E_2$, and the anomaly connections which gives $\frac{\partial v_2}{\partial q_i} = \frac{\partial M_s}{\partial q_i} + \frac{\partial v_0}{\partial q_i} + \frac{\partial v_1}{\partial q_i} + \frac{\partial v_2}{\partial q_i}$,

12. $\frac{\partial \psi_2}{\partial q_i}$ which gives $\frac{\partial \psi}{\partial q_i} = \frac{\partial \psi_s}{\partial q_i} + \frac{\partial \psi_0}{\partial q_i} + \frac{\partial \psi_1}{\partial q_i} + \frac{\partial \psi_2}{\partial q_i}$,

13. $\frac{\partial \chi}{\partial q_i}$,

14. $\frac{\partial \phi}{\partial q_i}$.

If everything is now substituted back into the equations for $\frac{\partial L_c}{\partial q_i}$,

$$\frac{\partial L_c}{\partial a} = K_{00} K_{20} + K_{01} K_{21} + K_{02} K_{22},$$

$$\frac{\partial L_c}{\partial e} = K_{00} K_{23} + K_{01} K_{24} + K_{02} K_{25},$$

$$\frac{\partial L_c}{\partial \eta_0} = K_{00} K_{26} + K_{01} K_{27} + K_{02} K_{28},$$

$$\frac{\partial L_c}{\partial \beta_1} = K_{00} K_{29} + K_{01} K_{30} + K_{02} K_{31},$$

$$\frac{\partial L_c}{\partial \beta_2} = K_{00} K_{32} + K_{01} K_{33} + K_{02} K_{34},$$

$$\frac{\partial L_c}{\partial \beta_3} = K_{00} K_{35} + K_{01} K_{36} + K_{02} K_{37}.$$

In computing these terms, in addition to the use of the mutual constants (Reference 2) the following expressions are utilized:

$$p = a(1 - e^2),$$

$$D = (ap - c^2)\left(ap - c^2\eta_0^2\right) + 4a^2c^2\eta_0^2,$$

$$D' = D + 4a^2c^2\left(1 - \eta_0^2\right),$$

$$A = -2ac^2D^{-1}\left(1 - \eta_0^2\right)\left(ap - c^2\eta_0^2\right),$$

$$B = c^2\eta_0^2D^{-1}D',$$

$$b_1 = -\frac{1}{2}A,$$

$$b_2 = B^{1/2},$$

$$\alpha_1 = -\frac{\mu}{2}(a + b_1)^{-1},$$

$$\alpha_2 = \left[\mu\left(a + b_1\right)^{-1}\right]^{1/2}\left[apD'D^{-1} - c^2\left(1 - \eta_0^2\right)\right]^{1/2},$$

$$\alpha_3 = \alpha_2\left\{1 - \frac{c^2\eta_0^2}{\left[apD'D^{-1} - c^2\left(1 - \eta_0^2\right)\right]}\right\}^{1/2}\left(1 - \eta_0^2\right)^{1/2},$$

$$\eta_2^{-2} = \frac{c^2D}{apD'},$$

$$q = \frac{\eta_0}{\eta_2},$$

$$K = \frac{c^2}{p^2},$$

where

$$K_{20} = \frac{\partial x}{\partial a}\cos\psi_A + \frac{\partial y}{\partial a}\sin\psi_A,$$

$$K_{21} = -\frac{\partial x}{\partial a}\sin\psi_A \sin\theta_D + \frac{\partial y}{\partial a}\cos\psi_A \sin\theta_D + \frac{\partial z}{\partial a}\cos\theta_D,$$

$$K_{22} = \frac{\partial x}{\partial a}\sin\psi_A \cos\theta_D - \frac{\partial y}{\partial a}\cos\psi_A \cos\theta_D + \frac{\partial z}{\partial a}\sin\theta_D,$$

$$K_{23} = \frac{\partial x}{\partial e}\cos\psi_A + \frac{\partial y}{\partial e}\sin\psi_A,$$

$$K_{24} = -\frac{\partial x}{\partial e}\sin\psi_A \sin\theta_D + \frac{\partial y}{\partial e}\cos\psi_A \sin\theta_D + \frac{\partial z}{\partial e}\cos\theta_D,$$

$$K_{25} = \frac{\partial x}{\partial e}\sin\psi_A \cos\theta_D - \frac{\partial y}{\partial e}\cos\psi_A \cos\theta_D + \frac{\partial z}{\partial e}\sin\theta_D,$$

$$K_{26} = \frac{\partial x}{\partial \eta_0}\cos\psi_A + \frac{\partial y}{\partial \eta_0}\sin\psi_A,$$

$$K_{27} = -\frac{\partial x}{\partial \eta_0}\sin\psi_A \sin\theta_D + \frac{\partial y}{\partial \eta_0}\cos\psi_A \sin\theta_D + \frac{\partial z}{\partial \eta_0}\cos\theta_D,$$

$$K_{28} = \frac{\partial x}{\partial \eta_0}\sin\psi_A \cos\theta_D - \frac{\partial y}{\partial \eta_0}\cos\psi_A \cos\theta_D + \frac{\partial z}{\partial \eta_0}\sin\theta_D.$$

In these equations

$$\frac{\partial x}{\partial a} = \frac{x}{\rho^2 + c^2}\, a(1 - e\cos E)\left(1 - e\cos E + a\, e\sin E\frac{\partial E}{\partial a}\right) - \frac{x\eta_0^2 \sin\psi\cos\psi}{1 - \eta_0^2\sin^2\psi}\frac{\partial\psi}{\partial a} - y\frac{\partial\phi}{\partial a},$$

$$\frac{\partial x}{\partial e} = \frac{x}{\rho^2 + c^2}\, a^2(1 - e\cos E)\left(e\sin E\frac{\partial E}{\partial e} - \cos E\right) - \frac{x\eta_0^2 \sin\psi\cos\psi}{1 - \eta_0^2\sin^2\psi}\frac{\partial\psi}{\partial e} - y\frac{\partial\phi}{\partial e},$$

$$\frac{\partial x}{\partial \eta_0} = \frac{x}{\rho^2 + c^2}\, a^2 e\sin E\,(1 - e\cos E)\frac{\partial E}{\partial \eta_0} - y\frac{\partial\phi}{\partial \eta_0} - \frac{x\eta_0^2 \sin\psi\cos\psi}{1 - \eta_0^2\sin^2\psi}\frac{\partial\psi}{\partial \eta_0} - \frac{x\eta_0\sin^2\psi}{1 - \eta_0^2\sin^2\psi},$$

$$\frac{\partial y}{\partial a} = \frac{y}{\rho^2 + c^2}\, a(1 - e\cos E)\left(1 - e\cos E + a\, e\sin E\frac{\partial E}{\partial a}\right) - \frac{y\eta_0^2 \sin\psi\cos\psi}{1 - \eta_0^2\sin^2\psi}\frac{\partial\psi}{\partial a} + x\frac{\partial\phi}{\partial a},$$

$$\frac{\partial y}{\partial e} = \frac{y}{\rho^2 + c^2} \quad a^2(1 - e \cos E)\left(e \sin E \frac{\partial E}{\partial e} - \cos E\right) - \frac{y \eta_0^2 \sin\psi \cos\psi}{1 - \eta_0^2 \sin^2\psi} \quad \frac{\partial\psi}{\partial e} + x \frac{\partial\phi}{\partial e},$$

$$\frac{\partial y}{\partial \eta_0} = \frac{y}{\rho^2 + c^2} \quad a^2 e \sin E(1 - e \cos E) \frac{\partial E}{\partial \eta_0} + x \frac{\partial\phi}{\partial \eta_0} - \frac{y \eta_0^2 \sin\psi \cos\psi}{1 - \eta_0^2 \sin^2\psi} \quad \frac{\partial\psi}{\partial \eta_0} - \frac{y \eta_0 \sin^2\psi}{1 - \eta_0^2 \sin^2\psi},$$

$$\frac{\partial z}{\partial a} = (1 - e \cos E)\left(\eta_0 \sin\psi + a \eta_0 \cos\psi \frac{\partial\psi}{\partial a}\right) + a e \eta_0 \sin\psi \sin E \frac{\partial E}{\partial a},$$

$$\frac{\partial z}{\partial e} = a \eta_0 (1 - e \cos E) \cos\psi \frac{\partial\psi}{\partial e} + a \eta_0 \sin\psi\left(e \sin E \frac{\partial E}{\partial e} - \cos E\right),$$

$$\frac{\partial z}{\partial \eta_0} = (1 - e \cos E)\left(a \sin\psi + a \eta_0 \cos\psi \frac{\partial\psi}{\partial \eta_0}\right) + a e \eta_0 \sin\psi \sin E \frac{\partial E}{\partial \eta_0}.$$

Now

$$\frac{\partial E}{\partial a} = \frac{\partial \mathcal{E}}{\partial a} + \frac{\partial E_1}{\partial a} + \frac{\partial E_2}{\partial a},$$

$$\frac{\partial E}{\partial e} = \frac{\partial \mathcal{E}}{\partial e} + \frac{\partial E_1}{\partial e} + \frac{\partial E_2}{\partial e},$$

$$\frac{\partial E}{\partial \eta_0} = \frac{\partial \mathcal{E}}{\partial \eta_0} + \frac{\partial E_1}{\partial \eta_0} + \frac{\partial E_2}{\partial \eta_0},$$

$$\frac{\partial \psi}{\partial a} = \frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a} + \frac{\partial \psi_1}{\partial a} + \frac{\partial \psi_2}{\partial a},$$

$$\frac{\partial \psi}{\partial e} = \frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e} + \frac{\partial \psi_1}{\partial e} + \frac{\partial \psi_2}{\partial e},$$

$$\frac{\partial \psi}{\partial \eta_0} = \frac{\partial \psi_s}{\partial \eta_0} + \frac{\partial \psi_0}{\partial \eta_0} + \frac{\partial \psi_1}{\partial \eta_0} + \frac{\partial \psi_2}{\partial \eta_0},$$

$$\frac{\partial \mathcal{E}}{\partial a} = \left(\frac{\partial M_s}{\partial a} + \frac{\partial e'}{\partial a} \sin \mathcal{E}\right) \frac{1}{1 - e' \cos \mathcal{E}},$$

$$\frac{\partial e'}{\partial a} = \frac{(a+b_1)e - a\,e\left(1 + \frac{\partial b_1}{\partial a}\right)}{(a+b_1)^2},$$

$$\frac{\partial b_1}{\partial a} = b_1\left[\frac{1}{a} + D\,\frac{\partial D^{-1}}{\partial a} + \frac{1}{ap - c^2\eta_0^2}\left(a\,\frac{\partial p}{\partial a} + p\right)\right],$$

$$\frac{\partial p}{\partial a} = 1 - e^2\,,$$

$$\frac{\partial D^{-1}}{\partial a} = -D^{-2}\,\frac{\partial D}{\partial a},$$

$$\frac{\partial D}{\partial a} = 8ac^2\eta_0^2 + \left(ap - c^2\eta_0^2 + ap - c^2\right)\left(a\,\frac{\partial p}{\partial a} + p\right),$$

$$\frac{\partial D'}{\partial a} = \frac{\partial D}{\partial a} + 8ac^2\left(1 - \eta_0^2\right).$$

Also

$$\frac{\partial M_s}{\partial a} = -2\pi\left[\nu_1 c^2\beta_2\eta_0^2\left(B_1B_2^{-1}\,\frac{\partial\alpha_2^{-1}}{\partial a} + \frac{1}{\alpha_2 B_2}\,\frac{\partial B_1}{\partial a} + \frac{B_1}{\alpha_2}\,\frac{\partial B_2^{-1}}{\partial a}\right) - \left(t + \beta_1 - c^2\beta_2\alpha_2^{-1}\eta_0^2 B_1B_2^{-1}\right)\frac{\partial\nu_1}{\partial a}\right],$$

$$\frac{\partial\alpha_2^{-1}}{\partial a} = -\alpha_2^{-2}\,\frac{\partial\alpha_2}{\partial a},$$

$$\frac{\partial\alpha_2}{\partial a} = \frac{1}{2}\left[\mu(a+b_1)^{-1}\right]^{1/2}\left(\frac{aD'}{D}\,\frac{\partial p}{\partial a} + \frac{pD'}{D} + \frac{ap}{D}\,\frac{\partial D'}{\partial a} + apD'\,\frac{\partial D^{-1}}{\partial a}\right)\left[apD'D^{-1} - c^2\left(1 - \eta_0^2\right)\right]^{-1/2}$$

$$- \frac{\mu(a+b_1)^{-2}}{2}\left[apD'D^{-1} - c^2\left(1 - \eta_0^2\right)\right]^{1/2}\left(1 + \frac{\partial b_1}{\partial a}\right)\left[\mu(a+b_1)^{-1}\right]^{-1/2},$$

$$\frac{\partial B_1}{\partial a} = \frac{3}{8}\,q\,\frac{\partial q}{\partial a} + \frac{15}{32}\,q^3\,\frac{\partial q}{\partial a},$$

$$\frac{\partial q}{\partial a} = -\frac{\eta_0}{\eta_2^2}\,\frac{\partial\eta_2}{\partial a},$$

$$\frac{\partial \eta_2}{\partial a} = \frac{1}{2\eta_2 c^2}\left(\frac{pD'}{D} + \frac{aD'}{D}\frac{\partial p}{\partial a} + \frac{ap}{D}\frac{\partial D'}{\partial a} + apD'\frac{\partial D^{-1}}{\partial a}\right),$$

$$\frac{\partial B_2^{-1}}{\partial a} = -B_2^{-2}\frac{\partial B_2}{\partial a},$$

$$\frac{\partial B_2}{\partial a} = \frac{1}{2}q\frac{\partial q}{\partial a} + \frac{9}{16}q^3\frac{\partial q}{\partial a},$$

$$\frac{\partial \nu_1}{\partial a} = \frac{1}{2\pi}\left\{-\left(a+b_1+A_1+c^2\eta_0^2 A_2 B_1 B_2^{-1}\right)^{-1}\frac{\mu}{2}(a+b_1)^{-2}\left[\mu(a+b_1)^{-1}\right]^{-1/2}\left(1+\frac{\partial b_1}{\partial a}\right)\right.$$

$$-\left[\mu(a+b_1)^{-1}\right]^{1/2}\left(a+b_1+A_1+c^2\eta_0^2 A_2 B_1 B_2^{-1}\right)^{-2}\left[1+\frac{\partial b_1}{\partial a}\right.$$

$$\left.\left.+\frac{\partial A_1}{\partial a}+c^2\eta_0^2 A_2\left(\frac{\partial B_1}{\partial a}B_2^{-1}+B_1\frac{\partial B_2^{-1}}{\partial a}\right)+c^2\eta_0^2 B_1 B_2^{-1}\frac{\partial A_2}{\partial a}\right]\right\},$$

$$\frac{\partial A_1}{\partial a} = \frac{A_1}{p}\frac{\partial p}{\partial a} + p(1-e^2)^{1/2}\sum_{n=2}^{\infty} n\left(\frac{b_2}{p}\right)^{n-1}\left(\frac{1}{p}\frac{\partial b_2}{\partial a} - \frac{b_2}{p^2}\frac{\partial p}{\partial a}\right)P_n\left(\frac{b_1}{b_2}\right)R_{n-2}\left[(1-e^2)^{1/2}\right]$$

$$+p(1-e^2)^{1/2}\sum_{n=2}^{\infty}\left(\frac{b_2}{p}\right)^n\frac{\partial}{\partial a}P_n\left(\frac{b_1}{b_2}\right)R_{n-2}\left[(1-e^2)^{1/2}\right],$$

$$\frac{\partial b_2}{\partial a} = \frac{c^2\eta_0^2}{2}\left(c^2\eta_0^2 D' D^{-1}\right)^{-1/2}\left(\frac{1}{D}\frac{\partial D'}{\partial a} + D'\frac{\partial D^{-1}}{\partial a}\right),$$

$$\frac{\partial}{\partial a}P_n\left(\frac{b_1}{b_2}\right) = P_n'\left(\frac{b_1}{b_2}\right)\frac{\partial}{\partial a}\left(\frac{b_1}{b_2}\right),$$

$$\frac{\partial}{\partial a}\left(\frac{b_1}{b_2}\right) = \frac{1}{b_2}\frac{\partial b_1}{\partial a} - \frac{b_1}{b_2^2}\frac{\partial b_2}{\partial a}.$$

By using

$$P_n\left(\frac{b_1}{b_2}\right) = \frac{1}{2^n}\sum_{r=0}^{\sigma}\frac{(-1)^r(2n-2r)!}{r!(n-r)!(n-2r)!}\left(\frac{b_1}{b_2}\right)^{n-2r},$$

where $\sigma = (1/2)$ if n is even, and $\sigma = (1/2)(n-1)$ if n is odd, then

$$\frac{\partial}{\partial a} P_n\left(\frac{b_1}{b_2}\right) = \left[\frac{1}{2^n} \sum_{r=0}^{\sigma} \frac{(-1)^r (2n-2r)!}{r!(n-r)!(n-2r)!} (n-2r) \left(\frac{b_1}{b_2}\right)^{n-2r-1}\right] \frac{\partial}{\partial a}\left(\frac{b_1}{b_2}\right)$$

The coefficient of $\frac{\partial}{\partial a}\left(\frac{b_1}{b_2}\right)$ is taken as $P_n'\left(\frac{b_1}{b_2}\right)$;

$$\frac{\partial A_2}{\partial a} = A_2 p \frac{\partial p^{-1}}{\partial a} + \frac{(1-e^2)^{1/2}}{p} \sum_{n=0}^{\infty} n\left(\frac{b_2}{p}\right)^{n-1} \frac{\partial}{\partial a}\left(\frac{b_2}{p}\right) P_n\left(\frac{b_1}{b_2}\right) R_n\left[(1-e^2)^{1/2}\right]$$

$$+ \frac{(1-e^2)^{1/2}}{p} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n \frac{\partial}{\partial a} P_n\left(\frac{b_1}{b_2}\right) R_n\left[(1-e^2)^{1/2}\right],$$

$$\frac{\partial}{\partial a}\left(\frac{b_2}{p}\right) = \frac{1}{p}\frac{\partial b_2}{\partial a} - \frac{b_2}{p^2}\frac{\partial p}{\partial a}$$

$$\frac{\partial p^{-1}}{\partial a} = -p^{-2}\frac{\partial p}{\partial a}.$$

Now

$$\frac{\partial \psi_s}{\partial a} = 2\pi \left\{\left[t + \beta_1 + \beta_2 \alpha_2^{-1}(a + b_1 + A_1)A_2^{-1}\right] \frac{\partial \nu_2}{\partial a} + \nu_2\left[\beta_2 \alpha_2^{-1}(a + b_1 + A_1) \frac{\partial A_2^{-1}}{\partial a}\right.\right.$$

$$\left.\left. + \beta_2 A_2^{-1}(a + b_1 + A_1) \frac{\partial \alpha_2^{-1}}{\partial a} + \beta_2 \alpha_2^{-1} A_2^{-1}\left(1 + \frac{\partial b_1}{\partial a} + \frac{\partial A_1}{\partial a}\right)\right]\right\},$$

$$\frac{\partial A_2^{-1}}{\partial a} = -A_2^{-2} \frac{\partial A_2}{\partial a},$$

$$\frac{\partial \nu_2}{\partial a} = \frac{1}{2\pi}\left\{-\left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} A_2 B_2^{-1} \left(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right)^{-2} \left[1 + \frac{\partial b_1}{\partial a} + \frac{\partial A_1}{\partial a}\right.\right.$$

$$\left. + c^2 \eta_0^2 B_1 B_2^{-1} \frac{\partial A_2}{\partial a} + c^2 \eta_0^2 A_2 \left(B_2^{-1}\frac{\partial B_1}{\partial a} + B_1 \frac{\partial B_2^{-1}}{\partial a}\right)\right]$$

$$+ \left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} B_2^{-1} \left(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right)^{-1} \frac{\partial A_2}{\partial a}$$

$$+ A_2 \eta_0^{-1} B_2^{-1} \left(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right)^{-1} \frac{\partial}{\partial a}\left[\left(\alpha_2^2 - \alpha_3^2\right)^{1/2}\right]$$

$$+ \left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} A_2 \left(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right)^{-1} \frac{\partial B_2^{-1}}{\partial a}\Bigg\},$$

$$\frac{\partial\left[\left(\alpha_2^2 - \alpha_3^2\right)^{1/2}\right]}{\partial a} = \left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\left(\alpha_2 \frac{\partial \alpha_2}{\partial a} - \alpha_3 \frac{\partial \alpha_3}{\partial a}\right),$$

$$\frac{\partial \alpha_3}{\partial a} = \left[1 - \frac{c^2 \eta_0^2}{apD'D^{-1} - c^2\left(1 - \eta_0^2\right)}\right]^{1/2} \Bigg\{\left(1 - \eta_0^2\right)^{1/2} \frac{\partial \alpha_2}{\partial a}$$

$$+ \left[1 - \frac{c^2 \eta_0^2}{apD'D^{-1} - c^2\left(1 - \eta_0^2\right)}\right] \frac{\alpha_2\left(1 - \eta_0^2\right)^{1/2}}{2} \frac{c^2 \eta_0^2\left(p \frac{D'}{D} + \frac{aD'}{D}\frac{\partial p}{\partial a} + apD' \frac{\partial D^{-1}}{\partial a} + \frac{ap}{D}\frac{\partial D'}{\partial a}\right)}{\left[apD'D^{-1} - c^2\left(1 - \eta_0^2\right)\right]^2}\Bigg\}.$$

Then

$$\frac{\partial \mathcal{E}}{\partial e} = \left(\frac{\partial M_s}{\partial e} + \sin \mathcal{E} \frac{\partial e'}{\partial e}\right) \frac{1}{1 - e' \cos \mathcal{E}},$$

$$\frac{\partial e'}{\partial e} = \frac{a}{a + b_1}\left[1 - \frac{e}{a + b_1} \frac{\partial b_1}{\partial e}\right],$$

$$\frac{\partial b_1}{\partial e} = \frac{a^2 c^2}{D}\left(1 - \eta_0^2\right) \frac{\partial p}{\partial e} + \left(1 - \eta_0^2\right)\left(ap - c^2 \eta_0^2\right) ac^2 \frac{\partial D^{-1}}{\partial e},$$

$$\frac{\partial p}{\partial e} = -2ae,$$

$$\frac{\partial D^{-1}}{\partial e} = -D^{-2} \frac{\partial D}{\partial e},$$

$$\frac{\partial D}{\partial e} = \left[2ap - c^2\left(\eta_0^2 + 1\right)\right] a \frac{\partial p}{\partial e}.$$

And

$$\frac{\partial M_s}{\partial e} = 2\pi\Bigg[\left(t + \beta_1 - c^2 \eta_0^2 \beta_2 \alpha_2^{-1} B_1 B_2^{-1}\right) \frac{\partial \nu_1}{\partial e}$$

$$- \nu_1 c^2 \beta_2 \eta_0^2 \left(\frac{B_1}{B_2} \frac{\partial \alpha_2^{-1}}{\partial e} + \frac{B_1}{\alpha_2} \frac{\partial B_2^{-1}}{\partial e} + \frac{1}{\alpha_2 B_2} \frac{\partial B_1}{\partial e} \right) \Bigg] ,$$

$$\frac{\partial \alpha_2^{-1}}{\partial e} = -\alpha_2^{-2} \frac{\partial \alpha_2}{\partial e} ,$$

$$\frac{\partial \alpha_2}{\partial e} = \frac{1}{2} \left[\mu (a + b_1)^{-1} \right]^{1/2} \left(\frac{aD'}{D} \frac{\partial p}{\partial e} + \frac{ap}{D} \frac{\partial D'}{\partial e} + apD' \frac{\partial D^{-1}}{\partial e} \right) \left[apD'D^{-1} - c^2 \left(1 - \eta_0^2 \right) \right]^{-1/2}$$

$$- \frac{\mu}{2} (a + b_1)^{-2} \left[\frac{apD'}{D} - c^2 \left(1 - \eta_0^2 \right) \right]^{1/2} \left[\mu (a + b_1)^{-1} \right]^{-1/2} \frac{\partial b_1}{\partial e} ,$$

$$\frac{\partial D'}{\partial e} = \frac{\partial D}{\partial e} ,$$

$$\frac{\partial B_1}{\partial e} = \frac{3}{8} q \frac{\partial q}{\partial e} + \frac{15}{32} q^3 \frac{\partial q}{\partial e} ,$$

$$\frac{\partial q}{\partial e} = - \frac{\eta_0}{\eta_2^2} \frac{\partial \eta_2}{\partial e} ,$$

$$\frac{\partial \eta_2}{\partial e} = \frac{a}{2 \eta_2 c^2} \left(\frac{D'}{D} \frac{\partial p}{\partial e} + \frac{p}{D} \frac{\partial D'}{\partial e} - \frac{pD'}{D^2} \frac{\partial D}{\partial e} \right) ,$$

$$\frac{\partial B_2^{-1}}{\partial e} = - B_2^{-2} \frac{\partial B_2}{\partial e} ,$$

$$\frac{\partial B_2}{\partial e} = \frac{1}{2} q \frac{\partial q}{\partial e} + \frac{9}{16} q^3 \frac{\partial q}{\partial e} ,$$

$$\frac{\partial \nu_1}{\partial e} = \frac{1}{2\pi} \Bigg\{ - \left(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1} \right)^{-1} \frac{\mu}{2} (a + b_1)^{-2} \frac{\partial b_1}{\partial e} \left[\mu (a + b_1)^{-1} \right]^{-1/2}$$

$$- \left[\mu (a + b_1)^{-1} \right]^{1/2} \left(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1} \right)^{-2} \Bigg[\frac{\partial b_1}{\partial e} + \frac{\partial A_1}{\partial e}$$

$$+ c^2 \eta_0^2 A_2 \left(\frac{\partial B_1}{\partial e} B_2^{-1} + B_1 \frac{\partial B_2^{-1}}{\partial e} \right) + c^2 \eta_0^2 \frac{B_1}{B_2} \frac{\partial A_2}{\partial e} \Bigg] \Bigg\} ,$$

$$\frac{\partial A_1}{\partial e} = \frac{A_1}{p}\frac{\partial p}{\partial e} - \frac{A_1 e}{1-e^2} + p(1-e^2)^{1/2} \sum_{n=2}^{\infty} n\left(\frac{b_2}{p}\right)^{n-1}\left(\frac{1}{p}\frac{\partial b_2}{\partial e} - \frac{b_2}{p^2}\frac{\partial p}{\partial e}\right) P_n\left(\frac{b_1}{b_2}\right) R_{n-2}\left[(1-e^2)^{1/2}\right]$$

$$+ p(1-e^2)^{1/2} \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n \frac{\partial}{\partial e} P_n\left(\frac{b_1}{b_2}\right) R_{n-2}\left[(1-e^2)^{1/2}\right]$$

$$+ p(1-e^2)^{1/2} \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) \frac{\partial}{\partial e} R_{n-2}\left[(1-e^2)^{1/2}\right],$$

$$\frac{\partial b_2}{\partial e} = \frac{c^2 \eta_0^2}{2}\left(D^{-1}\frac{\partial D'}{\partial e} + D'\frac{\partial D^{-1}}{\partial e}\right)\left(c^2 \eta_0^2 D' D^{-1}\right)^{-1/2},$$

$$\frac{\partial}{\partial e} P_n\left(\frac{b_1}{b_2}\right) = P_n'\left(\frac{b_1}{b_2}\right)\frac{\partial}{\partial e}\left(\frac{b_1}{b_2}\right),$$

$$\frac{\partial}{\partial e}\left(\frac{b_1}{b_2}\right) = \frac{1}{b_2}\frac{\partial b_1}{\partial e} - \frac{b_1}{b_2^2}\frac{\partial b_2}{\partial e},$$

$$\frac{\partial}{\partial e} R_{n-2}\left[(1-e^2)^{1/2}\right] = - P_{n-2}\left[(1-e^2)^{-1/2}\right](n-2)\left[(1-e^2)^{1/2}\right]^{n-3} e(1-e^2)^{-1/2}$$

$$+ \left[(1-e^2)^{1/2}\right]^{n-2}\frac{\partial}{\partial e} P_{n-2}\left[(1-e^2)^{-1/2}\right],$$

$$\frac{\partial}{\partial e} P_{n-2}\left[(1-e^2)^{-1/2}\right]$$

$$= \left\{\frac{1}{2^{n-2}} \sum_{r=0}^{\sigma} \frac{(-1)^r (2n-2r-4)!}{r!(n-r-2)!(n-2r-2)!}(n-2r-2)\left[(1-e^2)^{-1/2}\right]^{n-2r-3}\right\} \frac{\partial}{\partial e}\left[(1-e^2)^{-1/2}\right],$$

$$\frac{\partial}{\partial e}\left[(1-e^2)^{-1/2}\right] = e(1-e^2)^{-3/2},$$

$$\frac{\partial A_2}{\partial e} = A_2 p \frac{\partial p^{-1}}{\partial e} - \frac{eA_2}{1-e^2}$$

$$+ \frac{(1-e^2)^{1/2}}{p}\left[\sum_{n=0}^{\infty} n\left(\frac{b_2}{p}\right)^{n-1}\frac{\partial}{\partial e}\left(\frac{b_2}{p}\right) P_n\left(\frac{b_1}{b_2}\right) R_n\left[(1-e^2)^{1/2}\right]\right.$$

$$+ \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n \frac{\partial}{\partial e} P_n\left(\frac{b_1}{b_2}\right) R_n\left[(1-e^2)^{1/2}\right]$$

$$+ \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) \frac{\partial}{\partial e} R_n\left[(1-e^2)^{1/2}\right]\Bigg],$$

$$\frac{\partial}{\partial e}\left[(1-e^2)^{1/2}\right] = -e(1-e^2)^{-1/2},$$

$$\frac{\partial p^{-1}}{\partial e} = 2a^{-1}e(1-e^2)^{-2},$$

$$\frac{\partial}{\partial e}\left(\frac{b_2}{p}\right) = \frac{1}{p}\frac{\partial b_2}{\partial e} - \frac{b_2}{p^2}\frac{\partial p}{\partial e},$$

$$\frac{\partial}{\partial e} R_n\left[(1-e^2)^{1/2}\right] = \left[(1-e^2)^{1/2}\right]^n \frac{\partial}{\partial e} P_n\left[(1-e^2)^{-1/2}\right]$$

$$-e\left[(1-e^2)^{1/2}\right]^{n-1} n(1-e^2)^{-1/2} P_n\left[(1-e^2)^{-1/2}\right],$$

$$\frac{\partial}{\partial e} P_n\left[(1-e^2)^{-1/2}\right]$$

$$= \frac{1}{2^n} \sum_{r=0}^{\sigma} \frac{(-1)^r(2n-2r)!}{r!(n-r)!(n-2r)!} (n-2r)\left[(1-e^2)^{-1/2}\right]^{n-2r-1} \frac{\partial}{\partial e}\left[(1-e^2)^{-1/2}\right].$$

Also

$$\frac{\partial \psi_s}{\partial e} = 2\pi\Bigg\{\left[t + \beta_1 + \beta_2\alpha_2^{-1}(a + b_1 + A_1)A_2^{-1}\right]\frac{\partial \nu_2}{\partial e}$$

$$+ \beta_2\nu_2(a + b_1 + A_1)A_2^{-1}\frac{\partial \alpha_2^{-1}}{\partial e} + \nu_2\beta_2\alpha_2^{-1}\Bigg[(a + b_1 + A_1)\frac{\partial A_2^{-1}}{\partial e}$$

$$+ A_2^{-1}\left(\frac{\partial b_1}{\partial e} + \frac{\partial A_1}{\partial e}\right)\Bigg]\Bigg\},$$

$$\frac{\partial A_2^{-1}}{\partial e} = -A_2^{-2}\frac{\partial A_2}{\partial e},$$

where

$$\frac{\partial \nu_2}{\partial e} = \frac{1}{2\pi}\left\{-\left(\alpha_2^2-\alpha_3^2\right)^{1/2}\eta_0^{-1}A_2B_2^{-1}\left(a+b_1+A_1+c^2\eta_0^2A_2B_1B_2^{-1}\right)^{-2}\left[\frac{\partial b_1}{\partial e}+\frac{\partial A_1}{\partial e}\right.\right.$$

$$\left.+c^2\eta_0^2B_1B_2^{-1}\frac{\partial A_2}{\partial e}+c^2\eta_0^2A_2\left(B_2^{-1}\frac{\partial B_1}{\partial e}+B_1\frac{\partial B_2^{-1}}{\partial e}\right)\right]$$

$$+\left(\alpha_2^2-\alpha_3^2\right)^{1/2}\eta_0^{-1}B_2^{-1}\left(a+b_1+A_1+c^2\eta_0^2A_2B_1B_2^{-1}\right)^{-1}\frac{\partial A_2}{\partial e}$$

$$+A_2\eta_0^{-1}B_2^{-1}\left(a+b_1+A_1+c^2\eta_0^2A_2B_1B_2^{-1}\right)^{-1}\frac{\partial}{\partial e}\left[\left(\alpha_2^2-\alpha_3^2\right)^{1/2}\right]$$

$$\left.+\left(\alpha_2^2-\alpha_3^2\right)^{1/2}\eta_0^{-1}A_2\left(a+b_1+A_1+c^2\eta_0^2A_2B_1B_2^{-1}\right)^{-1}\frac{\partial B_2^{-1}}{\partial e}\right\},$$

$$\frac{\partial}{\partial e}\left[\left(\alpha_2^2-\alpha_3^2\right)^{1/2}\right] = \left(\alpha_2^2-\alpha_3^2\right)^{-1/2}\left(\alpha_2\frac{\partial \alpha_2}{\partial e}-\alpha_3\frac{\partial \alpha_3}{\partial e}\right)$$

$$\frac{\partial \alpha_3}{\partial e} = \left[1-\frac{c^2\eta_0^2}{\left[apD'D^{-1}-c^2\left(1-\eta_0^2\right)\right]}\right]^{1/2}\left(1-\eta_0^2\right)^{1/2}\frac{\partial \alpha_2}{\partial e}$$

$$+\frac{\alpha_2}{2}\left(1-\eta_0^2\right)^{1/2}\left[1-\frac{c^2\eta_0^2}{\left[apD'D^{-1}-c^2\left(1-\eta_0^2\right)\right]}\right]^{-1/2}\frac{c^2\eta_0^2\left(\frac{aD'}{D}\frac{\partial p}{\partial e}+\frac{ap}{D}\frac{\partial D'}{\partial e}+apD'\frac{\partial D^{-1}}{\partial e}\right)}{\left[apD'D^{-1}-c^2\left(1-\eta_0^2\right)\right]^2}.$$

Now

$$\frac{\partial \mathcal{E}}{\partial \eta_0} = \left(\frac{\partial M_s}{\partial \eta_0}+\frac{\partial e'}{\partial \eta_0}\sin\mathcal{E}\right)\frac{1}{1-e'\cos\mathcal{E}},$$

$$\frac{\partial e'}{\partial \eta_0} = \frac{-ae}{(a+b_1)^2}\frac{\partial b_1}{\partial \eta_0},$$

$$\frac{\partial b_1}{\partial \eta_0} = ac^2\left[\frac{\partial D^{-1}}{\partial \eta_0}\left(1-\eta_0^2\right)\left(ap-c^2\eta_0^2\right)-\frac{2\eta_0}{D}\left(ap-c^2\eta_0^2\right)-\frac{2c^2}{D}\left(1-\eta_0^2\right)\eta_0\right],$$

$$\frac{\partial D^{-1}}{\partial \eta_0} = -D^{-2}\frac{\partial D}{\partial \eta_0},$$

$$\frac{\partial D}{\partial \eta_0} = 8a^2c^2\eta_0 - 2(ap - c^2)\ c^2\eta_0,$$

$$\frac{\partial D'}{\partial \eta_0} = \frac{\partial D}{\partial \eta_0} - 8a^2c^2\eta_0.$$

And

$$\frac{\partial M_s}{\partial \eta_0} = \frac{M_s}{\nu_1}\frac{\partial \nu_1}{\partial \eta_0} - 2\pi\nu_1 c^2\beta_2\eta_0\left(2\alpha_2^{-1}B_1B_2^{-1}\right.$$

$$+ \alpha_2^{-1}\eta_0 B_2^{-1}\frac{\partial B_1}{\partial \eta_0} + \alpha_2^{-1}\eta_0 B_1\frac{\partial B_2^{-1}}{\partial \eta_0}$$

$$\left. + \eta_0 B_1 B_2^{-1}\frac{\partial \alpha_2^{-1}}{\partial \eta_0}\right),$$

$$\frac{\partial B_1}{\partial \eta_0} = \frac{3}{8}q\frac{\partial q}{\partial \eta_0} + \frac{15}{32}q^3\frac{\partial q}{\partial \eta_0},$$

$$\frac{\partial q}{\partial \eta_0} = \frac{1}{\eta_2} - \frac{\eta_0}{\eta_2^2}\frac{\partial \eta_2}{\partial \eta_0},$$

$$\frac{\partial \eta_2}{\partial \eta_0} = \frac{1}{2\eta_2}\left(\frac{ap}{c^2D}\frac{\partial D'}{\partial \eta_0} + \frac{apD'}{c^2}\frac{\partial D^{-1}}{\partial \eta_0}\right),$$

$$\frac{\partial \alpha_1}{\partial \eta_0} = \frac{\mu}{2(a+b_1)^2}\frac{\partial b_1}{\partial \eta_0},$$

$$\frac{\partial \alpha_2}{\partial \eta_0} = \frac{1}{2}\left[\mu(a+b_1)^{-1}\right]^{1/2}\left(\frac{ap}{D}\frac{\partial D'}{\partial \eta_0} + apD'\frac{\partial D^{-1}}{\partial \eta_0} + 2c^2\eta_0\right)\left[apD'D^{-1} - c^2\left(1-\eta_0^2\right)\right]^{-1/2}$$

$$-\frac{\mu}{2}\left[\mu(a+b_1)^{-1}\right]^{-1/2}(a+b_1)^{-2}\left[apD'D^{-1} - c^2(1-\eta_0^2)\right]^{1/2}\frac{\partial b_1}{\partial \eta_0},$$

$$\frac{\partial \alpha_3}{\partial \eta_0} = \frac{\alpha_3}{\alpha_2}\frac{\partial \alpha_2}{\partial \eta_0} - \eta_0\frac{\alpha_3}{1-\eta_0^2}$$

$$-\frac{c^2\eta_0\alpha_2}{2}\left(1-\eta_0^2\right)^{1/2}\frac{2\left[apD'D^{-1} - c^2\left(1-\eta_0^2\right)\right] - \eta_0\left(apD^{-1}\frac{\partial D'}{\partial \eta_0} + apD'\frac{\partial D^{-1}}{\partial \eta_0} + 2c^2\eta_0\right)}{\left[apD'D^{-1} - c^2\left(1-\eta_0^2\right)\right]^2}\left\{1 - \frac{c^2\eta_0^2}{\left[apD'D^{-1} - c^2\left(1-\eta_0^2\right)\right]}\right\}^{-1/2}$$

$$\frac{\partial B_2^{-1}}{\partial \eta_0} = -B_2^{-2} \frac{\partial B_2}{\partial \eta_0} ,$$

$$\frac{\partial B_2}{\partial \eta_0} = \frac{1}{2} q \frac{\partial q}{\partial \eta_0} + \frac{9}{16} q^3 \frac{\partial q}{\partial \eta_0} ,$$

$$\frac{\partial \alpha_2^{-1}}{\partial \eta_0} = -\alpha_2^{-2} \frac{\partial \alpha_2}{\partial \eta_0} .$$

Also

$$\frac{\partial \nu_1}{\partial \eta_0} = \frac{-1}{2\pi} \left\{ (-2\alpha_1)^{-1/2} \left(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right)^{-1} \frac{\partial \alpha_1}{\partial \eta_0} \right.$$

$$+ (-2\alpha_1)^{1/2} \left(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right)^{-2} \left[\frac{\partial b_1}{\partial \eta_0} + \frac{\partial A_1}{\partial \eta_0} \right.$$

$$\left. \left. + c^2 \left(2\eta_0 A_2 B_1 B_2^{-1} + \eta_0^2 B_1 B_2^{-1} \frac{\partial A_2}{\partial \eta_0} + \eta_0^2 A_2 B_2^{-1} \frac{\partial B_1}{\partial \eta_0} + \eta_0^2 A_2 B_1 \frac{\partial B_2^{-1}}{\partial \eta_0} \right) \right] \right\}$$

$$\frac{\partial A_1}{\partial \eta_0} = p(1 - e^2)^{1/2} \left\{ \sum_{n=2}^{\infty} n \left(\frac{b_2}{p}\right)^{n-1} \frac{1}{p} \frac{\partial b_2}{\partial \eta_0} P_n\left(\frac{b_1}{b_2}\right) R_{n-2} \left[(1 - e^2)^{1/2}\right] \right.$$

$$\left. + \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n \frac{\partial}{\partial \eta_0} P_n\left(\frac{b_1}{b_2}\right) R_{n-2} \left[(1 - e^2)^{1/2}\right] \right\} ,$$

$$\frac{\partial b_2}{\partial \eta_0} = \frac{1}{2} \left(c^2 \eta_0^2 D^{-1} D'\right)^{-1/2} \left(2c^2 D^{-1} D' \eta_0 + \frac{c^2 \eta_0^2}{D} \frac{\partial D'}{\partial \eta_0} + c^2 \eta_0^2 D' \frac{\partial D^{-1}}{\partial \eta_0} \right) ,$$

$$\frac{\partial}{\partial \eta_0} P_n\left(\frac{b_1}{b_2}\right) = P_n'\left(\frac{b_1}{b_2}\right) \frac{\partial}{\partial \eta_0} \left(\frac{b_1}{b_2}\right) ,$$

$$\frac{\partial}{\partial \eta_0} \left(\frac{b_1}{b_2}\right) = \frac{1}{b_2} \frac{\partial b_1}{\partial \eta_0} - \frac{b_1}{b_2^2} \frac{\partial b_2}{\partial \eta_0} ,$$

$$\frac{\partial A_2}{\partial \eta_0} = p^{-1} (1 - e^2)^{1/2} \left[\sum_{n=0}^{\infty} n \left(\frac{b_2}{p}\right)^{n-1} \frac{\partial}{\partial \eta_0} \left(\frac{b_2}{p}\right) P_n\left(\frac{b_1}{b_2}\right) R_n \left[(1 - e^2)^{1/2}\right] \right.$$

$$+ \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n \frac{\partial}{\partial \eta_0} P_n\left(\frac{b_1}{b_2}\right) R_n\left[(1-e^2)^{1/2}\right] \Bigg] ,$$

$$\frac{\partial}{\partial \eta_0}\left(\frac{b_2}{p}\right) = \frac{1}{p}\frac{\partial b_2}{\partial \eta_0} .$$

And

$$\frac{\partial \psi_s}{\partial \eta_0} = \frac{\psi_s}{\nu_2}\frac{\partial \nu_2}{\partial \eta_0} + 2\pi \Bigg\{ \beta_2 \nu_2 A_2^{-1}(a + b_1 + A_1)\frac{\partial a_2^{-1}}{\partial \eta_0}$$

$$+ \beta_2 \nu_2 \alpha_2^{-1}(a + b_1 + A_1)\frac{\partial A_2^{-1}}{\partial \eta_0} + \beta_2 \nu_2 \alpha_2^{-1} A_2^{-1}\left(\frac{\partial b_1}{\partial \eta_0} + \frac{\partial A_1}{\partial \eta_0}\right)\Bigg\} ,$$

$$\frac{\partial A_2^{-1}}{\partial \eta_0} = -A_2^{-2}\frac{\partial A_2}{\partial \eta_0} ,$$

$$\frac{\partial \nu_2}{\partial \eta_0} = \frac{1}{2\pi}\Bigg[-\left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} A_2 B_2^{-1}\left(a + b_1 + A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right)^{-2}\left(\frac{\partial b_1}{\partial \eta_0} + \frac{\partial A_1}{\partial \eta_0} + 2c^2 A_2 B_1 B_2^{-1}\eta_0 + c^2 \eta_0^2 B_1 B_2^{-1}\frac{\partial A_2}{\partial \eta_0}\right.$$

$$\left. + c^2 \eta_0^2 A_2 B_2^{-1}\frac{\partial B_1}{\partial \eta_0} + c^2 \eta_0^2 A_2 B_1 \frac{\partial B_2^{-1}}{\partial \eta_0}\right) + 2\pi \nu_2 \left(\frac{1}{A_2}\frac{\partial A_2}{\partial \eta_0} - \frac{1}{\eta_0} + B_2 \frac{\partial B_2^{-1}}{\partial \eta_0} + \frac{\alpha_2 \frac{\partial \alpha_2}{\partial \eta_0} - \alpha_3 \frac{\partial \alpha_3}{\partial \eta_0}}{\alpha_2^2 - \alpha_3^2}\right)\Bigg] .$$

Then

$$\frac{\partial v_0}{\partial a} = \frac{\sin \mathcal{E}\left(1 - e^2\right)\frac{\partial \mathcal{E}}{\partial a}}{\sin(M_s + v_0)(1 - e\cos \mathcal{E})^2} - \frac{\partial M_s}{\partial a} ,$$

$$\frac{\partial v_0}{\partial e} = \frac{(1 - e^2)\sin \mathcal{E}\frac{\partial \mathcal{E}}{\partial e} + \sin^2 \mathcal{E}}{\sin(M_s + v_0)(1 - e\cos \mathcal{E})^2} - \frac{\partial M_s}{\partial e} ,$$

$$\frac{\partial v_0}{\partial \eta_0} = \frac{\sin \mathcal{E}(1 - e^2)}{\sin(M_s + v_0)(1 - e\cos \mathcal{E})^2}\frac{\partial \mathcal{E}}{\partial \eta_0} - \frac{\partial M_s}{\partial \eta_0} .$$

$$\frac{\partial \psi_0}{\partial a} = \left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} A_2 B_2^{-1} v_0 \left(-2\alpha_1\right)^{-3/2} \frac{\partial \alpha_1}{\partial a}$$

$$+ \frac{\psi_0}{A_2} \frac{\partial A_2}{\partial a} + \frac{\psi_0}{\alpha_2^2 - \alpha_3^2} \left(\alpha_2 \frac{\partial \alpha_2}{\partial a} - \alpha_3 \frac{\partial \alpha_3}{\partial a}\right)$$

where

$$\frac{\partial \alpha_1}{\partial a} = \frac{\mu}{2(a + b_1)^2} \left(1 + \frac{\partial b_1}{\partial a}\right),$$

$$\frac{\partial \psi_0}{\partial e} = \left[\frac{1}{-2\alpha_1} \frac{\partial \alpha_1}{\partial e} + \frac{1}{A_2} \frac{\partial A_2}{\partial e} + \frac{1}{v_0} \frac{\partial v_0}{\partial e}\right.$$

$$\left. + \frac{1}{\alpha_2^2 - \alpha_3^2} \left(\alpha_2 \frac{\partial \alpha_2}{\partial e} - \alpha_3 \frac{\partial \alpha_3}{\partial e}\right) - \frac{1}{B_2} \frac{\partial B_2}{\partial e}\right] \psi_0 ,$$

where

$$\frac{\partial \alpha_1}{\partial e} = \frac{\mu}{2(a + b_1)^2} \frac{\partial b_1}{\partial e} ,$$

$$\frac{\partial \psi_0}{\partial \eta_0} = \psi_0 \left[\frac{1}{-2\alpha_1} \frac{\partial \alpha_1}{\partial \eta_0} + \frac{1}{A_2} \frac{\partial A_2}{\partial \eta_0} + \frac{1}{v_0} \frac{\partial v_0}{\partial \eta_0} + \frac{1}{\alpha_2^2 - \alpha_3^2} \left(\alpha_2 \frac{\partial \alpha_2}{\partial \eta_0} - \alpha_3 \frac{\partial \alpha_3}{\partial \eta_0}\right) - \frac{1}{\eta_0} + B_2 \frac{\partial B_2^{-1}}{\partial \eta_0}\right],$$

$$\frac{\partial M_1}{\partial a} = - \frac{M_1}{a + b_1} \left(1 + \frac{\partial b_1}{\partial a}\right) + (a + b_1)^{-1} \left\{ - \left(A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right) \frac{\partial v_0}{\partial a}\right.$$

$$\left. - v_0 \left[\frac{\partial A_1}{\partial a} + c^2 \eta_0^2 \left(\frac{B_1}{B_2} \frac{\partial A_2}{\partial a} + \frac{A_2}{B_2} \frac{\partial B_1}{\partial a} + A_2 B_1 \frac{\partial B_2^{-1}}{\partial a}\right)\right]\right.$$

$$+ \frac{c^2}{2}(-2\alpha_1)^{1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^3\left[\cos(2\psi_s + 2\psi_0)\left(\frac{\partial\psi_s}{\partial a} + \frac{\partial\psi_0}{\partial a}\right)\right.$$

$$\left. - \frac{1}{2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1}\sin(2\psi_s + 2\psi_0)\left(\alpha_2\frac{\partial\alpha_2}{\partial a} - \alpha_3\frac{\partial\alpha_3}{\partial a}\right) - \frac{1}{2}\sin(2\psi_s + 2\psi_0)(-2\alpha_1)^{-1}\frac{\partial\alpha_1}{\partial a}\right]\Bigg\},$$

$$\frac{\partial M_1}{\partial e} = -\frac{M_1}{a + b_1}\frac{\partial b_1}{\partial e} + (a + b_1)^{-1}\Bigg\{-\left(A_1 + c^2\eta_0^2 A_2 B_1 B_2^{-1}\right)\frac{\partial v_0}{\partial e}$$

$$- v_0\left[\frac{\partial A_1}{\partial e} + c^2\eta_0^2\left(\frac{B_1}{B_2}\frac{\partial A_2}{\partial e} + \frac{A_2}{B_2}\frac{\partial B_1}{\partial e} + A_2 B_1\frac{\partial B_2^{-1}}{\partial e}\right)\right]$$

$$+ \frac{c^2}{2}(-2\alpha_1)^{1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^3\left[\cos(2\psi_s + 2\psi_0)\left(\frac{\partial\psi_s}{\partial e} + \frac{\partial\psi_0}{\partial e}\right)\right.$$

$$\left. - \frac{1}{2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1}\sin(2\psi_s + 2\psi_0)\left(\alpha_2\frac{\partial\alpha_2}{\partial e} - \alpha_3\frac{\partial\alpha_3}{\partial e}\right) - \frac{1}{2}\sin(2\psi_s + 2\psi_0)(-2\alpha_1)^{-1}\frac{\partial\alpha_1}{\partial e}\right]\Bigg\},$$

$$\frac{\partial M_1}{\partial \eta_0} = \frac{-M_1}{a + b_1}\frac{\partial b_1}{\partial \eta_0} + (a + b_1)^{-1}\Bigg\{-\left(A_1 + c^2\eta_0^2 A_2 B_1 B_2^{-1}\right)\frac{\partial v_0}{\partial \eta_0}$$

$$- v_0\left(\frac{\partial A_1}{\partial \eta_0} + c^2\eta_0^2 B_1 B_2^{-1}\frac{\partial A_2}{\partial \eta_0} + 2c^2 B_1 B_2^{-1} A_2\eta_0 + c^2\eta_0^2 A_2 B_2^{-1}\frac{\partial B_1}{\partial \eta_0}\right.$$

$$\left. + c^2\eta_0^2 A_2 B_1\frac{\partial B_2^{-1}}{\partial \eta_0}\right) - \frac{c^2}{2}(-2\alpha_1)^{-1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^3\left[\frac{1}{2}\frac{\partial\alpha_1}{\partial \eta_0}\sin(2\psi_s + 2\psi_0)\right.$$

$$- \frac{3}{2}(-2\alpha_1)\frac{1}{\eta_0}\sin(2\psi_s + 2\psi_0) - (-2\alpha_1)\cos(2\psi_s + 2\psi_0)\left(\frac{\partial\psi_s}{\partial \eta_0} + \frac{\partial\psi_0}{\partial \eta_0}\right)$$

$$\left.\left. + \frac{1}{2}(-2\alpha_1)\left(\alpha_2^2 - \alpha_3^2\right)^{-1}\sin(2\psi_s + 2\psi_0)\left(\alpha_2\frac{\partial\alpha_2}{\partial \eta_0} - \alpha_3\frac{\partial\alpha_3}{\partial \eta_0}\right)\right]\right\}.$$

And,

$$\frac{\partial E_1}{\partial a} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_1}{\partial a} - M_1 (1 - e' \cos \mathcal{E})^{-2} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial a}\right.$$

$$\left. - \cos \mathcal{E} \frac{\partial e'}{\partial a}\right) - M_1 \sin \mathcal{E} \left(1 - e' \cos \mathcal{E}\right)^{-3} \left[\frac{M_1}{2} \frac{\partial e'}{\partial a}\right.$$

$$\left. - \frac{3}{2} e' M_1 (1 - e' \cos \mathcal{E})^{-1} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial a} - \cos \mathcal{E} \frac{\partial e'}{\partial a}\right) + \frac{e'}{2} M_1 \operatorname{ctn} \mathcal{E} \left(\frac{\partial \mathcal{E}}{\partial a} + e' \frac{\partial M_1}{\partial a}\right)\right],$$

$$\frac{\partial E_1}{\partial e} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_1}{\partial e} - M_1 (1 - e' \cos \mathcal{E})^{-2} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial e} - \cos \mathcal{E} \frac{\partial e'}{\partial e}\right)$$

$$- \frac{M_1^2}{2} (1 - e' \cos \mathcal{E})^{-3} \sin \mathcal{E} \frac{\partial e'}{\partial e} - e' (1 - e' \cos \mathcal{E}^{-3}) M_1 \sin \mathcal{E} \frac{\partial M_1}{\partial e}$$

$$- \frac{e'}{2} (1 - e' \cos \mathcal{E})^{-3} M_1^2 \cos \mathcal{E} \frac{\partial \mathcal{E}}{\partial e}$$

$$+ \frac{3}{2} e' (1 - e' \cos \mathcal{E})^{-4} M_1^2 \sin \mathcal{E} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial e} - \cos \mathcal{E} \frac{\partial e'}{\partial e}\right),$$

$$\frac{\partial E_1}{\partial \eta_0} = (1 - e' \cos \mathcal{E})^{-1} \frac{\partial M_1}{\partial \eta_0} - M_1 (1 - e' \cos \mathcal{E})^{-2} \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial \eta_0} - \cos \mathcal{E} \frac{\partial e'}{\partial \eta_0}\right)$$

$$- M_1 \sin \mathcal{E} (1 - e' \cos \mathcal{E})^{-3} \left[\frac{M_1}{2} \frac{\partial e'}{\partial \eta_0} + e' \frac{\partial M_1}{\partial \eta_0} + \frac{e'}{2} M_1 \operatorname{ctn} \mathcal{E} \frac{\partial \mathcal{E}}{\partial \eta_0}\right.$$

$$\left. - \frac{3}{2} e' (1 - e' \cos \mathcal{E})^{-1} M_1 \left(e' \sin \mathcal{E} \frac{\partial \mathcal{E}}{\partial \eta_0} - \cos \mathcal{E} \frac{\partial e'}{\partial \eta_0}\right)\right].$$

Then

$$\frac{\partial v_1}{\partial a} = \frac{(1-e^2)\sin(\mathcal{E}+E_1)\left(\frac{\partial \mathcal{E}}{\partial a}+\frac{\partial E_1}{\partial a}\right)}{\sin(M_s+v_0+v_1)\,[1-e\cos(\mathcal{E}+E_1)]^2} - \left(\frac{\partial M_s}{\partial a}+\frac{\partial v_0}{\partial a}\right),$$

$$\frac{\partial v_1}{\partial e} = \frac{(1-e^2)\sin(\mathcal{E}+E_1)\left(\frac{\partial \mathcal{E}}{\partial e}+\frac{\partial E_1}{\partial e}\right)+\sin^2(\mathcal{E}+E_1)}{\sin(M_s+v_0+v_1)\,[1-e\cos(\mathcal{E}+E_1)]^2} - \left(\frac{\partial M_s}{\partial e}+\frac{\partial v_0}{\partial e}\right),$$

$$\frac{\partial v_1}{\partial \eta_0} = \frac{(1-e^2)\sin(\mathcal{E}+E_1)\left(\frac{\partial \mathcal{E}}{\partial \eta_0}+\frac{\partial E_1}{\partial \eta_0}\right)}{\sin(M_s+v_0+v_1)\,[1-e\cos(\mathcal{E}+E_1)]^2} - \left(\frac{\partial M_s}{\partial \eta_0}+\frac{\partial v_0}{\partial \eta_0}\right).$$

Now

$$\frac{\partial \psi_1}{\partial a} = \left[\psi_1 - \frac{1}{8}q^2B_2^{-1}\sin(2\psi_s+2\psi_0)\right]\left[\frac{1}{-2\alpha_1}\frac{\partial \alpha_1}{\partial a} + \frac{1}{\alpha_2^2-\alpha_3^2}\left(\alpha_2\frac{\partial \alpha_2}{\partial a}-\alpha_3\frac{\partial \alpha_3}{\partial a}\right) - B_2^{-1}\frac{\partial B_2}{\partial a}\right]$$

$$+(-2\alpha_1)^{-1/2}\left(\alpha_2^2-\alpha_3^2\right)^{1/2}\eta_0^{-1}B_2^{-1}\left[A_2\frac{\partial v_1}{\partial a}+v_1\frac{\partial A_2}{\partial a}+A_{21}\cos(M_s+v_0)\left(\frac{\partial M_s}{\partial a}+\frac{\partial v_0}{\partial a}\right)\right.$$

$$+\sin(M_s+v_0)\frac{\partial A_{21}}{\partial a}+\sin(2M_s+2v_0)\frac{\partial A_{22}}{\partial a}$$

$$\left.+2A_{22}\cos(2M_s+2v_0)\left(\frac{\partial M_s}{\partial a}+\frac{\partial v_0}{\partial a}\right)\right]+\frac{1}{4}q^2B_2^{-1}\cos(2\psi_s+2\psi_0)\left(\frac{\partial \psi_s}{\partial a}+\frac{\partial \psi_0}{\partial a}\right)$$

$$+\frac{1}{4}qB_2^{-1}\sin(2\psi_s+2\psi_0)\frac{\partial q}{\partial a}-\frac{1}{8}q^2B_2^{-2}\sin(2\psi_s+2\psi_0)\frac{\partial B_2}{\partial a},$$

with

$$\frac{\partial A_{21}}{\partial a} = -\frac{A_{21}}{p}\frac{\partial p}{\partial a}+\frac{(1-e^2)^{1/2}}{p}e\left[-\frac{b_1}{p^2}\frac{\partial p}{\partial a}+\frac{1}{p}\frac{\partial b_1}{\partial a}-\frac{2}{p^3}\left(3b_1^2-b_2^2\right)\frac{\partial p}{\partial a}\right.$$

$$+\frac{1}{p^2}\left(6b_1\frac{\partial b_1}{\partial a}-2b_2\frac{\partial b_2}{\partial a}\right)+\frac{27}{2}b_1b_2^2\left(1+\frac{e^2}{4}\right)\frac{1}{p^4}\frac{\partial p}{\partial a}$$

$$-\frac{9}{2}\frac{1}{p^3}\left(1+\frac{e^2}{4}\right)b_2^2\frac{\partial b_1}{\partial a}-\frac{9}{p^3}\left(1+\frac{e^2}{4}\right)b_1 b_2\frac{\partial b_2}{\partial a}+\frac{3}{2}b_2^3(4+3e^2)\frac{1}{p^4}\frac{\partial b_2}{\partial a}$$

$$-\frac{3}{2}b_2^4(4+3e^2)\frac{1}{p^5}\frac{\partial p}{\partial a}\Bigg],$$

and

$$\frac{\partial A_{22}}{\partial a}=-\frac{A_{22}}{p}\frac{\partial p}{\partial a}+\frac{(1-e^2)^{1/2}}{p}\left[\frac{e^2}{8}p^{-2}\left(6b_1\frac{\partial b_1}{\partial a}-2b_2\frac{\partial b_2}{\partial a}\right)-\frac{e^2}{4p^3}\left(3b_1^2-b_2^2\right)\frac{\partial p}{\partial a}-\frac{9}{8}e^2b_2^2p^{-3}\frac{\partial b_1}{\partial a}\right.$$

$$\left.-\frac{9}{4}e^2b_1p^{-3}\frac{\partial b_2}{\partial a}b_2+\frac{27}{8}e^2b_1b_2^2p^{-4}\frac{\partial p}{\partial a}+\frac{3}{8}b_2^3(6e^2+e^4)\frac{1}{p^4}\frac{\partial b_2}{\partial a}-\frac{3}{8}b_2^4(6e^2+e^4)\frac{1}{p^5}\frac{\partial p}{\partial a}\right]$$

$$\frac{\partial\psi_1}{\partial e}=\left[\psi_1-\frac{1}{8}q^2B_2^{-1}\sin(2\psi_s+2\psi_0)\right]\left[\frac{1}{-2\alpha_1}\frac{\partial\alpha_1}{\partial e}+\frac{1}{\alpha_2^2-\alpha_3^2}\left(\alpha_2\frac{\partial\alpha_2}{\partial e}-\alpha_3\frac{\partial\alpha_3}{\partial e}\right)-B_2^{-1}\frac{\partial B_2}{\partial e}\right]$$

$$+\frac{1}{4}qB_2^{-1}\sin(2\psi_s+2\psi_0)\frac{\partial q}{\partial e}+(-2\alpha_1)^{-1/2}\left(\alpha_2^2-\alpha_3^2\right)^{1/2}\eta_0^{-1}B_2^{-1}\left[A_2\frac{\partial v_1}{\partial e}\right.$$

$$+v_1\frac{\partial A_2}{\partial e}+\sin(M_s+v_0)\frac{\partial A_{21}}{\partial e}+A_{21}\cos(M_s+v_0)\left(\frac{\partial M_s}{\partial e}+\frac{\partial v_0}{\partial e}\right)$$

$$\left.+\sin(2M_s+2v_0)\frac{\partial A_{22}}{\partial e}+2A_{22}\cos(2M_s+2v_0)\left(\frac{\partial M_s}{\partial e}+\frac{\partial v_0}{\partial e}\right)\right]$$

$$+\frac{1}{4}q^2B_2^{-1}\cos(2\psi_s+2\psi_0)\left(\frac{\partial\psi_s}{\partial e}+\frac{\partial\psi_0}{\partial e}\right)-\frac{1}{8}q^2B_2^{-2}\sin(2\psi_s+2\psi_0)\frac{\partial B_2}{\partial e},$$

with

$$\frac{\partial A_{21}}{\partial e}=-\frac{A_{21}}{1-e^2}e+\frac{A_{21}}{e}-\frac{A_{21}}{p}+\frac{\partial p}{\partial e}+\frac{(1-e^2)^{1/2}e}{p}\left[\frac{1}{p}\frac{\partial b_1}{\partial e}-\frac{b_1}{p^2}\frac{\partial p}{\partial e}\right.$$

$$-\frac{2}{p^3}\left(3b_1^2-b_2^2\right)\frac{\partial p}{\partial e}+\frac{1}{p^2}\left(6b_1\frac{\partial b_1}{\partial e}-2b_2\frac{\partial b_2}{\partial e}\right)-\frac{9}{2}b_2^2\left(1+\frac{e^2}{4}\right)\frac{1}{p^3}\frac{\partial b_1}{\partial e}$$

$$-9b_1b_2\left(1+\frac{e^2}{4}\right)\frac{1}{p^3}\frac{\partial b_2}{\partial e}-\frac{9}{4}b_1b_2^2p^{-3}e+\frac{27}{2}b_1b_2^2\left(1+\frac{e^2}{4}\right)\frac{1}{p^4}\frac{\partial p}{\partial e}$$

$$+\frac{3}{2}b_2^3\,(4+3e^2)\frac{1}{p^4}\frac{\partial b_2}{\partial e}-\frac{3}{2}b_2^4\frac{1}{p^5}\,(4+3e^2)\,\frac{\partial p}{\partial e}+\frac{9}{4}b_2^4\,ep^{-4}\Bigg],$$

and

$$\frac{\partial A_{22}}{\partial e}=-\frac{A_{22}\,e}{1-e^2}-\frac{A_{22}}{p}\frac{\partial p}{\partial e}+\frac{(1-e^2)^{1/2}}{p}\Bigg[\frac{1}{4}e\left(3b_1^2-b_2^2\right)\frac{1}{p^2}-\frac{e^2}{4}\left(3b_1^2-b_2^2\right)\frac{1}{p^3}\frac{\partial p}{\partial e}$$

$$+\left|\frac{e^2}{4p^2}\left(3b_1\frac{\partial b_1}{\partial e}-b_2\frac{\partial b_2}{\partial e}\right)+\frac{27}{8}e^2b_1b_2^2\frac{1}{p^4}\frac{\partial p}{\partial e}-\frac{9}{4}eb_1b_2^2\,p^{-3}-\frac{9}{8}e^2b_2^2p^{-3}\frac{\partial b_1}{\partial e}-\frac{9}{4}e^2b_1b_2p^{-3}\frac{\partial b_2}{\partial e}\right.$$

$$-\frac{3}{8}b_2^4\,(6e^2+e^4)\frac{1}{p^5}\frac{\partial p}{\partial e}+\frac{3}{8}(6e^2+e^4)\frac{1}{p^4}b_2^3\frac{\partial b_2}{\partial e}+\frac{3}{32}b_2^4p^{-4}(12e+4e^3)\Bigg];$$

$$\frac{\partial\psi_1}{\partial\eta_0}=\left[\psi_1-\frac{1}{8}q^2B_2^{-1}\sin(2\psi_s+2\psi_0)\right]\left[\frac{1}{-2\alpha_1}\frac{\partial\alpha_1}{\partial\eta_0}+\frac{1}{\alpha_2^2-\alpha_3^2}\left(\alpha_2\frac{\partial\alpha_2}{\partial\eta_0}-\alpha_3\frac{\partial\alpha_3}{\partial\eta_0}\right)-\frac{1}{\eta_0}-B_2^{-1}\frac{\partial B_2}{\partial\eta_0}\right]$$

$$+(-2\alpha_1)^{-1/2}\left(\alpha_2^2-\alpha_3^2\right)^{1/2}\eta_0^{-1}B_2^{-1}\Bigg[A_2\frac{\partial v_1}{\partial\eta_0}+v_1\frac{\partial A_2}{\partial\eta_0}+\sin(M_s+v_0)\frac{\partial A_{21}}{\partial\eta_0}$$

$$+A_{21}\cos(M_s+v_0)\left(\frac{\partial M_s}{\partial\eta_0}+\frac{\partial v_0}{\partial\eta_0}\right)+\sin(2M_s+2v_0)\frac{\partial A_{22}}{\partial\eta_0}$$

$$+2A_{22}\cos(2M_s+2v_0)\left(\frac{\partial M_s}{\partial\eta_0}+\frac{\partial v_0}{\partial\eta_0}\right)+\frac{1}{4}q\,B_2^{-1}\sin(2\psi_s+2\psi_0)\frac{\partial q}{\partial\eta_0}$$

$$+\frac{1}{4}q^2B_2^{-1}\cos(2\psi_s+2\psi_0)\left(\frac{\partial\psi_s}{\partial\eta_0}+\frac{\partial\psi_0}{\partial\eta_0}\right)\Bigg]-\frac{1}{8}q^2B_2^{-2}\sin(2\psi_s+2\psi_0)\frac{\partial B_2}{\partial\eta_0},$$

with

$$\frac{\partial A_{21}}{\partial\eta_0}=\frac{(1-e^2)^{1/2}e}{p}\Bigg[\frac{1}{p}\frac{\partial b_1}{\partial\eta_0}+\frac{1}{p^2}\left(6b_1\frac{\partial b_1}{\partial\eta_0}-2b_2\frac{\partial b_2}{\partial\eta_0}\right)$$

$$-\frac{9}{2}\left(1+\frac{e^2}{4}\right)\frac{1}{p^3}b_2^2\frac{\partial b_1}{\partial\eta_0}-9b_1\left(1+\frac{e^2}{4}\right)\frac{1}{p^3}b_2\frac{\partial b_2}{\partial\eta_0}$$

$$+\frac{3}{2}b_2^3\,(4+3e^2)\frac{1}{p^4}\frac{\partial b_2}{\partial\eta_0}\Bigg],$$

and

$$\frac{\partial A_{22}}{\partial \eta_0} = \left[\frac{(1-e^2)^{1/2}}{p}\left\{\frac{e^2}{8}\frac{1}{p^2}\left(6b_1\frac{\partial b_1}{\partial \eta_0} - 2b_2\frac{\partial b_2}{\partial \eta_0}\right) - \frac{9}{8}e^2 b_2^2 \frac{1}{p^3}\frac{\partial b_1}{\partial \eta_0}\right.\right.$$

$$\left.\left. - \frac{9}{4}e^2 b_1 b_2 \frac{1}{p^3}\frac{\partial b_2}{\partial \eta_0} + \frac{3}{8}b_2^3\,(6e^2 + e^4)\frac{1}{p^4}\frac{\partial b_2}{\partial \eta_0}\right\}\right].$$

Then

$$\frac{\partial M_2}{\partial a} - \frac{M_2}{a+b_1}\left(1 + \frac{\partial b_1}{\partial a}\right) - (a+b_1)^{-1}\left\{A_1\frac{\partial v_1}{\partial a} + v_1\frac{\partial A_1}{\partial a}\right.$$

$$+ \sin(M_s + v_0)\frac{\partial A_{11}}{\partial a} + A_{11}\cos(M_s + v_0)\left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a}\right)$$

$$+ 2A_{12}\cos(2M_s + 2v_0)\left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a}\right) + \sin(2M_s + 2v_0)\frac{\partial A_{12}}{\partial a}$$

$$- c^2(-2\alpha_1)^{-1/2}\frac{\partial \alpha_1}{\partial a}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^3\left[B_1\psi_1 - \frac{1}{2}\psi_1\cos(2\psi_s + 2\psi_0)\right.$$

$$\left. - \frac{1}{8}q^2\sin(2\psi_s + 2\psi_0) + \frac{1}{64}q^2\sin(4\psi_s + 4\psi_0)\right]$$

$$- c^2(-2\alpha_1)^{1/2}\eta_0^3\left(\alpha_2\frac{\partial \alpha_2}{\partial a} - \alpha_3\frac{\partial \alpha_3}{\partial a}\right)\left[B_1\psi_1 - \frac{1}{2}\psi_1\cos(2\psi_s + 2\psi_0)\right.$$

$$\left. - \frac{1}{8}q^2\sin(2\psi_s + 2\psi_0) + \frac{1}{64}q^2\sin(4\psi_s + 4\psi_0)\right]\left(\alpha_2^2 - \alpha_3^2\right)^{-3/2}$$

$$+ c^2(-2\alpha_1)^{1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^3\left[B_1\frac{\partial \psi_1}{\partial a} - \frac{1}{2}\frac{\partial \psi_1}{\partial a}\cos(2\psi_s + 2\psi_0)\right.$$

$$+ \psi_1\frac{\partial B_1}{\partial a} + \psi_1\sin(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a}\right)$$

$$- \frac{1}{4}q^2\cos(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a}\right) + \frac{1}{16}q^2\cos(4\psi_s + 4\psi_0)\left(\frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a}\right)$$

$$-\frac{1}{4}\, q \sin(2\psi_s + 2\psi_0)\frac{\partial q}{\partial a} + \frac{1}{32}\, q \sin(4\psi_s + 4\psi_0)\frac{\partial q}{\partial a}\Bigg]\Bigg\},$$

with

$$\frac{\partial A_{12}}{\partial a} = -\frac{9}{32}(1-e^2)^{1/2} b_2^4 e^2 p^{-4}\frac{\partial p}{\partial a} + \frac{3}{8}(1-e^2)^{1/2} b_2^3 e^2 p^{-3}\frac{\partial b_2}{\partial a},$$

and

$$\frac{\partial A_{11}}{\partial a} = -\frac{9}{4}(1-e^2)^{1/2} p^{-4} e\left(-2b_1 b_2^2 p + b_2^4\right)\frac{\partial p}{\partial a}$$

$$+\frac{3}{4}(1-e^2)^{1/2} p^{-3} e\left(-2b_2^2 p\frac{\partial b_1}{\partial a} - 4b_1 b_2 p\frac{\partial b_2}{\partial a} - 2b_1 b_2^2\frac{\partial p}{\partial a} + 4b_2^3\frac{\partial b_2}{\partial a}\right);$$

$$\frac{\partial M_2}{\partial e} = \frac{-M_2}{a+b_1}\frac{\partial b_1}{\partial e} - (a+b_1)^{-1}\Bigg\{A_1\frac{\partial v_1}{\partial e} + v_1\frac{\partial A_1}{\partial e} + \sin(M_s + v_0)\frac{\partial A_{11}}{\partial e}$$

$$+A_{11}\cos(M_s + v_0)\left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e}\right) + \sin(2M_s + 2v_0)\frac{\partial A_{12}}{\partial e}$$

$$+2A_{12}\cos(2M_s + 2v_0)\left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e}\right) - c^2(-2\alpha_1)^{-1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^3\frac{\partial \alpha_1}{\partial e}\Bigg[B_1\psi_1 - \frac{1}{2}\psi_1\cos(2\psi_s + 2\psi_0)$$

$$-\frac{1}{8}q^2\sin(2\psi_s + 2\psi_0) + \frac{1}{64}q^2\sin(4\psi_s + 4\psi_0)\Bigg] - c^2(-2\alpha_1)^{1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-3/2}\eta_0^3\left(\alpha_2\frac{\partial \alpha_2}{\partial e} - \alpha_3\frac{\partial \alpha_3}{\partial e}\right)\Bigg[B_1\psi_1$$

$$-\frac{1}{2}\psi_1\cos(2\psi_s + 2\psi_0) - \frac{1}{8}q^2\sin(2\psi_s + 2\psi_0) + \frac{1}{64}q^2\sin(4\psi_s + 4\psi_0)\Bigg]$$

$$+c^2(-2\alpha_1)^{1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^3\Bigg[B_1\frac{\partial \psi_1}{\partial e} + \psi_1\frac{\partial B_1}{\partial e} + \psi_1\sin(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e}\right)$$

$$-\frac{1}{2}\cos(2\psi_s + 2\psi_0)\frac{\partial \psi_1}{\partial e} - \frac{1}{4}q^2\cos(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e}\right)$$

$$-\frac{1}{4}q\sin(2\psi_s + 2\psi_0)\frac{\partial q}{\partial e} + \frac{1}{16}q^2\cos(4\psi_s + 4\psi_0)\left(\frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e}\right)$$

$$+\frac{1}{32}q\sin(4\psi_s + 4\psi_0)\frac{\partial q}{\partial e}\Bigg]\Bigg\},$$

with

$$\frac{\partial A_{11}}{\partial e} = -\frac{e}{1-e^2} A_{11} - \frac{3A_{11}}{p}\frac{\partial p}{\partial e} + \frac{A_{11}}{e} + \frac{A_{11}}{-2b_1b_2^2p + b_2^4}\left(-2b_2^2p\frac{\partial b_1}{\partial e} - 4b_1b_2p\frac{\partial b_2}{\partial e} - 2b_1b_2^2\frac{\partial p}{\partial e} + 4b_2^3\frac{\partial b_2}{\partial e}\right),$$

and

$$\frac{\partial A_{12}}{\partial e} = \frac{2A_{12}}{e} - \frac{3}{32}(1-e^2)^{-1/2} b_2^4 e^3 p^{-3} + \frac{3}{8}(1-e^2)^{1/2} b_2^3 e^2 p^{-3}\frac{\partial b_2}{\partial e} - \frac{9}{32}(1-e^2)^{1/2} b_2^4 e^2 p^{-4}\frac{\partial p}{\partial e};$$

$$\frac{\partial M_2}{\partial \eta_0} = \frac{-M_2}{a+b_1}\frac{\partial b_1}{\partial \eta_0}, - (a+b_1)^{-1}\left\{A_1\frac{\partial v_1}{\partial \eta_0} + v_1\frac{\partial A_1}{\partial \eta_0} + \sin(M_s + v_0)\frac{\partial A_{11}}{\partial \eta_0}\right.$$

$$+ A_{11}\cos(M_s + v_0)\left(\frac{\partial M_s}{\partial \eta_0} + \frac{\partial v_0}{\partial \eta_0}\right) + \sin(2M_s + 2v_0)\frac{\partial A_{12}}{\partial \eta_0} + 2A_{12}\cos(2M_s + 2v_0)\left(\frac{\partial M_s}{\partial \eta_0} + \frac{\partial v_0}{\partial \eta_0}\right)$$

$$- c^2(-2\alpha_1)^{-1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^3\frac{\partial \alpha_1}{\partial \eta_0}\left[B_1\psi_1 - \frac{1}{2}\psi_1\cos(2\psi_s + 2\psi_0) - \frac{1}{8}q^2\sin\left(2\psi_s + 2\psi_0\right)\right.$$

$$\left. + \frac{1}{64}q^2\sin(4\psi_s + 4\psi_0)\right] - c^2(-2\alpha_1)^{1/2}\eta_0^3\left(\alpha_2\frac{\partial \alpha_2}{\partial \eta_0} - \alpha_3\frac{\partial \alpha_3}{\partial \eta_0}\right)\left(\alpha_2^2 - \alpha_3^2\right)^{-3/2}\left[B_1\psi_1\right.$$

$$\left. - \frac{1}{2}\psi_1\cos(2\psi_s + 2\psi_0) - \frac{1}{8}q^2\sin(2\psi_s + 2\psi_0) + \frac{1}{64}q^2\sin(4\psi_s + 4\psi_0)\right]$$

$$+ 3c^2(-2\alpha_1)^{1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^2\left[B_1\psi_1 - \frac{1}{2}\psi_1\cos(2\psi_s + 2\psi_0)\right.$$

$$\left. - \frac{1}{8}q^2\sin(2\psi_s + 2\psi_0) + \frac{1}{64}q^2\sin(4\psi_s + 4\psi_0)\right]$$

$$+ c^2(-2\alpha_1)^{1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0^3\left[B_1\frac{\partial \psi_1}{\partial \eta_0} + \psi_1\frac{\partial B_1}{\partial \eta_0} + \psi_1\sin(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial \eta_0} + \frac{\partial \psi_0}{\partial \eta_0}\right)\right.$$

$$- \frac{1}{2}\cos(2\psi_s + 2\psi_0)\frac{\partial \psi_1}{\partial \eta_0} - \frac{1}{4}q\sin(2\psi_s + 2\psi_0)\frac{\partial q}{\partial \eta_0} - \frac{1}{4}q^2\cos(2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial \eta_0} + \frac{\partial \psi_0}{\partial \eta_0}\right)$$

$$\left.\left. + \frac{1}{32}q\sin(4\psi_s + 4\psi_0)\frac{\partial q}{\partial \eta_0} + \frac{1}{16}q^2\cos(4\psi_s + 4\psi_0)\left(\frac{\partial \psi_s}{\partial \eta_0} + \frac{\partial \psi_0}{\partial \eta_0}\right)\right]\right\}$$

with

$$\frac{\partial A_{11}}{\partial \eta_0} = \frac{3}{4} \frac{(1 - e^2)^{1/2}}{p^3} e \left(-2b_2^2 p \frac{\partial b_1}{\partial \eta_0} - 4pb_1b_2 \frac{\partial b_2}{\partial \eta_0} + 4b_2^3 \frac{\partial b_2'}{\partial \eta_0} \right),$$

and

$$\frac{\partial A_{12}}{\partial \eta_0} = \frac{3}{8} (1 - e^2)^{1/2} b_2^3 e^2 p^{-3} \frac{\partial b_2}{\partial \eta_0}.$$

Now

$$\frac{\partial E_2}{\partial a} = -M_2 \left[1 - e' \cos(\mathcal{E} + E_1)\right]^{-2} \left[\sin(\mathcal{E} + E_1) \left(\frac{\partial \mathcal{E}}{\partial a} + \frac{\partial E_1}{\partial a} \right) e' - \cos(\mathcal{E} + E_1) \frac{\partial e'}{\partial a} \right]$$

$$+ \left[1 - e' \cos(\mathcal{E} + E_1)\right]^{-1} \frac{\partial M_2}{\partial a},$$

$$\frac{\partial E_2}{\partial e} = -M_2 \left[1 - e' \cos(\mathcal{E} + E_1)\right]^{-2} \left[\left(\frac{\partial \mathcal{E}}{\partial e} + \frac{\partial E_1}{\partial e} \right) e' \sin(\mathcal{E} + E_1) - \cos(\mathcal{E} + E_1) \frac{\partial e'}{\partial e} \right]$$

$$+ \left[1 - e' \cos(\mathcal{E} + E_1)\right]^{-1} \frac{\partial M_2}{\partial e},$$

$$\frac{\partial E_2}{\partial \eta_0} = -M_2 \left[1 - e' \cos(\mathcal{E} + E_1)\right]^{-2} \left[\sin(\mathcal{E} + E_1) \left(\frac{\partial \mathcal{E}}{\partial \eta_0} + \frac{\partial E_1}{\partial \eta_0} \right) e' - \cos(\mathcal{E} + E_1) \frac{\partial e'}{\partial \eta_0} \right]$$

$$+ \left[1 - e' \cos(\mathcal{E} + E_1)\right]^{-1} \frac{\partial M_2}{\partial \eta_0}.$$

We now have $\frac{\partial E}{\partial a}$, $\frac{\partial E}{\partial e}$, and $\frac{\partial E}{\partial \eta_0}$. Also

$$\frac{\partial v_2}{\partial a} = \frac{(1 - e^2) \sin E}{\sin v (1 - e \cos E)^2} \frac{\partial E}{\partial a} - \left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a} + \frac{\partial v_1}{\partial a} \right),$$

$$\frac{\partial v_2}{\partial e} = \frac{(1 - e^2)\sin E \frac{\partial E}{\partial e} - \cos^2 E + 1}{(1 - e\cos E)^2 \sin v} - \left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e} + \frac{\partial v_1}{\partial e}\right),$$

$$\frac{\partial v_2}{\partial \eta_0} = \frac{(1 - e^2)\sin E}{\sin v\,(1 - e\cos E)^2} \frac{\partial E}{\partial \eta_0} - \left(\frac{\partial M_s}{\partial \eta_0} + \frac{\partial v_0}{\partial \eta_0} + \frac{\partial v_1}{\partial \eta_0}\right).$$

Since $v = M_s + v_0 + v_1 + v_2$,

$$\frac{\partial v}{\partial a} = \frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a} + \frac{\partial v_1}{\partial a} + \frac{\partial v_2}{\partial a},$$

$$\frac{\partial v}{\partial e} = \frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e} + \frac{\partial v_1}{\partial e} + \frac{\partial v_2}{\partial e},$$

$$\frac{\partial v}{\partial \eta_0} = \frac{\partial M_s}{\partial \eta_0} + \frac{\partial v_0}{\partial \eta_0} + \frac{\partial v_1}{\partial \eta_0} + \frac{\partial v_2}{\partial \eta_0}.$$

Now,

$$\frac{\partial \psi_2}{\partial a} = \left\{\psi_2 - \frac{1}{4} q^2 B_2^{-1}\left[\psi_1 \cos(2\psi_s + 2\psi_0) + \frac{3}{8} q^2 \sin(2\psi_s + 2\psi_0) - \frac{3}{64} q^2 \sin(4\psi_s + 4\psi_0)\right]\right\} \left[\frac{1}{-2\alpha_1} \frac{\partial \alpha_1}{\partial a}\right.$$

$$\left. + \frac{1}{\alpha_2^2 - \alpha_3^2}\left(\alpha_2 \frac{\partial \alpha_2}{\partial a} - \alpha_3 \frac{\partial \alpha_3}{\partial a}\right) - B_2^{-1} \frac{\partial B_2}{\partial a}\right]$$

$$+ (-2\alpha_1)^{-1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} B_2^{-1}\left[A_2 \frac{\partial v_2}{\partial a} + v_2 \frac{\partial A_2}{\partial a} + v_1 \cos(M_s + v_0) \frac{\partial A_{21}}{\partial a}\right.$$

$$+ A_{21} \cos(M_s + v_0) \frac{\partial v_1}{\partial a} - A_{21} v_1 \sin(M_s + v_0)\left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a}\right)$$

$$+ 2 A_{22} \cos(2M_s + 2v_0) \frac{\partial v_1}{\partial a} + 2v_1 \cos(2M_s + 2v_0) \frac{\partial A_{22}}{\partial a}$$

$$- 4 A_{22} v_1 \sin(2M_s + 2v_0)\left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a}\right) + \sin(3M_s + 3v_0) \frac{\partial A_{23}}{\partial a}$$

$$+ 3A_{23} \cos(3M_s + 3v_0)\left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a}\right) + \frac{\partial A_{24}}{\partial a} \sin(4M_s + 4v_0)$$

$$+ 4A_{24} \cos (4M_s + 4v_0)\left(\frac{\partial M_s}{\partial a} + \frac{\partial v_0}{\partial a}\right)\Bigg] + \frac{1}{2} q\left(\frac{\partial q}{\partial a}\right) B_2^{-1} \Big[\psi_1 \cos (2\psi_s + 2\psi_0)$$

$$+ \frac{3}{8} q^2 \sin (2\psi_s + 2\psi_0) - \frac{3}{64} q^2 \sin (4\psi_s + 4\psi_0)\Big]$$

$$- \frac{1}{4} q^2 B_2^{-2} \frac{\partial B_2}{\partial a} \Big[\psi_1 \cos (2\psi_s + 2\psi_0) + \frac{3}{8} q^2 \sin (2\psi_s + 2\psi_0)$$

$$- \frac{3}{64} q^2 \sin (4\psi_s + 4\psi_0)\Big] + \frac{1}{4} q^2 B_2^{-1}\Bigg[\frac{\partial \psi_1}{\partial a} \cos (2\psi_s + 2\psi_0) - 2\psi_1 \sin (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a}\right)$$

$$+ \frac{3}{4} q\left(\frac{\partial q}{\partial a}\right) \sin (2\psi_s + 2\psi_0) + \frac{3}{4} q^2 \cos (2\psi_s + 2\psi_0)\left(\frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a}\right)$$

$$- \frac{3}{32} q\left(\frac{\partial q}{\partial a}\right) \sin (4\psi_s + 4\psi_0) - \frac{3}{16} q^2 \cos (4\psi_s + 4\psi_0)\left(\frac{\partial \psi_s}{\partial a} + \frac{\partial \psi_0}{\partial a}\right)\Bigg],$$

where

$$\frac{\partial A_{23}}{\partial a} = -\frac{A_{23}}{p} \frac{\partial p}{\partial a} + \frac{(1 - e^2)^{1/2}}{p} \frac{e^3}{8}\left(- b_2^2 p^{-3} \frac{\partial b_1}{\partial a} - \frac{2b_1 b_2}{p^3} \frac{\partial b_2}{\partial a} + \frac{3b_1 b_2^2}{p^4} \frac{\partial p}{\partial a} + 4b_2^3 p^{-4} \frac{\partial b_2}{\partial a} - 4b_2^4 p^{-5} \frac{\partial p}{\partial a}\right),$$

$$\frac{\partial A_{24}}{\partial a} = \frac{4A_{24}}{b_2} \frac{\partial b_2}{\partial a} - \frac{5A_{24}}{p} \frac{\partial p}{\partial a}.$$

Also;

$$\frac{\partial \psi_2}{\partial e} = \left\{\psi_2 - \frac{1}{4} q^2 B_2^{-1} \left[\psi_1 \cos (2\psi_s + 2\psi_0) + \frac{3}{8} q^2 \sin (2\psi_s + 2\psi_0) - \frac{3}{64} q^2 \sin (4\psi_s + 4\psi_0)\right]\right\} \Bigg[\frac{1}{-2\alpha_1} \frac{\partial \alpha_1}{\partial e}$$

$$+ \frac{1}{\alpha_2^2 - \alpha_3^2} \left(\alpha_2 \frac{\partial \alpha_2}{\partial e} - \alpha_3 \frac{\partial \alpha_3}{\partial e}\right) - B_2^{-1} \frac{\partial B_2}{\partial e}\Bigg]$$

$$+ (-2\alpha_1)^{-1/2} \left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} B_2^{-1} \Bigg[A_2 \frac{\partial v_2}{\partial e} + v_2 \frac{\partial A_2}{\partial e} + v_1 \cos (M_s + v_0) \frac{\partial A_{21}}{\partial e}$$

$$+ A_{21} \cos (M_s + v_0) \frac{\partial v_1}{\partial e} - A_{21} v_1 \sin (M_s + v_0)\left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e}\right) + 2A_{22} \cos (2M_s + 2v_0) \frac{\partial v_1}{\partial e}$$

$$+2v_1 \cos(2M_s + 2v_0) \frac{\partial A_{22}}{\partial e} - 4A_{22} v_1 \sin(2M_s + 2v_0) \left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e}\right)$$

$$+ \sin(3M_s + 3v_0) \frac{\partial A_{23}}{\partial e} + 3A_{23} \cos(3M_s + 3v_0) \left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e}\right)$$

$$+ \frac{\partial A_{24}}{\partial e} \sin(4M_s + 4v_0) + 4A_{24} \cos(4M_s + 4v_0) \left(\frac{\partial M_s}{\partial e} + \frac{\partial v_0}{\partial e}\right)\Bigg]$$

$$+ \frac{1}{2} q \left(\frac{\partial q}{\partial e}\right) B_2^{-1} \left[\psi_1 \cos(2\psi_s + 2\psi_0) + \frac{3}{8} q^2 \sin(2\psi_s + 2\psi_0) - \frac{3}{64} q^2 \sin(4\psi_s + 4\psi_0)\right]$$

$$- \frac{1}{4} q^2 B_2^{-2} \frac{\partial B_2}{\partial e} \Bigg[\psi_1 \cos(2\psi_s + 2\psi_0) + \frac{3}{8} q^2 \sin(2\psi_s + 2\psi_0)$$

$$- \frac{3}{64} q^2 \sin(4\psi_s + 4\psi_0)\Bigg] + \frac{1}{4} q^2 B_2^{-1} \Bigg[\frac{\partial \psi_1}{\partial e} \cos(2\psi_s + 2\psi_0)$$

$$- 2\psi_1 \sin(2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e}\right) + \frac{3}{4} q \left(\frac{\partial q}{\partial e}\right) \sin(2\psi_s + 2\psi_0)$$

$$+ \frac{3}{4} q^2 \cos(2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e}\right) - \frac{3}{32} q \left(\frac{\partial q}{\partial e}\right) \sin(4\psi_s + 4\psi_0)$$

$$- \frac{3}{16} q^2 \cos(4\psi_s + 4\psi_0) \left(\frac{\partial \psi_s}{\partial e} + \frac{\partial \psi_0}{\partial e}\right)\Bigg],$$

where

$$\frac{\partial A_{23}}{\partial e} = - \frac{A_{23}}{1-e^2} e - \frac{A_{23}}{p} \frac{\partial p}{\partial e} + \frac{3A_{23}}{e} + \frac{(1-e^2)^{1/2}}{p} \frac{e^3}{8} \Bigg(-b_2^2 p^{-3} \frac{\partial b_1}{\partial e}$$

$$- \frac{2b_1 b_2}{p^3} \frac{\partial b_2}{\partial e} + \frac{3b_1 b_2^2}{p^4} \frac{\partial p}{\partial e} + 4b_2^3 p^{-4} \frac{\partial b_2}{\partial e} - 4p^{-5} b_2^4 \frac{\partial p}{\partial e}\Bigg),$$

$$\frac{\partial A_{24}}{\partial e} = - \frac{A_{24}}{1-e^2} e - \frac{5A_{24}}{p} \frac{\partial p}{\partial e} + \frac{4A_{24}}{b_2} \frac{\partial b_2}{\partial e} + \frac{4A_{24}}{e}.$$

And

$$\frac{\partial\psi_2}{\partial\eta_0} = \left\{\psi_2 - \frac{1}{4}q^2B_2^{-1}\left[\psi_1\cos(2\psi_s + 2\psi_0) + \frac{3}{8}q^2\sin(2\psi_s + 2\psi_0) - \frac{3}{64}q^2\sin(4\psi_s + 4\psi_0)\right]\right\}\left[\frac{1}{-2\alpha_1}\frac{\partial\alpha_1}{\partial\eta_0}\right.$$

$$\left. + \frac{1}{\alpha_2^2 - \alpha_3^2}\left(\alpha_2\frac{\partial\alpha_2}{\partial\eta_0} - \alpha_3\frac{\partial\alpha_3}{\partial\eta_0}\right) - B_2^{-1}\frac{\partial B_2}{\partial\eta_0} - \frac{1}{\eta_0}\right]$$

$$+ (-2\alpha_1)^{-1/2}\left(\alpha_2^2 - \alpha_3^2\right)^{1/2}\eta_0^{-1}B_2^{-1}\left[A_2\frac{\partial v_2}{\partial\eta_0} + v_2\frac{\partial A_2}{\partial\eta_0} + v_1\cos(M_s + v_0)\frac{\partial A_{21}}{\partial\eta_0}\right.$$

$$+ A_{21}\cos(M_s + v_0)\frac{\partial v_1}{\partial\eta_0} - A_{21}v_1\sin(M_s + v_0)\left(\frac{\partial M_s}{\partial\eta_0} + \frac{\partial v_0}{\partial\eta_0}\right)$$

$$+ 2A_{22}\cos(2M_s + 2v_0)\frac{\partial v_1}{\partial\eta_0} + 2v_1\cos(2M_s + 2v_0)\frac{\partial A_{22}}{\partial\eta_0}$$

$$- 4A_{22}v_1\sin(2M_s + 2v_0)\left(\frac{\partial M_s}{\partial\eta_0} + \frac{\partial v_0}{\partial\eta_0}\right) + \sin(3M_s + 3v_0)\frac{\partial A_{23}}{\partial\eta_0}$$

$$+ 3A_{23}\cos(3M_s + 3v_0)\left(\frac{\partial M_s}{\partial\eta_0} + \frac{\partial v_0}{\partial\eta_0}\right) + \frac{\partial A_{24}}{\partial\eta_0}\sin(4M_s + 4v_0)$$

$$\left. + 4A_{24}\cos(4M_s + 4v_0)\left(\frac{\partial M_s}{\partial\eta_0} + \frac{\partial v_0}{\partial\eta_0}\right)\right] + \left(\frac{1}{2}q\frac{\partial q}{\partial\eta_0}B_2^{-1} - \frac{1}{4}q^2B_2^{-2}\frac{\partial B_2}{\partial\eta_0}\right)\left[\psi_1\cos(2\psi_s + 2\psi_0)\right.$$

$$\left. + \frac{3}{8}q^2\sin(2\psi_s + 2\psi_0) - \frac{3}{64}q^2\sin(4\psi_s + 4\psi_0)\right] + \frac{1}{4}q^2B_2^{-1}\left[\frac{\partial\psi_1}{\partial\eta_0}\cos(2\psi_s + 2\psi_0)\right.$$

$$-2\psi_1\sin(2\psi_s + 2\psi_0)\left(\frac{\partial\psi_s}{\partial\eta_0} + \frac{\partial\psi_0}{\partial\eta_0}\right) + \frac{3}{4}q\left(\frac{\partial q}{\partial\eta_0}\right)\sin(2\psi_s + 2\psi_0)$$

$$\left. + \frac{3}{4}q^2\cos(2\psi_s + 2\psi_0)\left(\frac{\partial\psi_s}{\partial\eta_0} + \frac{\partial\psi_0}{\partial\eta_0}\right) - \frac{3}{32}q\left(\frac{\partial q}{\partial\eta_0}\right)\sin(4\psi_s + 4\psi_0) - \frac{3}{16}q^2\cos(4\psi_s + 4\psi_0)\left(\frac{\partial\psi_s}{\partial\eta_0} + \frac{\partial\psi_0}{\partial\eta_0}\right)\right],$$

where

$$\frac{\partial A_{23}}{\partial\eta_0} = \frac{(1-e^2)^{1/2}}{p}\frac{e^3}{8}\left(-\frac{b_2^2}{p^3}\frac{\partial b_1}{\partial\eta_0} - \frac{2b_1b_2}{p^3}\frac{\partial b_2}{\partial\eta_0} + \frac{4b_2^3}{p^4}\frac{\partial b_2}{\partial\eta_0}\right).$$

$$\frac{\partial A_{24}}{\partial \eta_0} = \frac{4A_{24}}{b_2} \frac{\partial b_2}{\partial \eta_0} .$$

We now have $\frac{\partial \psi}{\partial a}$, $\frac{\partial \psi}{\partial e}$, and $\frac{\partial \psi}{\partial \eta_0}$. Also

$$\frac{\partial \chi}{\partial a} = \frac{\left(1 - \eta_0^2\right) \sin \psi}{\sin \chi \left(1 - \eta_0^2 \sin^2 \psi\right)^{3/2}} \frac{\partial \psi}{\partial a},$$

$$\frac{\partial \chi}{\partial e} = \frac{\left(1 - \eta_0^2\right) \sin \psi}{\sin \chi \left(1 - \eta_0^2 \sin^2 \psi\right)^{3/2}} \frac{\partial \psi}{\partial e},$$

$$\frac{\partial \chi}{\partial \eta_0} = \frac{\left(1 - \eta_0^2\right) \sin \psi}{\sin \chi \left(1 - \eta_0^2 \sin^2 \psi\right)^{3/2}} \frac{\partial \psi}{\partial \eta_0} - \frac{\eta_0 \cos \psi \sin^2 \psi}{\sin \chi \left(1 - \eta_0^2 \sin^2 \psi\right)^{3/2}}$$

Then

$$\frac{\partial \phi}{\partial a} = \left(\alpha_2^2 - \alpha_3^2\right)^{-1/2} \eta_0 \left[(1 - \eta_0)^{-1/2} \left(1 - \eta_2^{-2}\right)^{-1/2} \chi + B_3 \psi + \frac{3}{32} \eta_0^2 \eta_2^{-4} \sin 2\psi\right] \frac{\partial \alpha_3}{\partial a}$$

$$- \frac{\alpha_3 \eta_0 \left[(1 - \eta_0)^{-1/2} \left(1 - \eta_2^{-2}\right)^{-1/2} \chi + B_3 \psi + \frac{3}{32} \eta_0^2 \eta_2^{-4} \sin 2\psi\right]}{\left(\alpha_2^2 - \alpha_3^2\right)^{3/2}} \left(\alpha_2 \frac{\partial \alpha_2}{\partial a} - \alpha_3 \frac{\partial \alpha_3}{\partial a}\right)$$

$$+ \alpha_3 \left(\alpha_2^2 - \alpha_3^2\right)^{-1/2} \eta_0 \left[\left(1 - \eta_0^2\right)^{-1/2} \left(1 - \eta_2^{-2}\right)^{-1/2} \frac{\partial \chi}{\partial a} - \frac{\left(1 - \eta_0^2\right)^{-1/2}}{\left(1 - \eta_2^{-2}\right)^{3/2}} \chi \eta_2^{-3} \frac{\partial \eta_2}{\partial a}\right.$$

$$\left. + B_3 \frac{\partial \psi}{\partial a} + \psi \frac{\partial B_3}{\partial a} - \frac{3}{8} \eta_0^2 \sin 2\psi \eta_2^{-5} \frac{\partial \eta_2}{\partial a} + \frac{3}{16} \frac{\eta_0^2 \cos 2\psi}{\eta_2^4} \frac{\partial \psi}{\partial a}\right]$$

$$- c^2 (-2\alpha_1)^{-1/2} \left(A_3 v + \sum_{n=1}^{4} A_{3n} \sin nv\right) \frac{\partial \alpha_3}{\partial a}$$

$$- c^2 \alpha_3 (-2\alpha_1)^{-3/2} \left(A_3 v + \sum_{n=1}^{4} A_{3n} \sin nv\right) \frac{\partial \alpha_1}{\partial a}$$

$$- c^2 \alpha_3 (-2\alpha_1)^{-1/2} \left[A_3 \frac{\partial v}{\partial a} + v \frac{\partial A_3}{\partial a} + \sum_{n=1}^{4} \left(n A_{3n} \cos nv \frac{\partial v}{\partial a} + \sin nv \frac{\partial A_{3n}}{\partial a}\right)\right],$$

with

$$\frac{\partial B_3}{\partial a} = \left(1 - \eta_2^{-2}\right)^{-3/2} \eta_2^{-3} \frac{\partial \eta_2}{\partial a} + \sum_{m=2}^{\infty} 2m\, \gamma_m\, \eta_2^{-2m-1} \frac{\partial \eta_2}{\partial a},$$

$$\frac{\partial A_3}{\partial a} = -\frac{3 A_3}{p} \frac{\partial p}{\partial a} + \frac{(1 - e^2)^{1/2}}{p^3} \sum_{m=0}^{\infty} R_{m+2} \left[\left(1 - e^2\right)^{1/2}\right] \frac{\partial D_m}{\partial a}.$$

If m is even

$$\frac{\partial D_m}{\partial a} = \frac{\partial D_{2i}}{\partial a} = \sum_{n=0}^{i} \left[(-1)^{i-n} (2i - 2n) \left(\frac{c}{p}\right)^{2i-2n-1} \left(-\frac{c}{p^2}\right) \frac{\partial p}{\partial a} \left(\frac{b_2}{p}\right)^{2n} P_{2n}\left(\frac{b_1}{b_2}\right)\right.$$

$$+ (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} 2n \left(\frac{b_2}{p}\right)^{2n-1} \left(\frac{1}{p} \frac{\partial b_2}{\partial a} - \frac{b_2}{p^2} \frac{\partial p}{\partial a}\right) P_{2n}\left(\frac{b_1}{b_2}\right)$$

$$\left. + (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n} \frac{\partial}{\partial a} P_{2n}\left(\frac{b_1}{b_2}\right)\right],$$

where

$$\frac{\partial}{\partial a} P_{2n}\left(\frac{b_1}{b_2}\right) = P'_{2n} \left(\frac{1}{b_2} \frac{\partial b_1}{\partial a} - \frac{b_1}{b_2^2} \frac{\partial b_2}{\partial a}\right).$$

Here

$$P'_{2n} = \frac{1}{2^{2n}} \sum_{r=0}^{\sigma} \frac{(-1)^r (4n - 2r)!}{r!\,(2n - r)!\,(2n - 2r)!} (2n - 2r) \left(\frac{b_1}{b_2}\right)^{2n-2r-1}$$

If m is odd

$$\frac{\partial D_m}{\partial a} = \frac{\partial D_{2i+1}}{\partial a} = \sum_{n=0}^{i} \left[(-1)^{i-n} (2i - 2n) \left(\frac{c}{p}\right)^{2i-2n-1} \left(-\frac{c}{p^2}\right) \frac{\partial p}{\partial a} \left(\frac{b_2}{p}\right)^{2n+1} P_{2n+1}\left(\frac{b_1}{b_2}\right)\right.$$

$$+ (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} (2n + 1) \left(\frac{b_2}{p}\right)^{2n} \left(\frac{1}{p} \frac{\partial b_2}{\partial a} - \frac{b_2}{p^2} \frac{\partial p}{\partial a}\right) P_{2n+1}\left(\frac{b_1}{b_2}\right)$$

$$\left. + (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n+1} P'_{2n+1} \left(\frac{1}{b_2} \frac{\partial b_1}{\partial a} - \frac{b_1}{b_2^2} \frac{\partial b_2}{\partial a}\right)\right],$$

where

$$P'_{2n+1} = \frac{1}{2^{2n+1}} \sum_{r=0}^{\sigma} \frac{(-1)^r (4n-2r+2)!}{r!(2n-r+1)!(2n-2r+1)!} (2n+1-2r) \left(\frac{b_1}{b_2}\right)^{2n-2r}$$

Also

$$\frac{\partial A_{31}}{\partial a} = -\frac{3A_{31}}{p} \frac{\partial p}{\partial a} + \frac{(1-e^2)^{1/2} e}{p^3} \left[-\frac{b_1}{p^2}\left(3+\frac{3}{4}e^2\right)\frac{\partial p}{\partial a} + \left(3+\frac{3}{4}e^2\right) p^{-1} \frac{\partial b_1}{\partial a} \right.$$

$$\left. + \frac{2}{p^3}\left(\frac{b_2^2}{2}+c^2\right)(4+3e^2)\frac{\partial p}{\partial a} - p^{-2}(4+3e^2) b_2 \frac{\partial b_2}{\partial a} \right]$$

$$\frac{\partial A_{32}}{\partial a} = -\frac{3A_{32}}{p}\frac{\partial p}{\partial a} + \frac{(1-e^2)^{1/2}}{p^3}\left[\frac{3}{4}p^{-1}\frac{\partial b_1}{\partial a} e^2 - \frac{3}{4}p^{-2} b_1 e^2 \frac{\partial p}{\partial a}\right.$$

$$\left. + 2p^{-3}\frac{\partial p}{\partial a}\left(\frac{b_2^2}{2}+c^2\right)\left(\frac{3}{2}e^2+\frac{e^4}{4}\right) - \frac{b_2}{p^2}\left(\frac{3}{2}e^2+\frac{e^4}{4}\right)\frac{\partial b_2}{\partial a}\right],$$

$$\frac{\partial A_{33}}{\partial a} = -\frac{3A_{33}}{p}\frac{\partial p}{\partial a} + \frac{e^3}{p^3}(1-e^2)^{1/2}\left[\frac{1}{12}\left(\frac{1}{p}\frac{\partial b_1}{\partial a} - \frac{b_1}{p^2}\frac{\partial p}{\partial a}\right)\right.$$

$$\left. + \frac{2}{3}p^{-3}\left(\frac{b_2^2}{2}+c^2\right)\frac{\partial p}{\partial a} - \frac{p^{-2}}{3} b_2 \frac{\partial b_2}{\partial a}\right],$$

$$\frac{\partial A_{34}}{\partial A} = -\frac{5A_{34}}{p}\frac{\partial p}{\partial a} - \frac{1}{32}\frac{(1-e^2)^{1/2}e^4}{p^5} b_2 \frac{\partial b_2}{\partial a}$$

Now

$$\frac{\partial \phi}{\partial e} = \left(\alpha_2^2-\alpha_3^2\right)^{-1/2} \eta_0 \left[(1-\eta_0)^{-1/2}\left(1-\eta_2^{-2}\right)^{-1/2}\chi + B_3\psi + \frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right]\frac{\partial \alpha_3}{\partial e}$$

$$- \alpha_3 \eta_0 \frac{\left[(1-\eta_0)^{-1/2}\left(1-\eta_2^{-2}\right)^{-1/2}\chi + B_3\psi + \frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right]}{\left(\alpha_2^2-\alpha_3^2\right)^{3/2}}\left(\alpha_2\frac{\partial \alpha_2}{\partial e} - \alpha_3\frac{\partial \alpha_3}{\partial e}\right)$$

$$+ \alpha_3\left(\alpha_2^2-\alpha_3^2\right)^{-1/2}\eta_0\left[\left(1-\eta_0^2\right)^{-1/2}\left(1-\eta_2^{-2}\right)^{-1/2}\frac{\partial \chi}{\partial e} - \frac{\left(1-\eta_0^2\right)^{-1/2}}{\left(1-\eta_2^{-2}\right)^{3/2}}\frac{\chi}{\eta_2^3}\frac{\partial \eta_2}{\partial e}\right.$$

$$+B_3 \frac{\partial \psi}{\partial e} + \psi \frac{\partial B_3}{\partial e} - \frac{3}{8} \eta_0^2 \sin 2\psi \, \eta_2^{-5} \frac{\partial \eta_2}{\partial e} + \frac{3}{16} \eta_0^2 \eta_2^{-4} \cos 2\psi \frac{\partial \psi}{\partial e}\Bigg]$$

$$-c^2 (-2\alpha_1)^{-1/2} \left(A_3 v + \sum_{n=1}^{4} A_{3n} \sin n v \right) \frac{\partial \alpha_3}{\partial e}$$

$$-c^2 \alpha_3 (-2\alpha_1)^{-3/2} \left(A_3 v + \sum_{n=1}^{4} A_{3n} \sin n v \right) \frac{\partial \alpha_1}{\partial e}$$

$$-c^2 \alpha_3 (-2\alpha_1)^{-1/2} \left[A_3 \frac{\partial v}{\partial e} + v \frac{\partial A_3}{\partial e} + \sum_{n=1}^{4} \left(n A_{3n} \cos n v \frac{\partial v}{\partial e} + \sin n v \frac{\partial A_{3n}}{\partial e} \right) \right],$$

with

$$\frac{\partial B_3}{\partial e} = \left(1 - \eta_2^2\right)^{-3/2} \eta_2^{-3} \frac{\partial \eta_2}{\partial e} + \sum_{m=2}^{\infty} 2m \, \gamma_m \eta_2^{-2m-1} \frac{\partial \eta_2}{\partial e},$$

$$\frac{\partial A_3}{\partial e} = -\frac{3 A_3}{p} \frac{\partial p}{\partial e} - \frac{A_3 e}{1 - e^2} + \frac{(1 - e^2)^{1/2}}{p^3} \sum_{m=0}^{\infty} R_{m+2} \left[(1 - e^2)^{1/2}\right] \frac{\partial D_m}{\partial e}$$

$$+ \frac{(1 - e^2)^{1/2}}{p^3} \sum_{m=0}^{\infty} D_m \frac{\partial}{\partial e} R_{m+2} \left[(1 - e^2)^{1/2}\right],$$

where

$$\frac{\partial}{\partial e} R_{m+2}\left[(1 - e^2)^{1/2}\right] = -P_{m+2} \left[(1 - e^2)^{-1/2}\right] (m + 2) \left[(1 - e^2)^{1/2}\right]^{m+1} e(1 - e^2)^{-1/2}$$

$$+ \left[(1 - e^2)^{1/2}\right]^{m+2} \frac{\partial}{\partial e} P_{m+2} \left[(1 - e^2)^{-1/2}\right];$$

$$\frac{\partial}{\partial e} P_{m+2} \left[(1 - e^2)^{-1/2}\right]$$

$$= \left\{ \frac{1}{2^{m+2}} \sum_{r=0}^{\sigma} \frac{(-1)^r (2m + 4 - 2r)!}{r!(m + 2 - r)!(m + 2 - 2r)!} (m + 2 - 2r) \left(\frac{b_1}{b_2}\right) \left[(1 - e^2)^{-1/2}\right]^{m-2r+1} \right\} e(1 - e^2)^{-3/2}$$

If m is even

$$\frac{\partial D_m}{\partial e} = \frac{\partial D_{2i}}{\partial e} = \sum_{n=0}^{i} \left[(-1)^{i-n} (2i-2n) \left(\frac{c}{p}\right)^{2i-2n-1} \left(-\frac{c}{p^2}\right) \frac{\partial p}{\partial e} \left(\frac{b_2}{p}\right)^{2n} P_{2n}\left(\frac{b_1}{b_2}\right) \right.$$

$$+ (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} 2n \left(\frac{b_2}{p}\right)^{2n-1} \left(\frac{1}{p}\frac{\partial b_2}{\partial e} - \frac{b_2}{p^2}\frac{\partial p}{\partial e}\right) P_{2n}\left(\frac{b_1}{b_2}\right)$$

$$\left. + (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n} \frac{\partial}{\partial e} P_{2n}\left(\frac{b_1}{b_2}\right) \right],$$

where

$$\frac{\partial}{\partial e} P_{2n} \frac{b_1}{b_2} = P'_{2n} \left(\frac{1}{b_2}\frac{\partial b_1}{\partial e} - \frac{b_1}{b_2^2}\frac{\partial b_2}{\partial e}\right).$$

If m is odd

$$\frac{\partial D_m}{\partial e} = \frac{\partial D_{2i+1}}{\partial e} = \sum_{n=0}^{i} \left[(-1)^{i-n} (2i-2n) \left(\frac{c}{p}\right)^{2i-2n-1} \left(-\frac{c}{p^2}\right) \frac{\partial p}{\partial e} \left(\frac{b_2}{p}\right)^{2n+1} P_{2n+1}\left(\frac{b_1}{b_2}\right) \right].$$

$$+ (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} (2n+1) \left(\frac{b_2}{p}\right)^{2n} \left(\frac{1}{p}\frac{\partial b_2}{\partial e} - \frac{b_2}{p^2}\frac{\partial p}{\partial e}\right) P_{2n+1}\left(\frac{b_1}{b_2}\right)$$

$$+ (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n+1} P'_{2n+1} \left(\frac{1}{b_2}\frac{\partial b_1}{\partial e} - \frac{b_1}{b_2^2}\frac{\partial b_2}{\partial e}\right).$$

Also

$$\frac{\partial A_{31}}{\partial e} = -\frac{3A_{31}}{p}\frac{\partial p}{\partial e} - \frac{A_{31}e}{1-e^2} + \frac{A_{31}}{e} + \frac{(1-e^2)^{1/2}e}{p^3} \left[-\frac{b_1}{p^2}\left(3+\frac{3}{4}e^2\right)\frac{\partial p}{\partial e} \right.$$

$$+ \left(3+\frac{3}{4}e^2\right)p^{-1}\frac{\partial b_1}{\partial e} + \frac{3}{2}e b_1 p^{-1} + \frac{2}{p^3}\left(\frac{b_2^2}{2}+c^2\right)(4+3e^2)\frac{\partial p}{\partial e}$$

$$\left. - \frac{b_2}{p^2}(4+3e^2)\frac{\partial b_2}{\partial e} - 6ep^{-2}\left(\frac{b_2^2}{2}+c^2\right) \right],$$

$$\frac{\partial A_{32}}{\partial e} = -\frac{3A_{32}}{p}\frac{\partial p}{\partial e} - A_{32}\, e\,(1-e^2)^{-1} + \frac{(1-e^2)^{1/2}}{p^3}\left[\frac{e}{2} + \frac{3}{4}\frac{e^2}{p}\frac{\partial b_1}{\partial e}\right.$$

$$+\frac{3}{2}\frac{e b_1}{p} - \frac{3}{4}\frac{e^2 b_1}{p^2}\frac{\partial p}{\partial e} + \frac{2}{p^3}\left(\frac{b_2^2}{2} + c^2\right)\left(\frac{3}{2}e^2 + \frac{e^4}{4}\right)\frac{\partial p}{\partial e}$$

$$\left. - \frac{b_2}{p^2}\left(\frac{3}{2}e^2 + \frac{e^4}{4}\right)\frac{\partial b_2}{\partial e} - p^{-2}\left(\frac{b_2^2}{2} + c^2\right)(3e + e^3)\right],$$

$$\frac{\partial A_{33}}{\partial e} = \frac{3A_{33}}{e} - \frac{A_{33}\, e}{1-e^2} + p^{-3}(1-e^2)^{1/2} e^3\left[\frac{1}{12}\left(\frac{1}{p}\frac{\partial b_1}{\partial e} - \frac{b_1}{p^2}\frac{\partial p}{\partial e}\right)\right.$$

$$\left. + \frac{2}{3}p^{-3}\left(\frac{b_2^2}{2} + c^2\right)\frac{\partial p}{\partial e} - \frac{b_2}{3p^2}\frac{\partial b_2}{\partial e}\right] - \frac{3A_{33}}{p}\frac{\partial p}{\partial e},$$

$$\frac{\partial A_{34}}{\partial e} = -\frac{A_{34}\, e}{1-e^2} - \frac{5A_{34}}{p}\frac{\partial p}{\partial e} + \frac{4A_{34}}{e} - \frac{e^4 b_2}{32\, p^5}(1-e^2)^{1/2}\frac{\partial b_2}{\partial e}.$$

Finally

$$\frac{\partial \phi}{\partial \eta_0} = \left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0\left[(1-\eta_0)^{-1/2}\left(1-\eta_2^{-2}\right)^{-1/2}\chi + B_3\psi + \frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right]\frac{\partial \alpha_3}{\partial \eta_0}$$

$$- \frac{\alpha_3\,\eta_0\left[(1-\eta_0)^{-1/2}\left(1-\eta_2^{-2}\right)^{-1/2}\chi + B_3\psi + \frac{3}{32}\eta_0^2\eta_2^{-4}\sin 2\psi\right]}{\left(\alpha_2^2 - \alpha_3^2\right)^{3/2}}\left(\alpha_2\frac{\partial \alpha_2}{\partial \eta_0} - \alpha_3\frac{\partial \alpha_3}{\partial \eta_0}\right)$$

$$+ \frac{\phi - \beta_3}{\eta_0} + \alpha_3\left(\alpha_2^2 - \alpha_3^2\right)^{-1/2}\eta_0\left[\frac{\left(1-\eta_2^{-2}\right)^{-1/2}}{\left(1-\eta_0^2\right)^{3/2}}\chi\,\eta_0 + \frac{\left(1-\eta_0^2\right)^{-1/2}}{\left(1-\eta_2^{-2}\right)^{1/2}}\frac{\partial \chi}{\partial \eta_0} - \frac{\left(1-\eta_0^2\right)^{-1/2}}{\left(1-\eta_2^{-2}\right)^{3/2}}\frac{\chi}{\eta_2^3}\frac{\partial \eta_2}{\partial \eta_0}\right.$$

$$\left. + B_3\frac{\partial \psi}{\partial \eta_0} + \psi\frac{\partial B_3}{\partial \eta_0} + \frac{3}{16}\eta_0\eta_2^{-4}\sin 2\psi - \frac{3}{8}\eta_0^2\eta_2^{-5}\sin 2\psi\frac{\partial \eta_2}{\partial \eta_0} + \frac{3}{16}\cos 2\psi\,\eta_0^2\eta_2^{-4}\frac{\partial \psi}{\partial \eta_0}\right]$$

$$- c^2(-2\alpha_1)^{-1/2}\frac{\partial \alpha_3}{\partial \eta_0}\left(A_3 v + \sum_{n=1}^{4} A_{3n}\sin n v\right) - c^2\alpha_3(-2\alpha_1)^{-3/2}\frac{\partial \alpha_1}{\partial \eta_0}\left[A_3 v + \sum_{n=1}^{4} A_{3n}\sin n v\right]$$

$$- c^2\alpha_3(-2\alpha_1)^{-1/2}\left[A_3\frac{\partial v}{\partial \eta_0} + v\frac{\partial A_3}{\partial \eta_0} + \sum_{n=1}^{4}\left(n A_{3n}\cos nv\frac{\partial v}{\partial \eta_0} + \sin nv\frac{\partial A_{3n}}{\partial \eta_0}\right)\right],$$

where

$$\frac{\partial B_3}{\partial \eta_0} = (1 - \eta_2^{-2})^{-3/2}\, \eta_2^{-3}\, \frac{\partial \eta_2}{\partial \eta_0} + \sum_{m=2}^{\infty} \left(2m\, \gamma_m\, \eta_2^{-2m-1}\, \frac{\partial \eta_2}{\partial \eta_0} - \eta_2^{-2m}\, \frac{\partial \gamma_m}{\partial \eta_0} \right),$$

with

$$\frac{\partial \gamma_m}{\partial \eta_0} = \frac{(2m)!}{2^{2m}\,(m!)^2} \sum_{n=1}^{m-1} \frac{(2n)!\,(2n)}{2^{2n}\,(n!)^2}\, \eta_0^{2n-1},$$

and

$$\frac{\partial A_3}{\partial \eta_0} = \frac{(1-e^2)^{1/2}}{p^3} \sum_{m=0}^{\infty} R_{m+2}\left[(1-e^2)^{1/2}\right] \frac{\partial D_m}{\partial \eta_0}.$$

If m is even

$$\frac{\partial D_m}{\partial \eta_0} = \frac{\partial D_{2i}}{\partial \eta_0} = \sum_{n=0}^{i} \left[(-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} (2n) \left(\frac{b_2}{p}\right)^{2n-1} \left(\frac{1}{p}\right) \frac{\partial b_2}{\partial \eta_0}\, P_{2n}\left(\frac{b_1}{b_2}\right) \right.$$

$$\left. + (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n} \frac{\partial}{\partial \eta_0} P_{2n}\left(\frac{b_1}{b_2}\right) \right],$$

with,

$$\frac{\partial}{\partial \eta_0} P_{2n}\left(\frac{b_1}{b_2}\right) = P'_{2n}\left(\frac{1}{b_2}\frac{\partial b_1}{\partial \eta_0} - \frac{b_1}{b_2^2}\, \frac{\partial b_2}{\partial \eta_0} \right).$$

If m is odd

$$\frac{\partial D_m}{\partial \eta_0} = \frac{\partial D_{2i+1}}{\partial \eta_0} = \sum_{n=0}^{i} \left[(-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} (2n+1) \left(\frac{b_2}{p}\right)^{2n} \frac{\partial b_2}{\partial \eta_0} \left(\frac{1}{p}\right) P_{2n+1}\left(\frac{b_1}{b_2}\right) \right.$$

$$\left. + (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n+1} \frac{\partial}{\partial \eta_0} P_{2n+1}\left(\frac{b_1}{b_2}\right) \right],$$

with

$$\frac{\partial}{\partial \eta_0} P_{2n+1}\left(\frac{b_1}{b_2}\right) = P'_{2n+1}\left(\frac{1}{b_2}\, \frac{\partial b_1}{\eta_0} - \frac{b_1}{b_2^2}\, \frac{\partial b_2}{\eta_0} \right).$$

Also

$$\frac{\partial A_{31}}{\partial \eta_0} = \frac{(1-e^2)^{1/2} e}{p^3} \left[\frac{1}{p}\left(3 + \frac{3}{4} e^2\right) \frac{\partial b_1}{\partial \eta_0} - \frac{b_2}{p^2} (4 + 3e^2) \frac{\partial b_2}{\partial \eta_0}\right]$$

$$\frac{\partial A_{32}}{\partial \eta_0} = \frac{(1-e^2)^{1/2}}{p^3} \left[\frac{3}{4} \frac{e^2}{p} \frac{\partial b_1}{\partial \eta_0} - \frac{b_2}{p^2}\left(\frac{3}{2} e^2 + \frac{e^4}{4}\right) \frac{\partial b_2}{\partial \eta_0}\right]$$

$$\frac{\partial A_{33}}{\partial \eta_0} = \frac{(1-e^2)^{1/2} e^3}{p^3} \left(\frac{1}{12p} \frac{\partial b_1}{\partial \eta_0} - \frac{b_2}{3p^2} \frac{\partial b_2}{\partial \eta_0}\right),$$

$$\frac{\partial A_{34}}{\partial \eta_0} = -\frac{1}{32} \frac{(1-e^2)^{1/2} e^4}{p^5} b_2 \frac{\partial b_2}{\partial \eta_0}.$$

And,

$$K_{29} = \frac{\partial x}{\partial \beta_1} \cos \psi_A + \frac{\partial y}{\partial \beta_1} \sin \psi_A,$$

$$K_{30} = -\frac{\partial x}{\partial \beta_1} \sin \psi_A \sin \theta_D + \frac{\partial y}{\partial \beta_1} \cos \psi_A \sin \theta_D + \frac{\partial z}{\partial \beta_1} \cos \theta_D,$$

$$K_{31} = \frac{\partial x}{\partial \beta_1} \sin \psi_A \cos \theta_D - \frac{\partial y}{\partial \beta_1} \cos \psi_A \cos \theta_D + \frac{\partial z}{\partial \beta_1} \sin \theta_D,$$

$$K_{32} = \frac{\partial x}{\partial \beta_2} \cos \psi_A + \frac{\partial y}{\partial \beta_2} \sin \psi_A,$$

$$K_{33} = -\frac{\partial x}{\partial \beta_2} \sin \psi_A \sin \theta_D + \frac{\partial y}{\partial \beta_2} \cos \psi_A \sin \theta_D + \frac{\partial z}{\partial \beta_2} \cos \theta_D,$$

$$K_{34} = \frac{\partial x}{\partial \beta_2} \sin \psi_A \cos \theta_D - \frac{\partial y}{\partial \beta_2} \cos \psi_A \cos \theta_D + \frac{\partial z}{\partial \beta_2} \sin \theta_D,$$

$$K_{35} = \frac{\partial x}{\partial \beta_3} \cos \psi_A + \frac{\partial y}{\partial \beta_3} \sin \psi_A,$$

$$K_{36} = -\frac{\partial x}{\partial \beta_3} \sin \psi_A \sin \theta_D + \frac{\partial y}{\partial \beta_3} \cos \psi_A \sin \theta_D + \frac{\partial z}{\partial \beta_3} \cos \theta_D$$

$$K_{37} = \frac{\partial x}{\partial \beta_3} \sin \psi_A \cos \theta_D - \frac{\partial y}{\partial \beta_3} \cos \psi_A \cos \theta_D + \frac{\partial z}{\partial \beta_3} \sin \theta_D$$

where

$$\frac{\partial x}{\partial \beta_1} = \frac{x}{\rho^2 + c^2} \quad a^2 e \sin E (1 - e \cos E) \frac{\partial E}{\partial \beta_1} - \frac{x}{1 - \eta_0^2 \sin^2 \psi} \eta_0^2 \sin \psi \cos \psi \frac{\partial \psi}{\partial \beta_1} - y \frac{\partial \phi}{\partial \beta_1},$$

$$\frac{\partial x}{\partial \beta_2} = \frac{x}{\rho^2 + c^2} \quad a^2 e \sin E (1 - e \cos E) \frac{\partial E}{\partial \beta_2} - \frac{x}{1 - \eta_0^2 \sin^2 \psi)} \eta_0^2 \sin \psi \cos \psi \frac{\partial \psi}{\partial \beta_2} - y \frac{\partial \phi}{\partial \beta_2},$$

$$\frac{\partial x}{\partial \beta_3} = - y,$$

$$\frac{\partial y}{\partial \beta_1} = \frac{y}{\rho^2 + c^2} \quad a^2 e \sin E (1 - e \cos E) \frac{\partial E}{\partial \beta_1} - \frac{y}{1 - \eta_0^2 \sin^2 \psi} \eta_0^2 \sin \psi \cos \psi \frac{\partial \psi}{\partial \beta_1} + x \frac{\partial \phi}{\partial \beta_1},$$

$$\frac{\partial y}{\partial \beta_2} = \frac{y}{\rho^2 + c^2} \quad a^2 e \sin E (1 - e \cos E) \frac{\partial E}{\partial \beta_2} - \frac{y}{1 - \eta_0^2 \sin^2 \psi} \eta_0^2 \sin \psi \cos \psi \frac{\partial \psi}{\partial \beta_2} + x \frac{\partial \phi}{\partial \beta_2},$$

$$\frac{\partial y}{\partial \beta_3} = x,$$

$$\frac{\partial z}{\partial \beta_1} = a \eta_0 \left[(1 - e \cos E) \cos \psi \frac{\partial \psi}{\partial \beta_1} + e \sin \psi \sin E \frac{\partial E}{\partial \beta_1} \right],$$

$$\frac{\partial z}{\partial \beta_2} = a \eta_0 \left[(1 - e \cos E) \cos \psi \frac{\partial \psi}{\partial \beta_2} + e \sin \psi \sin E \frac{\partial E}{\partial \beta_2} \right],$$

$$\frac{\partial z}{\partial \beta_3} = 0 .$$

In these equations

$$\frac{\partial E}{\partial \beta_1} = \frac{\partial M_s}{\partial \beta_1} + \frac{\partial E_0}{\partial \beta_1} + \frac{\partial E_1}{\partial \beta_1} + \frac{\partial E_2}{\partial \beta_1} ,$$

$$\frac{\partial E}{\partial \beta_2} = \frac{\partial M_s}{\partial \beta_2} + \frac{\partial E_0}{\partial \beta_2} + \frac{\partial E_1}{\partial \beta_2} + \frac{\partial E_2}{\partial \beta_2} ,$$

$$\frac{\partial E}{\partial \beta_3} = 0 ,$$

$$\frac{\partial \psi}{\partial \beta_1} = \frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1} + \frac{\partial \psi_1}{\partial \beta_1} + \frac{\partial \psi_2}{\partial \beta_1} ,$$

$$\frac{\partial \psi}{\partial \beta_2} = \frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2} + \frac{\partial \psi_1}{\partial \beta_2} + \frac{\partial \psi_2}{\partial \beta_2} ,$$

$$\frac{\partial \psi}{\partial \beta_3} = 0 ,$$

where

$$\frac{\partial M_s}{\partial \beta_1} = 2 \pi \nu_1 ,$$

$$\frac{\partial M_s}{\partial \beta_2} = - \frac{2 \pi \nu_1 c^2 \eta_0^2 B_1}{\alpha_2 B_2} ,$$

$$\frac{\partial M_s}{\partial \beta_3} = 0 ,$$

$$\frac{\partial \psi_s}{\partial \beta_1} = 2 \pi \nu_2 ,$$

$$\frac{\partial \psi_s}{\partial \beta_2} = \frac{2 \pi \nu_2 (a + b_1 + A_1)}{\alpha_2 A_2} ,$$

$$\frac{\partial \psi_s}{\partial \beta_3} = 0 ,$$

$$\frac{\partial E_0}{\partial \beta_1} = \frac{e' \cos \mathcal{E}}{1 - e' \cos \mathcal{E}} \frac{\partial M_s}{\partial \beta_1} ,$$

$$\frac{\partial E_0}{\partial \beta_2} = \frac{e' \cos \mathcal{E}}{1 - e' \cos \mathcal{E}} \frac{\partial M_s}{\partial \beta_2} .$$

Then

$$\frac{\partial \psi_0}{\partial \beta_1} = \left[(-2\alpha_1)^{-1/2} \left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} A_2 B_2^{-1}\right] \frac{\partial v_0}{\partial \beta_1} ,$$

$$\frac{\partial \psi_0}{\partial \beta_2} = (-2\alpha_1)^{-1/2} \left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} A_2 B_2^{-1} \frac{\partial v_0}{\partial \beta_2} ,$$

where

$$\frac{\partial v_0}{\partial \beta_1} = \frac{\sin \mathcal{E} \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial E_0}{\partial \beta_1}\right)(1 - e^2)}{\sin(M_s + v_0)(1 - e \cos \mathcal{E})^2} - \frac{\partial M_s}{\partial \beta_1} ,$$

$$\frac{\partial v_0}{\partial \beta_2} = - \frac{\sin \mathcal{E} \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial E_0}{\partial \beta_2}\right)(1 - e^2)}{\sin(M_s + v_0)(1 - e \cos \mathcal{E})^2} - \frac{\partial M_s}{\partial \beta_2} ,$$

and

$$\frac{\partial E_1}{\partial \beta_1} = \frac{\left(1 - e' \cos \mathcal{E}\right) \frac{\partial M_1}{\partial \beta_1} - M_1 e' \sin \mathcal{E} \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial E_0}{\partial \beta_1}\right)}{(1 - e' \cos \mathcal{E})^2}$$

$$- \frac{e'}{2} \left[2M_1 (1 - e' \cos \mathcal{E})^{-3} \sin \mathcal{E} \frac{\partial M_1}{\partial \beta_1} \right.$$

$$+ M_1^2 (1 - e' \cos \mathcal{E})^{-3} \cos \mathcal{E} \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial E_0}{\partial \beta_1}\right)$$

$$\left. - 3e' (1 - e' \cos \mathcal{E})^{-4} M_1^2 \sin^2 \mathcal{E} \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial E_0}{\partial \beta_1}\right)\right] ,$$

$$\frac{\partial E_1}{\partial \beta_2} = \frac{\left(1 - e' \cos \mathcal{E}\right) \frac{\partial M_1}{\partial \beta_2} - M_1 e' \sin \mathcal{E} \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial E_0}{\partial \beta_2}\right)}{(1 - e' \cos \mathcal{E})^2}$$

$$- \frac{e'}{2} \left[2M_1 (1 - e' \cos \mathcal{E})^{-3} \sin \mathcal{E} \frac{\partial M_1}{\partial \beta_2} \right.$$

$$+ M_1^2 (1 - e' \cos \mathcal{E})^{-3} \cos \mathcal{E} \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial E_0}{\partial \beta_2}\right)$$

$$\left. - 3e' (1 - e' \cos \mathcal{E})^{-4} M_1^2 \sin^2 \mathcal{E} \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial E_0}{\partial \beta_2}\right) \right].$$

Here

$$\frac{\partial M_1}{\partial \beta_1} = (a + b_1)^{-1} \left[- \left(A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right) \frac{\partial v_0}{\partial \beta_1} \right.$$

$$\left. + \frac{c^2}{2} (-2\alpha_1)^{1/2} \left(\alpha_2^2 - \alpha_3^2\right)^{-1/2} \eta_0^3 \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1}\right) \right],$$

$$\frac{\partial M_1}{\partial \beta_2} = (a + b_1)^{-1} \left[- \left(A_1 + c^2 \eta_0^2 A_2 B_1 B_2^{-1}\right) \frac{\partial v_0}{\partial \beta_2} \right.$$

$$\left. + \frac{c^2}{2} (-2\alpha_1)^{1/2} \left(\alpha_2^2 - \alpha_3^2\right)^{-1/2} \eta_0^3 \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2}\right) \right].$$

Also

$$\frac{\partial \psi_1}{\partial \beta_1} = (-2\alpha_1)^{-1/2} \left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} B_2^{-1} \left[A_2 \frac{\partial v_1}{\partial \beta_1} + A_{21} \cos (M_s + v_0) \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1}\right) \right.$$

$$\left. + 2A_{22} \cos (2M_s + 2v_0) \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1}\right) \right]$$

$$+ \frac{1}{4} q^2 B_2^{-1} \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_1} + \frac{\partial \psi_0}{\partial \beta_1}\right),$$

$$\frac{\partial \psi_1}{\partial \beta_2} = (-2\alpha_1)^{-1/2} \left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} B_2^{-1} \left[A_2 \frac{\partial v_1}{\partial \beta_2} + A_{21} \cos (M_s + v_0) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} \right) \right.$$

$$\left. + 2A_{22} \cos (2M_s + 2v_0) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} \right) \right]$$

$$+ \frac{1}{4} q^2 B_2^{-1} \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2} \right),$$

where

$$\frac{\partial v_1}{\partial \beta_1} = \frac{(1 - e^2) \sin (\mathcal{E} + E_1) \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial E_0}{\partial \beta_1} + \frac{\partial E_1}{\partial \beta_1} \right)}{(1 - e \cos (\mathcal{E} + E_1))^2 \sin (M_s + v_0 + v_1)} - \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1} \right),$$

$$\frac{\partial v_1}{\partial \beta_2} = \frac{(1 - e^2) \sin (\mathcal{E} + E_1) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial E_0}{\partial \beta_2} + \frac{\partial E_1}{\partial \beta_2} \right)}{(1 - e \cos (\mathcal{E} + E_1))^2 \sin (M_s + v_0 + v_1)} - \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} \right)$$

Finally,

$$\frac{\partial E_2}{\partial \beta_1} = \frac{\left[1 - e' \cos (\mathcal{E} + E_1)\right] \frac{\partial M_2}{\partial \beta_1} - M_2 e' \sin (\mathcal{E} + E_1) \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial E_0}{\partial \beta_1} + \frac{\partial E_1}{\partial \beta_1} \right)}{\left[1 - e' \cos (\mathcal{E} + E_1)\right]^2},$$

$$\frac{\partial E_2}{\partial \beta_2} = \frac{\left[1 - e' \cos (\mathcal{E} + E_1)\right] \frac{\partial M_2}{\partial \beta_2} - M_2 e' \sin (\mathcal{E} + E_1) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial E_0}{\partial \beta_2} + \frac{\partial E_1}{\partial \beta_2} \right)}{\left[1 - e' \cos (\mathcal{E} + E_1)\right]^2},$$

where

$$\frac{\partial M_2}{\partial \beta_1} = -(a+b_1)^{-1}\left\{A_1 \frac{\partial v_1}{\partial \beta_1} + A_{11}\cos(M_s+v_0)\left(\frac{\partial M_s}{\partial \beta_1}+\frac{\partial v_0}{\partial \beta_1}\right)\right.$$

$$+ 2A_{12}\cos(2M_s+2v_0)\left(\frac{\partial M_s}{\partial \beta_1}+\frac{\partial v_0}{\partial \beta_1}\right)$$

$$+ c^2(-2\alpha_1)^{1/2}\left(\alpha_2^2-\alpha_3^2\right)^{-1/2}\eta_0^3\left[B_1\frac{\partial \psi_1}{\partial \beta_1}+\psi_1\sin(2\psi_s+2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_1}+\frac{\partial \psi_0}{\partial \beta_1}\right)\right.$$

$$-\frac{1}{2}\cos(2\psi_s+2\psi_0)\frac{\partial \psi_1}{\partial \beta_1}-\frac{1}{4}q^2\cos(2\psi_s+2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_1}+\frac{\partial \psi_0}{\partial \beta_1}\right)+\frac{1}{16}q^2\cos(4\psi_s+4\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_1}+\frac{\partial \psi_0}{\partial \beta_1}\right)$$

$$\frac{\partial M_2}{\partial \beta_2} = -(a+b_1)^{-1}\left\{A_1 \frac{\partial v_1}{\partial \beta_2} + A_{11}\cos(M_s+v_0)\left(\frac{\partial M_s}{\partial \beta_2}+\frac{\partial v_0}{\partial \beta_2}\right)\right.$$

$$+ 2A_{12}\cos(2M_s+2v_0)\left(\frac{\partial M_s}{\partial \beta_2}+\frac{\partial v_0}{\partial \beta_2}\right)$$

$$+ c^2(-2\alpha_1)^{1/2}\left(\alpha_2^2-\alpha_3^2\right)^{-1/2}\eta_0^3\left[B_1\frac{\partial \psi_1}{\partial \beta_2}+\psi_1\sin(2\psi_s+2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_2}+\frac{\partial \psi_0}{\partial \beta_2}\right)\right.$$

$$-\frac{1}{2}\cos(2\psi_s+2\psi_0)\frac{\partial \psi_1}{\partial \beta_2}-\frac{1}{4}q^2\cos(2\psi_s+2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_2}+\frac{\partial \psi_0}{\partial \beta_2}\right)+\frac{1}{16}q^2\cos(4\psi_s+4\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_2}+\frac{\partial \psi_0}{\partial \beta_2}\right)$$

$$\frac{\partial \psi_2}{\partial \beta_1} = (-2\alpha_1)^{-1/2}\left(\alpha_2^2-\alpha_3^2\right)^{1/2}\eta_0^{-1}B_2^{-1}\left\{A_2\frac{\partial v_2}{\partial \beta_1}-A_{21}\left[v_1\sin(M_s+v_0)\left(\frac{\partial M_s}{\partial \beta_1}+\frac{\partial v_0}{\partial \beta_1}\right)\right.\right.$$

$$\left.-\frac{\partial v_1}{\partial \beta_1}\cos(M_s+v_0)\right]+2A_{22}\left[\frac{\partial v_1}{\partial \beta_1}\cos(2M_s+2v_0)-2v_1\sin(2M_s+2v_0)\left(\frac{\partial M_s}{\partial \beta_1}+\frac{\partial v_0}{\partial \beta_1}\right)\right]$$

$$\left.+3A_{23}\cos(3M_s+3v_0)\left(\frac{\partial M_s}{\partial \beta_1}+\frac{\partial v_0}{\partial \beta_1}\right)+4A_{24}\cos(4M_s+4v_0)\left(\frac{\partial M_s}{\partial \beta_1}+\frac{\partial v_0}{\partial \beta_1}\right)\right]\right\}$$

$$+\frac{1}{4}q^2B_2^{-1}\left[\frac{\partial \psi_1}{\partial \beta_1}\cos(2\psi_s+2\psi_0)-2\psi_1\sin(2\psi_s+2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_1}+\frac{\partial \psi_0}{\partial \beta_1}\right)\right.$$

$$\left.+\frac{3}{4}q^2\cos(2\psi_s+2\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_1}+\frac{\partial \psi_0}{\partial \beta_1}\right)-\frac{3}{16}q^2\cos(4\psi_s+4\psi_0)\left(\frac{\partial \psi_s}{\partial \beta_1}+\frac{\partial \psi_0}{\partial \beta_1}\right)\right],$$

$$\frac{\partial \psi_2}{\partial \beta_2} = (-2\alpha_1)^{-1/2} \left(\alpha_2^2 - \alpha_3^2\right)^{1/2} \eta_0^{-1} B_2^{-1} \Bigg\{ A_2 \frac{\partial v_2}{\partial \beta_2} - A_{21} \Bigg[v_1 \sin (M_s + v_0) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} \right)$$

$$- \frac{\partial v_1}{\partial \beta_2} \cos (M_s + v_0) \Bigg] + 2A_{22} \Bigg[\frac{\partial v_1}{\partial \beta_2} \cos (2M_s + 2v_0) - 2v_1 \sin (2M_s + 2v_0) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} \right) \Bigg]$$

$$+ 3A_{23} \cos (3M_s + 3v_0) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} \right) + 4A_{24} \cos (4M_s + 4v_0) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} \right) \Bigg\}$$

$$+ \frac{1}{4} q^2 B_2^{-1} \Bigg[\frac{\partial \psi_1}{\partial \beta_2} \cos (2\psi_s + 2\psi_0) - 2\psi_1 \sin (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2} \right)$$

$$+ \frac{3}{4} q^2 \cos (2\psi_s + 2\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2} \right) - \frac{3}{16} q^2 \cos (4\psi_s + 4\psi_0) \left(\frac{\partial \psi_s}{\partial \beta_2} + \frac{\partial \psi_0}{\partial \beta_2} \right) \Bigg]$$

where

$$\frac{\partial v_2}{\partial \beta_1} =$$

$$\frac{(1 - e^2) \sin (\mathcal{E} + E_1 + E_2) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial E_0}{\partial \beta_1} + \frac{\partial E_1}{\partial \beta_1} + \frac{\partial E_2}{\partial \beta_1} \right)}{\left[1 - e \cos (\mathcal{E} + E_1 + E_2)\right]^2 \sin (M_s + v_0 + v_1 + v_2)} - \left(\frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1} + \frac{\partial v_1}{\partial \beta_1} \right),$$

$$\frac{\partial v_2}{\partial \beta_2} =$$

$$\frac{(1 - e^2) \sin (\mathcal{E} + E_1 + E_2) \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial E_0}{\partial \beta_2} + \frac{\partial E_1}{\partial \beta_2} + \frac{\partial E_2}{\partial \beta_2} \right)}{\left[1 - e \cos (\mathcal{E} + E_1 + E_2)\right]^2 \sin (M_s + v_0 + v_1 + v_2)} - \left(\frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} + \frac{\partial v_1}{\partial \beta_2} \right).$$

Also

$$\frac{\partial \phi}{\partial \beta_1} = \alpha_3 \left(\alpha_2^2 - \alpha_3^2\right)^{-1/2} \eta_0 \left[\left(1 - \eta_0^2\right)^{-1/2} \left(1 - \eta_2^{-2}\right)^{-1/2} \frac{\partial \chi}{\partial \beta_1} + B_3 \frac{\partial \psi}{\partial \beta_1} + \frac{3}{16} \eta_0^2 \eta_2^{-4} \cos 2\psi \frac{\partial \psi}{\partial \beta_1}\right]$$

$$- c^2 \alpha_3 (-2\alpha_1)^{-1/2} \left(A_3 \frac{\partial v}{\partial \beta_1} + \sum_{n=1}^{4} n A_{3n} \cos nv \frac{\partial v}{\partial \beta_1}\right),$$

$$\frac{\partial \phi}{\partial \beta_2} = \alpha_3 \left(\alpha_2^2 - \alpha_3^2\right)^{-1/2} \eta_0 \left[\left(1 - \eta_0^2\right)^{-1/2} \left(1 - \eta_2^{-2}\right)^{-1/2} \frac{\partial \chi}{\partial \beta_2} + B_3 \frac{\partial \psi}{\partial \beta_2} + \frac{3}{16} \eta_0^2 \eta_2^{-4} \cos 2\psi \frac{\partial \psi}{\partial \beta_2}\right]$$

$$- c^2 \alpha_3 (-2\alpha_1)^{-1/2} \left(A_3 \frac{\partial v}{\partial \beta_2} + \sum_{n=1}^{4} n A_{3n} \cos nv \frac{\partial v}{\partial \beta_2}\right),$$

$$\frac{\partial \phi}{\partial \beta_3} = 1,$$

where

$$\frac{\partial \chi}{\partial \beta_1} = \frac{\left(1 - \eta_0^2\right) \sin \psi}{\left(1 - \eta_0^2 \sin^2 \psi\right)^{3/2} \sin \chi} \frac{\partial \psi}{\partial \beta_1},$$

$$\frac{\partial \chi}{\partial \beta_2} = \frac{\left(1 - \eta_0^2\right) \sin \psi}{\left(1 - \eta_0^2 \sin^2 \psi\right)^{3/2} \sin \chi} \frac{\partial \psi}{\partial \beta_2},$$

$$\frac{\partial v}{\partial \beta_1} = \frac{\partial M_s}{\partial \beta_1} + \frac{\partial v_0}{\partial \beta_1} + \frac{\partial v_1}{\partial \beta_1} + \frac{\partial v_2}{\partial \beta_1},$$

$$\frac{\partial v}{\partial \beta_2} = \frac{\partial M_s}{\partial \beta_2} + \frac{\partial v_0}{\partial \beta_2} + \frac{\partial v_1}{\partial \beta_2} + \frac{\partial v_2}{\partial \beta_2}.$$

Similarly,

$$M_c = \frac{y_m}{\sqrt{x_m^2 + y_m^2 + z_m^2}},$$

$$\frac{\partial M_c}{\partial q_i} = \frac{\partial M_c}{\partial x_m}\frac{\partial x_m}{\partial q_i} + \frac{\partial M_c}{\partial y_m}\frac{\partial y_m}{\partial q_i} + \frac{\partial M_c}{\partial z_m}\frac{\partial z_m}{\partial q_i},$$

$$\frac{\partial M_c}{\partial x_m} = \frac{-x_m y_m}{\left(x_m^2 + y_m^2 + z_m^2\right)^{3/2}} = K_{01},$$

$$\frac{\partial M_c}{\partial y_m} = \frac{1}{\left(x_m^2 + y_m^2 + z_m^2\right)^{1/2}} - \frac{y_m^2}{\left(x_m^2 + y_m^2 + z_m^2\right)^{3/2}} = K_{04},$$

$$\frac{\partial M_c}{\partial z_m} = \frac{-y_m z_m}{\left(x_m^2 + y_m^2 + z_m^2\right)^{3/2}} = K_{03}.$$

Proceeding as before, we obtain the six first partial derivatives of the 'M' equation of condition:

$$\frac{\partial M_c}{\partial a} = K_{01} K_{20} + K_{04} K_{21} + K_{03} K_{22},$$

$$\frac{\partial M_c}{\partial e} = K_{01} K_{23} + K_{04} K_{24} + K_{03} K_{25},$$

$$\frac{\partial M_c}{\partial \eta_0} = K_{01} K_{26} + K_{04} K_{27} + K_{03} K_{28},$$

$$\frac{\partial M_c}{\partial \beta_1} = K_{01} K_{29} + K_{04} K_{30} + K_{03} K_{31},$$

$$\frac{\partial M_c}{\partial \beta_2} = K_{01} K_{32} + K_{04} K_{33} + K_{03} K_{34},$$

$$\frac{\partial M_3}{\partial \beta_3} = K_{01} K_{35} + K_{04} K_{36} + K_{03} K_{37} .$$

REMARKS

Preliminary tests for Explorer XI (1961 $\nu 1$) on the IBM 7090 computer indicate that the orbit generator portion can compute approximately 1600 minute points (time, x, y, z, $\dot{x}$, $\dot{y}$, and $\dot{z}$, 1600 times) per minute of computer operation, while simultaneously producing BCD tape. Consequently, since the differential correction for the orbit is an analytic one, it is expected to go proportionately as fast as the orbit generator. Also, such expressions as $2A_{12}/e$ and $-A_{23}e/1 - e^2$, for example, are actually computed as

$$\frac{3}{16} (1 - e^2)^{1/2} b_2^4 e p^{-3} \text{ and } -(1 - e^2)^{-1/2} p^{-1} \frac{e^4}{8} (-b_1 b_2^2 p^{-3} + b_2^4 p^{-4}),$$

in order to avoid the possibility of indeterminate forms.

The computation of coordinates and the corresponding differential correction for the case of equatorial orbits has been completed, as have the first partials for the range and range rate tracking method (Appendix A). Both are to be incorporated into the general Vinti satellite orbit computation program, along with the coordinate changes due to the effects of the residual oblateness potential now unaccounted for. A refined version of this satellite orbit computation is to contain corrections for aerodynamic and electromagnetic drag and lunar-solar forces.

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APPENDIX A

Computation with Range and Range Rate Data

Since the partial expressions $\partial E/\partial q_i$, $\partial\psi/\partial q_i$ and $\partial\phi/\partial q_i$ are derived from Vinti's kinetic equations, to handle range and range rate data in the orbit improvement method the following procedure is utilized.

Denote the slant range from the point of observation as

$$R_c = \sqrt{x_m^2 + y_m^2 + z_m^2} \ ,$$

and the range rate as

$$\dot{R}_c = \frac{x_m \dot{x}_m + y \dot{y}_m + z \dot{z}_m}{\sqrt{x_m^2 + y_m^2 + z_m^2}} \ .$$

The equations of condition now become

$$R_0 - R_c = \frac{\partial R_c}{\partial a} \Delta a + \quad . \ . \ . \quad + \frac{\partial R_c}{\partial \beta_3} \Delta \beta_3 \ ,$$

$$\dot{R}_0 - \dot{R}_c = \frac{\partial \dot{R}_c}{\partial a} \Delta a + \quad . \ . \ . \quad + \frac{\partial \dot{R}_c}{\partial \beta_3} \Delta \beta_3 \ ,$$

for a given time of observation. Now

$$\frac{\partial R_c}{\partial a} = \frac{\partial R_c}{\partial x_m} \frac{\partial x_m}{\partial a} + \frac{\partial R_c}{\partial y_m} \frac{\partial y_m}{\partial a} + \frac{\partial R_c}{\partial z_m} \frac{\partial z_m}{\partial a} ,$$

$$\frac{\partial R_c}{\partial e} = \frac{\partial R_c}{\partial x_m} \frac{\partial x_m}{\partial e} + \cdot \ \cdot \ \cdot + \frac{\partial R_c}{\partial z_m} \frac{\partial z_m}{\partial e} ,$$

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$$\frac{\partial R_c}{\partial \beta_3} = \frac{\partial R_c}{\partial x_m}\frac{\partial x_m}{\partial \beta_3} + \cdot \cdot \cdot + \frac{\partial R_c}{\partial z_m}\frac{\partial z_m}{\partial \beta_3} .$$

Also

$$\frac{\partial \dot{R}_c}{\partial a} = \frac{\partial \dot{R}_c}{\partial x_m}\frac{\partial x_m}{\partial a} + \frac{\partial \dot{R}_c}{\partial y_m}\frac{\partial y_m}{\partial a} + \frac{\partial \dot{R}_c}{\partial z_m}\frac{\partial z_m}{\partial a} + \frac{\partial \dot{R}_c}{\partial \dot{x}_m}\frac{\partial \dot{x}_m}{\partial a} + \frac{\partial \dot{R}_c}{\partial \dot{y}_m}\frac{\partial \dot{y}_m}{\partial a} + \frac{\partial \dot{R}_c}{\partial \dot{z}_m}\frac{\partial \dot{z}_m}{\partial a} ,$$

$$\frac{\partial \dot{R}_c}{\partial e} = \frac{\partial \dot{R}_c}{\partial x_m}\frac{\partial x_m}{\partial e} + \cdot \cdot \cdot + \frac{\partial \dot{R}_c}{\partial \dot{z}_m}\frac{\partial \dot{z}_m}{\partial e} ,$$

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$$\frac{\partial \dot{R}_c}{\partial \beta_3} = \frac{\partial \dot{R}_c}{\partial x_m}\frac{\partial x_m}{\partial \beta_3} + \cdot \cdot \cdot + \frac{\partial \dot{R}_c}{\partial z_m}\frac{\partial \dot{z}_m}{\partial \beta_3} .$$

Here

$$\frac{\partial R_c}{\partial x_m} = \frac{x_m}{\sqrt{x_m^2 + y_m^2 + z_m^2}} ,$$

$$\frac{\partial R_c}{\partial y_m} = \frac{y_m}{\sqrt{x_m^2 + y_m^2 + z_m^2}} ,$$

$$\frac{\partial R_c}{\partial z_m} = \frac{z_m}{\sqrt{x_m^2 + y_m^2 + z_m^2}} .$$

and

$$\frac{\partial \dot{R}_c}{\partial x_m} = \frac{1}{R_c}\left(\dot{x}_m - \frac{\dot{R}_c\, x_m}{R_c}\right),$$

$$\frac{\partial \dot{R}_c}{\partial y_m} = \frac{1}{R_c}\left(\dot{y}_m - \frac{\dot{R}_c\, y}{R_c}\right),$$

$$\frac{\partial \dot{R}_c}{\partial z_m} = \frac{1}{R_c}\left(\dot{z}_m - \frac{\dot{R}_c\, z_m}{R_c}\right),$$

$$\frac{\partial \dot{R}_c}{\partial \dot{x}_m} = R_c\, x_m,$$

$$\frac{\partial \dot{R}_c}{\partial \dot{y}_m} = R_c\, y_m,$$

$$\frac{\partial \dot{R}_c}{\partial \dot{z}_m} = R_c\, z_m,$$

Also

$$\begin{pmatrix} \dfrac{\partial \dot{x}_m}{\partial q_i} \\ \\ \dfrac{\partial \dot{y}_m}{\partial q_i} \\ \\ \dfrac{\partial \dot{z}_m}{\partial q_i} \end{pmatrix} = (A_I)^{-1} \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial q_i} \\ \\ \dfrac{\partial \dot{y}}{\partial q_i} \\ \\ \dfrac{\partial \dot{z}}{\partial q_i} \end{pmatrix} + (\dot{A}_I)^{-1} \begin{pmatrix} \dfrac{\partial x}{\partial q_i} \\ \\ \dfrac{\partial y}{\partial q_i} \\ \\ \dfrac{\partial z}{\partial q_i} \end{pmatrix}$$

Now

$$\frac{\partial \dot{x}}{\partial a} = \left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1-\eta^2}\right)\frac{\partial x}{\partial a} + x \left[\frac{(\rho^2 + c^2)\left(\dot{\rho}\,\dfrac{\partial \rho}{\partial a} + \rho\,\dfrac{\partial \dot{\rho}}{\partial a}\right) - 2\rho^2\dot{\rho}\,\dfrac{\partial \rho}{\partial a}}{(\rho^2 + c^2)^2}\right.$$

$$\left. - \frac{(1-\eta^2)\left(\dot{\eta}\,\dfrac{\partial \eta}{\partial a} + \eta\,\dfrac{\partial \dot{\eta}}{\partial a}\right) + 2\eta^2\dot{\eta}\,\dfrac{\partial \eta}{\partial a}}{(1-\eta^2)^2}\right] - \left(\frac{\alpha_3}{h_\phi^2}\,\frac{\partial y}{\partial a} + \frac{y}{h_\phi^2}\,\frac{\partial \alpha_3}{\partial a} + y\alpha_3\,\frac{\partial h_\phi^{-2}}{\partial a}\right).$$

Since $\rho = a(1 - e \cos E)$ and $\eta = \eta_0 \sin \psi$, then

$$\frac{\partial \rho}{\partial a} = 1 - e \cos E + a e \sin E \frac{\partial E}{\partial a},$$

$$\frac{\partial \eta}{\partial a} = \eta_0 \cos \psi \frac{\partial \psi}{\partial a},$$

with

$$\frac{\partial h_\phi^{-2}}{\partial a} = - 2 h_\phi^{-3} \frac{\partial h_\phi}{\partial a},$$

$$\frac{\partial h_\phi}{\partial a} = \left[- \eta (\rho^2 + c^2) \frac{\partial \eta}{\partial a} + \rho (1 - \eta^2) \frac{\partial \rho}{\partial a} \right] \frac{1}{h_\phi}.$$

Also

$$\frac{\partial \dot{\rho}}{\partial a} = \frac{\dot{\rho}}{a} + \frac{a e \sin E}{h_\rho^2 (\rho^2 + c^2)} \quad \frac{1}{2} \left[- 2 \alpha_1 (\rho^2 + A\rho + B) \right]^{-1/2} \quad \left[- (\rho^2 + A\rho + B) 2 \frac{\partial \alpha_1}{\partial a} \right.$$

$$\left. - 2 \alpha_1 \left(2 \rho \frac{\partial \rho}{\partial a} + A \frac{\partial \rho}{\partial a} + \rho \frac{\partial A}{\partial a} + \frac{\partial B}{\partial a} \right) \right]$$

$$+ \frac{a e \sqrt{-2 \alpha_1 (\rho^2 + A\rho + B)}}{h_\rho^2 (\rho^2 + c^2)} \cos E \frac{\partial E}{\partial a} - \frac{2 \dot{\rho}}{h_\rho} \frac{\partial h_\rho}{\partial a} - \frac{2 \dot{\rho} \rho}{\rho^2 + c^2} \frac{\partial \rho}{\partial a},$$

where

$$\frac{\partial h_\rho}{\partial a} = \frac{1}{h_\rho} \left(\rho \frac{\partial \rho}{\partial a} + c^2 \eta \frac{\partial \eta}{\partial a} - \frac{\rho^2 + c^2 \eta^2}{\rho^2 + c^2} \rho \frac{\partial \rho}{\partial a} \right) \frac{1}{\rho^2 + c^2},$$

$$\frac{\partial A}{\partial a} = - 2 \frac{\partial b_1}{\partial a},$$

$$\frac{\partial B}{\partial a} = \frac{\partial b_2^2}{\partial a} = 2 b_2 \frac{\partial b_2}{\partial a}.$$

And

$$\frac{\partial \dot{\eta}}{\partial a} = - \frac{c\,\eta_0 \sqrt{-2\alpha_1 \left(\eta_2^2 - \eta^2\right)}}{h_\eta^2 \left(1 - \eta^2\right)} \sin\psi \quad \frac{\partial \psi}{\partial a}$$

$$+ \frac{c\,\eta_0}{2} \quad \cos\psi \quad \frac{\left[-2\alpha_1 \left(2\eta_2 \frac{\partial \eta_2}{\partial a} - 2\eta \frac{\partial \eta}{\partial a}\right) - \left(\eta_2^2 - \eta^2\right) 2 \frac{\partial \alpha_1}{\partial a}\right]}{h_\eta^2 (1-\eta^2) \sqrt{-2\alpha_1 \left(\eta_2^2 - \eta^2\right)}}$$

$$- \frac{2\dot{\eta}}{h_\eta} \quad \frac{\partial h_\eta}{\partial a} + \frac{2\dot{\eta}\eta}{1-\eta^2} \quad \frac{\partial \eta}{\partial a},$$

with

$$\frac{\partial h_\eta}{\partial a} = \frac{1}{h_\eta} \left(\rho \frac{\partial \rho}{\partial a} + \eta c^2 \frac{\partial \eta}{\partial a} + \frac{\rho^2 + c^2 \eta^2}{1 - \eta^2} \quad \eta \frac{\partial \eta}{\partial a}\right) \quad \frac{1}{1 - \eta^2}.$$

Then

$$\frac{\partial \dot{y}}{\partial a} = \left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1-\eta^2}\right) \frac{\partial y}{\partial a} + y \quad \left[\frac{(\rho^2 + c^2)\left(\dot{\rho} \frac{\partial \rho}{\partial a} + \rho \frac{\partial \dot{\rho}}{\partial a}\right) - 2\rho^2 \dot{\rho} \frac{\partial \rho}{\partial a}}{(\rho^2 + c^2)^2}\right.$$

$$\left. - \frac{(1 - \eta^2)\left(\dot{\eta} \frac{\partial \eta}{\partial a} + \eta \frac{\partial \dot{\eta}}{\partial a}\right) + 2\eta^2 \dot{\eta} \frac{\partial \eta}{\partial a}}{(1 - \eta^2)^2}\right] + \frac{\alpha_3}{h_\phi^2} \frac{\partial x}{\partial a} + \frac{x}{h_\phi^2} \frac{\partial \alpha_3}{\partial a} + x\alpha_3 \frac{\partial h_\phi^{-2}}{\partial a},$$

$$\frac{\partial \dot{z}}{\partial a} = \rho \frac{\partial \dot{\eta}}{\partial a} + \dot{\eta} \frac{\partial \rho}{\partial a} + \eta \frac{\partial \dot{\rho}}{\partial a} + \dot{\rho} \frac{\partial \eta}{\partial a},$$

$$\frac{\partial \dot{x}}{\partial e} = \left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1-\eta^2}\right) \frac{\partial x}{\partial e} + x \quad \left[\frac{(\rho^2 + c^2)\left(\dot{\rho} \frac{\partial \rho}{\partial e} + \rho \frac{\partial \dot{\rho}}{\partial e}\right) - 2\rho^2 \dot{\rho} \frac{\partial \rho}{\partial e}}{(\rho^2 + c^2)^2}\right.$$

$$- \left[\frac{(1 - \eta^2)\left(\dot{\eta} \frac{\partial \eta}{\partial e} + \eta \frac{\partial \dot{\eta}}{\partial e}\right) + 2\eta^2 \dot{\eta} \frac{\partial \eta}{\partial e}}{(1 - \eta^2)^2}\right] - \left(\frac{\alpha_3}{h_\phi^2} \frac{\partial y}{\partial e} + \frac{y}{h_\phi^2} \frac{\partial \alpha_3}{\partial e} + y\alpha^3 \frac{\partial h_\phi^{-2}}{\partial e}\right),$$

where

$$\frac{\partial \rho}{\partial e} = a \left(e \sin E \frac{\partial E}{\partial e} - \cos E \right),$$

$$\frac{\partial \eta}{\partial e} = \eta_0 \cos \psi \frac{\partial \psi}{\partial e},$$

$$\frac{\partial h_\phi^{-2}}{\partial e} = - 2 h_\phi^{-3} \frac{\partial h_\phi}{\partial e},$$

with

$$\frac{\partial h_\phi}{\partial e} = \left[- (\rho^2 + c^2) \eta \frac{\partial \eta}{\partial e} + (1 - \eta^2) \rho \frac{\partial \rho}{\partial e} \right] \frac{1}{h_\phi} .$$

Now

$$\frac{\partial \dot{\rho}}{\partial e} = \frac{\dot{\rho}}{e} + \frac{a e \sqrt{-2\alpha_1 (\rho^2 + A\rho + B)}}{h_\rho^2 (\rho^2 + c^2)} \cos E \frac{\partial E}{\partial e}$$

$$+ \frac{a e \sin E}{2 h_\rho^2 (\rho^2 + c^2)} \left[-2\alpha_1 (\rho^2 + A\rho + B) \right]^{-1/2} \left[-2 \frac{\partial \alpha_1}{\partial e} (\rho^2 + A\rho + B) - 2\alpha_1 \left(2\rho \frac{\partial \rho}{\partial e} + A \frac{\partial \rho}{\partial e} \right.\right.$$

$$\left.\left. + \rho \frac{\partial A}{\partial e} + \frac{\partial B}{\partial e} \right) \right] - \frac{2\dot{\rho}}{h_\rho} \frac{\partial h_\rho}{\partial e} - \frac{2\rho\dot{\rho}}{\rho^2 + c^2} \frac{\partial \rho}{\partial e}$$

with

$$\frac{\partial h_\rho}{\partial e} = \frac{1}{h_\rho} \left(\rho \frac{\partial \rho}{\partial e} + \eta c^2 \frac{\partial \eta}{\partial e} - \frac{\rho^2 + c^2 \eta^2}{\rho^2 + c^2} \rho \frac{\partial \rho}{\partial e} \right) \frac{1}{\rho^2 + c^2},$$

$$\frac{\partial A}{\partial e} = - 2 \frac{\partial b_1}{\partial e},$$

$$\frac{\partial B}{\partial e} = \frac{\partial b_2^2}{\partial e} = 2 b_2 \frac{\partial b_2}{\partial e}$$

Also

$$\frac{\partial \dot{\eta}}{\partial e} = -\frac{c\,\eta_0 \sqrt{-2\alpha_1\left(\eta_2^2 - \eta^2\right)}}{h_\eta^2\,(1-\eta^2)} \sin\psi \frac{\partial \psi}{\partial e} - \frac{2\dot{\eta}}{h_\eta}\frac{\partial h_\eta}{\partial e} + \frac{2\dot{\eta}\eta}{1-\eta^2}\frac{\partial \eta}{\partial e}$$

$$-\frac{c\,\eta_0 \cos\psi}{h_\eta^2\,(1-\eta^2)} \frac{\frac{\partial \alpha_1}{\partial e}\left(\eta_2^2 - \eta^2\right) + \alpha_1\left(2\eta_2 \frac{\partial \eta_2}{\partial e} - 2\eta \frac{\partial \eta}{\partial e}\right)}{\sqrt{-2\alpha_1\left(\eta_2^2 - \eta^2\right)}},$$

where

$$\frac{\partial h_\eta}{\partial e} = \frac{1}{h_\eta}\left(\rho \frac{\partial \rho}{\partial e} + \eta c^2 \frac{\partial \eta}{\partial e} + \frac{\rho^2 + c^2\eta^2}{1-\eta^2}\,\eta \frac{\partial \eta}{\partial e}\right)\frac{1}{1-\eta^2}.$$

Then

$$\frac{\partial \dot{y}}{\partial e} = \left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1-\eta^2}\right)\frac{\partial y}{\partial e} + y\left[\frac{(\rho^2 + c^2)\left(\dot{\rho}\frac{\partial \rho}{\partial e} + \rho\frac{\partial \dot{\rho}}{\partial e}\right) - 2\rho^2\dot{\rho}\frac{\partial \rho}{\partial e}}{(\rho^2 + c^2)^2}\right.$$

$$\left. - \frac{(1-\eta^2)\left(\dot{\eta}\frac{\partial \eta}{\partial e} + \eta\frac{\partial \dot{\eta}}{\partial e}\right) + 2\eta^2\dot{\eta}\frac{\partial \eta}{\partial e}}{(1-\eta^2)^2}\right] + \frac{\alpha_3}{h_\phi^2}\frac{\partial x}{\partial e} + \frac{x}{h_\phi^2}\frac{\partial \alpha_3}{\partial e} + x\,\alpha_3 \frac{\partial h_\phi^{-2}}{\partial e}$$

$$\frac{\partial \dot{z}}{\partial e} = \rho\frac{\partial \dot{\eta}}{\partial e} + \dot{\eta}\frac{\partial \rho}{\partial e} + \eta\frac{\partial \dot{\rho}}{\partial e} + \dot{\rho}\frac{\partial \eta}{\partial e}.$$

Finally

$$\frac{\partial \dot{x}}{\partial \eta_0} = \left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1-\eta^2}\right)\frac{\partial x}{\partial \eta_0} + x\left[\frac{(\rho^2 + c^2)\left(\dot{\rho}\frac{\partial \rho}{\partial \eta_0} + \rho\frac{\partial \dot{\rho}}{\partial \eta_0}\right) - 2\rho^2\dot{\rho}\frac{\partial \rho}{\partial \eta_0}}{(\rho^2 + c^2)^2}\right.$$

$$\left. - \frac{(1-\eta^2)\left(\dot{\eta}\frac{\partial \eta}{\partial \eta_0} + \eta\frac{\partial \dot{\eta}}{\partial \eta_0}\right) + 2\eta^2\dot{\eta}\frac{\partial \eta}{\partial \eta_0}}{(1-\eta^2)^2}\right] - \left(\frac{\alpha_3}{h_\phi^2}\frac{\partial y}{\partial \eta_0} + \frac{y}{h_\phi^2}\frac{\partial \alpha_3}{\partial \eta_0} + y\,\alpha_3 \frac{\partial h_\phi^{-2}}{\partial \eta_0}\right),$$

With

$$\frac{\partial \rho}{\partial \eta_0} = a\,e\,\sin E\,\frac{\partial E}{\partial \eta_0},$$

$$\frac{\partial \eta}{\partial \eta_0} = \sin \psi + \psi_0 \cos \psi\,\frac{\partial \psi}{\partial \eta_0},$$

$$\frac{\partial h_\phi^{-2}}{\partial \eta_0} = -2h_\phi^{-3}\,\frac{\partial h_\phi}{\partial \eta_0},$$

where

$$\frac{\partial h_\phi}{\partial \eta_0} = \left[-(\rho^2 + c^2)\,\eta\,\frac{\partial \eta}{\partial \eta_0} + \rho(1-\eta^2)\,\frac{\partial \rho}{\partial \eta_0}\right]\frac{1}{h_\phi}.$$

Now

$$\frac{\partial \dot{\rho}}{\partial \eta_0} = \frac{a\,e\sqrt{-2\alpha_1(\rho^2 + A\rho + B)}}{h_\rho^2\quad(\rho^2 + c^2)}\cos E\,\frac{\partial E}{\partial \eta_0}$$

$$+\frac{a\,e\sin E}{2h_\phi^2(\rho^2 + c^2)}\left[-2\alpha_1(\rho^2 + A\rho + B)\right]^{-1/2}\left[-2\,\frac{\partial \alpha_1}{\partial \eta_0}\,(\rho^2 + A\rho + B) - 2\alpha_1\left(2\rho\,\frac{\partial \rho}{\partial \eta_0}\right.\right.$$

$$\left.\left. + A\,\frac{\partial \rho}{\partial \eta_0} + \rho\,\frac{\partial A}{\partial \eta_0} + \frac{\partial B}{\partial \eta_0}\right)\right] - \frac{2\dot{\rho}}{h_\rho}\,\frac{\partial h_\rho}{\partial \eta_0} - \frac{2\rho\,\dot{\rho}}{(\rho^2 + c^2)}\,\frac{\partial \rho}{\partial \eta_0},$$

with

$$\frac{\partial h_\rho}{\partial \eta_0} = \frac{1}{h_\rho}\left(\rho\,\frac{\partial \rho}{\partial \eta_0} + \eta\,c^2\,\frac{\partial \eta}{\partial \eta_0} - \frac{\rho^2 + c^2\,\eta^2}{\rho^2 + c^2}\,\rho\,\frac{\partial \rho}{\partial \eta_0}\right)\frac{1}{\rho^2 + c^2},$$

$$\frac{\partial A}{\partial \eta_0} = -2\,\frac{\partial b_1}{\partial \eta_0},$$

$$\frac{\partial B}{\partial \eta_0} = \frac{\partial b_2^2}{\partial \eta_0} = 2b_2\,\frac{\partial b_2}{\partial \eta_0}.$$

Also

$$\frac{\partial \dot{\eta}}{\partial \eta_0} = \frac{\dot{\eta}}{\eta_0} - \frac{c\sqrt{-2\alpha_1\left(\eta_2^2 - \eta^2\right)}}{h_\eta^2\left(1 - \eta^2\right)} \sin\psi \frac{\partial \psi}{\partial \eta_0}$$

$$+ \frac{c\,\eta_0}{2} \frac{\cos\psi}{h_\eta^2\left(1 - \eta^2\right)} \frac{\left(-2\alpha_1\right)\left(2\eta_2 \frac{\partial \eta_2}{\partial \eta_0} - 2\eta \frac{\partial \eta}{\partial \eta_0}\right) - \left(\eta_2^2 - \eta^2\right) 2 \frac{\partial \alpha_1}{\partial \eta_0}}{\sqrt{-2\alpha_1\left(\eta_2^2 - \eta^2\right)}}$$

$$- \frac{2\dot{\eta}}{h_\eta} \frac{\partial h_\eta}{\partial \eta_0} + \frac{2\dot{\eta}\eta}{1 - \eta^2} \frac{\partial \eta}{\partial \eta_0},$$

where

$$\frac{\partial h_\eta}{\partial \eta_0} = \frac{1}{h_\eta} \left(\rho \frac{\partial \rho}{\partial \eta_0} + \eta c^2 \frac{\partial \eta}{\partial \eta_0} + \frac{\rho^2 + c^2\eta^2}{1 - \eta^2} \eta \frac{\partial \eta}{\partial \eta_0}\right) \frac{1}{1 - \eta^2}.$$

Then

$$\frac{\partial \dot{y}}{\partial \eta_0} = \left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1 - \eta^2}\right) \frac{\partial x}{\partial \eta_0} + y \left[\frac{\left(\rho^2 + c^2\right)\left(\dot{\rho} \frac{\partial \rho}{\partial \eta_0} + \rho \frac{\partial \dot{\rho}}{\partial \eta_0}\right) - 2\rho^2\dot{\rho} \frac{\partial \rho}{\partial \eta_0}}{\left(\rho^2 + c^2\right)^2}\right.$$

$$\left. - \frac{\left(1 - \eta^2\right)\left(\dot{\eta} \frac{\partial \eta}{\partial \eta_0} + \eta \frac{\partial \dot{\eta}}{\partial \eta_0}\right) + 2\eta^2\dot{\eta} \frac{\partial \eta}{\partial \eta_0}}{\left(1 - \eta^2\right)^2}\right] + \frac{\alpha_3}{h_\phi^2} \frac{\partial x}{\partial \eta_0} + \frac{x}{h_\phi^2} \frac{\partial \alpha_3}{\partial \eta_0} + x\,\alpha_3 \frac{\partial h_\phi^{-2}}{\partial \eta_0},$$

$$\frac{\partial \dot{z}}{\partial \eta_0} = \rho \frac{\partial \dot{\eta}}{\partial \eta_0} + \dot{\eta} \frac{\partial \rho}{\partial \eta_0} + \eta \frac{\partial \dot{\rho}}{\partial \eta_0} + \dot{\rho} \frac{\partial \eta}{\partial \eta_0}.$$

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—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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