TECHNICAL MEMORANDUM NO. 36

A PRELIMINARY STUDY OF SINE WAVE PLUS NOISE FOR APPLICATION TO INSTRUMENTATION TAPE RECORDER PERTURBATIONS

Gerald Weiss

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ABSTRACT

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A model of a sine wave plus noise is presented to show the interrelationship between phase correlation, time displacement error and power spectrum of instrument type tape recorders. It is shown that for small ovservation times, during which the output phase is highly correlated, the time displacement error increases directly with observation time. For large observation times, however, phase correlation no longer obtains and time displacement error is constant with time. In addition, it is shown that a knowledge of time displacement error is sufficient to determine the output power spectrum.

author

TABLE OF CONTENTS

	Page
MAIN BODY	
1. INTRODUCTION	1
2. DISCUSSION	3
2.1 Basic Model	3
2.2 The Phase Function	5
2.3 Time Displacement Error	8
2.4 Phase Correlation and Time Displacement Error	9
2.5 Output Power Spectrum	11
CONCLUSIONS	14
PROGRAM FOR THE NEXT REPORTING INTERVAL	15
APPENDIX 1	17
APPENDIX 2	19
APPENDIX 3	20
APPENDIX 4	24
REFERENCES	26

LIST OF ILLUSTRATIONS

Figure		Page
1	Signal and Noise Model	4
2	Composite Phase Function	7
3	Mean Square Time Displacement Error with Observation Time	10

A PRELIMINARY STUDY OF SINE WAVE PLUS NOISE FOR APPLICATION TO INSTRUMENTATION TAPE RECORDER PERTURBATIONS

1. INTRODUCTION

A study has been initiated to determine the causes of noise in the recorded output of instrumentation tape recorders. As a preliminary to this study, some consideration is given to the relationship among the various parameters which are commonly used to describe the tape recorder output variations. The parameters are: (1) phase correlation, (2) time displacement error, and (3) output power spectrum. A model which aids in formulating a mathematical description of these parameters is one which contains a sine wave and added noise. In the model, the sine wave represents the unperturbed recorder output and the noise, unwanted disturbances. Initially, it is assumed that the noise spectrum is symmetrically located about the frequency of the sine wave, and contains a non-zero component at this frequency. The way in which the noise perturbs the phase angle of the sine wave is developed for two cases: the case where the phase angle is directly proportional to the noise (phase modulation), and the case where the angle is proportional to the integral of the noise (frequency modulation). A time displacement error, δt, is derived in terms of the perturbed phase function. The phase function enables a determination of the time period between zero crossings of the sine wave, since the period corresponds to an accumulation of 2π radians in the phase function. The time displacement error is then shown to be the difference in time corresponding to the difference between the

perturbed and nonperturbed phase functions. It is also shown that the time displacement error is a function of the time period between the two measurements of the phase angle of the sine wave to a point after which it is a constant function of the time period. The mechanism which produces this division in the phenomena of time displacement error is the statistical correlation of the phase function at the two points of measure. This, in turn, is related to the reciprocal of the bandwidth of the noise. It is also shown that a knowledge of the time displacement characteristics is sufficient to determine the output spectrum of the sine wave when acted on by the noise.

2. DISCUSSION

2.1 Basic Model

A model is sought for the manner in which random noise causes time and frequency errors in the output of instrument type tape recorders using speed lock control. Initially, the only aspect of the problem to be pursued now will be the determination of the relationship among time displacement error, phase correlation, and output power spectrum. After this is accomplished, the model will be expanded to include the complexities introduced by the speed lock mechanism. The effect of speed lock is intended to suppress the effects of the low frequency components of the noise at the cost of follow up error. The model now under consideration consists of a sine wave signal generator which is fixed in amplitude and frequency and has added to it a band limited Gaussian noise voltage. This model is shown in Fig. 1. What is required is the manner in which the noise voltage, e_n affects the signal voltage, e_s, to produce phase perturbations in the output voltage e_o. If the signal voltage is expressed as:

$$e_s = E \cos \omega_c t \quad ; \tag{1}$$

and the noise voltage as the sum of cosine waves with random phase angles:

$$e_n(t) = \sum_{i=1}^{N} c_i \cos (\omega_i t - \varphi_i) \quad , \tag{2}$$

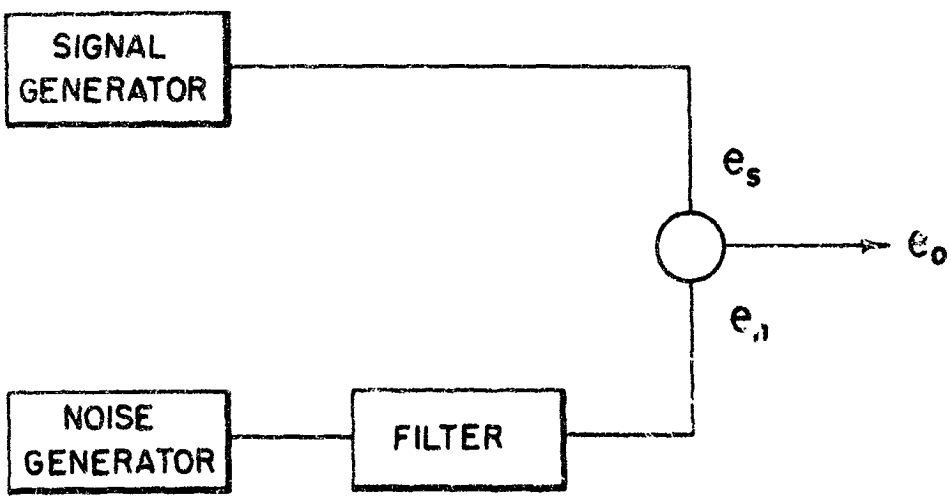

FIG. 1

SIGNAL AND NOISE MODEL

then* the output voltage is:

$$e_o(t) = A(t) \cos [\omega_o t - \theta(t)] \quad . \tag{3}$$

In the above equation, the envelope function may be expressed as:

$$A(t) = [E + x(t)]^2 + y^2(t) \quad ; \tag{4}$$

and the phase function as:

$$\theta(t) = \tan^{-1} \left[\frac{- y(t)}{E + x(t)} \right] \quad . \tag{5}$$

The quantity $x(t)$ is the noise voltage in-phase with $\cos \omega_o t$, and $y(t)$ is the noise voltage in quadrature. The spectrum of $x(t)$ and $y(t)$ are both the low pass equivalent of the spectrum of the noise located at ω_o. If the mean square noise voltage in phase with the signal is small compared to the mean square signal voltage, then amplitude perturbations in the output may be ignored and the following approximations result:

$$A(t) \approx E \quad , \tag{6}$$

$$\theta(t) \approx - y(t)/E \quad . \tag{7}$$

2.2 The Phase Function

The perturbations in the phase function are directly proportional to the noise. This process corresponds to a random phase modulation. The power spectrum of the quadrature component of the noise, $W_y(\omega)$, is

*See Appendix 1.

assumed to have a non zero d.c. component, and is narrow band relative to ω_0. Consequently, the phase exhibits slow changes relative to $\omega_0 t$. The total phase function may be written as:

$$\gamma(t) = \omega_0 t - y(t)/E \quad , \tag{8}$$

and is plotted in Fig. 2 for variations in θ with time very close to ω_0 because of the narrow band approximation.

A measurement of the total phase angle γ, corresponds to a time t_1 if $\theta_1 = 0$. For a given perturbation θ_1, a time displacement error δt_1 occurs which may be approximated by:

$$\delta t_1 \approx \frac{\theta_1}{\omega_0} \quad , \tag{9}$$

if

$$\frac{d\theta_1}{dt} \approx \omega_0 \quad .$$

The mean square time displacement error is then proportional to the mean square perturbation in the phase angle:

$$\overline{\delta t_1^2} = \overline{\theta_1^2}/\omega_0^2 = \frac{\overline{y^2}/E^2}{\omega_0^2} \quad . \tag{10}$$

Therefore, if some aspect of the sine wave signal is measured, such as a zero crossing corresponding to a phase angle of $\gamma_1 = m2\pi$ radians, then the mean square time displacement error, relative to the start of the process at $t = 0$, is given by (10). The time displacement error is linearly related to the noise producing it.

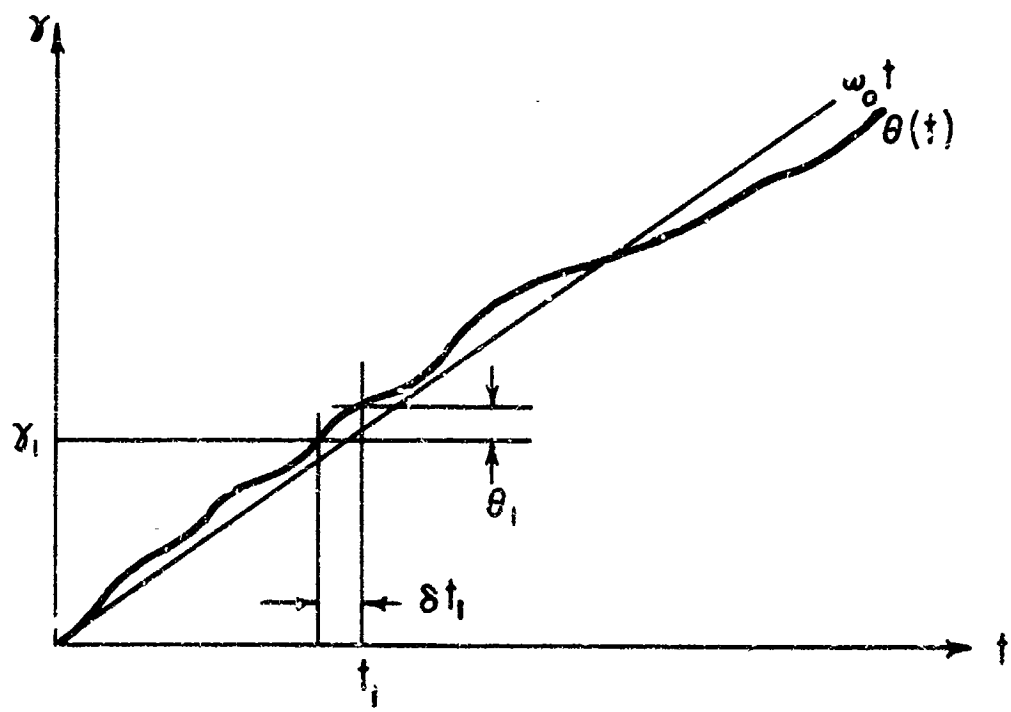

FIG. 2

COMPOSITE PHASE FUNCTION

2.3 Time Displacement Error

In general, it is not convenient to relate the time error to the beginning of the process ($t=0$), but rather to take two measurements displaced by an interval of time τ. An error in time δt_1 will then exist at the beginning of the measure and one at the end δt_2, where $t_2 - t_1 = \tau$. Under the same narrow band limitations on θ_1 as imposed above, the time displacement error accrued over the interval τ is:

$$\Delta\tau = \delta t_2 - \delta t_1 = \theta_2 - \theta_1/\omega_0 \quad . \tag{11}$$

The mean square value of $\Delta\tau$ is:

$$\overline{\Delta\tau^2} = \overline{(\theta_2 - \theta_1)^2}/\omega_0^2 \quad . \tag{12}$$

The quantity $\overline{(\theta_2 - \theta_1)^2}$ may be expanded* in terms of the noise quadrature component $y(t)$:

$$\overline{(\theta_2 - \theta_1)^2} = \frac{2}{E^2}\left[\psi_0 - R_y(\tau)\right] \quad ; \tag{13}$$

where ψ_0 is the mean square value of the noise,

$R_y(\tau)$ is the autocorrelation function of the noise,

i.e.: $\psi_0 = R_y(0)$.

Therefore:

$$\overline{\Delta\tau^2} = 2/\omega_0^2\, E^2\left[\psi_0 - R_y(\tau)\right] \quad . \tag{14}$$

*See Appendix 2.

The autocorrelation function may in turn be related to the power spectrum of the noise by the Wiener-Khinchin[1] relations:

$$R_y(\tau) = \int_0^\infty W_y(f) \cos \omega\tau df \quad . \tag{15}$$

As an illustration of the above, assume that the noise is shaped by an RLC filter with outer frequency ω_o. The location of this filter in the model is shown in Fig. 1. Using expressions (14) and (15):

$$\overline{\Delta\tau^2} = \frac{2\psi_o}{\omega_o^2 E^2} (1 - \epsilon^{-\omega_a \tau}) \quad , \tag{16}$$

where ω_a is the half power radian frequency of the low pass equivalent of the RLC filter. A plot of (16) is shown in Fig. 3.

2.4 Phase Correlation and Time Displacement Error

In view of the previous discussion, the relationship between the time displacement error and the statistical correlation of the noise may be demonstrated. From expression (16) it is seen that for a small observation time the following approximation results:

$$\overline{\Delta\tau^2} \approx \frac{2\psi_o}{\omega_o^2 E^2} \tau \quad , \tag{17}$$

$$\text{for } \tau \ll 1/\omega_a \quad .$$

Consequently, at $\tau = 0$, $\overline{\Delta\tau^2} = 0$ since complete correlation of the noise exists. As τ increases slightly from zero, the phase angle at the

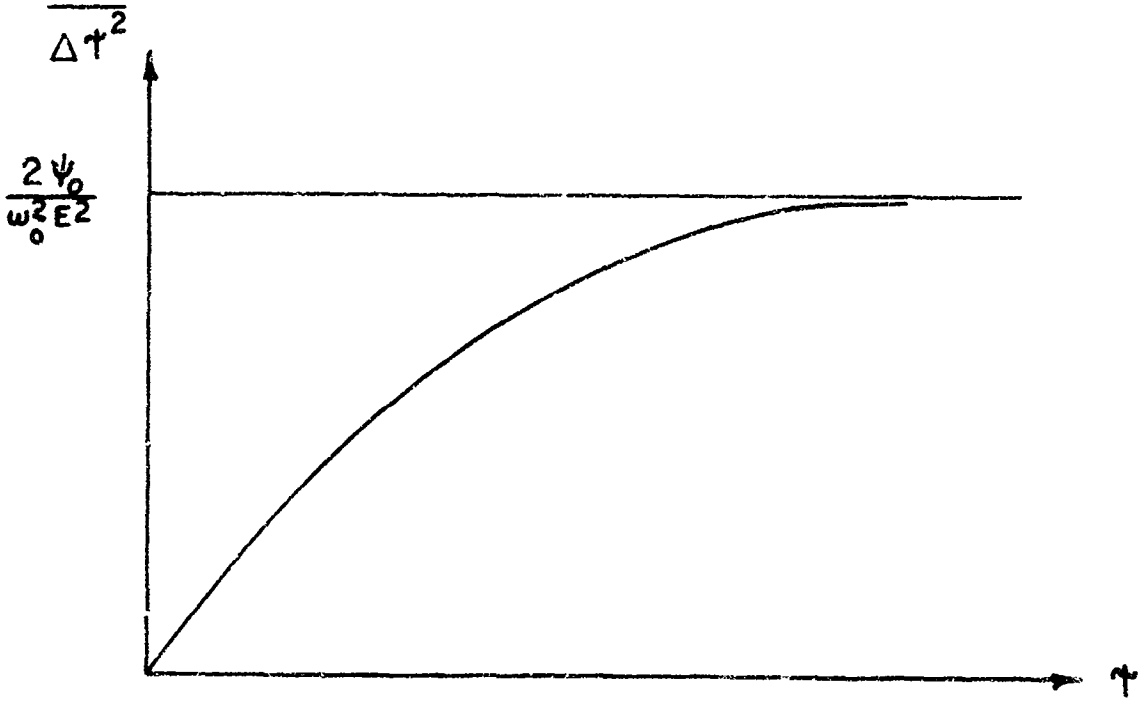

FIG. 3

MEAN SQUARE TIME DISPLACEMENT ERROR WITH OBSERVATION TIME

beginning and at the end of the period τ has not changed to any large degree, since correlation still exists in the phase angle for very small τ. As τ is increased towards $1/\omega_a$, correlation in the phase angle at the end of the period, relative to the beginning, decreases with a resulting increase in the time displacement error. Finally, when the period τ exceeds $1/\omega_a$ no correlation exists. A measurement at the beginning of the interval is statistically independent of one taken at the end. The error is additive and becomes twice that obtained at the beginning, it being equal to $2\psi_o/\omega_o^2 E^2$.

2.5 Output Power Spectrum

What is desired is an expression for the power spectrum of the voltage e_o based on the model which appears in Fig. 1. The approach is to obtain the autocorrelation function of e_o and then, using the Wiener-Khinchin[1] relation

$$W_{e_o}(f) = 4 \int_0^\infty R_{e_o}(\tau) \cos \omega\pi d\tau \quad , \tag{18}$$

obtain the power spectrum. The autocorrelation function of e_o may be obtained by taking the time average of the output separated by a time interval τ:

$$R_{e_o}(\tau) = \overline{e_o(t)e_o(t + \tau)} \quad . \tag{19}$$

The bar over an expression will signify a time average. An expression for the output e_o is given by (3). Substitution of (3) into (19), evaluating the indicated time averages, and invoking a narrow band

approximation* result in:

$$R_{e_o}(\tau) \approx E^2/2 \quad e\left\{\epsilon^{-j\omega_o\tau}\,\overline{\epsilon^{j[\theta(t)\,-\,\theta(t+\tau)]}}\right\} \quad . \tag{20}$$

In Appendix 3, it is shown that:

$$\overline{\epsilon^{j[\theta(t)\,-\,\theta(t+\tau)]}} = \epsilon^{-[R_\theta(0)\,-\,R_\theta(\tau)]} \quad . \tag{21}$$

$$[R_\theta(0) - R_\theta(\tau)] = \frac{\overline{(\theta_2-\theta_1)^2}}{2} = \frac{\overline{\Delta\tau^2}\,\omega_o^2}{2} \quad . \tag{22}$$

Consequently, the power spectrum in terms of the time displacement error is:

$$W_{e_o}(f) = 4(E^2/2)\int_0^\infty \epsilon^{-\frac{\overline{\Delta\tau^2}\omega_o^2}{2}} \cos\omega_o\tau \cos\omega\tau \, d\tau \quad . \tag{23}$$

If the contribution to the power spectrum at $2f_o$ is assumed to be small, then

$$W_{e_o}(f) \approx E^2\int_0^\infty \epsilon^{-\frac{\overline{\Delta\tau^2}\omega_o^2}{2}} \cos(\omega-\omega_o)\tau \, d\tau \quad . \tag{24}$$

An expression for the power spectrum may be obtained in terms of the quadrature component of the noise, y(t), from (14) and (15):

*See Appendix 3.

$$\overline{\Delta\tau^2} = \frac{2}{\omega_o^2 E^2} \int_0^\infty W_y(f)(1 - \cos \omega\tau)\, df \quad . \tag{25}$$

If the random noise is producing equivalent phase modulation, then

$$W_{e_o}(f) \approx E^2 \int_0^\infty \exp\left[- 1/E^2 \int_0^\infty W_y(f)(1 - \cos \omega\tau) df\right] \cos (\omega - \omega_o)\tau \, d\tau \quad . \tag{26}$$

For equivalent frequency modulation*:

$$W_{e_o}(f) \approx E^2 \int_0^\infty \exp\left[- 1/E^2 \int_0^\infty \frac{W_y(f)}{\omega^2} (1 - \cos \omega\tau) df\right] \cos (\omega - \omega_o)\tau \, d\tau \quad . \tag{27}$$

*See Appendix 4.

CONCLUSIONS

Utilizing a basic model of signal and noise, the relationship between phase correlation and time displacement error can be formulated in terms of signal and noise parameters. The time displacement error dependence upon time of observation is illustrated for the case where the noise spectrum is shaped by an RLC filter, and the composite phase angle is directly proportional to the quadrature component of the noise. For this equivalent phase modulation case, the results appear to be in agreement with reported[2] time displacement errors with respect to the general shape of the time characteristic.

Theoretically, a knowledge of the time displacement error is sufficient to determine the output power spectrum. In addition, the output power spectrum may be obtained by considering the basic model as a mechanism for producing equivalent phase and frequency modulation due to a random modulation process. Consequently, known analytical techniques[3] used for handling mixed random and deterministic angle modulation processes may be applied to the problem of spectrum determination of the tape recorder output.

PROGRAM FOR THE NEXT REPORTING INTERVAL

1. Experimental Phase

A report is in progress on data gathered using the experimental auction discriminator at New York University. It includes curves showing probability of detection versus false alarm rates, probability of detection over the comb filter frequency range, and additional curves of interest. This report concludes the present stage of experimental work at New York University.

2. Residue Arithmetic

A report is completed which summarizes a survey on residue arithmetic. This report includes a review of the residue number systems, its advantages and the problems involved in using it.

3. Tape Recorder Study

An analysis of a servo tape speed control system is now being conducted. In this study, it is assumed that a sinusoidal signal is recorded on tape and then played back. The phase variation between the playback signal and a fixed reference will be obtained. The object of the study is to determine time displacement errors, phase perturbations and frequency perturbations under the influence of a speed control mechanism.

4. PCM Synchronization Analysis

Calculations are in progress for the probability analysis of the method planned to be used in synchronizing a PCM signal with the data processing system. Analytic relations have been determined for the probabilities of correct and incorrect decision at the counter outputs for both the acquisition and lock modes. These probabilities are expressed as functions of sample size, bit error probability or signal-to-noise ratio, tolerable bit errors per frame synchronization word and counter settings. Performance curves will be obtained to determine the most desirable settings for the frame sync pattern detector and for the counters in the search and lock unit.

APPENDIX 1

The output voltage resulting from the process of signal plus noise is:

$$e_o(t) = e_s(t) + e_n(t) \quad , \tag{1-1}$$

where the signal voltage is:

$$e_s(t) = E \cos \omega_o t \quad , \tag{1-2}$$

and the noise voltage represented[1] by:

$$e_n(t) = \sum_{i=1}^{N} c_i \cos (\omega_i t - \varphi_i) \quad , \tag{1-3}$$

$$e_n(t) = \sum_{i=1}^{N} c_i \cos (\omega_i t + \omega_o t - \omega_o t - \varphi_i) \quad , \tag{1-4}$$

$$e_n(t) = x(t) \cos \omega_o t - y(t) \sin \omega_o t \quad , \tag{1-5}$$

which expresses the noise voltage in terms of the component $x(t)$ in-phase with $\cos \omega_o t$, and $y(t)$ in quadrature with $\cos \omega_o t$. If the power spectrum of the noise is $W_n(\omega-\omega_o)$, then the power spectrum of the quadrature component and in-phase component of the noise is:

$$W_y(\omega) = W_x(\omega) \quad . \tag{1-6}$$

The output voltage may then be written as:

$$e_o(t) = [E + x(t)] \cos \omega_o t - y(t) \sin \omega_o t \quad , \tag{1-7}$$

$$e_o(t) \approx A(t) \cos [\omega_o t - \theta(t)] \quad ; \tag{1-8}$$

where

$$A(t) = [E = x(t)]^2 + y^2(t) \quad ,$$

$$\theta(t) = \tan^{-1} \frac{y(t)}{E + x(t)} \quad . \tag{1-9}$$

APPENDIX 2

The phase angle θ is measured at time t_1 and t_2, where the time interval between measure is τ. Using the notation $\theta_1 = \theta(t_1)$ and $\theta_2 = \theta(t_2) = \theta(t+\tau)$, the mean square value of the phase difference may be expanded as:

$$\overline{(\theta_2 - \theta_1)^2} = \overline{\theta_2^2 + \theta_1^2 - 2\theta_1\theta_2} \quad , \tag{2-1}$$

$$\overline{(\theta_2 - \theta_1)^2} = \overline{\theta_2^2} + \overline{\theta_1^2} - 2\overline{\theta_1\theta_2} \quad .$$

However:

$$\overline{\theta_2^2} = \overline{\theta_1^2} = \overline{y^2}/E^2 = \psi_0/E^2 \quad , \tag{2-2}$$

and

$$\overline{\theta_1\theta_2} = R_\theta(\tau) = R_y(\tau)/E^2 \quad . \tag{2-3}$$

The quantity ψ_0 is the mean square value of the noise:

$$\psi_0 = \int_0^\infty W_y(f)df \quad , \tag{2-4}$$

and $R_0(\tau)$ is the autocorrelation function of the phase angle, which is directly proportional to the autocorrelation function of the quadrature component of the noise, $R_y(\tau)$. Therefore:

$$\overline{(\theta_2 - \theta_1)^2} = 2/E^2\,[\psi_0 - R_y(\tau)] \quad . \tag{2.5}$$

APPENDIX 3

$$e_o(t) = E \cos [\omega_o t + \theta(t)] \quad , \tag{3-1}$$

and

$$R_{e_o}(\tau) = \overline{e_o(t)\ e_o(t+\tau)} \quad , \tag{3-2}$$

$$R_{e_o}(\tau) = E^2 \overline{\left\{ \cos [\omega_o t + \theta(t)] \cos [\omega_o(t+\tau) + \theta(t+\tau)] \right\}} \quad , \tag{3-3}$$

$$R_{e_o}(\tau) = E^2/2 \overline{\left\{ \cos [2\omega_o t + \omega_o \tau + \theta(t) + \theta(t+\tau)] \right.}$$

$$\overline{\left. + \cos [-\omega_o \tau + \theta(t) - \theta(t+\tau)] \right\}} \tag{3-4}$$

$$R_{e_o}(\tau) = E^2/2 \lim_{T \to \infty} \left\{ \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos [2\omega_o t + \omega_o \tau + \theta(t) + \theta(t+\tau)] dt \right.$$

$$\left. + \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos [-\omega_o \tau + \theta(t) - \theta(t+\tau)] dt \right\} \quad . \tag{3-5}$$

In expression (3-5) above:

$$\cos [2\omega_o t + \omega_o \tau + \theta(t) + \theta(t+\tau)]$$

$$= \cos [f(t,\tau)] \cos 2\omega_o t - \sin [f(t,\tau)] \sin 2\omega_o t \quad . \tag{3-6}$$

The functions cos $[f(t,\tau)]$ and sin $[f(t,\tau)]$ are slowly varying functions compared to cos $2\omega_0 t$ and sin $2\omega_0 t$. Therefore:

$$\lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\,[2\omega_0 t + \omega_0\tau + \theta(t) + \theta(t+\tau)]dt \approx 0 \quad , \tag{3-7}$$

and

$$R_{e_0}(\tau) \approx E^2/2 \lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\,[\omega_0\tau + \theta(t) - \theta(t-\tau)]dT \quad , \tag{3-8}$$

$$R_{e_0}(\tau) \approx E^2/2 \quad e\left\{ \epsilon^{-j\omega_0\tau}\, \epsilon^{\overline{j[\theta(t) - \theta(t+\tau)]}} \right\} \quad . \tag{3-9}$$

Consider the quantity:

$$\epsilon^{\overline{j[\theta_1 - \theta_2]}} \quad ,$$

where

$$\theta_1 = \theta(t)$$

and

$$\theta_2 = \theta(t+\tau) \quad .$$

The time average of the exponential functional may be determined by considering the ensemble average. The expected value of the exponential functional may then be expressed[3] in terms of the characteristic function:

$$E\left\{ \epsilon^{j\xi_1\theta_1 + j\xi_2\theta_2} \right\} = F_{\theta_1\theta_2}(\xi_1,\xi_2) \quad , \tag{3-10}$$

$$E\left\{ \epsilon^{j\theta_1 + j\theta_2} \right\} = F_{\theta_1\theta_2}(+1,-1) \quad . \tag{3-11}$$

For a Gaussian joint distribution in θ:

$$F_{\theta_1,\theta_2}(\xi_1,\xi_2) = \epsilon^{-\frac{1}{2}\tilde{\xi}K\xi} \quad ; \tag{3-12}$$

where it is assumed that the mean value of θ is zero. In (3-12):

ξ is the matrix $\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$,

$\tilde{\xi}$ is the transpose matrix $[\xi_1 \ \xi_2]$,

K is the covariance matrix $\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$.

Replacing the covariance by a corresponding time average:

$$K_{11} = \overline{\theta_1^2} \ ,$$

$$K_{12} = \overline{\theta_1\theta_2} = K_{21} \ ,$$

$$K_{22} = \theta_2^2 \ .$$

Therefore:

$$\tilde{\xi}K\xi = \xi_1^2\,\overline{\theta_1^2} + 2\xi_1\xi_2\,\overline{\theta_1\theta_2} + \xi_2^2\,\overline{\theta_2^2} \ ,$$

and

$$F_{\theta_1,\theta_2}(+1,-1) = e^{-\frac{1}{2}(\overline{\theta_1^2} - 2\overline{\theta_1\theta_2} + \overline{\theta_2^2})} \quad ; \tag{3-13}$$

since

$$\overline{\theta_1^2} = \overline{\theta_2^2} \quad ,$$

then

$$F_{\theta_1,\theta_2}(+1,-1) = \epsilon^{-(\overline{\theta_2} - \overline{\theta_1\theta_2})} \quad , \tag{3-14}$$

$$F_{\theta_1,\theta_2}(+1,-1) = \epsilon^{-[R_\theta(0) - R_\theta(\tau)]} \quad . \tag{3-15}$$

Therefore:

$$\epsilon^{j\overline{[\theta(t) - \theta(t+\tau)]}} = \epsilon^{-[R_\theta(0) - R_\theta(\tau)]} \quad . \tag{3-16}$$

APPENDIX 4

The output voltage of the composite signal acted upon by noise is:

$$e_o = E \cos [\omega t + \theta(t)] \quad . \tag{4-1}$$

If the angle θ is proportional to some modulating noise $y(t)$, then the process is phase modulation by noise:

$$\theta(t) = D_\varphi y(t) \quad . \tag{4-2}$$

The power spectrum corresponding to the noise $y(t)$ is $W_y(f)$. For an equivalent frequency modulation process by the noise, θ is proportional to the integral of the noise:

$$\theta(t) = D_f \int_0^t y(\alpha)\, d\alpha \quad . \tag{4-3}$$

Let it be assumed that the Fourier transform of $y(t)$ is given by:

$$\mathcal{F}\{y(t)\} = Y(j\omega) \quad , \tag{4-4}$$

and

$$|Y(j\omega)|^2 = W_y(\omega) \quad . \tag{4-5}$$

The Fourier transform of the integral of $y(t)$ is:

$$\mathcal{F}\left\{ \int_0^t y(\alpha)\, d\alpha \right\} = \frac{Y(j\omega)}{j\omega} \quad ,$$

and

$$\left| \frac{Y(j\omega)}{j\omega} \right|^2 = W(\omega)/\omega^2 \quad . \tag{4-6}$$

Consequently, the power spectrum of the process producing equivalent frequency modulation may be obtained by dividing the power spectrum of the process producing equivalent phase modulation by ω^2.

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