NASA CONTRACTOR REPORT

NASA CR-110

NASA CR-110

N64-31206
ACCESSION NUMBER
49
PAGES
(THRU)
1
(CODE)
NASA CR OR TMX OR AD NUMBER
(CATEGORY)
FACILITY FORM 602

THE THREE BODY PROBLEM

by C. G. Lemaitre

Prepared under Grant Nos. NsG-243-62 and NsG-122-61 *by*
UNIVERSITY OF CALIFORNIA
Berkeley, Calif.
for

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • OCTOBER 1964

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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THE THREE BODY PROBLEM

Introduction

ABSTRACT

N64-31206

The motion of three material points under their mutual attraction from Newton's law is called the Three Body Problem.

I propose to obtain the equations of this problem in the Hamiltonian form. I introduce relative axes which are the principal axis of inertia of the system of the three masses. They are related to fixed axis by Euler's angles ϕ, ψ, θ in the customary manner.

The cartesian coordinates of the masses relative to the principal axes are defined by way of three coordinates r, B, L, which are related in a simple way to the three distances r_1, r_2, r_3 of the masses.

These formulae involve three angles μ_1 μ_2 μ_3 depending on the masses. They are the angles of the triangle of the masses in the particular configuration for which the ellipse of inertia is a circle.

The elimination of the nodes is made in the usual way by elimination of the Eulerian angle θ, and the Hamiltonian (12.12) is obtained. The relation of our coordinates with a variable introduced by Lagrange in 1772 (Oeuvres, Vol VI) is shown, leading to simplification in the work of Lagrange.

Author

The result of Lagrange, in the same paper , on the equilateral configuration is deduced from our Hamiltonian.

Emphasis is put on the fact that the variables r and its conjugate momentum p_r occur in our Hamiltonian in a form essentially the same as in the Keplerian motion. It is, in fact, this circumstance which brings Lagrange's result.

But this leads to extensions to the general case. It leads to a simplified demonstration of the theorem of Sundman-Weierstrass that triple collisions can only occur when the total angular momentum vanishes.

In fact, our r is essentially the variable used by Sundman.

The coordinates B, L define the shape of the triangle irrespective of its size given by r. They may be thought of as polar coordinates, latitude B and longitude L, of a sphere of unit radius.

The special configuration with undetermined L is the pole.

It is shown that these variables may be transformed into new variables β, λ which can also be understood as points on another unit sphere.

This defines a conformal transformation of one sphere on the other.

This can be done in such a way that the points which represent the binary encounters and lie on the equator in the B L representation are transformed into the three vertices of a triangle with three right angles in the β λ representation.

The transformation is singular for these points and the singularity so introduced just removes the singularity in the original formulation of the problem.

This provides for a symmetrical presentation of the geometrical aspect of the regularization which has been introduced by Levi-Civita.

This present work is a greatly improved version of a work which has been summarized already in:

> Vistas in Astronomy, 1955: Regularization of the Three-Body Problem. (Vol. I, p. 207)

and had been published in 1952 "Coordonnées Symétriques dans le Probleme des Trois Corps," Bulletin de l'Académie Royale de Belgique (Classe des Sciences) vol. 28, 582-91 and 1218-34. Some additions had already been given in:

> Régularisation dans la problème des trois corps. Bulletin de l'Académie royale de Belgique (Classe des Sciences) vol 40, 759, 767 (1954).

In this former work the author had been able to use the principal axis only in the case where the three masses were equal.

Many useless complications result from this shortcoming.

The theory can be illustrated by using some actual computation. We have used a rather extensive computation by Carl Burrau Astr. Nachr n° 4662 - 1913 for masses $m_1 = 5$, $m_2 = 4$, $m_3 = 3$. The $p_\phi = 0$ because the masses start without speed.

A B L diagram shows a near encounter around the origin while the $\beta\lambda$ curve gives a much more simple curve. The $\beta\lambda$ have been defined in a somewhat different way as in the text and are computed by

$$\sin\beta = \pm q_1/q$$

$$\tan\lambda = \pm q_2/q_3.$$

Figures 1 and 2 show respectively the B L diagram and the $\beta\lambda$ one.

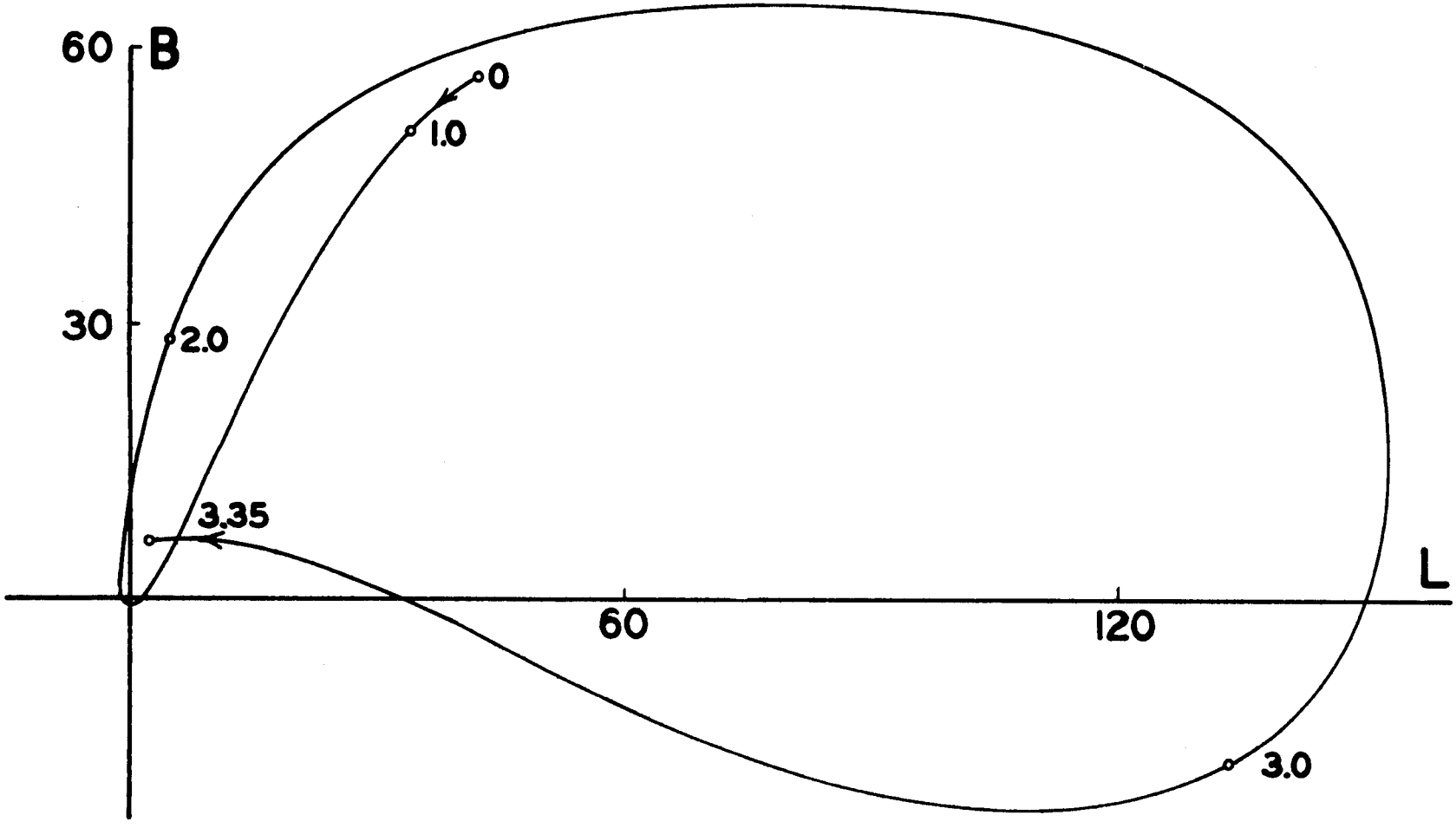

FIGURE 1

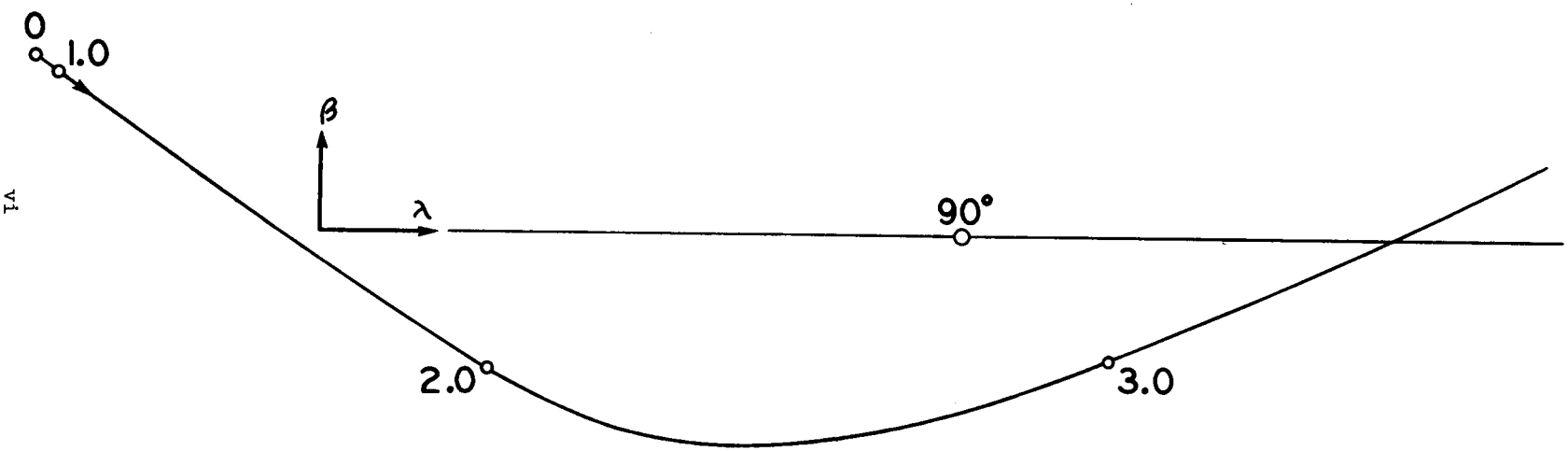

FIGURE 2

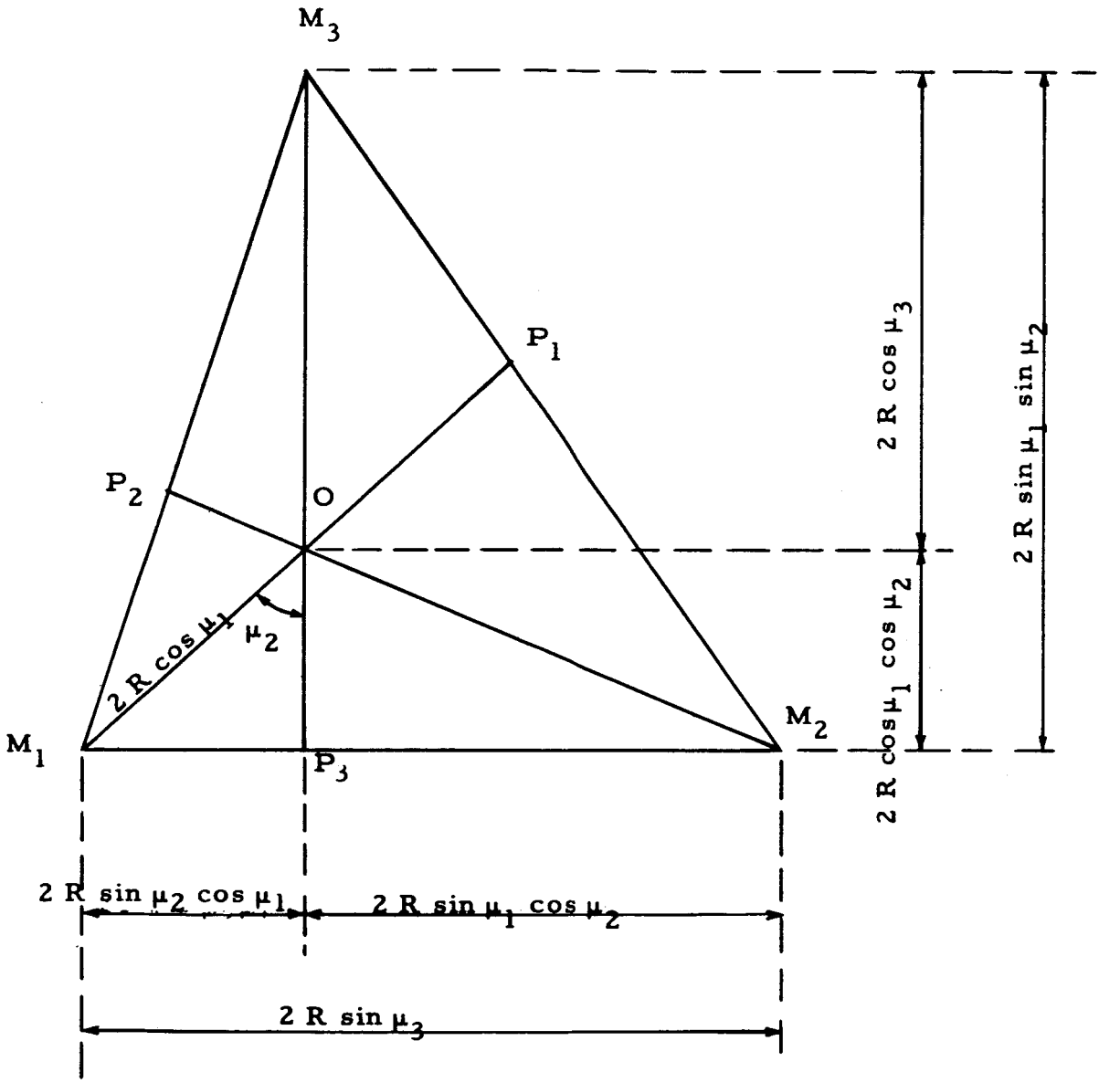
M3
P1
P2
O
2 R cos μ3
2 R sin μ1 sin μ2
2 R cos μ1
μ2
2 R cos μ1 cos μ2
M1
P3
M2
2 R sin μ2 cos μ1
2 R sin μ1 cos μ2
2 R sin μ3

FIGURE 3

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THE THREE BODY PROBLEM

1. Geometrical preliminaries

In a triangle $M_1M_2M_3$ (figure 3) the three angles μ_1, μ_2, μ_3 are related to the sizes by formulae such as

$$M_1M_2 = 2R \sin \mu_3 \tag{1.1}$$

where R is the radius of the circle through $M_1M_2M_3$.

If we draw M_3P_3 perpendicular to M_1M_2 we have

$$M_1P_3 = 2R \sin \mu_2 \cos \mu_1 \tag{1.2}$$

and

$$M_1O = 2R \cos \mu_1 \tag{1.3}$$

where M_1P_1 and M_3P_3 cut in O at angle μ_2.

Similarly

$$P_3O = 2R \cos \mu_1 \cos \mu_2 \tag{1.4}$$

and

$$P_3M_3 = 2R \sin \mu_1 \sin \mu_2. \tag{1.5}$$

2. A special configuration of three masses

Suppose that three masses $m_1m_2m_3$ are located at $M_1M_2M_3$ and that the orthocenter O of the preceding section is the center of masses.

We must have from (1.4) and (1.3)

$$(m_1 + m_2) \cos \mu_1 \cos \mu_2 = m_3 \cos \mu_3 \tag{2.1}$$

or from (1.5)

$$(m_1 + m_2 + m_3)\cos \mu_1 \cos \mu_2 = m_3 \sin \mu_1 \sin \mu_2 \tag{2.2}$$

which can be written in a more symmetric way

$$\frac{\tan \mu_1}{m_1} = \frac{\tan \mu_2}{m_2} = \frac{\tan \mu_3}{m_3} \tag{2.3}$$

This shows that the masses are proportional to the area of the three triangles OM_2M_3, OM_3M_1, OM_1M_2.

These ratios are equal to

$$\frac{\tan\mu_1 + \tan\mu_2 + \tan\mu_3}{m_1 + m_2 + m_3} = \sqrt[3]{\frac{\tan\mu_1 \tan\mu_2 \tan\mu_3}{m_1 m_2 m_3}} \qquad 2.4$$

and we have

$$\tan\mu_1 + \tan\mu_2 + \tan\mu_3 = \tan\mu_1 \tan\mu_2 \tan\mu_3 \qquad 2.5$$

because

$$\mu_1 + \mu_2 + \mu_3 = \pi. \qquad 2.6$$

It follows that the angles μ_i are

$$\tan\mu_i = m_i \sqrt{\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3}}. \qquad 2.7$$

Let us consider the moment of inertia

$$(m_1 + m_2)\,\overline{P_3O}^2 + m_3\,\overline{OM_3}^2$$

relative to a line through O, parallel to M_1M_2.

Its value, by (1.4) and (1.3) is

$$4R^2\left[(m_1 + m_2)\cos^2\mu_1 \cos^2\mu_2 + m_3 \cos^2\mu_3\right] \qquad 2.8$$

From 1 and 2

$$(m_1 + m_2 + m_3)\cos\mu_3 = (m_1 + m_2)\sin\mu_1 \sin\mu_2. \qquad 2.9$$

We have therefore

$$4R^2 \frac{m_1 + m_2 + m_3}{\sin\mu_1 \sin\mu_2} \cos\mu_1 \cos\mu_2 \cos\mu_3 \left[\cos\mu_1 \cos\mu_2 + \cos\mu_3\right] \qquad 2.10$$

but $\cos\mu_3 = \cos(\pi - \mu_1 - \mu_2) = \sin\mu_1 \sin\mu_2 - \cos\mu_1 \cos\mu_2$.

We therefore obtain

$$4R^2(m_1 + m_2 + m_3)\cos\mu_1 \cos\mu_2 \cos\mu_3 . \qquad 2.11$$

which is a symmetrical expression. It follows that, for our special configuration, the ellipse of inertia is a circle.

3. Introduction of rectangular axes.

Using rectangular axes, center at O with OM_3 as y axis, we have

$$x_1 = -2R \cos \mu_1 \sin \mu_2 \qquad y_1 = -2R \cos \mu_1 \cos \mu_2$$

$$x_2 = 2R \cos \mu_2 \sin \mu_1 \qquad y_2 = -2R \cos \mu_1 \cos \mu_2 \qquad 3.1$$

$$x_3 = 0 \qquad y_3 = 2R \cos \mu_3$$

These equations can be written

$$x_i = 2R \cos \mu_i \cos \frac{\pi - L_i}{2}$$

$$y_i = 2R \cos \mu_i \sin \frac{\pi - L_i}{2} \qquad 3.2$$

with

$$\frac{L_1}{2} = \mu_2 - \pi \qquad \frac{\pi}{2} - \frac{L_1}{2} = \frac{3\pi}{2} - \mu_2$$

$$\frac{L_2}{2} = \pi - \mu_1 \qquad \frac{\pi}{2} - \frac{L_2}{2} = \frac{-\pi}{2} + \mu_1 \qquad 3.3$$

$$\frac{L_3}{2} = 0.$$

If the axes are rotated by an arbitrary angle $\frac{L}{2} - \frac{\pi}{2}$ we should have $(i = 1, 2, 3)$

$$x_i = 2R \cos \mu_i \cos \frac{L - L_i}{2}$$

$$y_i = 2R \cos \mu_1 \sin \frac{L - L_i}{2} \quad . \qquad 3.4$$

A more symmetrical notation is obtained [by a rotation] with

$$\begin{cases} \frac{L_1}{2} = \frac{\mu_2}{3} - \frac{\mu_3}{3} - \frac{2\pi}{3} \\ \frac{L_2}{2} = \frac{\mu_3}{3} - \frac{\mu_1}{3} - \frac{4\pi}{3} \\ \frac{L_3}{2} = \frac{\mu_1}{3} - \frac{\mu_2}{3} \end{cases} \qquad 3.5$$

i.e.

$$\frac{L_i}{2} = \frac{\mu_{i+1}}{3} - \frac{\mu_{i-1}}{3} - \frac{2\pi i}{3} \quad . \qquad (\mu_{1+3} = \mu_1) \qquad 3.6$$

We have then

$$-\frac{L_{i+1}+L_{i-1}}{2} = \frac{\mu_{i+1}}{3} - \frac{\mu_{i-1}}{3} + \frac{2\pi i}{3} = \frac{L_i}{2} + \frac{2\pi i}{3} \qquad 3.7$$

$$\frac{L_{i+1}-L_{i-1}}{2} = \pi - \mu_i . \qquad 3.8$$

And also

$$\frac{L_i}{2} - \mu_i = \frac{2}{3}(\mu_{i+1} - \mu_i) - \frac{\pi}{3}(2i+1) \qquad 3.9$$

$$\frac{L_i}{2} + \mu_i = \frac{2}{3}(\mu_i - \mu_{i-1}) - \frac{\pi}{3}(2i-1). \qquad 3.10$$

4. The fundamental identities

The properties of the special configuration could have been obtained from the following identities which are valid for any integer n.

$$\sum_i \sin n\mu_i \cos\frac{n}{2}(L - L_i) = 0 \qquad 4.1$$

$$\sum_i \sin n\mu_i \sin\frac{n}{2}(L - L_i) = 0. \qquad 4.2$$

Consider, for instance, 1; we have to show that

$$\frac{1}{2}\sum_i \left[\sin n\left(\frac{L}{2} - \frac{L_i}{2} + \mu_i\right) - \sin n\left(\frac{L}{2} - \frac{L_i}{2} - \mu_i\right)\right] = 0 \qquad 4.3$$

This follows from (3,9) and (3,10). For n = 1 we have

$$\sum m_i x_i = 0 \quad \sum m_i y_i = 0 \qquad 4.4$$

and for n = 2

$$\sum m_i x_i y_i = 0. \qquad 4.5$$

We have also the identity

$$\sum \sin 2n\mu_i = 4(-1)^{n+1} \sin n\mu_i \sin n\mu_2 \sin n\mu_3. \qquad 4.6$$

We have, in fact, using $\mu_3 = \pi - \mu_1 - \mu_2$

$$\sin 2n\mu_1 (1 - \cos 2n\mu_2) + \sin 2n\mu_2 (1 - \cos 2n\mu_1)$$

$$= 4 \sin n\mu_1 \cos n\mu_1 \sin^2\mu_2 + 4 \sin n\mu_2 \cos n\mu_1 \sin^2\mu_1$$

$$= 4 \sin n\mu_1 \sin n\mu_2 \sin n(\mu_1 + \mu_2).$$

We obtained the announced identity by noting that

$$\sin n(\mu_1 + \mu_2) = \sin(n\pi - n\mu_3) = (-1)^n \sin n\mu_3.$$

5. The principal axes in the general case

Principal axes through the center of mass O are such that

$$\sum m_i x_i y_i = 0. \qquad 5.1$$

Let the principal momenta be

$$A = \sum m_i y_i^2 = C \sin^2 \frac{B}{2} \qquad 5.2$$

$$B = \sum m_i x_i^2 = C \cos^2 \frac{B}{2} \qquad 5.3$$

$$C = \sum m_i (x_i^2 + y_i^2). \qquad 5.4$$

This differs from the special case by the introduction of factors $\sin\frac{B}{2}$ in y and $\cos\frac{B}{2}$ in x. We obtain the general case by modifying (3.4) into

$$x_i = 2R \cos\mu_i \cos\frac{B}{2} \cos\frac{L - L_i}{2} \qquad 5.5$$

$$y_i = 2R \cos\mu_i \sin\frac{B}{2} \sin\frac{L - L_i}{2} \qquad 5.6$$

and we have as before from (2.11)

$$C = 4R^2(m_1 + m_2 + m_3) \cos\mu_1 \cos\mu_2 \cos\mu_3. \qquad 5.7$$

It will be convenient to replace R by r by

$$2R = r\sqrt{2} \qquad 5.8$$

and to write

$$C = Mr^2 \qquad 5.9$$

with

$$M = 2(m_1 + m_2 + m_3) \cos \mu_1 \cos \mu_2 \cos \mu_3 \qquad 5.10$$

or alternatively from (2.7)

$$M = 2 \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}} \sin \mu_1 \sin \mu_2 \sin \mu_3. \qquad 5.11$$

From (2.7) and (5.11) we may obtain

$$\frac{1}{m_1} + \frac{1}{m_2} = \frac{2}{M} \sin^2 \mu_3 \qquad 5.12$$

and therefore

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = \frac{1}{M} (\sin^2 \mu_1 + \sin^2 \mu_2 + \sin^2 \mu_3) \qquad 5.13$$

or

$$\sin^4 \mu_1 + \sin^4 \mu_2 + \sin^4 \mu_3 = \frac{M^2}{2} \left(\sum \frac{1}{m_1^2} + \sum \frac{1}{m_1 m_2}\right) \qquad 5.14$$

$$\sin^2 \mu_2 \sin^2 \mu_3 + \sin^2 \mu_3 \sin^2 \mu_1 + \sin^2 \mu_1 \sin^2 \mu_2 = \frac{M^2}{4} \left(\sum \frac{1}{m_1^2} + 3 \sum \frac{1}{m_1 m_2}\right) \qquad 5.15$$

$$\sin^2 \mu_1 \sin^2 \mu_2 \sin^2 \mu_3 = \frac{M^3}{8} \left[\frac{2}{m_1 m_2 m_3} + \sum \frac{1}{m_1^2} \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\right]$$

$$= \frac{M^3}{8} \frac{1}{m_1 m_2 m_3} \left[2 + \frac{m_2 + m_3}{m_1} + \frac{m_3 + m_1}{m_2} + \frac{m_1 + m_2}{m_3}\right]$$

$$= \frac{M^3}{8} \frac{1}{m_1 m_2 m_3} \left[-1 + (m_1 + m_2 + m_3) \left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right)\right].$$

Comparing with (5.11) we get

$$\frac{2}{M} = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} - \frac{1}{m_1 + m_2 + m_3}. \qquad 5.16$$

We have also

$$\sin^2 \mu_1 \sin^2 \mu_2 \sin^2 \mu_3 = \frac{M^2}{4} \sum \frac{1}{m_1 m_2} \qquad 5.17$$

and

$$\frac{1}{\tan \mu_1} + \frac{1}{\tan \mu_2} + \frac{1}{\tan \mu_3} = \frac{1 + \cos \mu_1 \cos \mu_2 \cos \mu_3}{\sin \mu_1 \sin \mu_2 \sin \mu_3} \qquad 5.18$$

and from (2.2) and (10)

$$\frac{1}{m_3} = \frac{2}{M} \sin \mu_1 \sin \mu_2 \cos \mu_3. \qquad 5.19$$

6. Relative coordinates

We have by (3.4) and (3.8)

$$x_3 = 2R \cos \mu_3 \cos \frac{L-L_3}{2} = 2R \cos \mu_3 \cos \left(\frac{L-L_1}{2} + \mu_2 - \pi\right)$$

$$= -2R \cos \frac{B}{2} \cos \mu_3 \cos\left(\frac{L-L_1}{2} + \mu_2\right)$$

$$= -R \cos \frac{B}{2} \left[\cos \left(\frac{L-L_1}{2} + \mu_2 + \mu_3\right) + \cos \left(\frac{L-L_1}{2} + \mu_2 - \mu_3\right)\right].$$

Similarly

$$x_2 = -2R \cos \frac{B}{2} \cos \mu_2 \cos \left(\frac{L-L_1}{2} - \mu_3\right)$$

$$= -R \cos \frac{B}{2} \left[\cos \left(\frac{L-L_1}{2} - \mu_3 + \mu_2\right) + \cos\left(\frac{L-L_1}{2} - \mu_2 - \mu_3\right)\right].$$

Therefore

$$x_3 - x_2 = -R \cos \frac{B}{2} \left[\cos \left(\frac{L-L_1}{2} + \mu_2 + \mu_3\right) - \cos\left(\frac{L-L_1}{2} - \mu_2 - \mu_3\right)\right]$$

$$= R \cos \frac{B}{2} \left[\cos\left(\frac{L-L_1}{2} - \mu_1\right) - \cos \left(\frac{L-L_1}{2} + \mu_1\right)\right]$$

$$= 2R \sin \mu_1 \cos \frac{B}{2} \sin \frac{L-L_1}{2}.$$

The computation for y can be done by replacing $\frac{L}{2}$ by $\frac{L}{2} + \frac{\pi}{2}$.

We have therefore

$$x_{23} = x_3 - x_2 = 2R \sin\mu_1 \cos\frac{B}{2} \sin\frac{L - L_1}{2} = r_1 \cos\zeta_1$$

$$y_{23} = y_3 - y_2 = -2R \sin\mu_1 \sin\frac{B}{2} \cos\frac{L - L_1}{2} = r_1 \sin\zeta_1 \qquad 6.1$$

We have for the slope ζ_i

$$\tan\zeta_i = -\frac{\tan\frac{B}{2}}{\tan\frac{L - L_i}{2}} \qquad 6.2$$

and for the length of the side r_i of the triangle (using (5.8)),

$$r_i^2 = 2r^2 \sin^2\mu_i \left[\cos^2\frac{B}{2} \sin^2\frac{L - L_i}{2} + \sin^2\frac{B}{2} \cos^2\frac{L - L_i}{2}\right] \qquad 6.3$$

and therefore

$$r_i = r \sin\mu_i \sqrt{1 - \cos B \cos(L - L_i)} \quad . \qquad 6.4$$

By differentiation of (2) we have

$$\frac{d\zeta_i}{\sin\zeta_i \cos\zeta_i} = \frac{dB}{\sin B} - \frac{dL}{\sin(L - L_i)} \qquad 6.5$$

or by (1)

$$d\zeta_i = -\frac{r^2 \sin^2\mu_i \sin B \sin(L - L_i)}{2r_i^2} \left(\frac{dB}{\sin B} - \frac{dL}{\sin(L - L_i)}\right)$$

or by (4)

$$2d\zeta_i = \frac{\sin B dL - \sin(L - L_i) dB}{1 - \cos B \cos(L - L_i)} \quad . \qquad 6.6$$

The angles of the triangles may be obtained from the slopes of the sides. One finds, for instance, writing

$$\chi_i = \frac{L - L_i}{2}$$

$$\tan(\zeta_3 - \zeta_2) = -\sin B \cos B \frac{\cos \chi_3 \sin \chi_2 - \cos \chi_2 \sin \chi_3}{\sin^2 B \sin \chi_2 \sin \chi_3 + \cos^2 B \cos \chi_2 \cos \chi_3}$$

$$= 2 \sin B \cos B \frac{\sin(\chi_3 - \chi_2)}{\cos(\chi_3 - \chi_2) + \cos 2B \cos(\chi_3 + \chi_2)}$$

or using the symmetric definition of the L_i (3.5)

$$\tan(\zeta_3 - \zeta_2) = \sin 2B \frac{\sin(\pi - \mu_1)}{\cos(\pi - \mu_1) + \cos 2B \cos\left(L - \frac{L_2 + L_3}{2}\right)} \qquad 6.7$$

which gives for $B = \pi/4$

$$\zeta_3 - \zeta_2 = \pi - \mu_1 \quad .$$

Let us write

$$\begin{aligned} r_1 &= q_2^2 + q_3^2 \\ r_2 &= q_3^2 + q_1^2 \\ r_3 &= q_1^2 + q_2^2 \end{aligned} \qquad 6.8$$

and therefore

$$q^2 = q_1^2 + q_2^2 + q_3^2 = \frac{1}{2}(r_1 + r_2 + r_3) \qquad 6.9$$

and

$$\begin{aligned} q_1^2 &= \frac{1}{2}(-r_1 + r_2 + r_3) \\ q_2^2 &= \frac{1}{2}(r_1 - r_2 + r_3) \\ q_3^2 &= \frac{1}{2}(r_1 + r_2 - r_3) \quad . \end{aligned} \qquad 6.10$$

We know that

$$16 q^2 q_1^2 q_2^2 q_3^2 = 2r_1^2 r_2^2 + 2r_2^2 r_3^2 + 2r_3^2 r_1^2 - r_1^4 - r_2^4 - r_3^4 \qquad 6.11$$

is sixteen times the square of the area of the triangle.

Now the triangle has been obtained from a special configuration of Section 2 by a process which involves a multiplication of the x and y by $\cos\frac{B}{2}$ and $\sin\frac{B}{2}$. This involves multiplying the area of the triangle by 1/2 sin B.

The area of the triangle for the special configuration and therefore also for the actual triangle does not depend on L. It follows that

$$\frac{4q\, q_1 q_2 q_3}{r^2 \sin B} \tag{6.12}$$

does not depend on the coordinates r, B, L, but only on the masses.

In order to find its value we can choose special values $r = 1$, $B = \pi/2$, and for these values we have

$$r_i = \sin \mu_i \quad . \tag{6.13}$$

We find

$$\frac{16 q^2 q_1^2 q_2^2 q_3^2}{r^4 \sin^2 B} = 2 \sum \sin^2\mu_2 \sin^2\mu_3 - \sum \sin^4 \mu_1$$

and by (5.15) and (5.14)

$$= \frac{M^2}{2}\left[\sum \frac{1}{m_1^2} + 3\sum \frac{1}{m_1 m_2} - \sum \frac{1}{m_1 m_2}\right] = \frac{M}{2}\sum \frac{1}{m_1 m_2}$$

$$= 4 \sin^2\mu_1 \sin^2\mu_2 \sin^2\mu_3$$

by (5.17).

We have therefore

$$r^2 \sin B = \frac{2q\, q_1 q_2 q_3}{\sin\mu_1 \sin\mu_2 \sin\mu_3} \quad . \tag{6.14}$$

Alternatively, we have from the fundamental identity (4.7) applied to (4)

$$2 \sin \mu_1 \sin \mu_2 \sin \mu_3 \, r^2 = \sum_i \frac{r_i^2}{\tan \mu_i} \qquad 6.15$$

and therefore

$$\sin B = \frac{4 q q_1 q_2 q_3}{r_1^2/\tan\mu_1 + r_2^2/\tan\mu_2 + r_3^2/\tan\mu_3} \quad . \qquad 6.16$$

When we know the three distances $r_1 r_2 r_3$ it is possible, from the formulae we have just given, to find r and B.

We shall see in the next section that L can be determined also.

Then we are able to find the rectangular coordinates of the three points from (5.5) and (5.6).

We have also

$$\frac{r_1^2}{m_1} + \frac{r_2^2}{m_2} + \frac{r_3^2}{m_3} = \sqrt{\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3}} \left(\frac{r_1^2}{\tan\mu_1} + \frac{r_2^2}{\tan\mu_2} + \frac{r_3^2}{\tan\mu_3} \right)$$

$$= \sqrt{\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3}} \, 2 \sin\mu_1 \sin\mu_2 \sin\mu_3 \, r^2 \qquad 6.17$$

$$= \frac{4 \sin^2\mu_1 \sin^2\mu_2 \sin^2\mu_3}{M} r^2 = M \sum \frac{1}{m_1 m_2} r^2 \quad .$$

7. Kinetic Energy

Let v_1 be the relative velocity of M_3 as regards M_2. Then the kinetic energy relative to the principle axis is

$$2T = \frac{m_2 m_3 v_1^2 + m_3 m_1 v_2^2 + m_1 m_2 v_3^2}{m_1 + m_2 + m_3} \quad . \qquad 7.1$$

In fact, we have

$$0 = (m_1\dot{x}_1 + m_2\dot{x}_2 + m_3\dot{x}_3)^2$$

$$= (m_1 + m_2 + m_3)(m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2) - m_2m_3(\dot{x}_3 - \dot{x}_2)^2$$

$$- m_3m_1(\dot{x}_1 - \dot{x}_3)^2 - m_1m_2(\dot{x}_2 - \dot{x}_1)^2 \qquad 7.2$$

from which the expression of T follows easily.

v_i has two components, one radial

$$\frac{dr_i}{dt}, \qquad 7.3$$

the other transversal,

$$r_i \frac{d\zeta_i}{dt} . \qquad 7.4$$

If we want to consider, not velocities relative to the principle axes, but absolute velocities, we must add the velocities due to the angular rotation

$$\omega_x \; \omega_y \; \omega_z$$

of the principle axis.

This introduces components normal to the plane of the triangle and also an additional component

$$r_i \omega_z$$

in the transverse direction. Writing

$$T = T' + T'' + \omega_z K' \qquad 7.5$$

where

$$2T'' = A\omega_x^2 + B\omega_y^2 + C\omega_z^2 \qquad 7.6$$

we have to compute the kinetic energy T' relative to the principle axis and the relative angular momentum

$$K' = \sum_i \frac{\partial T'}{\partial \dot{\zeta}_i} = \frac{m_1m_2m_3}{m_1 + m_2 + m_3} \sum \frac{\dot{\zeta}_i r_i^2}{m_i} . \qquad 7.7$$

8. The Relative Angular Momentum K'

From (6.6) we have

$$\sum \frac{\dot{\zeta}_i r_i^2}{m_i} = \sum (\sin B \dot{L} - \sin(L - L_i) \dot{B})\, r^2 \frac{\sin^2 \mu_i}{2m_i} . \qquad 8.1$$

But, from (2.7)

$$\frac{\sin^2 \mu_i}{m_i} = \sqrt{\frac{m_1 + m_2 + m_3}{m_1 m_2 m_3}} \frac{\sin 2\mu_i}{2} \qquad 8.2$$

and, therefore, the fundamental identity (4.2) with n = 1 gives

$$\sum \sin 2\mu_i \sin (L - L_i) = 0 \qquad 8.3$$

while (4.7)

$$\sum \sin 2\mu_i = 4 \sin\mu_1 \sin \mu_2 \sin \mu_3 . \qquad 8.4$$

We have, therefore,

$$K' = \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}}\, r^2 \sin B \;\; \dot{L} \sin \mu_2 \sin \mu_3 \qquad 8.5$$

or using (5.11)

$$K' = \frac{Mr^2}{2} \sin B \;\; \dot{L} . \qquad 8.6$$

9. The Relative Kinetic Energy T'

We have to compute

$$2T' = \frac{m_1 m_2 m_3}{m_1 + m_2 + m_3} \sum \frac{1}{M_i} (\dot{r}_i^2 + r_i^2 \dot{\zeta}_i^2) . \qquad 9.1$$

From (6.4) we have

$$\frac{\dot{r}_i}{r_i} = \frac{\dot{r}}{r} + \frac{1}{2} \frac{\sin B \cos(L - L_i) \dot{B} + \cos B \sin (L - L_i) \dot{L}}{1 - \cos B \cos (L - L_i)} . \qquad 9.2$$

We obtain

$$2T' = \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}} \sum \frac{\sin 2\mu_i}{2} \left\{ \dot{r}^2 (1 - \cos B \cos (L - L_i) \right.$$

$$+ \dot{r} \left[\sin B \cos (L - L_i) \dot{B} + \cos B \sin (L - L_i) \dot{L} \right]$$

$$\left. + \frac{r^2}{4} \frac{\left[\sin B \cos (L - L_i) \dot{B} + \cos B \sin (L - L_i) \dot{L} \right]^2 + \left[\sin B \dot{L} - \sin (L - L_i) \dot{B} \right]^2}{1 - \cos B \cos (L - L_i)} \right\} . \qquad 9.3$$

The last fraction can be simplified and becomes

$$\frac{r^2}{4} \left[(1 + \cos B \cos (L - L_i)) \dot{L}^2 - 2 \sin B \sin (L - L_i) \dot{L} \dot{B} + \dot{B}^2 \right]. \qquad 9.4$$

Using the fundamental identity we obtain

$$T' = \frac{M}{2} (\dot{r}^2 + \frac{r^2}{4} (\dot{B}^2 + \dot{L}^2)) . \qquad 9.5$$

10. The Eulerian Angles

When the position of the principle axes is referred to fixed axes through the Eulerian angles θ, ϕ, ψ, the angular velocity is

$$\begin{aligned} \omega_x &= -\dot{\phi} \sin \theta \cos \psi + \dot{\theta} \sin \psi \\ \omega_y &= -\dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \omega_z &= \dot{\phi} \cos \theta + \dot{\psi} \end{aligned} \qquad 10.1$$

The angular momentum K is fixed in place and it is convenient to adopt its direction for the z component of the fixed axes.

Then the angular momentum has relative components

$$\begin{aligned} K_x &= -K \sin \theta \cos \psi \\ K_y &= K \sin \theta \sin \psi \\ K_z &= K \cos \theta \quad . \end{aligned} \qquad 10.2$$

This is the sum of the relative angular momentum K' along oz and of the component

$$A\omega_x \ , \ B\omega_y \ , \ C\omega_z \qquad 10.3$$

due to the motion of the axes.

These quantities have such simple expressions because the axes are principle axes of inertia. Comparing with (7.5) and (7. 6) we see that

$$K_x = \frac{\partial T}{\partial \omega_x}$$

$$K_y = \frac{\partial T}{\partial \omega_y}$$

$$K_z = \frac{\partial T}{\partial \omega_z} \qquad 10.4$$

11. The Elimination of the Nodes

Introducing canonical moments conjugated to the Eulerian axes, and using (10. 4) and (10. 1) we have

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = -K_x \sin\theta \cos\psi + K_y \sin\theta \sin\psi + K_z \cos\theta$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = -K_x \sin\psi + K_y \cos\psi$$

$$p_\psi = \frac{\partial T}{\partial \dot{\psi}} = K_z \quad . \qquad 11.1$$

Comparing with (10. 2) we obtain

$$p_\phi = K \qquad 11.2$$

$$p_\theta = 0 \qquad 11.3$$

$$p_\psi = K \cos\theta \qquad 11.4$$

Let us replace in H, θ from Eq. (4). Let us call for a moment $\overline{H}$ the result of this elimination. We have

$$\frac{\partial \overline{H}}{\partial p_\psi} = \frac{\partial H}{\partial p_\psi} - \frac{1}{p_\phi \sin\theta} \frac{\partial H}{\partial \theta} \qquad 11.5$$

$$\frac{d\overline{H}}{dp_\phi} = \frac{\partial H}{\partial p_\phi} + \frac{\cos\theta}{p_\phi \sin\theta} \frac{\partial H}{\partial \theta} \quad . \qquad 11.6$$

But we wish to use this relation for the particular choice we have made of the fixed z axis. In that case the canonical equation

$$\frac{dp_\theta}{dt} = -\frac{\partial H}{\partial \theta} \qquad 11.7$$

together with (3) shows that

$$\frac{\partial H}{\partial \theta} = 0. \qquad 11.8$$

Then the derivatives of H and $\overline{H}$ are the same and there is no reason to maintain a difference of notation any more.

12. The Hamiltonian

Besides eliminating θ from (11.4) we have also to replace in T the derivatives by the momenta.

We have, using the expression (8.6) and (9.1) of K' and T'

$$p_r = \frac{\partial T}{\partial \dot{r}} = \frac{\partial T'}{\partial \dot{r}} = M\dot{r} \qquad 12.1$$

$$p_B = \frac{\partial T}{\partial \dot{B}} = \frac{\partial T'}{\partial \dot{B}} = \frac{M}{4} r^2 \dot{B} \qquad 12.2$$

$$p_L = \frac{\partial T}{\partial \dot{L}} = \frac{\partial T'}{\partial \dot{L}} + \omega_z \frac{\partial K'}{\partial \dot{L}} = \frac{M}{4} r^2 \dot{L} + \frac{M}{2} r^2 \sin B\, \omega_z . \qquad 12.3$$

In order to compute

$$T' + \frac{C}{2}\omega_z^2 + \omega_z K' = T' + \frac{C}{2}\left(\omega_z + \frac{K'}{C}\right)^2 - \frac{K'^2}{2C}$$

$$= \frac{M}{2}\left[\dot{r}^2 + \frac{r^2}{4}(\dot{B}^2 + \cos^2 B\, \dot{L}^2)\right] + \frac{C}{2}\left(\omega_z + \frac{K'}{C}\right)^2 \qquad 12.4$$

we have to use (11.4)

$$K_z = K' + C\omega_z = p_\psi \qquad 12.5$$

which gives

$$\omega_z + \frac{K'}{C} = \frac{p_\psi}{C} . \qquad 12.6$$

It remains to compute $\dot{L}$. We must eliminate ω_z from (3) and (5). This gives

$$p_L - \frac{M}{2C} r^2 \sin B \, p_\psi = \frac{M}{4} \left(1 - \frac{Mr^2}{C} \sin^2 B\right) \dot{L} \qquad 12.7$$

or using (5.9)

$$p_L - \frac{p_\psi}{2} \sin B = \frac{M}{4} \cos^2 B \, \dot{L} \; . \qquad 12.8$$

The remaining terms are

$$\frac{1}{2} A\omega_x^2 + \frac{1}{2} B\omega_y^2 = \frac{1}{2} \left(\frac{K_x^2}{A} + \frac{K_y^2}{B} \right) = \frac{p_\phi^2 \sin^2\theta}{2C} \left(\frac{\cos^2\psi}{\sin^2 \frac{B}{2}} + \frac{\sin^2\psi}{\cos^2 \frac{B}{2}} \right) \qquad 12.9$$

from (5.2), (5.3) and (10.2), or

$$\frac{p_\phi^2 - p_\psi^2}{4Mr^2} \; \frac{\cos^2\psi \cos^2 \frac{B}{2} + \sin^2\psi \sin^2 \frac{B}{2}}{\sin^2 \frac{B}{2} \cos^2 \frac{B}{2}}$$

$$= \frac{p_\phi^2 - p_\psi^2}{Mr^2 \sin^2 B} \left[(1 + \cos\psi)(1 + \cos B) + (1 - \cos\psi)(1 - \cos B) \right]$$

$$= \frac{p_\phi^2 - p_\psi^2}{Mr^2 \sin^2 B} (1 + \cos B \cos\psi) \; . \qquad 12.10$$

Collecting our results the Hamiltonian becomes

$$H = T - \mathcal{U} \qquad 12.11$$

$$H = \frac{1}{2M} \left[p_r^2 + \frac{4}{r^2} \left(p_B^2 + \frac{(p_L - \frac{p_\psi}{2} \sin B)^2}{\cos^2 B} \right) \right] + \frac{p_\psi^2}{2Mr^2} + \frac{p_\phi^2 - p_\psi^2}{Mr^2 \sin^2 B} (1 + \cos B \cos\psi) - \mathcal{U} \qquad 12.12$$

where

$$\mathcal{U} = \sum \frac{Gm_2m_3}{r_1} = Gm_1m_2m_3 \sum \frac{1}{m_1 r_1}$$

$$= \frac{Gm_1m_2m_3}{r} \sum \frac{1}{m_i \sin\mu_i \sqrt{1 - \cos B \cos(L - L_i)}} \qquad 12.13$$

13. Canonical Equations for the Eulerian Angles

When the motion relative to the principle axis, i.e., the relative coordinates r, L, B, have been found the Eulerian angles are obtained from the canonical equation

$$\frac{d\psi}{dt} = \frac{\partial H}{\partial p_\psi} = -\frac{2\sin B}{Mr^2\cos^2 B}\left(p_L - \frac{p_\psi}{2}\sin B\right) + \frac{p_\psi}{Mr^2} - \frac{p_\psi(1+\cos B\cos\psi)}{Mr^2\sin^2 B} \qquad 13.1$$

$$\frac{dp_\psi}{dt} = -\frac{\partial H}{\partial\psi} = \frac{p_\phi^2 - p_\psi^2}{Mr^2\sin^2 B}\cos B\sin\psi \qquad 13.2$$

and

$$\frac{d\phi}{dt} = \frac{\partial H}{\partial p_\phi} = \frac{p_\psi}{Mr^2\sin^2 B}(1+\cos B\cos\psi) \qquad 13.3$$

$$\frac{dp_\phi}{dt} = -\frac{\partial H}{\partial\phi} = 0 \qquad 13.4$$

together with

$$p_\psi = p_\phi\cos\theta \qquad 13.$$

which gives θ.

It will be convenient to introduce modified momenta such as

$$\bar{p}_L = p_L - \frac{p_\psi}{2}\sin B\,. \qquad 13.6$$

Then if $\overline{H}$ is the Hamiltonian H written with the modified momenta we shall have

$$\frac{d\psi}{dt} = \frac{\partial\overline{H}}{\partial p_\psi} + \frac{\partial\overline{H}}{\partial\bar{p}_L}\frac{\partial\bar{p}_L}{\partial p_\psi} = \frac{\partial\overline{H}}{\partial p_\psi} - \frac{\partial\overline{H}}{\partial\bar{p}_L}\frac{\sin B}{2} \qquad 13.7$$

as modified canonical equation following (1).

14. Use of Modified Momenta

For the sake of generality, let us write $q^1 q^2 q^3$ for $r\ L\ B$

$p_1 p_2 p_3$ for $p_r p_L p_B$

and let us define modified momenta

$$\bar{p}_i = p_i - A_i p_\psi \ , \qquad i = 1, 2, 3 \ . \tag{14.1}$$

Here we have

$$A_1 = 0 \qquad A_2 = 0 \qquad A_3 = \frac{\sin B}{2} \ . \tag{14.2}$$

Let $\bar{H}$ be the Hamiltonian H written with the modified momenta $\bar{p}_i$. Then the canonical equations are

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} = \frac{\partial \bar{H}}{\partial \bar{p}_i} \ . \tag{14.3}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} = -\frac{\partial \bar{H}}{\partial q^i} - \frac{\partial \bar{H}}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial q^i} = -\frac{\partial \bar{H}}{\partial q^i} + p_\psi \frac{\partial \bar{H}}{\partial \bar{p}_k}\frac{\partial A_k}{\partial q^i} \ . \tag{14.4}$$

From these equations we deduce the modified canonical equations. We have

$$\frac{d\bar{p}_i}{dt} = \frac{dp_i}{dt} - \frac{d(A_i p_\psi)}{dt} = \frac{dp_i}{dt} - A_i \frac{dp_\psi}{dt} - p_\psi \frac{\partial A_i}{\partial q^k}\frac{dq^k}{dt} \tag{14.5}$$

or

$$\frac{d\bar{p}_i}{dt} = -\frac{\partial \bar{H}}{\partial q^i} - A_i \frac{dp_\psi}{dt} + p_\psi \left(\frac{\partial A_k}{\partial q^i} - \frac{\partial A_i}{\partial q^i}\right)\frac{dq^k}{dt} \ . \tag{14.6}$$

We may write A_{ik} or more explicitly

$$A_{q^i q^k} = \frac{\partial A_k}{\partial q^i} - \frac{\partial A_i}{\partial q^k} \tag{14.7}$$

and we obtain the modified canonical equations

$$\frac{dq^i}{dt} = \frac{\partial \overline{H}}{\partial \overline{p}_i}$$

$$\frac{d\overline{p}_i}{dt} = -\frac{\partial \overline{H}}{\partial q^i} - A_i \frac{dp_\psi}{dt} + p_\psi A_{q^i q^k} \frac{dq^k}{dt} \qquad 14.8$$

Actually all derivatives vanish except

$$\frac{\partial A_3}{\partial q^2} = \frac{d}{dB} \frac{\sin B}{2} = \frac{\cos B}{2} \qquad 14.9$$

and all A_{ik} vanish except

$$A_{BL} = A_{q^2 q^3} = -A_{q^3 q^2} = \frac{\cos B}{2} \quad . \qquad 14.10$$

In general, (13.7) should be written

$$\frac{d\psi}{dt} = \frac{\partial \overline{H}}{\partial p_\psi} - \frac{\partial \overline{H}}{\partial \overline{p}_k} A_k \quad . \qquad 14.11$$

In the following we shall generally suppress the bars and write p_i for the modified momenta.

15. Change of Coordinates

Let us replace q^i $(i = 1, 2, 3)$ by a new variable $q^{i'}$. We can define new modified momenta $p_{i'}$ by

$$p_i dq^i = p_{i'} dq^{i'} \ , \ A_i dq^i = A_{i'} dq^{i'} \qquad 15.1$$

and the new $A_{q^{i'} q^{k'}}$ will be obtained by

$$A_{q^i q^k} dq^i \wedge dq^k = A_{q^{i'} q^{k'}} dq^{L'} \wedge dq^{k'} \qquad 15.2$$

which means

$$A_{q^i q^k}(dq^i \delta q^k - \delta q^i dq^k) = A_{q^{i'} q^{k'}} (dq^{i'} \delta q^{k'} - \delta q^{i'} dq^{k'}) \quad . \qquad 15.3$$

16. Restriction to Motion in a Plane

The equation (13.2) can be written using (13.5)

$$\frac{d\theta}{dt} = -\frac{p_\phi \sin\theta \cos B \sin\psi}{Mr^2 \sin^2 B} \qquad 16.1$$

This shows that if at some time $\theta = 0$ it will remain null at any time.

This means that the three bodies will remain in a plane.

We can therefore restrict the problem to plane motion and such a restriction is justified, i.e., when imposed it will maintain itself automatically.

In the case of the restricted plane motion it is no use to know the A_i but simply the $A_{q_i q_k}$.

Also the angles ϕ and ψ are not defined but only their sum and we have from (13.2) and (13.3) with $(p_\phi = p_\psi)$

$$\frac{d(\phi + \psi)}{dt} = \frac{p_\phi \cos^2 B - 2p_L \sin B}{Mr^2 \cos^2 B} \quad . \qquad 16.2$$

17. A Variable used by Lagrange (1772)

The velocity of M_1 as regards to M_3 has rectangular components

$$u_{31} = \frac{dx_{31}}{dt} - \omega_z y_{31}$$

$$v_{31} = \frac{dy_{31}}{dt} + \omega_z y_{31}$$

$$w_{31} = \omega_x y_{31} - \omega_y x_{31} \quad . \qquad 17.1$$

The variable introduced by Lagrange has been put in a more symmetrical way by the editor of his work, Serret, Oeuvres vol VI, p. 329. It is shown to be the same for any permutation of the indices and may be written

$$\rho = x_{23}\, u_{31} - x_{31}\, u_{23} + y_{23}\, v_{31} - y_{31}\, v_{23}$$

We shall find its expression is our notation. We have from (6.1) for the term independent of ω_z

$$2r^2 \sin\mu_1 \sin\mu_2 (\cos^2\frac{B}{2} + \sin^2\frac{B}{2}) \begin{vmatrix} \sin\frac{L-L_1}{2} & \cos\frac{L-L_2}{2} \\ -\sin\frac{L-L_2}{2} & \cos\frac{L-L_1}{2} \end{vmatrix} \frac{dL}{2dt}$$

$$= - r^2 \sin\mu_1 \sin\mu_2 \sin\mu_3 \frac{dL}{dt} \qquad 17.3$$

because (3.8)

$$\sin\frac{L_2-L_1}{2} = \sin(\mu_3 - \pi) = - \sin\mu_3 . \qquad 17.4$$

The term in ω_z is minus twice

$$-2r^2 \sin\mu_1 \sin\mu_2 \sin\frac{B}{2} \cos\frac{B}{2} \sin\frac{L_2-L_1}{2} .$$

We have therefore

$$\rho = -r^2 \sin\mu_1 \sin\mu_2 \sin\mu_3 \left(\frac{dL}{dt} + 2\omega_z \sin B\right) \qquad 17.5$$

$$= -r^2 \sin\mu_1 \sin\mu_2 \sin\mu_3 \left(\frac{dL}{dt}\cos^2 B + \frac{2p_\psi \sin B}{Mr^2}\right) \qquad 17.6$$

from (12.6), (8.6) and (5.9) .

18. Lagrange's Variable in Terms of r_i and their Derivatives

By differentiation of the expression (6.4) of r^2, one obtains

$$\frac{2r_1 dr_1}{r^2 \sin^2\mu_1} = 2(1 - \cos B \cos(L-L_1))\frac{dr}{r} + \sin B \cos(L-L_1)dB + \cos B \sin(L-L_1)dL . \qquad 18.1$$

By multiplying by

$$\cos(L-L_3) - \cos(L-L_2)$$

and adding the three analogous equations we have

$$dL = \sum \frac{\cos(L-L_3) - \cos(L-L_2)}{2\sin\mu_1 \sin\mu_2 \sin\mu_3} \frac{r_1 dr_1}{r^2 \sin^2\mu_1 \cos B} \qquad 18.2$$

or using (6.4)

$$dL = \sum \frac{(r_2^2/\sin^2\mu_2) - (r_3^2/\sin^2\mu_3)}{2r^2 \sin\mu_1 \sin\mu_2 \sin\mu_3} \qquad \frac{r_1 dr_1}{r^2 \sin^2\mu_1 \cos^2 B} \tag{18.3}$$

or

$$= \sum \frac{(r_2^2 \sin^2\mu_3 - r_3^2 \sin^2\mu_2)\, r_1 dr_1}{r^4 \sin^3\mu_1 \sin^3\mu_2 \sin^3\mu_3 \cos^2 B} \quad . \tag{18.4}$$

This gives for Lagrange's variable

$$-\rho = \sum \frac{(r_2^2 \sin^2\mu_3 - r_3^2 \sin\mu_2)\, r_1 dr_1/dt}{2r^2 \sin^2\mu_1 \sin^2\mu_2 \sin^2\mu_3} - \frac{2 \sin\mu_1 \sin\mu_2 \sin\mu_3}{M}\, p_\psi \sin B \; . \tag{18.5}$$

This expression is much more simple than the one (p. 329), formula (21) given by Lagrange.

19. Lagrange's Equilateral Configuration

In the same paper (1772) Lagrange has investigated the case where the shape of the triangle remains invariable, i.e., in our notation when B and L do not vary, although the scale r may be variable. These are plane configurations. They may be obtained immediately from our Hamiltonian (12.12).

The conditions to be fulfilled (with $p_\psi = p_\phi$) are obviously

$$\frac{\partial U}{\partial L} = 0 \quad , \quad \frac{\partial U}{\partial B} = 0 \quad . \tag{19.1}$$

Quite generally, we have

$$\frac{\partial U}{\partial B} = \frac{G}{r} \sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)} \sum_i \frac{\sin 2\mu_i \sin B \cos(L - L_i)}{4 r_i^3} \tag{19.2}$$

$$\frac{\partial U}{\partial L} = \frac{G}{r} \sqrt{m_1 m_2 m_3 (m_1 + m_2 + m_3)} \sum_i \frac{\sin 2\mu_i \cos B \sin(L - L_i)}{4 r_i^3} \quad . \tag{19.3}$$

When $r_1 = r_2 = r_3$, these expressions vanish from the fundamental identities (4.1) and (4.2) (for n = 2).

Then the motion is restricted to a variation r and ϕ with Hamiltonian

$$H = \frac{1}{2M} p_r^2 + \frac{p_\phi^2}{2Mr^2} + \frac{U_o}{r} \qquad 19.4$$

where U_o is a constant.

This is the Hamiltonian of the Keplerian motion. It follows that in this case each of the three bodies describes elliptic motion with the same eccentricity and axis while the triangle $M_1M_2M_3$ remains an equilateral triangle.

The values B_o, L_o in this case are given by

$$\sin B_o = \sqrt{3}\, \frac{\sin \mu_1 \sin \mu_2 \sin \mu_3}{1 + \cos\mu_1 \cos\mu_2 \cos\mu_3} \qquad 19.5$$

which follows from (6.14).

We have used

$$r_1 = r_2 = r_3 = 2q_1^2 = 2q_2^2 = 2q_3^2 = \frac{2}{3} q^2$$

and (6.15) and (5.18). L_o can be obtained from

$$\cos B_o \sin(L - L_3) = \frac{\sin\mu_1 \sin \mu_2 \sin(\mu_2 - \mu_1)}{1 + \cos\mu_1 \cos\mu_2 \cos \mu_3} \quad . \qquad 19.6$$

20. Canonical Transformation for the Keplerian Motion

We shall need the theory of the canonical reduction of the Keplerian motion.

Let the Hamiltonian be written

$$H = \frac{p_r^2}{2M} + \frac{\mathcal{A}}{r^2} - \frac{\mathcal{B}}{r} \qquad 20.1$$

We had in Sec. 19

$$\mathcal{A} = \frac{p_\phi^2}{2M} \ , \ \mathcal{B} = U_o \ . \qquad 20.2$$

Let us introduce a new variable q, p in place of r, p_r .

We must introduce a function S (the determining function) of r and p and then we have

$$p_r = \frac{\partial S}{\partial r} \qquad 20.3$$

$$q = \frac{\partial S}{\partial p} \qquad 20.4$$

so that

$$p_r dr = pdq - d(pq) + dS \qquad 20.5$$

which is the condition for the transformation to be a canonical transformation.

We choose p such that

$$H = \frac{p_r^2}{2M} + \frac{\mathcal{A}}{r^2} - \frac{\mathcal{B}}{r} = \frac{p}{r} + h \ . \qquad 20.6$$

h is a constant: the energy constant.

Then

$$S = \int p_r dr = \sqrt{2M} \int_{r_o} \sqrt{\frac{-\mathcal{A}}{r^2} + \frac{\mathcal{B} + p}{r} + h} \ dr \ . \qquad 20.7$$

The limit of integration r_o is a constant or a root of the radical. Then we have

$$q = \frac{\partial S}{\partial p} = \sqrt{\frac{M}{2}} \int_{r_o} \frac{dr}{\sqrt{-\mathcal{A} + (\mathcal{B} + p)\, r + hr^2}} \qquad 20.8$$

The inversion of the integral gives

$$r = \frac{\mathcal{B} + p}{-2h} \left(1 + e \cos q \frac{\sqrt{-2h}}{M}\right) \qquad 20.9$$

with

$$e = \sqrt{1 + \frac{4\mathcal{A}h}{(\mathcal{B} + p)^2}} \quad . \qquad 20.10$$

We notice that

$$E = q\sqrt{\frac{-2h}{M}} \qquad 20.11$$

is the eccentric anomaly, and

$$a = \frac{\mathcal{B} + p}{-2h} \qquad 20.12$$

is the semi-major axis.

We must take into account that $\mathcal{A}$ is a function of p_ϕ .

In order to do this we must admit that ϕ and p_ϕ will also be transformed into new variables, say, ϕ_o, p_{ϕ_o} , and that

$$S(r, p, \phi, p_{\phi_o})$$

contains also the old coordinates ϕ and the new momentum p_{ϕ_o} .

The corresponding canonical transformation will be the same as far as r, p_r, q, p are concerned, but it will be extended to a transformation of ϕ and p_ϕ by

$$p_\phi = p_{\phi_o} + \frac{\partial S}{\partial \phi} \qquad 20.13$$

$$\phi_o = \phi + \frac{\partial S}{\partial p_{\phi_o}} \quad . \qquad 20.14$$

This gives

$$p_\phi = p_{\phi_o} \qquad 20.1$$

but

$$\phi_o = \phi - \frac{1}{\sqrt{2M}} \int \frac{(\partial \mathcal{A}/\partial p_{\phi_o})\, dr}{r^2 \sqrt{-\frac{\mathcal{A}}{r^2} + \frac{\mathcal{B} + p}{r} + h}} \qquad 20.16$$

therefore

$$\phi_o = \phi - \frac{p_{\phi_o}}{\sqrt{2M}} \int \frac{dq}{r} \qquad 20.17$$

The Hamiltonian reduces to

$$H = \frac{p}{a(1 + e \cos q \sqrt{\frac{-2h}{M}})} + h \qquad 20.18$$

with p = 0 and

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = \frac{1}{r} \qquad 20.19$$

which gives Kepler equation

$$nt = E - e \sin E \qquad 20.20$$

with

$$n = \sqrt{\frac{-2h}{M}} \; . \qquad 20.21$$

ϕ_o is a constant

$$\phi - \phi_o = \frac{p_{\phi_o}}{\sqrt{2M}} \int \frac{dt}{r^2} \qquad 20.22$$

21. Extensions to the General Case

In the general three-body problem the Hamiltonian (12.12) is still of the Keplerian type (20.1) as far as the variable r and p_r are concerned, but the expressions $\mathcal{A}$ and $\mathcal{B}$ are more complicated functions of the other variables.

We have

$$\mathcal{A} = \frac{p\psi^2}{2M} + \frac{2}{M}\left(p_B^2 + \frac{(p_L - \frac{p\psi}{2}\sin B)^2}{\cos^2 B}\right) + \frac{p_\phi^2 - p_\psi^2}{M \sin^2 B}(1 + \cos B \cos \psi) \qquad 21.1$$

and

$$\mathcal{B} = Gm_1m_2m_3 \sum \frac{1}{m_i \sin \mu_i \sqrt{1 - \cos B \cos (L - L_i)}} \; . \qquad 21.2$$

Let us call q^i these variables other than r, and the corresponding momenta p_i. Then the $\mathcal{A}$ and $\mathcal{B}$ are functions of these variables:

$$\mathcal{A}(q^i, p_i) \tag{21.3}$$

$$\mathcal{B}(q^i, p_i) . \tag{21.4}$$

We shall make the Keplerian canonical transformation of the preceding section but we shall imagine that the $\mathcal{A}$ and $\mathcal{B}$ are

$$\mathcal{A}(q_o^i, p_i^o) \tag{21.5}$$

$$\mathcal{B}(q^i, p_i^o) \tag{21.6}$$

i.e., we have replaced the old momenta p_i by the new momenta p_i^o.

The new coordinates q_o^i and the corresponding new momenta p_i^o shall be defined by

$$q_o^i = \frac{\partial S}{\partial p_i^o} = q^i - \int \frac{1}{r} \frac{\partial \mathcal{A}}{\partial p_i^o} dq \tag{21.7}$$

and

$$p_i = p_i^o + \int \left(\frac{\partial \mathcal{B}}{\partial q^i} - \frac{1}{r} \frac{\partial \mathcal{A}}{\partial q^i} \right) dq . \tag{21.8}$$

22. The Sundman-Weierstrass Theorem

According to the preceding section, we see that the formulae (9) and (10) of Sec. 20 are valid in the general case with the understanding that $\mathcal{A}$ and $\mathcal{B}$ may be variable according to (21.1) and 21.2).

Then formula (20.9) shows that r cannot vanish except if e becomes equal to its limiting value ± 1. And according to (20.10) this can only occur if $\mathcal{A}$ vanishes.

Now $\mathcal{A}$ from (21.1) is the sum of non-negative terms and it can vanish only if each of the terms vanishes.

This means that

$$p_\psi = 0$$

and also

$$p_B = 0, \; p_L - \frac{p_\psi}{2} \sin B = 0 .$$

Now

$$\frac{1 + \cos B \cos \psi}{\sin^2 B}$$

was obtained as

$$\frac{\cos^2\psi \cos^2 \frac{B}{2} + \sin^2\psi \sin^2 \frac{B}{2}}{2 \sin^2 \frac{B}{2} \cos^2 \frac{B}{2}}$$

and is essentially positive; therefore we must have

$$p_\phi = \pm p_\psi = 0 .$$

This is Sundman's theorem.

23. Change of Coordinates -- Cartan's Result

Let us take $r_1 r_2 r_3$ as coordinates in place of r, B, L. Then from

$$2 \frac{dr_i}{r_i} = 2 \frac{dr}{r} + \frac{\sin B \cos (L - L_i) dB + \cos B \sin (L - L_i) dL}{1 - \cos B \cos (L - L_i)} \qquad 23.1$$

obtained by derivation of (6. 3) and the relation

$$p_{r_1} dr_1 + p_{r_2} dr_2 + p_{r_3} dr_3 = p_r dr + p_B dB + p_L dL \qquad 23.2$$

between the momenta or the generalized momenta we have

$$p_r = \sum \frac{r_i p_{r_i}}{r} , \; p_B = \sum \frac{r_i p_{r_i}}{2} \frac{\sin B \cos (L-L_i)}{1-\cos B \cos (L-L_i)} , \; p_L = \sum \frac{r_i p_{r_i}}{2} \frac{\cos B \sin (L-L_i)}{1-\cos B \cos (L-L_i)} \qquad 23.3$$

from which we deduce

$$p_B^2 + \frac{p_L^2}{\cos^2 B} = \sum_i \sum_k \frac{\sin^2 B \cos(L-L_i) \cos(L-L_k) + \sin(L-L_i) \sin(L-L_k)}{(1 - \cos B \cos (L-L_i))(1 - \cos B \cos (L-L_k))} \frac{r_i r_k p_{r_i} p_{r_k}}{4} .$$

$$= \sum\sum \frac{\cos(L_i - L_k) - \cos^2 B \cos(L - L_i)\cos(L - L_k)}{(1 - \cos B \cos(L - L_i))(1 - \cos B \cos(L - L_k))} \cdot \frac{r_i r_k r_{p_i} r_{p_k}}{4} \qquad 23.4$$

Using

$$\cos B \cos(L - L_i) = 1 - \frac{r_i^2}{r^2 \sin^2 \mu_i} \qquad 23.5$$

we obtain

$$\sum\sum \left[\frac{\cos(L_i - L_k) - 1 + (r_i^2 / r^2 \sin^2 \mu_i) + (r_k^2 / r^2 \sin^2 \mu_k)}{r_i^2 r_k^2 / r^4 \sin^2 \mu_i \sin^2 \mu_k} - 1 \right] \frac{r_i p_k p_{r_i} p_{r_k}}{4} \qquad 23.6$$

and therefore

$$\frac{p_r^2}{2M} + \frac{2}{Mr^2}\left(p_B^2 + \frac{p_L^2}{\cos^2 B}\right) = \frac{1}{M} \sum_i \sin^2 \mu_i p_{r_i}^2$$

$$+ \frac{1}{M} \sum_{(ijk)} (-2r^2 \sin^2 \mu_1 \sin^2 \mu_2 \sin^2 \mu_3 + r_i^2 \sin^2 \mu_k + r_k^2 \sin^2 \mu_i) \frac{p_{r_i} p_{r_k}}{r_i r_k} . \qquad 23.7$$

The last term using (6.4) and the relation between the μ_i becomes

$$\frac{1}{M} \sum_{(ijk)} \sin \mu_i \cos \mu_j \sin \mu_k (r_i^2 - r_j^2 + r_k^2) \frac{p_{r_i} p_{r_k}}{r_i r_k} . \qquad 23.8$$

Then using (5.12) and (5.18) we have

$$\frac{p_R^2}{2M} + \frac{2}{Mr^2}\left(p_B^2 + \frac{p_L^2}{\cos^2 B}\right) = \sum_{(ijk)} \frac{1}{2M_j} \left(p_{r_i}^2 + p_{r_k}^2 + \frac{r_i^2 - r_j^2 + r_k^2}{r_i r_k} p_{r_i} p_{r_k}\right), \qquad 23.9$$

This expression has been given by Elie Cartan ("Les Invariants Intégraux," p. 183, 1922) in the case $p_\phi = p_\psi = 0$.

It has been found without elaborate computation in the following way which gives a nice geometrical interpretation of the results.

The momentum mV_j of the mass M_j is shown to be a geometrical sum of two vectors, one called p_{r_i} along M_iM_j, the other, p_{r_j} along M_kM_j .

In this way the systems of all momenta is equivalent to zero as it should. If p_ϕ is not assumed to be zero, it would be equivalent to a couple -- the theory of Cartan applies to that case but Cartan did not carry through the calculation.

Now from geometry

$$\cos(M_iM_jM_k) = \frac{r_i^2 - r_j^2 + r_k^2}{2r_ir_k} \qquad 23.10$$

so that

$$V_j^2 = \left(\frac{p_{r_i}}{m_j}\right)^2 + \left(\frac{p_{r_k}}{m_j}\right)^2 + \frac{2p_{r_i}}{m_j}\,\frac{p_{r_k}}{m_j}\cos(M_iM_jM_k) \ . \qquad 23.11$$

The expression of the kinetic energy follows immediately.

It is obvious that the transformation is the same for momenta or modified momenta so that the general case follows from the case when $p_\phi = 0$.

We could therefore dispense with the computation at the beginning of this section by using Cartan's result.

24. The Kinetic Energy in Terms of the Derivatives of the Distances

We shall transform Cartan's expression (23.9) of the kinetic energy (or more strictly, of the part of the kinetic energy not involving p_ϕ and p_ψ) by introducing the derivatives of the velocities obtained from the canonical equation

$$\dot{r}_i = \frac{\partial H}{\partial p_{r_i}} = \left(\frac{1}{m_j} + \frac{1}{m_k}\right)p_{r_i} + \frac{\cos M_k}{m_k}\,p_{r_j} + \frac{\cos M_j}{m_j}\,p_{r_k} \ . \qquad 24.1$$

In order to solve for the p_{r_i} we have to consider the determinants

$$g = \begin{vmatrix} \frac{1}{m_2} + \frac{1}{m_3} & \frac{\cos M_3}{m_3} & \frac{\cos M_2}{m_2} \\ \frac{\cos M_3}{m_3} & \frac{1}{m_3} + \frac{1}{m_1} & \frac{\cos M_1}{m_1} \\ \frac{\cos M_2}{m_2} & \frac{\cos M_1}{m_1} & \frac{1}{m_1} + \frac{1}{m_2} \end{vmatrix} \qquad 24.2$$

or developing

$$\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\left(\frac{\sin^2 M_1}{m_1^2} + \frac{1}{m_2 m_3} + \frac{1}{m_3 m_1} + \frac{1}{m_1 m_2}\right)$$

$$+ \frac{\cos M_3}{m_3}\left(\frac{\cos M_1 \cos M_2}{m_1 m_2} - \frac{\cos M_3}{m_1 m_3} - \frac{\cos M_3}{m_1 m_2}\right)$$

$$+ \frac{\cos M_2}{m_2}\left(\frac{\cos M_1 \cos M_3}{m_1 m_3} - \frac{\cos M_2}{m_2 m_3} - \frac{\cos M_2}{m_2 m_1}\right) \qquad 24.3$$

or

$$\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\left(\frac{\sin^2 M_1}{m_1^2} + \sum \frac{1}{m_2 m_3}\right)$$

$$+ \frac{\cos M_3}{m_3}\left(\frac{\sin M_1 \sin M_2}{m_1 m_2} - \cos M_3 \sum \frac{1}{m_2 m_3}\right)$$

$$+ \frac{\cos M_2}{m_2}\left(\frac{\sin M_1 \sin M_3}{m_1 m_3} - \cos M_2 \sum \frac{1}{m_2 m_3}\right) . \qquad 24.4$$

Thus

$$g = \sum \frac{1}{m_2 m_3}\left(\frac{\sin^2 M_1}{m_1} + \frac{\sin^2 M_2}{m_2} + \frac{\sin^2 M_3}{m_3}\right) , \qquad 24.5$$

or using the notation (6.8) and elementary properties of triangles

$$g = \frac{4q^2 q_1^2 q_2^2 q_3^2}{r_1^2 r_2^2 r_3^2} \left(\frac{r_1^2}{m_1} + \frac{r_2^2}{m_2} + \frac{r_3^2}{m_3} \right) \left(\sum \frac{1}{m_2 m_3} \right) , \qquad 24.6$$

or from (6.17)

$$g = 4Mr^2 \frac{q^2 q_1^2 q_2^2 q_3^2}{r_1^2 r_2^2 r_3^2} \left(\sum \frac{1}{m_2 m_3} \right)^2 . \qquad 24.7$$

Then the kinetic energy is

$$\frac{1}{2g} \left[\sum \frac{\sin^2 M_1}{m_1^2} (\dot{r}_1)^2 + 2 \sum \frac{\sin M_2 \sin M_3}{m_2 m_3} \dot{r}_1 \dot{r}_2 \right]$$

$$+ \frac{1}{2g} \sum \frac{1}{m_2 m_3} \left[\sum \dot{r}_1^2 - 2 \sum \cos M_3 \, \dot{r}_1 \, \dot{r}_2 \right] . \qquad 24.8$$

The first term is

$$\frac{1}{2 \sum (1/m_2 m_3)} \frac{\left((r_1 \dot{r}_1 / m_1) + (r_2 \dot{r}_2 / m_2) + (r_3 \dot{r}_3 / m_3) \right)^2}{(r_1^2/m_1) + (r_2^2/m_2) + (r_3^2/m_3)} \qquad 24.9$$

or from (6.17)

$$\frac{M}{2} \dot{r}^2 , \qquad 24.10$$

while the second term is (using (5.17))

$$\frac{M}{2r^2} \frac{r_1^2 r_2^2 r_3^2}{16 q^2 q_1^2 q_2^2 q_3^2} \frac{\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_1^3 - 2\dot{r}_1 \dot{r}_2 \cos M_3 - 2\dot{r}_2 \dot{r}_3 \cos M_1 - 2\dot{r}_3 \dot{r}_1 \cos M_2}{\sin^2 \mu_1 \, \sin^2 \mu_2 \, \sin^2 \mu_3} \qquad 24.11$$

This expression can be more conveniently written using $q_1 q_2 q_3$ (and q) in place of $r_1 r_2 r_3$. It gives for the kinetic energy

$$\frac{M}{2}\dot{r}^2 + \frac{M}{2r^2}\,\frac{r_1 r_2 r_3 (\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 - \dot{q}^2)}{\sin^2\mu_1 \sin^2\mu_2 \sin^2\mu_3} \quad . \tag{24.12}$$

Of course

$$q\dot{q} = q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3 \quad . \tag{24.13}$$

Introducing polar coordinates, for instance,

$$q_1 = q \cos\beta \cos\lambda$$

$$q_2 = q \cos\beta \sin\lambda$$

$$q_3 = q \sin\beta \tag{24.14}$$

we obtain

$$\frac{M}{2}\dot{r}^2 + \frac{M}{2r^2}\,\frac{r_1 r_2 r_3 q^2 (\dot{\beta}^2 + \cos^2\beta\,\dot{\lambda}^2)}{\sin^2\mu_1 \sin^2\mu_2 \sin^2\mu_3} \quad . \tag{24.15}$$

25. Regularization of Binary Encounters

Let us use r, β, λ in place of r, B, L. Let

$$\frac{r_1 r_2 r_3 q^2}{r^4} = f(\beta, \lambda) \tag{25.1}$$

be expressed in terms of β λ from formulae (24.14), (6.8) and (6.15).

The kinetic energy is

$$\frac{M}{2}\dot{r}^2 + \frac{Mr^2}{2} f\,\frac{(\beta_1\lambda)(\dot{\beta}^2 + \cos^2\beta\,\dot{\lambda}^2)}{\sin^2\mu_1 \sin^2\mu_2 \sin^2\mu_3} \tag{25.2}$$

Turning back to momenta (or generalized momenta) and introducing the other terms of the Hamiltonian (2.11) we obtain

$$H = \frac{p_r^2}{2M} + \frac{\sin^2\mu_1 \sin^2\mu_2 \sin^2\mu_3}{2Mr^2 f(\beta, \lambda)}\left(p_\beta^2 + \frac{p_\lambda^2}{\cos^2\beta}\right) \tag{25.3}$$

$$+ \frac{p_\psi^2}{2Mr^2} + \frac{p_\phi^2 - p_\psi^2}{Mr^2} \quad \frac{1 + \cos B \cos \psi}{\sin^2 B}$$

$$- \frac{G\, m_1 m_2 m_3}{r} \left(\frac{1}{m_1} \left(\frac{r}{r_1}\right) + \frac{1}{m_2} \left(\frac{r}{r_2}\right) + \frac{1}{m_3} \left(\frac{r}{r_3}\right) \right)$$

of course, B and L must be given in terms of β λ using (6.14) and the ratios r_i/r must be given in terms of these same variables. Let

$$\mathcal{H} = (H - h)\, f(\beta_1 \lambda) \qquad 25.4$$

where h is the constant value of H because

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0 . \qquad 25.5$$

Then generally

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} = \frac{1}{f(\beta_1 \lambda)} \frac{\partial \mathcal{H}}{\partial p_i} + \mathcal{H} \frac{f(\beta_1 \lambda)}{p_i} \qquad 25.6$$

and the last term vanishes in the equation of motion.

If we write

$$dt = f(\beta_1 \lambda)\, ds \qquad 25.7$$

we have

$$\frac{dq^i}{ds} = \frac{\partial \mathcal{H}}{\partial p_i} \qquad 25.8$$

and similarly

$$\frac{dp_i}{ds} = - \frac{\partial \mathcal{H}}{\partial q^i} \qquad 25.9$$

The simplified Hamiltonian $\mathcal{H}$ can be used with the understanding that it is used with a different independent variable s which may be called the regularizing time.

The real time can be obtained afterwards by computing

$$t = \int f(\beta_1 \lambda)\, ds \qquad 25.10$$

It is obvious that singularities of binary encounters have disappeared from the new Hamiltonian.

$$\mathcal{H} = f(\beta_1 \lambda)\left(\frac{p_r^2}{2M} + \frac{p_\psi^2}{2Mr^2} + \frac{p_\phi^2 - p_\psi^2}{Mr^2}\ \frac{1 + \cos B \cos \psi}{\sin^2 B}\right)$$

$$+ \frac{\sin^2\mu_1 \sin^2\mu_2 \sin^2\mu_3}{2Mr^2}\left(p_\beta^2 + \frac{p_\lambda^2}{\cos^2\beta}\right) - \frac{G}{r} F(\beta_1 \lambda) \qquad 25.11$$

where

$$F(\beta_1 \lambda) = \frac{(m_2 m_3 r_2 r_3 + m_3 m_1 r_3 r_1 + m_1 m_2 r_1 r_2)\, q^2}{r^3} \qquad 25.12$$

26. Geometry of the Transformation BL $\longrightarrow$ $\beta\lambda$

The coordinates BL can be considered as polar coordinates on a sphere of radius one. B is the latitude, L the longitude. This interpretation is the obvious one. The special configuration of Sec. 2 for which $B = \pi/2$ and L is undetermined is represented by the pole of the sphere.

Every triangle is represented by a point with $B > 0$, i.e., by a point of the half North Sphere.

But the same values of the distances r_i could be obtained by reversing the sign of B so that negative values of B could be used with the understanding that the representation of the sphere is twofold.

The singularities of the problem are the binary encounters for which one of the distances r_i vanishes.

From (6.1) it is obvious that the representative points lie on the equator $B = 0$ with longitudes L_1, L_2, L_3.

Similarly, the coordinates $\beta\lambda$ can be understood as points on a sphere of radius one.

The equation (24.14) shows that

$$\frac{q_1}{q}, \frac{q_2}{q}, \frac{q_3}{q}$$

are the corresponding Cartesian coordinates.

In fact the coordinates β, λ may be defined from these Cartesian coordinates after an arbitrary rotation without introducing any change in the form of the Hamiltonian.

The value $r_1 = 0$ is obtained only when $q_2 = q_3 = 0$. Similarly, the other binary encounters are represented on the sphere $\beta\lambda$ by three points which are the vertices of a triangle with three angles of $\pi/2$.

Every configuration is represented inside of the triangle, but an equivalent representation occurs when the sign of any of the q_i is changed so that each configuration is represented in eight different places.

It is obvious that the transformation of one sphere on the other just defined is a conformal transformation; the change of scale from one to the other may vary from place to place but is indifferent to orientation.

The points which represent the binary collisions are singular points of the transformation. In fact, a point which runs along the equator in the BL sphere describes the sides of the triangle in the $\beta\lambda$ sphere. At the vertices the angle is reduced to the half.

It follows that points which pass through the vertices along one of the sides and pass through the singular point correspond to points which reach the point of longitude L_i and rebound, making a complete change of direction of 2π.

This type of regularization is well known from the work of Thiele in special cases, and of Levi-Civita in the general problem of the three bodies. If we consider sufficiently small regions we may say that elliptic motions with the singular point as a focus in the BL sphere are transformed into ellipses with center at the singular point on the $\beta\lambda$ sphere.

27. The Modifying Terms in the Momenta

What we have obtained applies to modified canonical momenta as was explained in Sec. 14. We have explained in Sec. 15 how the modification A_i or the associated $A_{q_1 q_2}$ is transformed when we change coordinates.

The latter quantities are the only ones which are needed when the problem is restricted to plane motion.

Let us consider first the use of $r_1 r_2 r_3$ as coordinates as in Sec. 23. We have from (15.1) and (14.2)

$$\frac{1}{2} \sin B \, dL = A_{r_1} dr_1 + A_{r_2} dr_2 + A_{r_3} dr_3 \qquad 27.1$$

which gives from (18.4), for instance,

$$A_{r_1} = \frac{r_2^2 \sin^2 \mu_3 - r_3^2 \sin^2 \mu_2}{4r^4 \sin^3 \mu_1 \sin^3 \mu_2 \sin^3 \mu_3} \; \frac{\sin B}{\cos^2 B} \; r_1 \; . \qquad 27.2$$

The value of B in terms of $r_1 r_2 r_3$ is given in (6.14).

If we use the q_1, q_2, q_3 of (6.8) with

$$dr_1 = 2q_2 dq_2 + 2q_3 dq_3$$

we find

$$A_{q_3} = 2(A_{r_1} + A_{r_2}) q_3 \; . \qquad 27.3$$

For the variables $q \; \beta \; \lambda$ we have

$$\begin{aligned} A_q dq + A_\beta d\beta + A_\lambda d\lambda &= Aq_1(dq \cos\beta \cos\lambda - q \sin\beta \cos\lambda \, d\beta - q \cos\beta \sin\lambda \, d\lambda) \\ &+ Aq_2(dq \cos\beta \sin\lambda - q \sin\beta \sin\lambda \, d\beta + q \cos\beta \cos\lambda \, d\lambda) \\ &+ Aq_3(dq \sin\beta + q \cos\beta \, d\beta) \end{aligned} \qquad 27.4$$

which gives

$$A_q = 0 \qquad A_\lambda = -A_{q_1} q_2 + A_{q_2} q_1$$

$$A_\beta = q(-A_{q_1} \sin\beta \cos\lambda - A_{q_2} \sin\beta \sin\lambda + A_{q_3} \cos\beta) . \qquad 27.5$$

When the variables r, β, λ are used as in Sec. 25 the values of A_β and A_λ are not modified. This can be seen through a transformation $q_1\beta\lambda \longrightarrow r\beta\lambda$.

Writing for a moment $A_\beta^{(r)}$ when the variable r is used we have

$$A_q dq + A_\beta d\beta + A_\lambda d\lambda = A_r dr + A_\beta^{(r)} d\beta + A_\lambda^{(r)} (d\lambda) \qquad 27.6$$

but $A_q = 0$, therefore

$$A_r = 0 \qquad A_\beta^{(r)} = A_\beta \qquad A_\lambda^{(r)} = A_\lambda \quad . \qquad 27.7$$

Similarly the $A_{q^i q^i}$ are found from (15.2).

One finds

$$dL \wedge dB = \frac{r_1 r_2 r_3}{\sin^3\mu_1 \sin^3\mu_2 \sin^3\mu_3} \; \frac{\sum r_3 dr_1 \wedge dr_2}{r^6 \sin B \cos B} \quad . \qquad 27.8$$

From

$$A_{LB} = -\frac{\partial A_L}{\partial B} = -\frac{1}{2}\cos B \qquad 27.9$$

one finds using (7.14)

$$A_{r_1 r_2} = -\frac{r_1 r_2 r_3}{q q_1 q_2 q_3} \; \frac{r_3}{4r^4 \sin^2\mu_1 \sin^2\mu_2 \sin^2\mu_3} \qquad 27.10$$

and from

$$dr_1 \wedge dr_2 = -4q_1 q_2 dq_1 \wedge dq_2 + 4q_2 q_3 dq_2 \wedge dq_3 + 4q_3 q_1 dq_3 \wedge dq_1$$

$$A_{q_1 q_2} = - \frac{2 r_1 r_2 r_3 \quad q_3}{q r^4 \sin^2 \mu_1 \sin^2 \mu_2 \sin^2 \mu_3} \tag{27.11}$$

and finally from

$$dq_1 \wedge dq_2 = - q\, q_3 \cos \beta \; d\beta \wedge d\lambda \; . \; . \; . \tag{27.12}$$

$$A_{\beta\lambda} = \frac{2 q^2 r_1 r_2 r_3 \cos \beta}{r^4 \sin^2 \mu_1 \sin^2 \mu_2 \sin^2 \mu_3} = \frac{2 f(\beta, \lambda) \cos \beta}{\sin^2 \mu_1 \sin^2 \mu_2 \sin^2 \mu_3} \tag{27.13}$$

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