UNPUBLISHED PRELIMINARY DATA

FINITE SLUMP STRAINS IN VERTICAL, INFINITELY LONG, HOLLOW, ELASTIC CYLINDER EXTERNALLY CASE-BONDED TO AN ELASTIC TANK

FACILITY FORM 602

N65 17530 (ACCESSION NUMBER)

(THRU)

20 (PAGES)

1 (CODE)

CR 60845 (NASA CR OR TMX OR AD NUMBER)

32 (CATEGORY)

by

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GPO PRICE $ _____

OTS PRICE(S) $ _____

Hard copy (HC) 2.00

Microfiche (MF) .50

Technical Report No. 7

National Aeronautics and Space Administration

Research Grant No. NsG-125-61 (Suppl. 3)

DECEMBER 1964

THE CATHOLIC UNIVERSITY OF AMERICA

WASHINGTON, D.C. 20017

Vaishnav, R.N.: FINITE SLUMP STRAINS IN VERTICAL, INFINITELY LONG, HOLLOW, ELASTIC CYLINDER EXTERNALLY CASE-BONDED TO AN ELASTIC TANK.

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Abstract

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The problem of finite slump strains in a vertical, infinite, hollow, elastic cylinder, case-bonded to an elastic case, is solved assuming the material of the cylinder to be incompressible, and further, to possess a strain energy density function of the Mooney type. The problem reduces to that of solution of a trancendental equation in a geometrical parameter. Numerical results for a set of realistic data is obtained and the effect of change in bore radius, case stiffness and the specific weight of the material and the departure of the results from those of the linear theory are discussed in detail.

Author

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Introduction

It is not infrequent that solid propellant rocket motors are stored for a considerable length of time in a vertical position. Under the action of the gravity forces, stresses and strains are induced in the propellant material. In general, these stresses and strains, increase with time, as the propellants are viscoelastic in nature. For a rational design of a rocket motor it is necessary to have at hand an estimate of these stresses and strains. Usually such analysis are based on the assumption, among others, of the propellant being a linearly viscoelastic material. However, these materials usually are highly compliant so that these analyses are inadequate to represent accurately the states of stress and deformation in an actual rocket motor. A more accurate estimate could be achieved by an appropriate nonlinear theory of viscoelasticity. Such theories have been developed during the last ten years, however, they are relatively complex and the application of an appropriate theory to most problems of practical interest remains a hitherto unachieved goal. A great deal of useful information can, however, be obtained through an analysis wherein the material is considered, as a first approximation, to be elastic, but capable of sustaining rather large deformations. The present problem represents just such an attempt to gain useful insight into the behavior of the core material of a vertical rocket motor under gravity loading. The analysis is made tractable through several idealizations and assumptions, such as those of infinitude of the length of the rocket motor, homogeneity and isotropy of the core material, ideal core-case bond, and incompressibility of the core material. In addition, several minor, realistic assumptions have been made; these are stated in context. As in a previous report[1]*, wherein

* Numbers in brackets designate References at the end of the paper.

a pressurization problem was solved, the treatment here follows the approach and notation of Green and Zerna[2], as far as possible. The results are specialized to the case where the core material is taken to be a Mooney Material[3]. Application of the solution method is demonstrated for a real set of data wherein several parameters are varied. The results are compared with the analogous, almost trivial, infinitesimal theory solution. Several nonlinear, second order effects, absent in the latter theory, emerge from the solution as expected. These have been discussed in detail.

The General Theory

The initial portion of the general theory follows the initial portion of the previous report[2] on the pressurization problem. However, for reasons of completeness, the present study is entirely self-contained. The spirit of a problem solved by Green and Zerna* is followed in the following to a certain extent.

Consider a vertical, infinitely long, hollow, circular cylinder of initial external radius ρ_1 and initial internal radius ρ_2. Let the cylinder be ideally bonded to a thin elastic casing.

Let the cylinder be referred to a rectangular Cartesian reference frame (X_1, X_2, X_3) with the X_3 -axis directed vertically downwards, the origin being arbitrarily located on the axis of the cylinder. We refer to the undeformed cylinder as the body B_o.

Suppose, now, that we permit the weight of core material to act gradually, till the full weight is acting, resulting in the deformed configuration, body B, of the body B_o. In view of the axial symmetry of the geometry and loads and the assumed homogeneity of the material, the deformed body B also will be a circular cylinder. Let it be referred to another rectangular Cartesian frame (y^1, y^2, y^3) which we take to coincide with the frame (x^1, x^2, x^3), index to index. Let the outer and the inner radii of body B be r_1 and r_2, respectively. We recall that the infinitesimal theory would predict the state of strain in B as that of simple, nonhomogeneous shearing strain. Each particle of B, accordingly, would move axially only. We expect, however, in reality, a

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* See p. 95 ff. of Ref. [2].

departure from the state of simple shear. Maintenance of a state of simple finite shear, for the given body forces, would require a finite radial pressure at the inner free boundary of the core, which, in fact, is traction-free.

Let a polar cylindrical system (r, θ, z) be imbedded in the strained body B so that

$$y^1 = r\cos\theta, \quad y^2 = r\sin\theta, \quad y^3 = z \tag{1}$$

We shall designate the system (r, θ, z) alternatively as $(\theta^1, \theta^2, \theta^3)$. As a result of axisymmetry of deformation, the system of curvilinear coordinates, which is convected into (r, θ, z) as a result of deformation, is also polar cylindrical, say (ρ, θ, ξ), wherein we take $\rho = \rho(\xi)$ to be a function of r only. Thus, the rectangular Cartesian coordinates of the point $P_o(\rho, \theta, \xi)$ in B_o, corresponding to the point $P(\xi, \theta, z)$ in B, are given by

$$x^1 = \rho\cos\theta \quad ; \quad x^2 = \rho\sin\theta \quad ; \quad x^3 = \xi \tag{2}$$

where $\xi = \xi(r)$ is a function of r only. The metric tensors G_{ij}, for the body B, and g_{ij}, for the body B_o, can be obtained from Eqs. (1) and (2) as follows:

$$\left[G_{ij}\right] = \left[\frac{\partial y^p}{\partial \theta^i}\frac{\partial y^p}{\partial \theta^j}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

and

$$\left[g_{ij}\right] = \left[\frac{\partial x^p}{\partial \theta^i} \quad \frac{\partial x^p}{\partial \theta^j}\right] \begin{bmatrix} (\rho')^2 + (w')^2 & 0 & -w' \\ 0 & \rho^2 & 0 \\ -w' & 0 & 1 \end{bmatrix} \tag{4}$$

where $w = z - \xi = w(r)$, a function of r only, denotes the axial displacement of the point P_0 , and a prime denotes total derivative with respect to r. The usual summation convention is, of course, used in the above equation and in all of the following work unless explicitly stated otherwise.

The contravariant metric tensors G^{ij} and g^{ij} can be easily obtained from Eqs. (3) and (4). G^{ij} is trivially obtained; g^{ij} is obtained by inverting the matrix of elements of g_{ij} in Eq. (4) in a routine manner. Thus we have,

$$\left[G^{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

and

$$\left[g^{ij}\right] = \begin{bmatrix} \frac{1}{(\rho')^2} & 0 & \frac{w'}{(\rho')^2} \\ 0 & \frac{1}{\rho^2} & 0 \\ \frac{w'}{(\rho')^2} & 0 & 1 + \left(\frac{w'}{\rho'}\right)^2 \end{bmatrix} \tag{6}$$

We now have a complete metric description of bodies B_o and B. We may mention here that we chose to imbed the (r, θ, z) system in the strained body rather than the unstrained body for then the equilibrium equations retain their customary form.

As is reasonable for solid propellant materials, we make the assumption that the core material in question is incompressible. Mathematically this corresponds to the constraint that I_3, the third strain invariant, should maintain a unit value. As $I_3 = G/g$ also, where G and g are the determinants of the matrices in Eqs. (3) and (4), respectively, we have the following incompressibility condition:

$$(\rho\rho')^2 = r^2$$

$$\text{or } (\rho\rho') = \pm r \tag{7}$$

Simple integration of Eq. (7) and subsequent evolution gives

$$\rho = (K \pm r^2)^{1/2} \tag{8}$$

wherein the negative root is discarded for obvious reasons and K is an arbitrary constant of integration. From Eq. (8) we readily obtain

$$r_1^2 - r_2^2 = \pm(\rho_1^2 - \rho_2^2)$$

and it is immediately clear that we should discard the negative sign since we do not expect the rocket motor to turn inside out under the given set of body forces.

We see that a positive value of K implies an inward slump and a negative value the opposite. K is clearly zero in the infinitesimal strain theory solution.

For an incompressible material the first two strain invariants I_1 and I_2 are respectively given by

$$I_1 = g^{rs} G_{rs} \tag{10a}$$

and

$$I_2 = g_{rs} G^{rs} \tag{10b}$$

which become, on using Eqs. (3) to (6),

$$I_1 = 1 + \frac{1 + (\omega')^2}{(\rho')^2} + \frac{\eta^2}{\rho^2} \tag{11a}$$

and

$$I_2 = 1 + (\omega')^2 + (\rho')^2 + \frac{\rho^2}{\eta^2} \tag{11b}$$

We shall need the contravariant tensor B^{ij} given by

$$B^{ij} = I_1 g^{ij} - g^{ir} g^{js} G_{rs} \tag{12}$$

The matrix of the components of B^{ij} takes the following form on using Eqs. (3), (6) and (11a):

$$\left[B^{ij}\right] = \begin{bmatrix} \frac{1}{(\rho')^2}\left(1+\frac{r^2}{\rho^2}\right) & 0 & \frac{r^2 w'}{\rho^2(\rho')^2} \\ & \frac{1}{\rho^2}\left(1+\frac{1}{(\rho')^2}+\frac{(w')^2}{(\rho')^2}\right) & \\ \frac{r^2 w'}{\rho'(\rho')^2} & 0 & \frac{1}{(\rho')^2}+\frac{r^2}{\rho^2}+\frac{r^2}{\rho^2}\frac{(w')^2}{(\rho')^2} \end{bmatrix} \tag{13}$$

For convenience, let

$$\rho = r Q \tag{14}$$

where Q = Q(r) is a dimensionless function of r only. From Eq. (7) we obtain

$$\rho' = \frac{r}{\rho}$$

wherein we have discarded the negative sign. Then, from Eq. (14) it follows that

$$\rho' = \frac{1}{Q} \tag{15}$$

Equation (13), for the incompressible core, simplifies to

$$\left[B^{ij}\right] = \begin{bmatrix} Q^2+1 & 0 & w' \\ 0 & \frac{1}{r^2Q^2}+\frac{1}{r^2}+\frac{(w')^2}{r^2} & 0 \\ w' & 0 & Q^2+\frac{1}{Q^2}+(w')^2 \end{bmatrix} \tag{16}$$

The contravariant stress tensor components are formally given by

$$\tau^{ij} = \Phi g^{ij} + \Psi B^{ij} + p G^{ij} \tag{17}$$

wherein, for $I_3 = 1$,

$$\Phi = 2\frac{\partial W}{\partial I_1} \quad , \quad \Psi = 2\frac{\partial W}{\partial I_2} \tag{18}$$

$W = W(I_1, I_2)$ being the strain energy density function giving the elastic potential per unit volume of the unstrained body; B^{ij} is the tensor given by Eq. (16) and p is a uniform hydrostatic pressure which must be calculated from the equilibrium equations and the boundary conditions for the problem.

In view of Eqs. (5), (6), (14), (15), and (16), from Eqs. (17) we obtain the following stress tensor components:

$$\tau^{11} = \frac{\Phi}{(\rho')^2} + \frac{\Psi}{(\rho')^2}\left(1 + \frac{r^2}{\rho^2}\right) + p = \Phi Q^2 + \Psi(1 + Q^2) + p$$

$$\tau^{22} = \frac{\Phi}{\rho^2} + \frac{\Psi}{\rho^2}\left[1 + \frac{1}{(\rho')^2} + \frac{(w')^2}{(\rho')^2}\right] + \frac{p}{r^2} = \frac{\Phi}{r^2 Q^2} + \frac{\Psi}{r^2 Q^2}\left[1 + Q^2 + (w')^2 Q^2\right] + \frac{p}{r^2}$$

$$\left.\begin{aligned}
\tau^{33} &= \Phi\left[1+\frac{(\omega')^2}{(\rho')^2}\right] + \Psi\left[\frac{1}{(\rho')^2}+\frac{r^2}{\rho^2}+\frac{r^2}{\rho^2}\frac{(\omega')^2}{(\rho')^2}\right]+p \\
&= \Phi\left[1+(\omega')^2 Q^2\right] + \Psi\left[Q^2+\frac{1}{Q^2}+(\omega')^2\right]+p \\
\tau^{12} &= \tau^{21} = 0 \\
\tau^{31} &= \tau^{13} = \Phi\frac{\omega'}{(\rho')^2} + \frac{\Psi r^2\omega'}{\rho^2(\rho')^2} = \Phi\,\omega' Q^2 + \Psi\,\omega' \\
\tau^{23} &= \tau^{32} = 0
\end{aligned}\right\} \quad (19)$$

The equilibrium equations are given

$$D_p\,\tau^{pj} + \mu F^j = 0 \qquad (20)$$

where D_i denotes covariant differention with respect to the variables θ^i, μ is the local density in the deformed body and F^j is the contravariant component of the body force vector per unit mass.

We recall that the covariant derivative $D_r A^{ij}$ of a contravariant tensor A^{ij} is given by

$$D_r A^{ij} = A^{ij}{}_{,r} + \Gamma^{i}_{rp} A^{pj} + \Gamma^{j}_{rp} A^{ip} \tag{21}$$

where the Christoffel symbols Γ^{r}_{ij} are given by

$$\Gamma^{r}_{ij} = \frac{1}{2} G^{rp} \left(G_{ip,j} + G_{jp,i} - G_{ij,p} \right) \tag{22}$$

For the polar cylindrical coordinate system whose metric tensor is described by Eq. (3), Eq. (22) gives the only non-zero Γ^{r}_{ij} to be

$$\Gamma^{1}_{22} = -r \; ; \; \Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{r} \tag{23}$$

Using Eqs. (21), the equilibrium equations (20) become

$$\tau^{pj}{}_{,p} + \Gamma^{q}_{qp} \tau^{pj} + \Gamma^{j}_{qp} \tau^{pq} + \mu F^{j} = 0 \qquad (j=1,2,3) \tag{24}$$

where Γ^{s}_{s} have the values given by Eq. (23). The inertia terms in Eqs. (20) and (24) have been neglected as a result of our assumption of the body force being applied gradually only. Equation (24) reduce to the following

$$\tau^{11}{}_{,r} + \frac{\tau^{11}}{r} - r\tau^{22} = 0$$

$$\tau^{13}{}_{,r} + \frac{\tau^{13}}{r} + \mu a = 0 , \tag{25a,b}$$

since the equation corresponding to j = 2 is identically satisfied, all quantities are independent of z and the only non-zero body force component per unit mass is the acceleration due to gravity a.

Equation (25b) can be easily solved on realizing that for an incompressible material the density μ does not change. Thus we have, on integration,

$$\tau^{13} = \frac{2D}{r} - \frac{\gamma r}{2} \tag{26}$$

where $\gamma = \mu a$ and D is an arbitrary constant of integration.

Substituting for τ^{13} from Eq. (19) and solving for w' we obtain

$$w' = \frac{4D - \gamma r^2}{2r(\Phi Q^2 + \Psi)} \tag{27}$$

If Eq.(27) could be integrated, the form of w would be known to within two arbitrary constants. This, however, can not be done till the exact forms of Φ and Ψ are known. If we assume, now, that the strain energy density function W is described by the Mooney[4] equation,

$$W = W(I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{28}$$

where C_1 and C_2 are constants, we obtain, from Eq. (18),

$$\Phi = 2C_1 \qquad \Psi = 2C_2 \tag{29}$$

Equation (27) now becomes

$$w' = \frac{4D - \gamma r^2}{4r(C_1 Q^2 + C_2)} \tag{30}$$

which can be integrated since from Eqs. (9) and (14) we obtain

$$Q^2 = \frac{r^2 + K}{r^2} \tag{31}$$

On carrying out the integration it is possible, after a few steps, to arrive at the following expression for w:

$$w = \left[\frac{4D(C_1 + C_2) + \gamma C_1 K}{8(C_1 + C_2)^2}\right] \ln \frac{C_1 Q^2 + C_2}{C_1 + C_2} r^2 - \frac{\gamma r^2}{8(C_1 + C_2)} + E \tag{32}$$

where E is an arbitrary constant of integration. Constants E and D will have to be evaluated from the boundary conditions at the inside and the outside surfaces.

Before attempting to apply the boundary conditions, we need to consider the first equilibrium equation, Eq. (25a), which has not been used so far. Substituting in Eq. (25a) from Eq. (19) and (29) gives, after a few algebraic steps,

$$\frac{dp}{dr} = -2\frac{d}{dr}\left[C_1 Q^2 + C_2(1 + Q^2)\right] + \frac{2}{r}(C_1 + C_2)\left(\frac{1}{Q^2} - Q^2\right) + \frac{2C_2 w'^2}{r} \tag{33}$$

On integration of Eq. (33) we have

$$p = -2\left[C_1 Q^2 + C_2(1 + Q^2)\right] + 2\int\left[(C_1 + C_2)\left(\frac{1}{Q^2} - Q^2\right) + C_2 w'^2\right]\frac{dr}{r} + F \tag{34}$$

where F is an arbitrary constant of integration. As the forms of the functions

w' and Q are known through Eqs. (30) and (31), the integral on the right hand side of Eq. (34) can be evaluated. Omitting details of the elementary but tedious process of integration involved, we write down the following result:

$$\phi = -2\left[C_1 Q^2 + C_2(1+Q^2)\right] - 2(C_1+C_2)\left[\ln r - \frac{1}{2}\ln(r^2+K) - \frac{K}{2r^2}\right]$$

$$+ \frac{C_2}{8(C_1+C_2)^2}\left\{\frac{\gamma^2 r^2}{2} + \beta\left[\ln\frac{\sqrt{r^2+\alpha^2}}{|\alpha|} - \frac{r^2}{2(r^2+\alpha^2)}\right] - \frac{\delta}{2(r^2+\alpha^2)}\right\} + F \tag{35}$$

where

$$\left.\begin{aligned} \beta &= -8D\gamma - 2\gamma^2\alpha^2 \\ \delta &= 16D^2 - \gamma^2\alpha^4 \\ \text{and} \quad \alpha^2 &= \frac{C_1 K}{C_1 + C_2} \end{aligned}\right\} \quad (36\ a,b,c)$$

Let us now assume that the circumferential strains in the outer part of the core will remain sufficiently small so that the circumferential strain in the case remains within the limits of applicability of the Hooke's law. As it is reasonable to expect the outer part of the core to experience very small strains even when the core strains are rather large in the interior part, our assumption is realistic.

In view of the above, the boundary conditions for the problem may be written down, in words, as:

1. Axial displacement should vanish at the outer boundary of the core,

2. The radial component of traction on the outside deformed surface should be proportional to the local radial displacement, and

3. The inner boundary of the core should be traction-free.

The first boundary condition, which eliminates rigid body translation in the axial direction can be rewritten as

$$w\Big|_{r=r_1} = 0 \tag{37}$$

Using Eq. (32) it is not difficult to obtain the following value for the constant E:

$$E = \frac{\gamma r_1^2}{8(C_1+C_2)} - \left[\frac{4D(C_1+C_2)+\gamma C_1 K}{8(C_1+C_2)^2}\right] \ln \frac{C_1 Q_1^2 + C_2}{C_1+C_2}\, r_1^2 \tag{38}$$

where $Q_1^2 = \frac{\rho_1^2}{r_1^2} = \frac{r_1^2+k}{r_1^2}$, Q_1 being the value of Q at $r = r_1$. We shall use, in what follows, the notation Q_2 for the value of Q at $r = r_2$, defined in a similar manner.

In view of Eq. (38), Eq. (32) now becomes

$$w = \left[\frac{4D(C_1+C_2)+\gamma C_1 K}{8(C_1+C_2)^2}\right] \ln \frac{(C_1 Q^2 + C_2) r^2}{(C_1 Q_1^2 + C_2) r_1^2} - \frac{\gamma(r^2 - r_1^2)}{8(C_1+C_2)} \tag{39}$$

D and K are the only two unknown quantities in Eq. (39). Their determination obtains through the use of the remaining two boundary conditions.

If we denote the traction p^k on the outside and the inside curved surfaces by subscripts 1 and 2 in parentheses, respectively, we have,

$$p^k_{(2)} = \left[\tau^{ik} n_i \right]_{r=r_2}$$

$$= \left[\tau^{1k} n_1 + \tau^{2k} n_2 + \tau^{3k} n_3 \right]_{r=r_2} \qquad (40)$$

where n_1, n_2, n_3 are the components of the unit surface normal to the deformed body B. In this case

$$n_1 = 1, \quad n_2 = n_3 = 0$$

Hence, from Eq. (40),

$$p^k_{(2)} = \tau^{1k}_{(2)} \quad , (k = 1, 2, 3) \qquad (41)$$

where $\tau^{1k}_{(2)} = \left[\tau^{1k}\right]_{r=r_2}$. Since the inner cylindrical surface of the core must be traction-free, it follows from Eq. (41) that

$$p^2_{(2)} = \tau^{12}_{(2)} = 0 \qquad (42)$$

$$p^1_{(2)} = \tau^{11}_{(2)} = 2C_1 Q_2^2 + 2C_2(1 + Q_2^2) + p_2 = 0 \qquad (43a)$$

$$p^3_{(2)} = \tau^{13}_{(2)} = 2C_1 w'_{(2)} Q_2^2 + 2C_2 w'_{(2)} = 0 \qquad (43b)$$

where Eqs. (19) and (29) have been used along with the following notations:

$$p_2 = [p]_{r=r_2}$$
$$w'_{(2)} = [w']_{r=r_2} \qquad (44)$$

Substituting in Eq. (43a) from Eq. (35), the value of the constant F can be evaluated as

$$F = 2(C_1 + C_2)\left[\ln r_2 - \frac{1}{2}\ln(r_2^2 + K) - \frac{K}{2r_2^2}\right]$$
$$- \frac{C_2}{8(C_1+C_2)^2}\left\{\frac{\gamma^2 r_2^2}{2} + \beta\left[\ln\frac{\sqrt{r_2^2+\alpha^2}}{|\alpha|} - \frac{r_2^2}{2(r_2^2+\alpha^2)}\right] - \frac{\delta}{2(r_2^2+\alpha^2)}\right\} \qquad (45)$$

With this value of F, Eq. (35) for p contains only two unknowns, D and K, as does Eq. (39) for w.

Eq. (43b) is ambiguous as it stands. It implies either of the following two conditions:

$$w'_{(2)} = 0$$

or (46 a, b)

$$C_1 Q_2^2 + C_2 = 0$$

We ignore Eq. (46b) on the following grounds. Eq. (46b) implies that

$$Q_2^2 = -\frac{C_2}{C_1} = \frac{\rho_2^2}{\rho_2^2 - K}$$

$$K = \rho_2^2\left(1 + \frac{C_1}{C_2}\right)$$

This asserts that once the initial inner radius and the material constants are known, the radial change in geometry is decided, regardless of the value of γ, the specific weight of the material. Equation (46b) is thus discarded by reductio ad absurdum, leaving Eq. (46a) as the only boundary condition resulting from Eq. (43b). Imposing this condition on Eq. (30), we can solve for D. We get thereby,

$$D = \frac{1}{4}\gamma r_2^2 \tag{47a}$$

and, in turn, from Eq. (36a, b),

$$\beta = -2\gamma^2(r_2^2 + \alpha^2) \tag{47b}$$

and

$$\delta = \gamma^2(r_2^4 - \alpha^4) \tag{47c}$$

The only unknown at this stage is K, which can be evaluated by the use of the remaining boundary condition, namely, the one stating that the radial component of traction on the deformed outside surface should be proportional to the local radial displacement. The radial displacement at the exterior boundary of the core is

$$u\big|_{r=r_1} \equiv u_1 = r_1 - \rho_1 = r_1(1 - Q_1) \tag{48}$$

The radial component of the surface traction at $r = r_1$, is

$$P^{1}_{(1)} = \tau^{11}_{(1)} = 2C_1 Q_1^2 + 2C_2(1+Q_1^2) + p_1 \tag{49}$$

As before, the parenthetical subscript unity designates the fact that the value of the quantity is computed at $r = r_1$. The rightmost term of Eq. (49) is based on Eqs. (19) and (29).

The physical components of τ^{ij}, according to Green and Zerna, are given by

$$\sigma_{ij} = \left(\frac{G_{jj}}{G^{ii}}\right)^{\frac{1}{2}} \tau^{ij} \tag{50}$$

As $G^{11} = G_{11} = 1$, we have

$$\sigma_{11} = \sigma_r = \tau^{11} \tag{51}$$

and thus,

$$\sigma_r \Big|_{r=r_1} = 2C_1 Q_1^2 + 2C_2(1+Q_1^2) + p_1 \tag{52}$$

The stated boundary condition can now be expressed as

$$\sigma_r \Big|_{r=r_1} = -k\, u_1 \tag{53}$$

and, using Eqs. (48) and (52), it can be recast in the form

$$2C_1Q_1^2 + 2C_2(1+Q_1^2) + p_1 = -kr_1(1-Q_1) \tag{54}$$

The quantity k in Eqs. (53) and (54) is the constant of proportionality, called the radial stiffness of the casing. Equation (53) defines k ; it is the internal pressure needed to cause a unit increase in the case radius. The negative sign in Eq. (53) takes into account the fact that if the displacement u_1 , is positive, the associated radial stress will be negative.

As the only unknown in Eq. (54) is K, which enters p_1, Q_1 and r_1 implicitly, Eq. (54) constitutes an equation in K. Using Eqs. (35), (45) and (47), we may rewrite Eq. (54) explicitly in terms of K as

$$k(r_1-\rho_1)+2(C_1+C_2)\left[\ln\frac{r_2\sqrt{r_1^2+K}}{r_1\sqrt{r_2^2+K}}+\frac{K}{2}\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)\right.$$
$$+\frac{C_2}{8(C_1+C_2)^2}\left\{\frac{\gamma^2}{2}(r_1^2-r_2^2)+\beta\left[\ln\frac{\sqrt{r_1^2+\alpha^2}}{\sqrt{r_2^2+\alpha^2}}+\frac{r_2^2}{2(r_2^2+\alpha^2)}-\frac{r_1^2}{2(r_1^2+\alpha^2)}\right]\right.$$
$$\left.\left.-\frac{\delta}{2}\left(\frac{1}{r_1^2+\alpha^2}-\frac{1}{r_2^2+\alpha^2}\right)\right\}\right]=0 \tag{55}$$

where α is given by Eq. (36c) and β and δ are given by Eqs. (47b,c). The possibility of our being able to solve the problem hinges entirely on our being able to solve the above transcendental equation in K. Once K is determined, Q and w can be determined from Eqs. (31) and (39), r itself being determinate through Eq. (8). The quantity ρ' can be computed by using Eq. (15) and w' can be computed through the use of Eqs. (30) and (47a). The metric

tensors g_{ij} and g^{ij} can then be evaluated through Eqs. (4) and (6), respectively.

Other quantities of interest can be calculated easily also. Radial displacement, at $r = r_1$ for example, is easy to find through Eq. (48); that at $r = r_2$ can be found by using a similar obvious formula.

The physical stress components $\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}$ can be obtained through Eqs. (19), (29) and (50). The quantity p appearing in Eq. (19) can be obtained from Eq. (35). The constant F in Eq. (35) has to be evaluated as in Eq. (45).

Application of the Theory

To illustrate the use of the above theory, let us choose the following data:

Tensile modulus of the fiberglas case, $E_c = 5 \times 10^6$ psi

Poisson's ratio of the fiberglas case, $\nu = 0.2$

Case thickness, $t = 0.125$ in.

Initial internal radius of the case, $= 18.625$ in.

Initial internal radius of the core, $= 1.3, 7, 10$ ins.

Equilibrium tensile modulus for the core, $E_o = 600$ psi

Specific weight of the propellant, $= 0.063$ lbs/in^3

The three valued of ρ_2 correspond to different transverse sections of the rocket motor. The above data are the same as those used in the previous report. They are typical of rocket motor XM-57 of the Hercules Powder Company, Salt Lake City, Utah.

Before the theory of the preceding section can be applied, we must calculate the values of k, C_1, C_2. Assuming that the strains in the case will be infinitesimal, the results of the classical theory of elasticity can be applied to evaluate k using the following[1] equation:

$$k = \frac{E_c t}{\rho_1^2 (1 - \nu^2)} \tag{56a}$$

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[1] See, for example, Reference [4]

For the chosen data we have

$$k = 1877 \text{ psi/inch} \tag{56 b}$$

As regards the values of C_1 and C_2 to be used, we must indulge in some estimation for the lack of actual data based on finite deformation theory concerning the material in question. Since the Mooney expression, Eq. (28), must be valid for the infinitesimal strain regime also, it can be proven easily, from an analysis of a simple state of tensile stress, that the quantity $C_1 + C_2$ and the modulus of elasticity E_o are related by the expression

$$E_o = 6(C_1 + C_2) \tag{57}$$

For the assumed value of $E_o = 600$ psi, we have

$$C_1 + C_2 = 100 \text{ psi} \tag{58}$$

Now, if a reasonable ratio of magnitudes can be estimated, we can obtain values of C_1 and C_2 individually. Choosing $C_2 = \frac{1}{5} C_1$, for example, we obtain, from Eq. (58),

$$\begin{aligned} C_1 &= 83.3 \text{ psi} \\ C_2 &= 16.7 \text{ psi} \end{aligned} \tag{59}$$

We shall use these values in subsequent computations.

As it stands, Eq. (55) contains the following variables:

$$K, k, r, \rho_1, C_1, C_2, r_2, \gamma, \beta, \delta, \alpha$$

Out of these, ρ_1 and γ are given and k, C_1, C_2 are quantities calculated above from given quantities. The remaining variables, namely, r_1, r_2, α, β, δ, directly or indirectly, involve the unknown K alongwith other given quantities. Once a value of ρ_2 is chosen, all these quantities can be evaluated in terms of K. Equation (55) then would contain K as the only unknown. The equation was solved using IBM digital computers 1620 and 7090 for the three values of ρ_2 given above. Various quantities of interest were computed for a number of points through the core thickness. Calculations were repeated using factors of 10, 100 and 1000 to multiply the specific weight in order to study the effect of inertia forces under accelerations. The stiffness of the the case was changed through various multiples, namely, 10^{-6}, 10^{-3}, 10^{-1}, 10, 10^3 and 10^6, thus ranging almost from no case to a completely rigid case.

Discussion of Results

We shall refer to the problem data with k = 1877 psi/in and γ = 0.063 lbs/in^3 as the basic data and the corresponding problem, the basic problem. Let us define two dimensionless factors: M, the case stiffness factor, and N, the gravity factor. The first one indicates the ratio of the stiffness of the casing considered to the case stiffness in the basic problem; the second one indicates the ratio of the body force per unit mass considered to that in the basic problem. Introduction of these two factors will ease our discussion of the effects of changes in casing stiffness and body force. In addition, we shall refer to the common logarithms of M and N, namely, $\log_{10} M$ and $\log_{10} N$, as logarithmic case stiffness factor M' and logarithmic gravity factor N', respectively. Although there is nothing intrinsically fundamental about the basic problem, we choose to use this terminology for the sake of convenience.

The figures may be divided into three series. Figures 1-6 describe variation of w, u, σ_r, σ_θ, σ_z and τ_{rz} with radius for the case of M' = 0 and N' = 3. Figures 7-10 describe the dependence of the maximum values of u, σ_r, σ_θ and σ_z with M' for N' = 3. Finally, in Figs. 11-16 are plotted certain functions of w, u, σ_r, σ_θ, σ_z and τ_{rz} against the gravity factor N. In all the figures three curves are given corresponding to ρ_2 = 1.3,7 and 10 inches.

In Figs. 1-10 we have chosen a gravity factor of one thousand, which is a rather high value. However, as slump problems are particularly important under large gravity factors, we considered it desirable to demonstrate the results for such a large value of N. Besides, it was found that all the quantities were of rather small value for N = 1 and the second order effects associated with large strains were not too marked. A larger value of N was thus considered more appropriate for the purpose of demonstration of results. As Figs. 10-16 correlate the maximum values of various quantities for different values of N, only one value of N has been chosen in Figs. 1-10 for demonstration. It should be pointed out here that in practice, before applying the theory to such high values of N, one would be required to check the case and core strains

to see if the strains are within the limits of applicability of the linear theory for the case and the Mooney expression for the core.

The graphs of w versus ρ in Fig. 1, show the typical parabolic variation of the linear theory. As a matter of fact, the values of w from the finite strain theory agree with those from the linear theory of elasticity to within one percent, the agreement being even better for smaller values of N. An important conclusion is that, in general, if the axial displacements are the only quantities of interest, the nonlinear analysis is unnecessary, even for rather high values of gravity factor. This, however, is not so in Fig. 2 where u is shown as a function of ρ. We recall that according to the linear theory there should be no radial displacement at all. Thus the magnitude of u denotes the departure from the linear theory. We note that the displacement is inwards and increases rather rapidly towards the perforation of the core. Notice that the magnitude of u for a given value of ρ is the smallest for $\rho_2 = 10''$, the largest for $\rho_2 = 7''$, that for $\rho_2 = 1.3''$ being intermediate. This curious behavior can be seen to be plausible by observing the fact that compared with the case of $\rho_2 = 7''$, there is not enough material to cause inward slump in the case of $\rho_2 = 10''$, while in the case of $\rho_2 = 1.3''$, there is enough material to resist large inward slump. However, we should still consider $\rho_2 = 1.3''$ as the worst case as the magnitude of the maximum inward displacement is by far the largest in this case.

Figures 3 and 4 show the radial variations in the radial and circumferential stresses for $N' = 3$. As expected for a traction-free boundary, the radial stresses are zero at the bore. The core-case interface is, in general, not free of radial stress. The maximum radial stress does not occur at the outer core boundary, but occurs, instead, at some intermediate value of ρ. This value is closer to ρ_2, the smaller the value of ρ_2. It is surprising that the radial stresses in all cases are compressive, except for the case of $\rho_2 = 1.3''$ where a zone near the case is subjected to tensile stresses. The circumferential stress does not vanish at the inner boundary, although for $\rho_2 = 7''$ and $10''$, its magnitude is rather small at the bore. The stresses are compressive in the in the vicinity of the bore but tensile in the vicinity of the case. The

maximum values of the radial and circumferential stresses are of the same order of magnitude.

Axial stresses assume rather large values, as can be seen in Fig. 5. This is particularly noteworthy in view of the fact that the axial stress vanishes in the linear theory solution. We notice that the axial stresses are primarily tensile except for a rather narrow band in the vicinity of the bore. Their maximum values can be seen to be at the outer radius of the core.

The final figure of the series, Fig. 6, shows the radial variation of τ_{rz}. The values of τ_{rz} are almost, but not quite, equal to those obtained by the linear theory which gives values not more than one percent smaller. This is in agreement with what could be expected from very simple considerations. The shear stress at any radius can be calculated by uniformly distributing the weight of the core interior to a cylinder of that radius over the area of the curved surface of that cylinder. According to the linear theory the radius of the cylinder would be taken as the initial value ρ while the finite theory would take it as r. As ρ and r differ only slightly, the predictions of the theories are close. The largest shear stress is at the core-case interface as would be expected.

The next series of figures shows the manner in which the maximum magnitudes of various quantities, indicated by suffix "max", depend upon the case stiffness. As the axial displacement and the axial shear stress are almost insensitive to a change in M', their plots are not included. We observe in Fig. 7 that, for very small values of M', u_{max} decreases very slightly with increasing M', almost suddenly starts decreasing in magnitude at almost constant rate for intermediate values of M' and again starts decreasing very slowly for rather high values of M', until it vanishes for $M' = 6$. Thus, we observe distinct plateaus in the curves for low values of M' corresponding to soft cases and for high values of M' corresponding to stiff cases, the plateaus being joined by a smooth curve with u_{max} decreasing monotonically. The fact that the plateau for high M' corresponds to $u_{max} = 0$ is a consequence of the assumed incompressibility and plane

strain. An infinitely rigid case would permit no radial displacement at the core-case interface and to preserve the cross-sectional area, no radial displacement anywhere in the core. We notice again that for low values of M', u_{max} for $\rho_2 = 7''$ is the largest. The two-plateau behavior is again manifest in the remaining figures of the series, Figs. 8-10, for the maximum values of σ_r, σ_θ and σ_z.

The last series of figures, Figs. 11-16, show the variation of the maximum values of the displacements and stresses with the gravity factor. Whereas N' is taken as the abscissa, the ordinates are chosen to represent quantities of the type $\log_{10}\left|\frac{w_{max}}{N}\right|$. This is done with the following simple rationale in mind. Let V be a quantity which is dependent on N and vanishes when N does. If, now, the dependence of V on N is linear, then

$$V = aN$$

where a is a constant. Further,

$$\log\left|\frac{V}{N}\right| = \log |a| = b$$

where b is a constant also.
The graph of $\log \left|\frac{V}{N}\right|$ vs. N or log N should, hence, be a horizontal straight line. If it is known for a fact that V = 0 when N = 0, any departure from horizontality represents a departure from linearity in the V - N relation. Logarithms are used here to shrink the graphs. Absolute values are used to guarantee the existence of logarithms. As all the variables considered vanish for N = 0, the above reasoning applies to all of the Figs. 11 - 16.

We observe in Figs 11 and 16, respectively, that w_{max} and $(\tau_{rz})_{max}$ are linearly related to N as they are almost exactly the same as in the linear theory. The relation of all the other variables considered is nonlinear with respect to N.

In the case of Figs. 12 - 15, inclusive, we observe that although the plotted graphs are not horizontal, they are linear, or almost linear, for these cases. A little analysis shows that this implies that V and N are related by the power law

$$V = cN^{d+1}$$

where V is the variable in question (in this case, either u_{max}, $(\sigma_r)_{max}$, $(\sigma_\theta)_{max}$ or $(\sigma_z)_{max}$) and c and d are constants, d being the slope of the graph of log $\left|\frac{V}{N}\right|$ vs. log N. As for all the latter figures d $\approx$ 1, the V - N relation can be approximated by

$$|V| = cN^2,$$

which is a useful result in arriving at values of V for a given N from the knowledge of the value for another N.

Acknowledgments

The research was carried out under the supervision and guidance of Professor J. H. Baltrukonis.

Thanks are acknowledged to Mr. Edward Magrab for direction and supervising the computational aspects of the problem, Mr. Myoung Sohn for carrying out the computations and plotting the graphs, and Mr. Gerald Smith for drawing the graphs in ink. Thanks are also due to Mrs. Evelyn Mead and Miss Sara McCornack for typing this report and Mr. Amir Tuteja for entering the equations.

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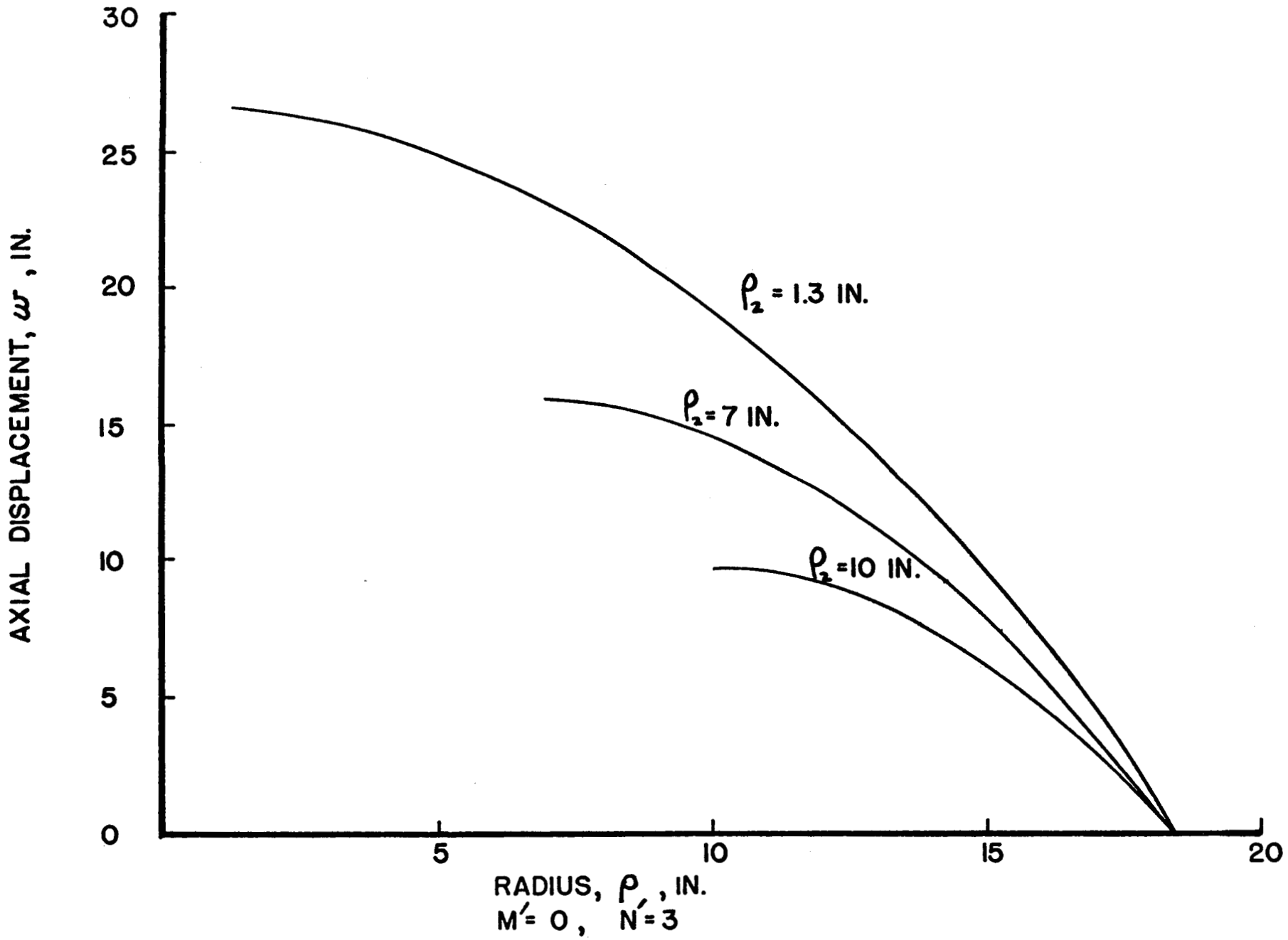
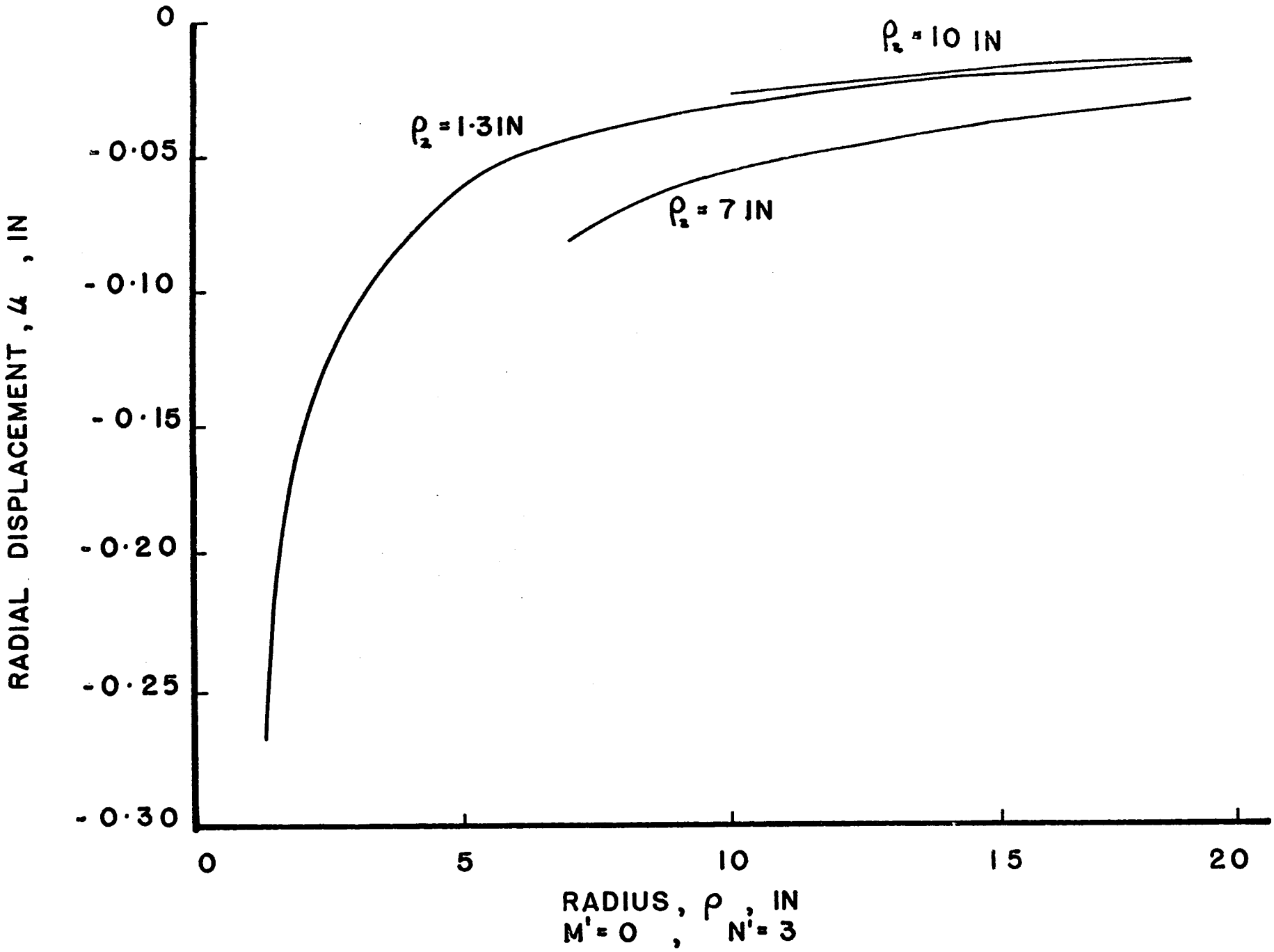
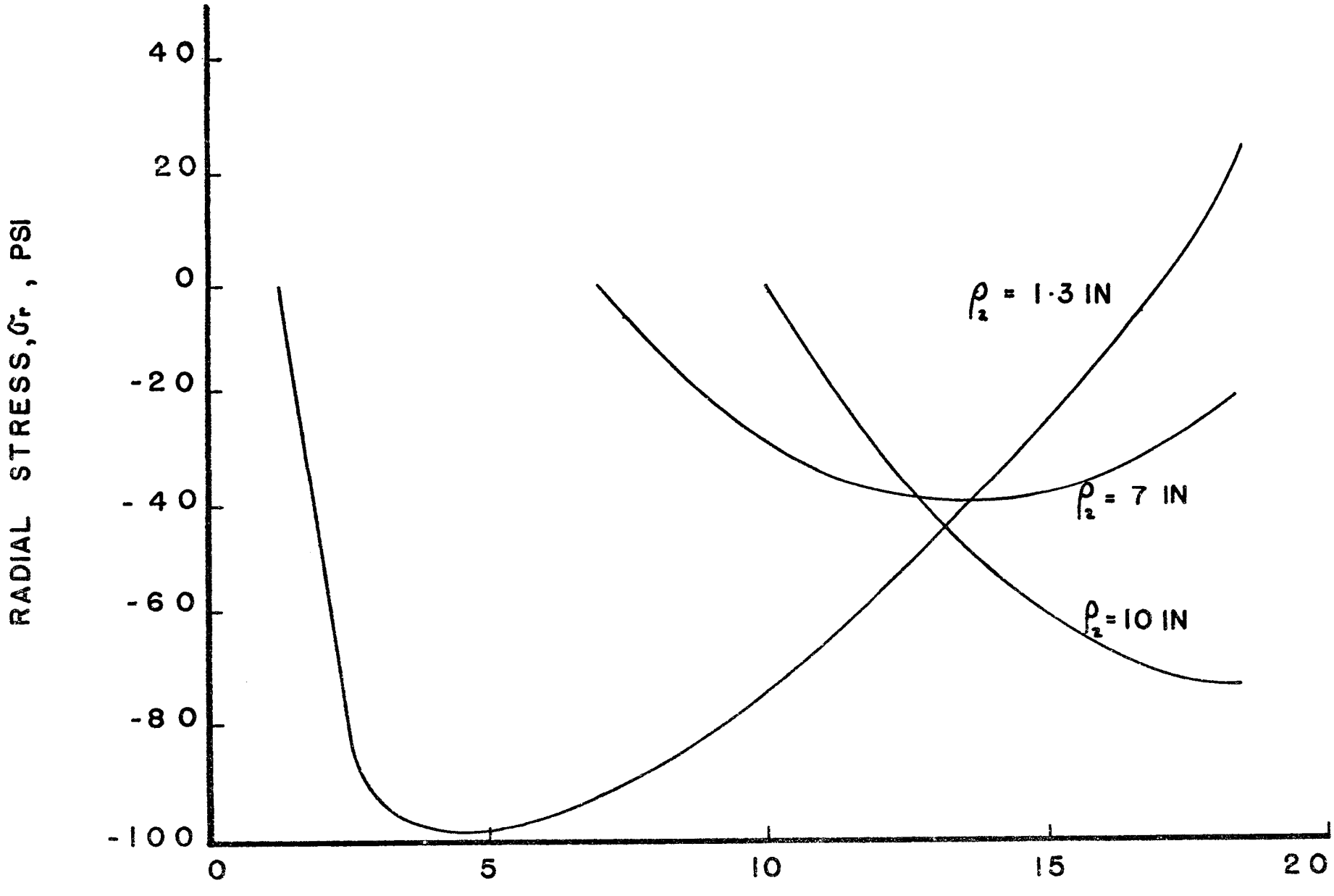


FIG. 1 AXIAL DISPLACEMENT AS A FUNCTION OF RADIUS

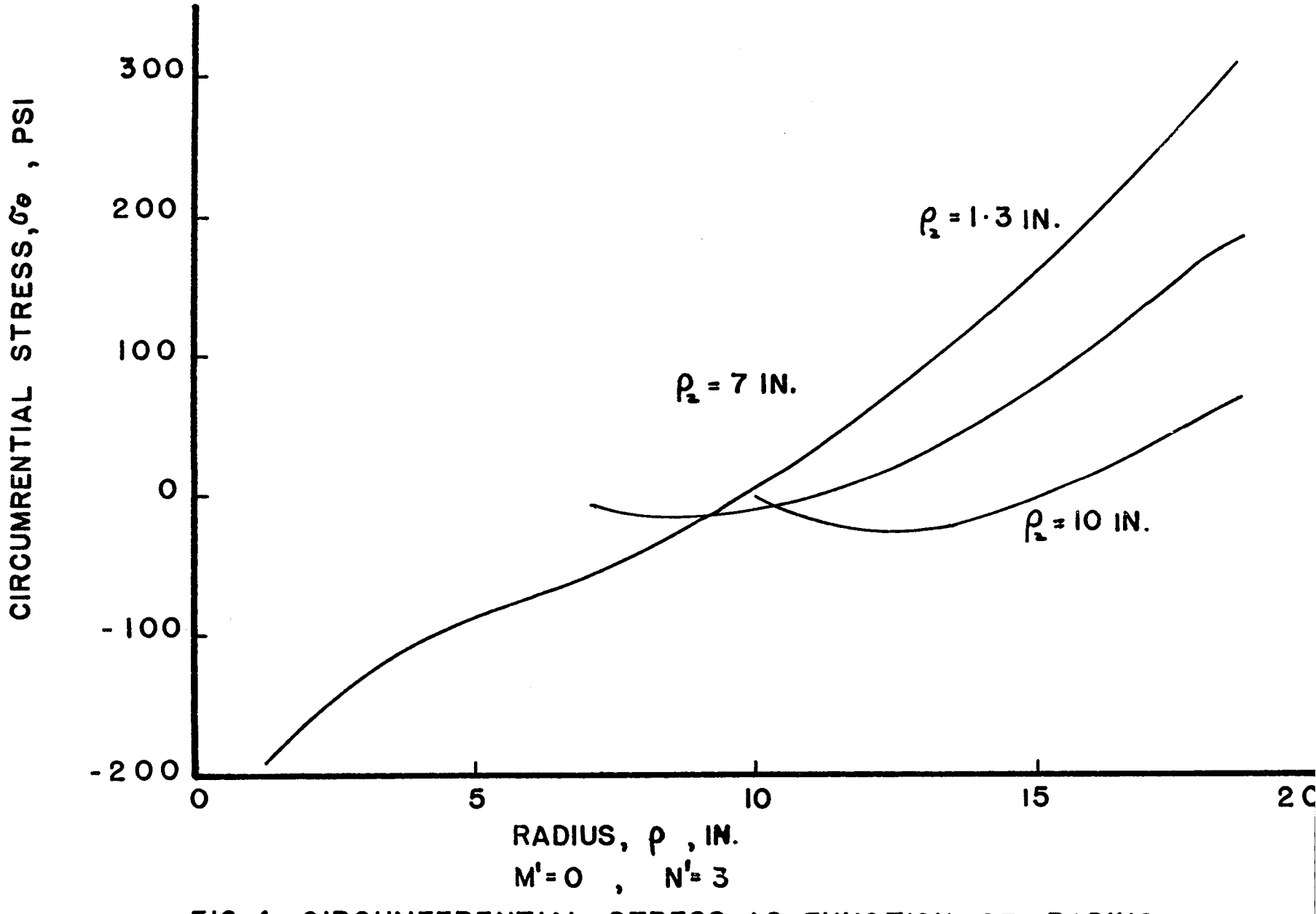


FIG· 2 RADIAL DISPLACEMENT AS FUNCTION OF RADIUS

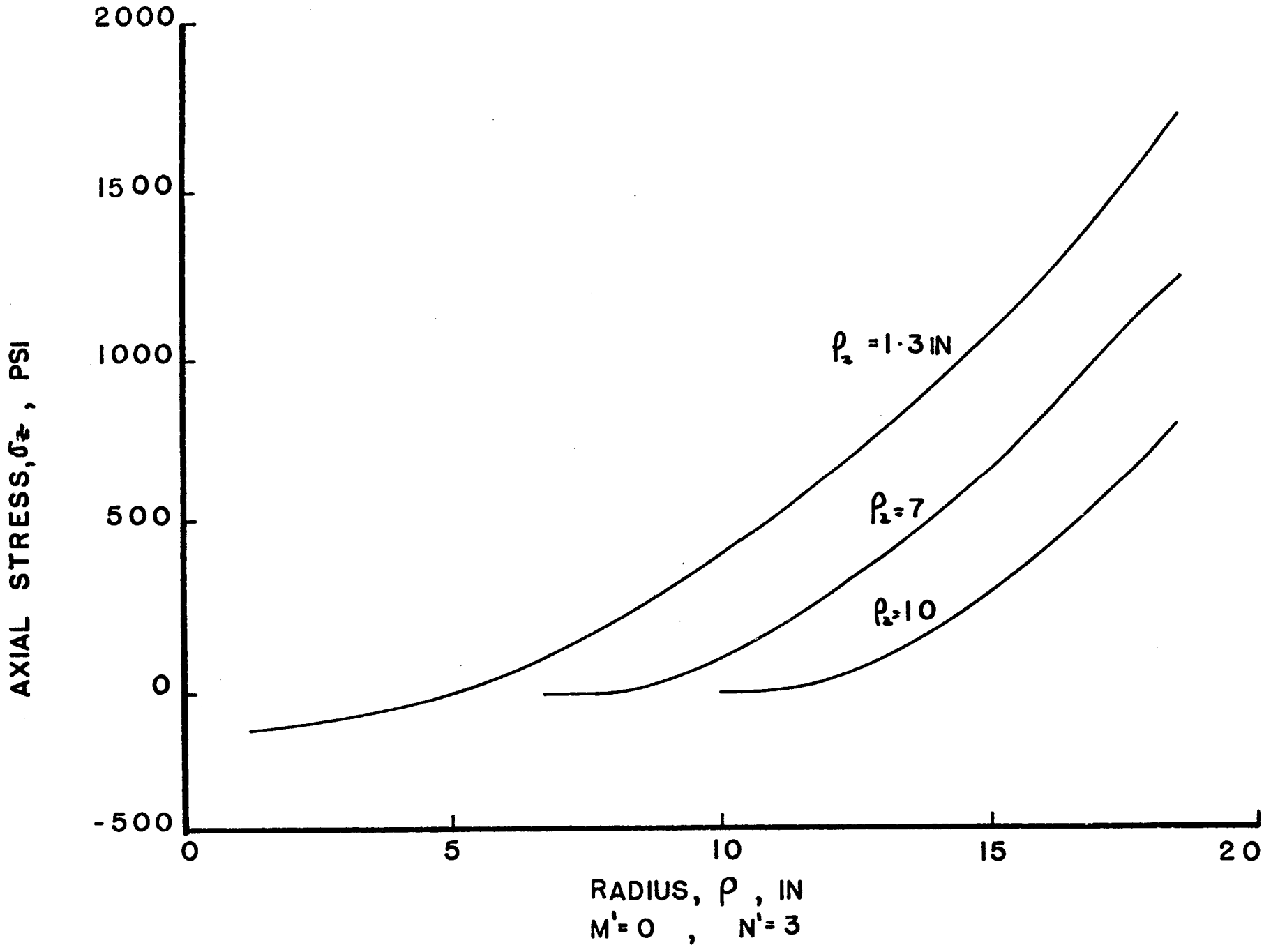


$M' = 0$, $N' = 3$

FIG·3 RADIAL STRESS AS FUNCTION OF RADIUS



FIG·4 CIRCUMFERENTIAL STRESS AS FUNCTION OF RADIUS



FIG· 5 AXIAL STRESS AS FUNCTION OF RADIUS

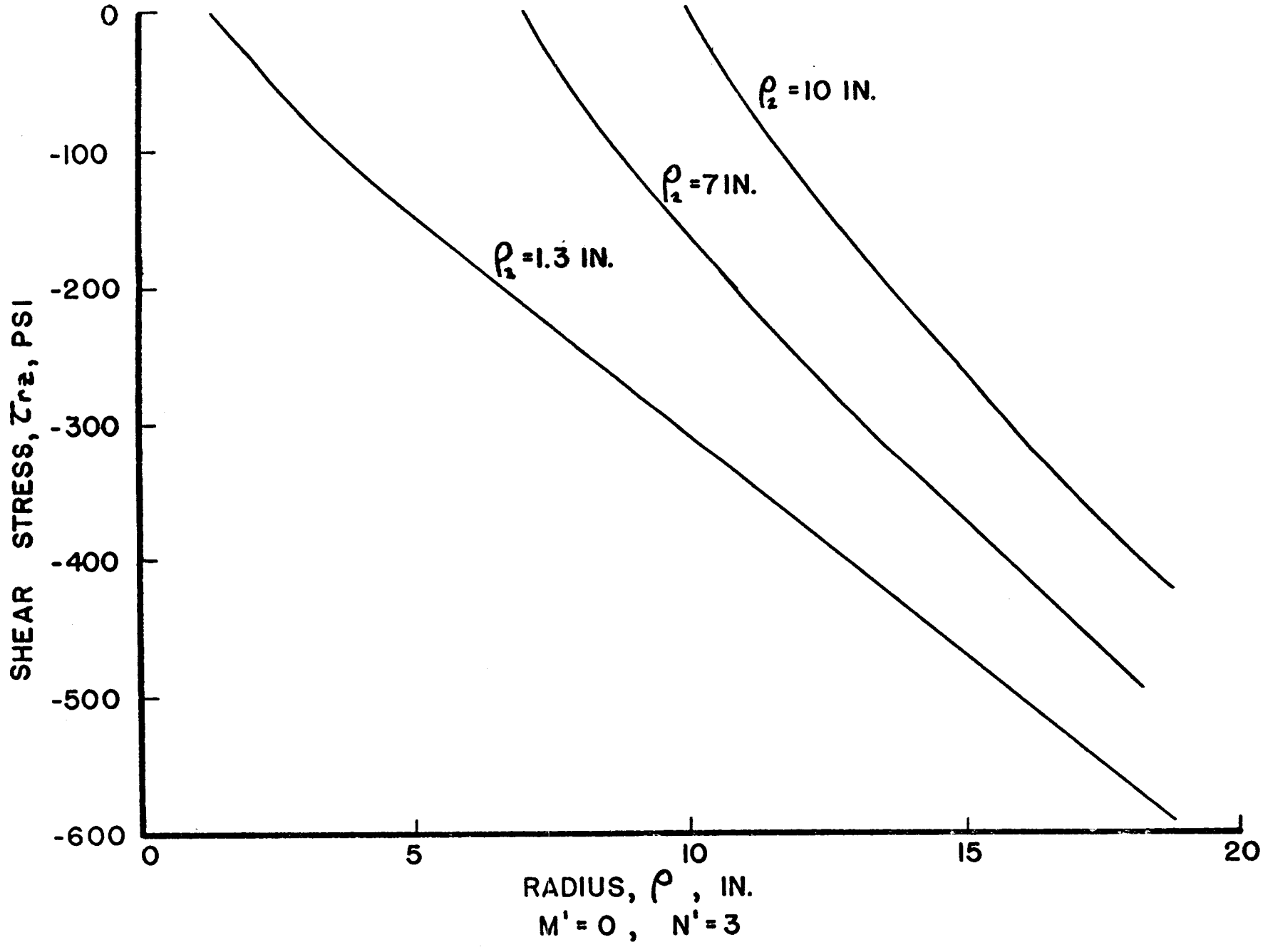


FIG. 6 SHEAR STRESS AS A FUNCTION OF RADIUS

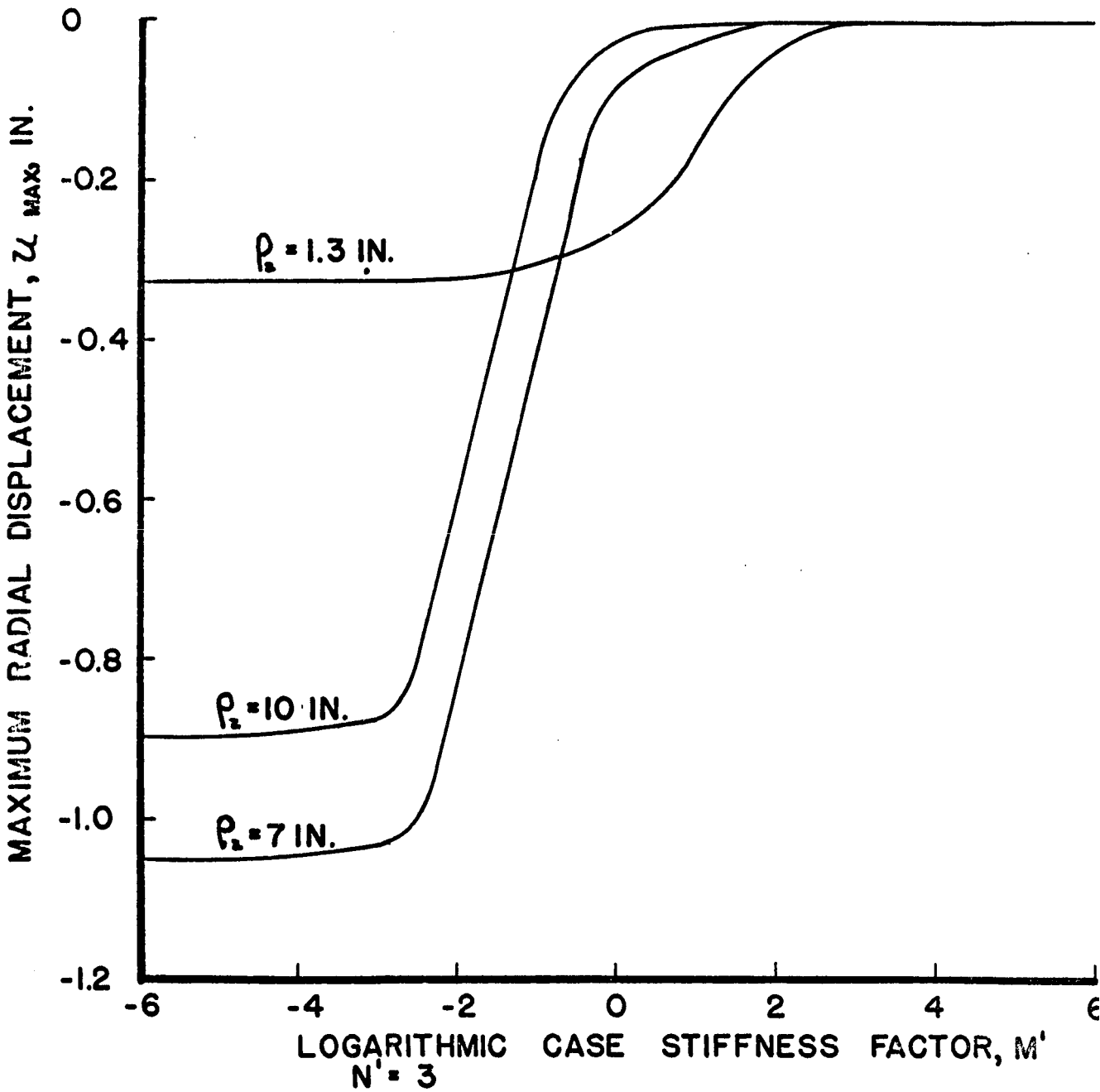


FIG. 7 VARIATION OF MAXIMUM RADIAL DISPLACEMENT WITH LOGARITHMIC CASE STIFFNESS FACTOR FOR LOGARITHMIC GRAVITY FACTOR EQUAL TO THREE

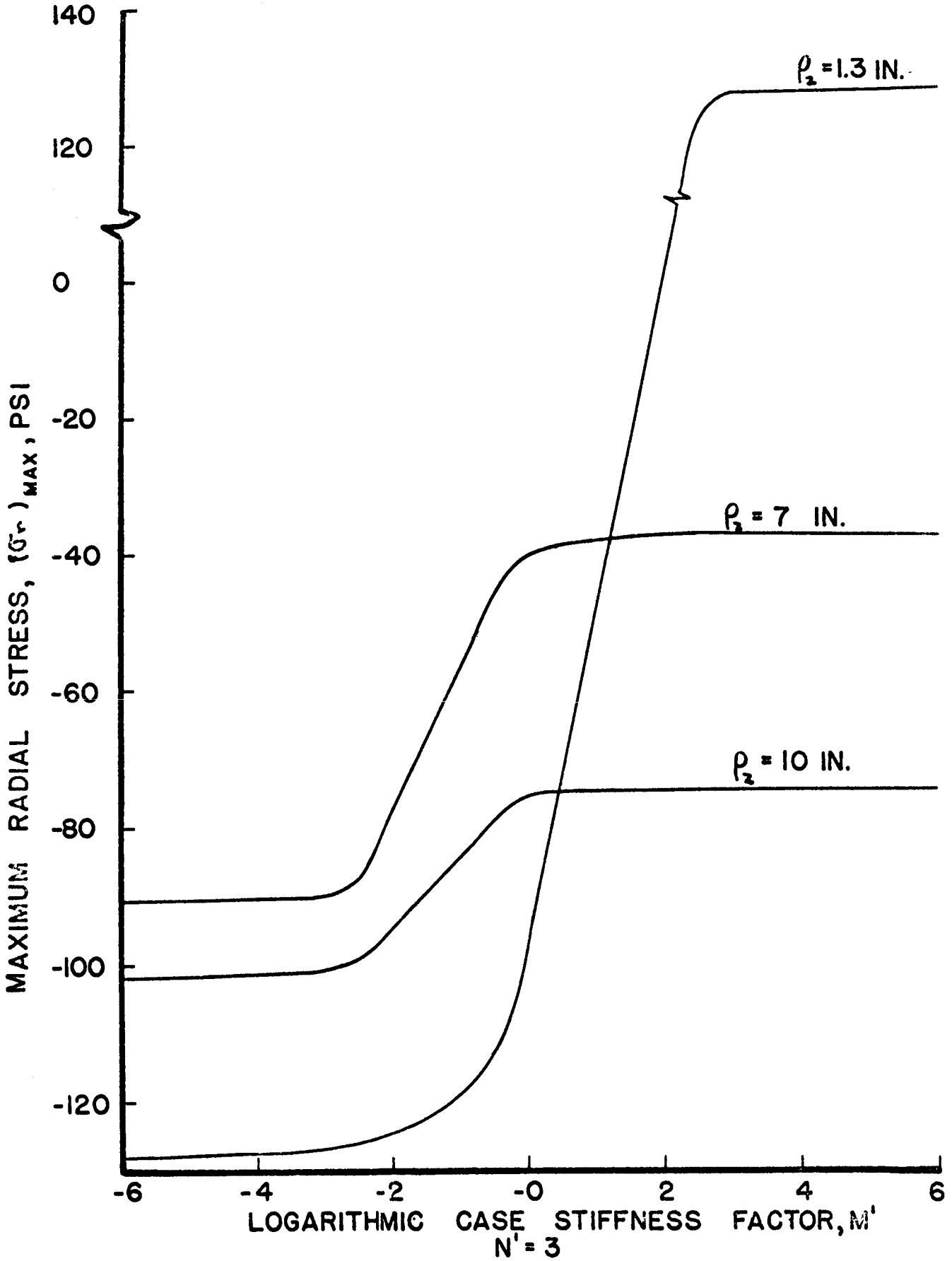


FIG. 8 VARIATION OF MAXIMUM RADIAL STRESS WITH LOGARITHMIC CASE STIFFNESS FACTOR FOR LOGARITHMIC GRAVITY FACTOR EQUAL TO THREE

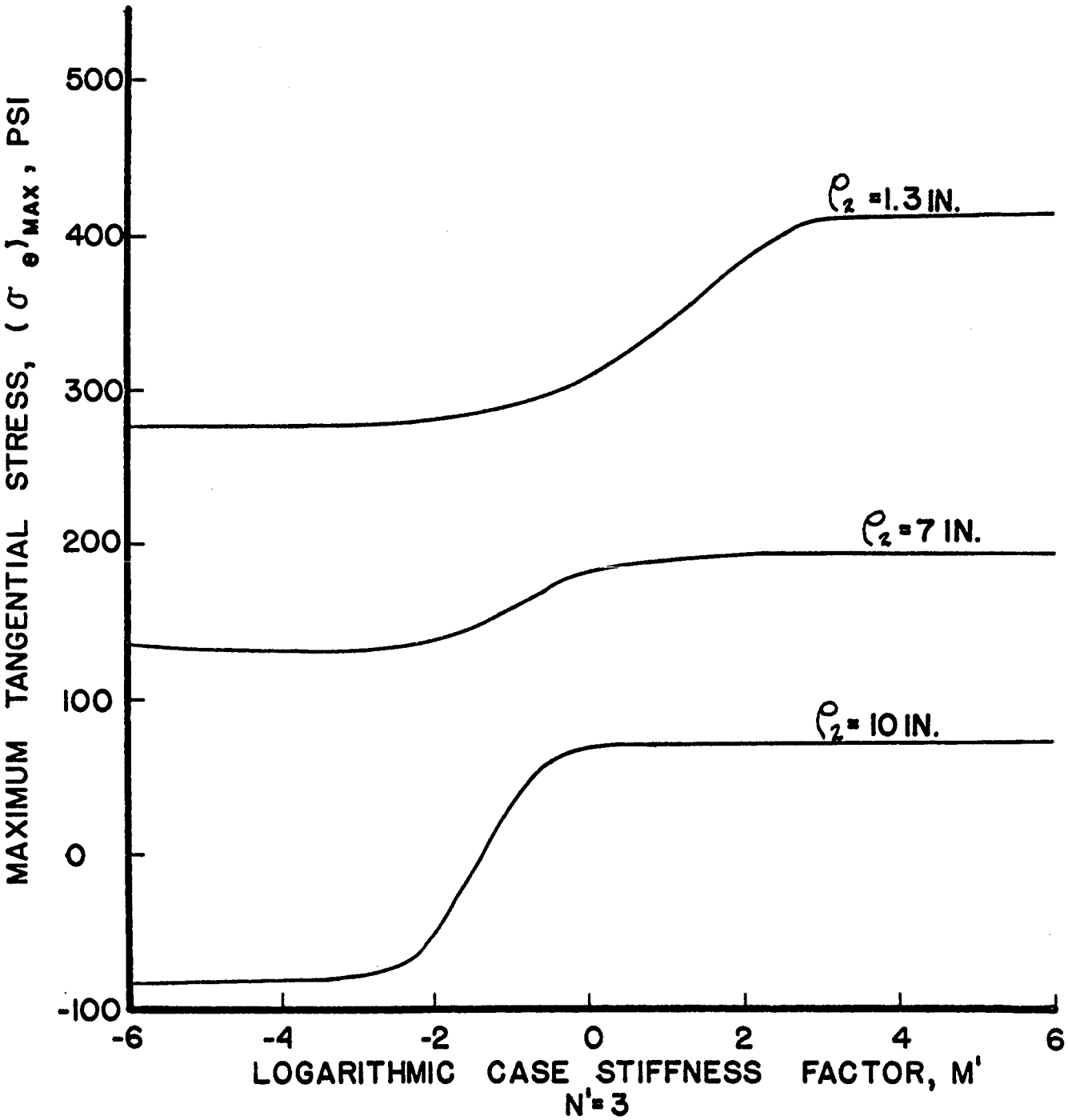


FIG. 9 VARIATION OF MAXIMUM TANGENTIAL STRESS WITH LOGARITHMIC CASE STIFFNESS FACTOR FOR LOGARITHMIC GRAVITY FACTOR EQUAL TO THREE

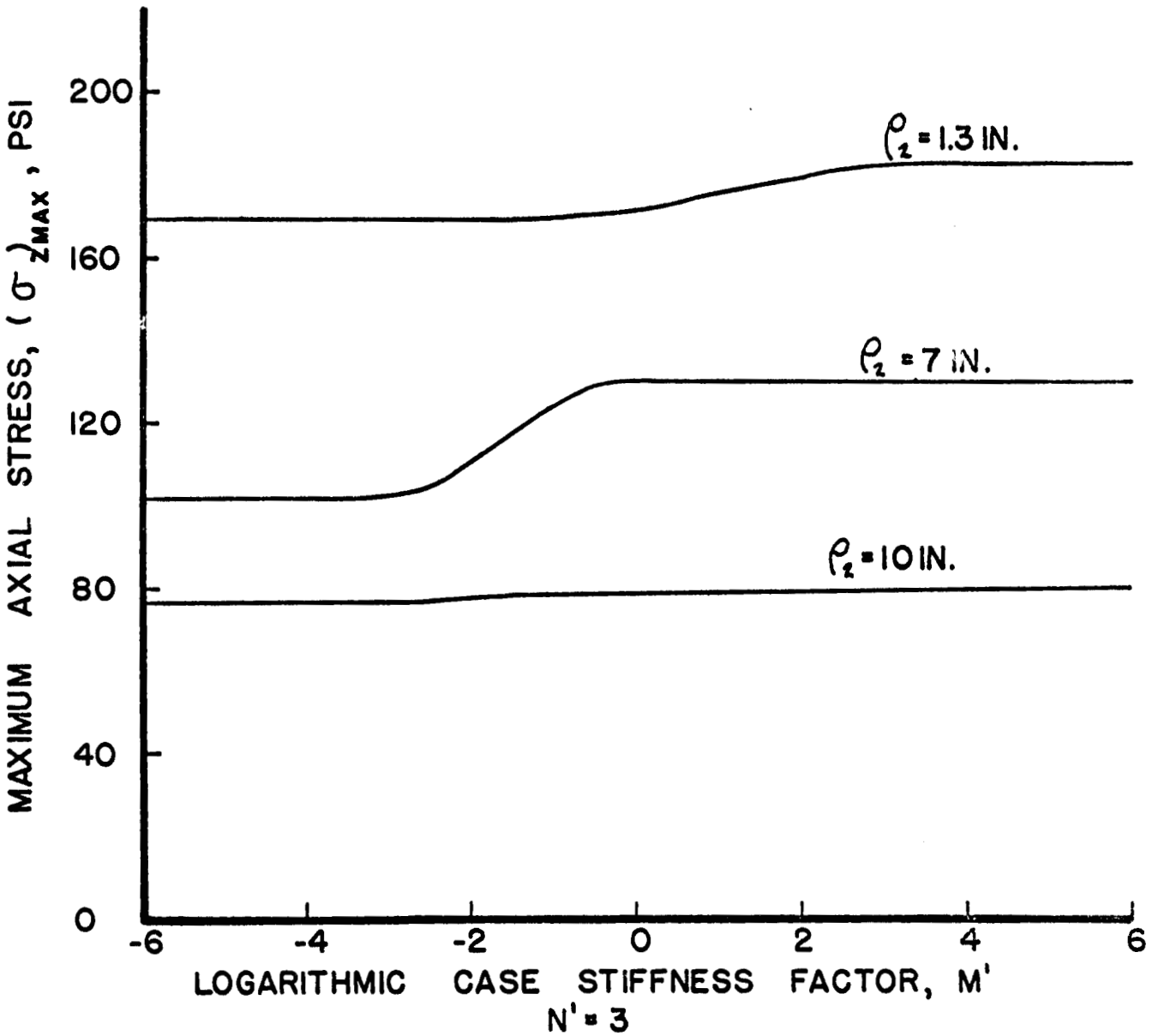


FIG. 10 VARIATION OF MAXIMUM AXIAL STRESS WITH LOGARITHMIC CASE STIFFNESS FACTOR FOR LOGARITHMIC GRAVITY FACTOR EQUAL TO THREE

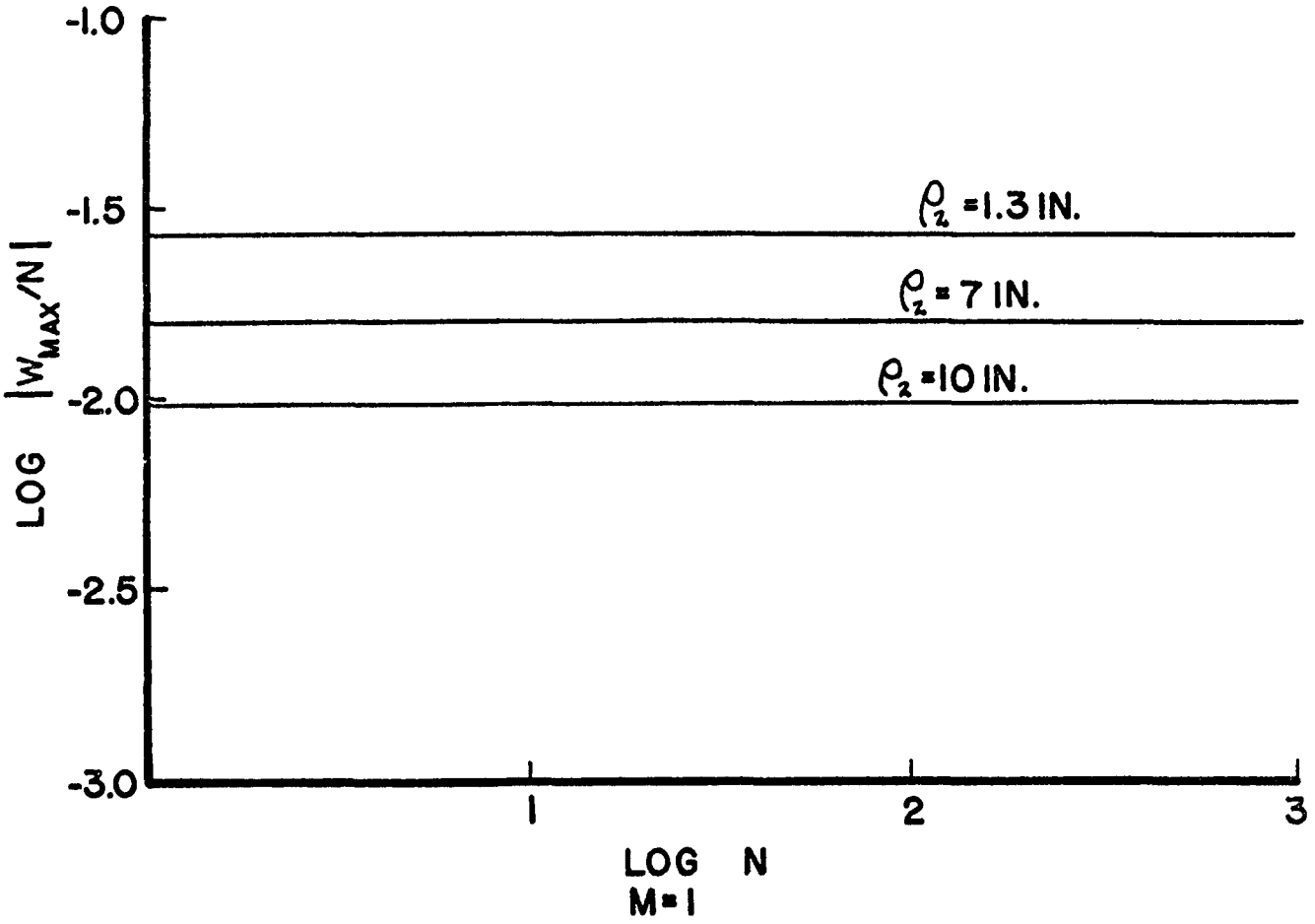


FIG. 11 LOGARITHMIC PLOT OF $|W_{MAX}/N|$ VS. N FOR M=1

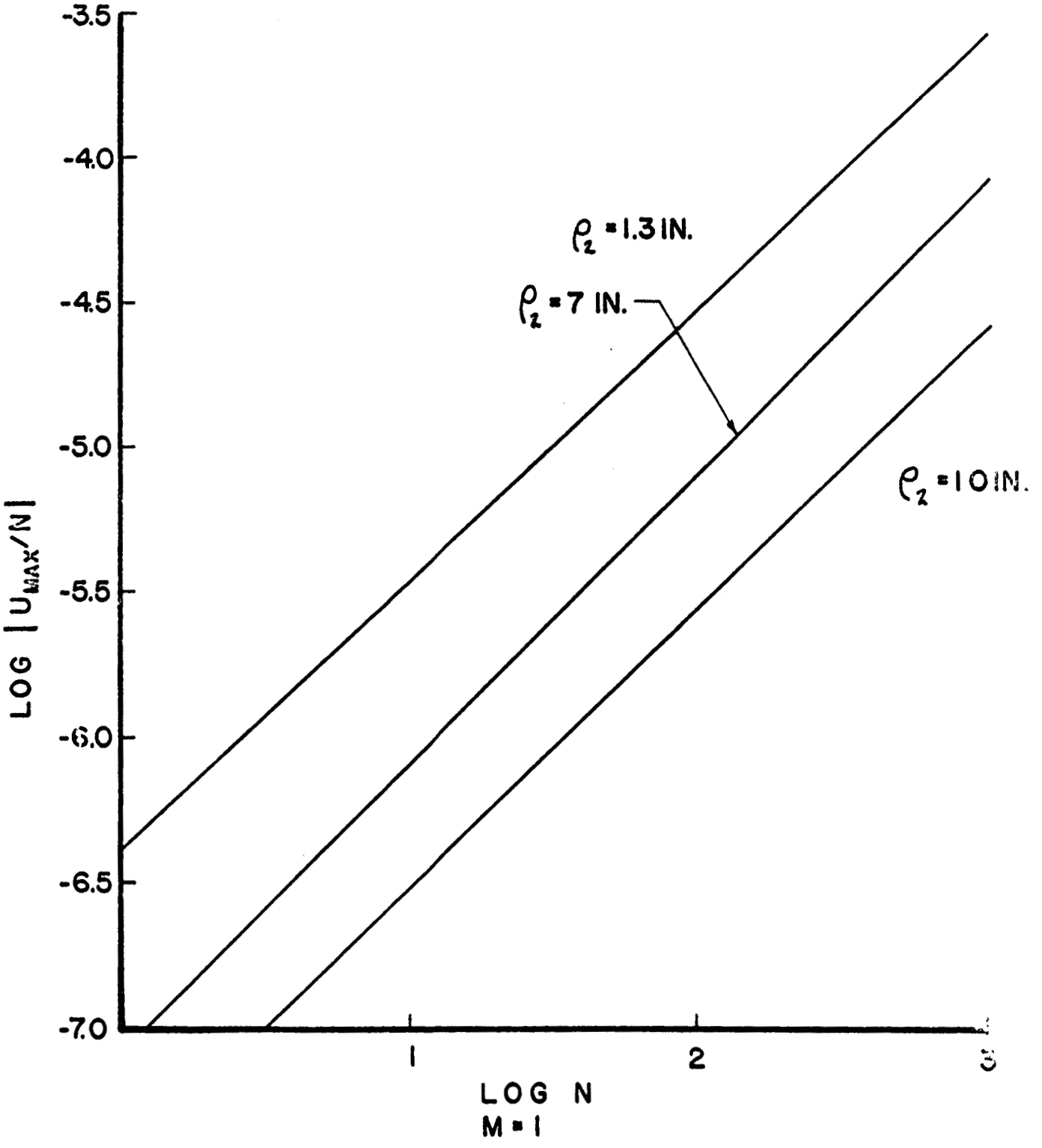


FIG. 12 LOGARITHMIC PLOT OF $|(U_{MAX})/N|$ VS. N FOR M = 1

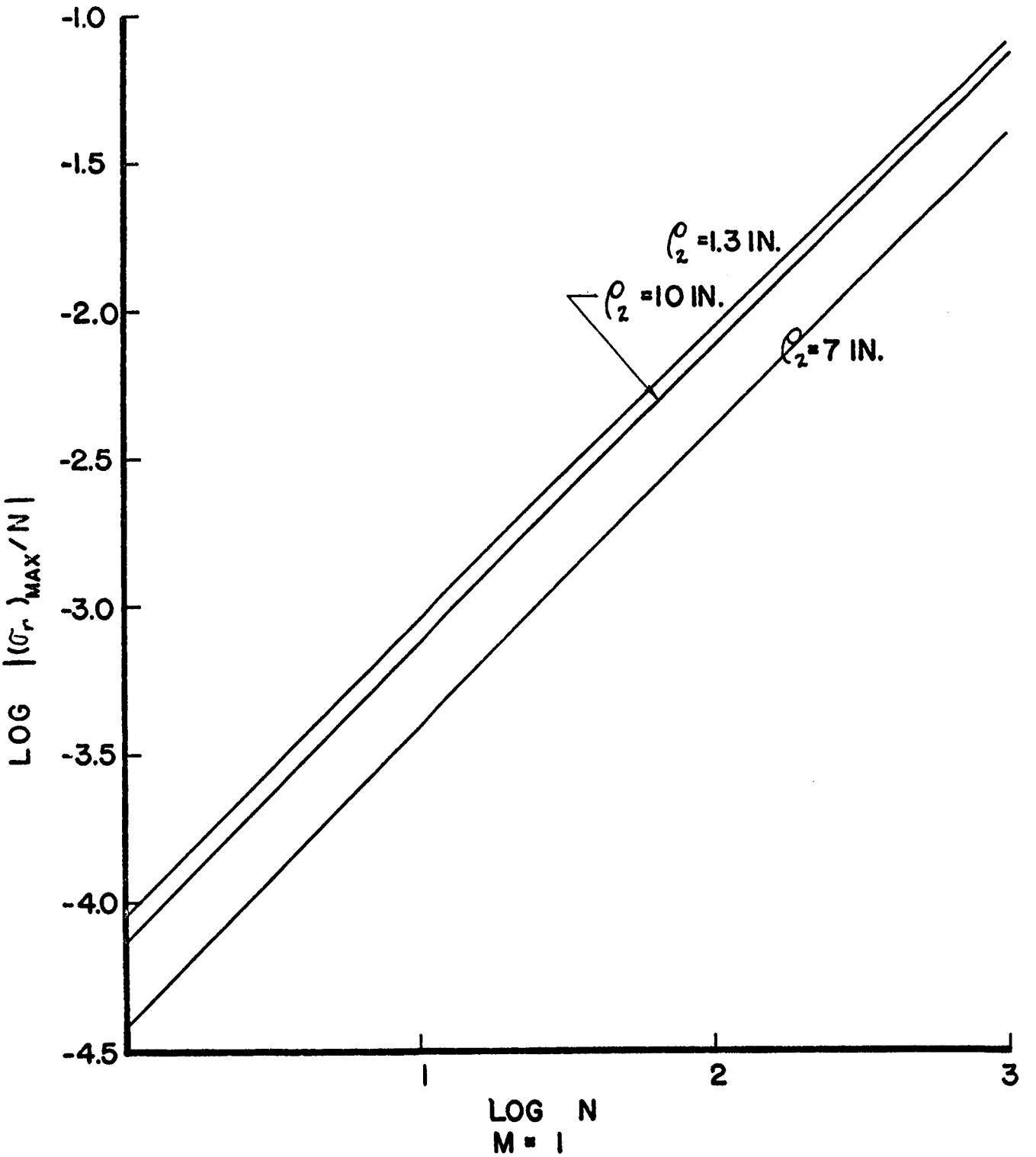


FIG. 13 LOGARITHMIC PLOT OF $|(\sigma_r)_{MAX}/N|$ VS. N FOR M=1

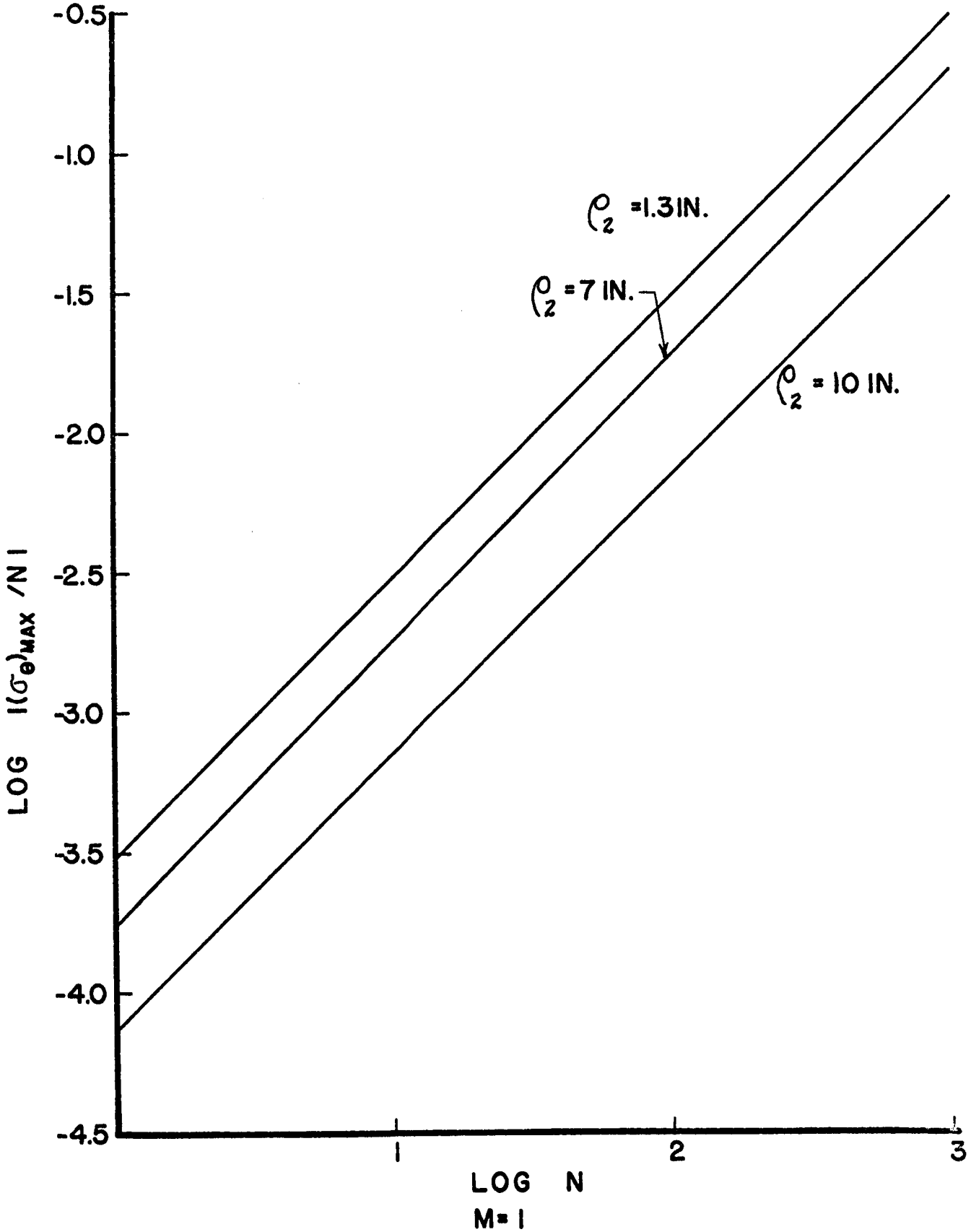


FIG. 14 LOGARITHMIC PLOT OF $|(\sigma_{\theta})_{MAX}/N|$ VS. N FOR M= 1

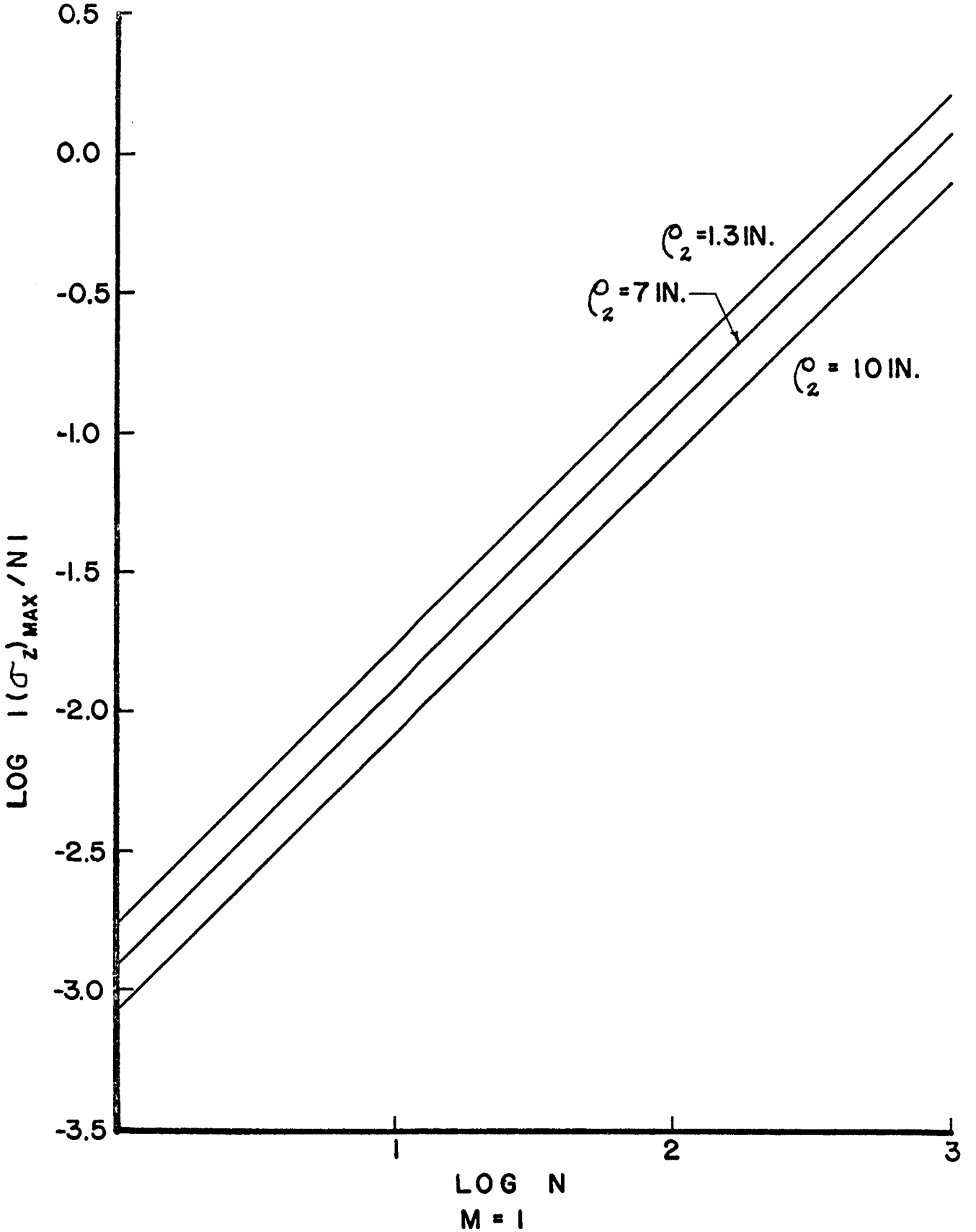


FIG. 15 LOGARITHMIC PLOT OF $I(\sigma_z)_{MAX}/N I$ VS. N FOR M=1

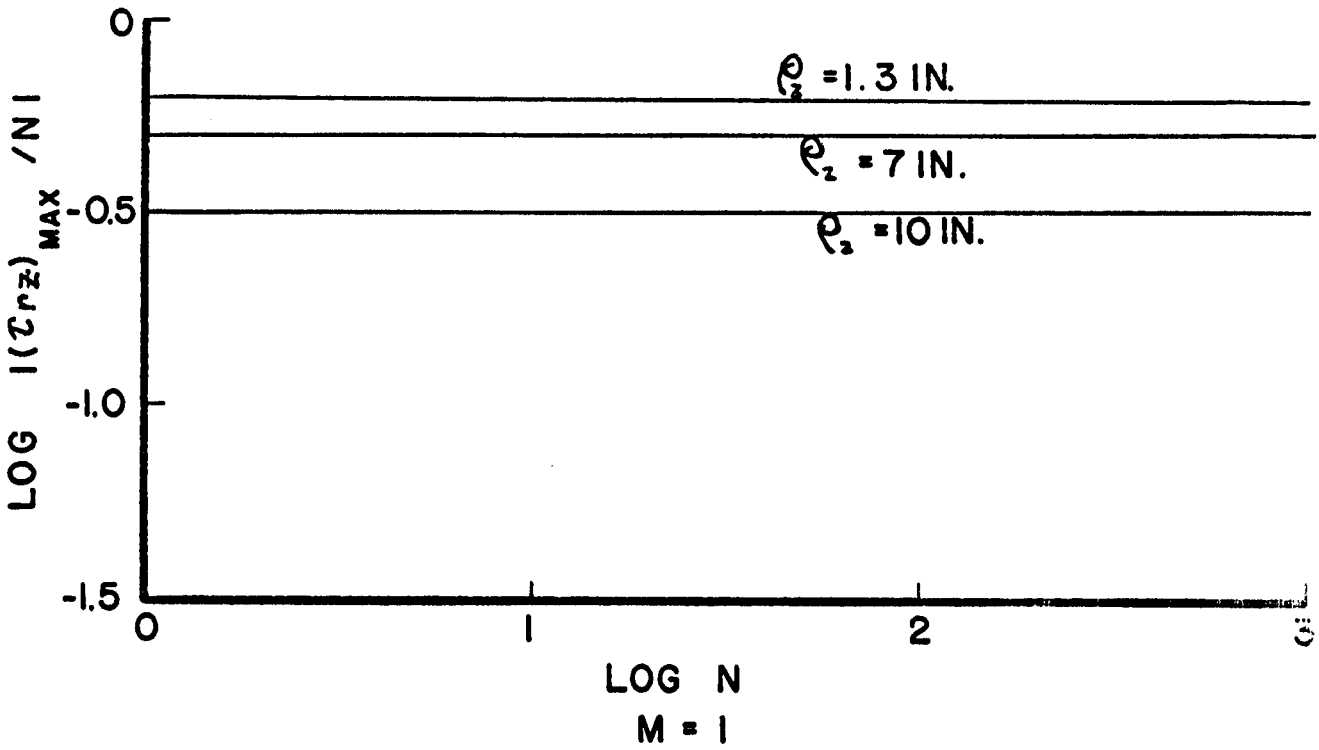


FIG. 16 LOGARITHMIC PLOT OF $|(\tau_{rz})_{MAX}/N|$ VS. N FOR M = 1