ILLINOIS INSTITUTE OF TECHNOLOGY
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DEPARTMENT OF MECHANICAL AND AEROSPACE ENGINEERING

Ref: SC-NsG-561

30 June, 1965

Office of Grants and Research Contracts
National Aeronautics and Space Administration
Washington 25, D.C.

Attn: Code SC

Gentlemen:

This is written in compliance with instructions contained in your letter dated 30 March 1964 which initiated a two-year study entitled "A Study of Nonlinear Stability of Motion of Finite Satellites". It may be considered to provide a complete resume of efforts to date, both those pertaining directly to this study and also others which will give an impression of the breadth of the evolving program of work which can be classified under the general heading of astrodynamics at the Illinois Institute of Technology. The primary role of the referenced NASA grant will be evident: related research topics, the emphasis of study programs of NASA and other predoctoral trainees, closer collaboration with program-oriented university groups as well as space-science agencies and industries, invaluable enhancement of useful exchanges on the international level, -- all can be directly traced to the initial support in this area provided by the present grant.

The work herein reported constitutes fourteen months of effort from notification date 30 March 1964 to 31 May 1965, including a no-cost time extension.

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The effort may be conveniently traced in chronological form. The original viewpoint was with regard to problems of attitude stabilization of artificial satellites, a problem of personal research interest to the undersigned for roughly two years before NASA or other research sponsorship was sought and received. Several months before the present grant became a fact, technical discussions between Dr. Wilson and the undersigned raised questions of the dynamical analogy with lunar librations and the known serious deficiencies with that theory. Dr. Wilson is to be thanked for an introduction to other astronomers deeply concerned over this problem who were anxious to secure an improved theory as basis for ephemeris calculations. In the succeeding months of waiting for the NASA grant to become official, undersigned devoted considerable attention and effort to assisting by formulating a more accurate librations theory, on a purely voluntary basis. Subsequently the undersigned, who had already half a dozen publications in this field, has cooperated in like manner with astronomers and space physicists having a broad range of interests but all dependent on an improved librations theory. The U.S. Naval Observatory, most directly and deeply concerned, later initiated a formal program in this area on a small scale and for a limited time, which has nonetheless provided invaluable aid in access to specialized literature and occasional discussions with astronomer specialists.

It is appropriate at this point to recall that librations theory is a very old problem which strained the resources of the greatest celestial mechanicians of all time -- it is physically recognized as an extended form of the three-body problem, in which one body is of finite extent. Lagrange and Laplace gave separate linearized analyses which cannot be reconciled with each other, nor the latter with current theories developed for artificial satellites. It is the undersigned's main contribution to this field to have demonstrated that the universally accepted theory of Laplace is seriously incomplete, and that Lagrange's study forms a more suitable starting point for a quantitatively useful theory. The crucial importance of accurate stability criteria for an understanding of physical librations has justified the deviation from originally planned direction of effort, in order to construct an improved nonlinear librations theory. By this means the original goal of elucidating the nonlinear stability theory has been materially aided (for more complete discussion, see Enclosure 7), so that future effort will deal directly with this question in a manner that aids most importantly the astronomical problem as a special case.

Documentation of accomplishment is represented in part by the several enclosures, in NASA report form, journal reprints form, abstract of paper to be presented at XVIth International Astronautical Congress, and a lengthy and detailed analysis of the dynamics of a librating rigid body in general orbital motion. This nonlinear treatment, which will serve as a base for

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the accurate stability discussion, is not for publication yet or for general circulation. As the enclosures speak for themselves, no further discussion here is required. Two other fundamentally related but entirely distinct efforts which are not documented with enclosures deserve brief mention. The first one carries into tidal theory the same correction required in Laplace's libration study; this work has been submitted for journal publication. The second is a re-examination of a question posed by Dr. Wilson himself, dealing with the motion of ecliptic nodes on the invariable plane. After special search and calculation, undersigned was led only to negative findings, warranting no addition to literature at this time. All of the effort described above may be attributed to NASA grant, with the exception of early librations analysis for ephemeris purposes performed for USNO.

Contemporary studies are closely described by the most recent of the enclosures, which is the abstract of an invited paper to be delivered at the above-mentioned Congress (Enclosure 8), as well as analysis of rigorous stability of motions in the vicinity of relative equilibria in rotating systems.

A related study has been initiated on a small scale with the support of the National Science Foundation and full information of other agencies, which ought to be mentioned. This attempts to complement the purely dynamical approach to the NASA-supported study of the Moon as an object of stability analysis, by following the lead of the physical chemists who deal with cosmic origins. Specifically, Lyapunov methods are also appropriate for analysis of bodies with the discontinuous density variations that are believed to exist. It is hoped that the totally different viewpoints will eventually be merged in a satisfactory knowledge of lunar structure. Although this auxiliary and inter-disciplinary effort was originally intended to provide support for a pre-doctoral student, it happens that a pre-doctoral fellow has been engaged in this work at no cost to the program which is therefore permitted to function for a longer period than the one year that was intended.

On the initiative of the USNO and Air Force cartographers and geodescists, discussions have been held to consider how specific needs of the military agencies may be related to studies now underway at IIT. Although large-scale computation programs appear to be more appropriately referred to commercial computing facilities with scientific liaison, more direct research and training collaboration at IIT is contemplated for selected military personnel.

With regard to content of future effort, as well as its duration and cost, this is to state principal investigator's belief that no substantial change is required in the statement of the formal proposal earlier submitted

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for the entire two-year program. The IIT business office is being asked to furnish you with the financial report which you require.

Yours very truly,

Irving Michelson

Irving Michelson
Professor of Aerospace Engineering.

IM:jh
Enc.

1. Nonlinear Analysis of Physical Librations. I. New Formulation and Equilibrium Orientations Deduced From an Integral of Motion, R.L. Duncombe and I. Michelson, NASA Contractor Report CR-168, February 1965.

2. Reply by Author to T.R. Kane and P.W. Likins (Continuation of "Equilibrium Orientations of Gravity-Gradient Satellites" I. Michelson, AIAA Journal Vol. 2, No. 7, 1357 (1964).

3. Euler Angles for Libration Analysis, Irving Michelson, AIAA Journal Vol. 2, No. 8, 1469 (1964).

4. Libration Damping and Lunar Mass, Irving Michelson, AIAA Journal, Vol. 3, No. 1, 172 (1965).

5. Theory of Lunar Librations Re-Examined, Irving Michelson, Astronomical Journal, Vol. 70, No. 2, March 1965.

6. Abstract to Symposium on Celestial Mechanics, Yale University, February 1965 (No Written Copy Available).

7. Lunar Librations Theory -- Critique and Reformulation, I. Michelson, 1965.

8. Deductions from Nonlinear Analysis of Satellite Physical Librations, Irving Michelson, Abstract of Paper to be Presented at the XVIth International Astronautical Congress, Athens, Greece, September 18, 1965.

Encl. 2

N65-38285

Reply by Author to T. R. Kane and P. W. Likins

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ALTHOUGH the result (5)† was obtained by direct integration of the approximate differential system (1–3), a more fundamental demonstration may be preferred by readers who are unfamiliar with analogous conclusions in celestial mechanics. The fact that an integral of energy is available leads at once to a complete map of equilibrium orientations in angular coordinate space, as shown below. A further refinement is the assignment of accurate limits to regions of possible equilibrium orientation, by recourse to the full nonlinear dynamical formulation. The argument that follows is motivated in part by Hill's deductions based on the use of Jacobi's integral in the restricted problem of three bodies.

The usual set of detailed libration equations (1–3) exhibits the motion as a conventional sixth-order differential system, there being three degrees of rotational freedom. A better point of departure is available, however, whenever the Lagrangian function is not an explicit function of time. This is just the case for circular orbits, and an integral of energy is written at once as

$$\dot{\alpha}\frac{\partial L}{\partial \dot{\alpha}} + \dot{\beta}\frac{\partial L}{\partial \dot{\beta}} + \dot{\gamma}\frac{\partial L}{\partial \dot{\gamma}} - L = \text{const} \qquad (6)$$

where L denotes the Lagrangian function. Equation (6) is then the complete expression of the dynamical (e.g., Hamiltonian) principle, and being of third order only, is advantageous for present purposes, as compared with (1–3). It is noted that condition (4), earlier obtained by three integrations of the original system, is also of third order, conforming with (6).

By setting the angular velocities all simultaneously equal to zero, $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$, (6) becomes the defining relationship for "permanent configurations" here sought as librational equilibrium orientations. Configuration coordinates are then given by (6) as a function of total system energy. Owing to the presence of gyroscopic terms in the kinetic energy, the left member of (6) takes a form slightly different from the potential energy itself, giving finally the equation

$$(3a_3^2 - a_1^2)A + (3b_3^2 - b_1^2)B + (3c_3^2 - c_1^2)C = \text{const} \qquad (7)$$

The symbols $a_1, \ldots, c_3$ are introduced as principal axis (x_1, x_2, x_3) direction cosines, determined by the ordered sequence of finite rotations α, β, γ starting from the reference axes (X_1, X_2, X_3) themselves rotating with orbital angular speed Ω; in matrix form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \qquad (8)$$

The more accurate and complete form (7) of the equations defining equilibrium orientations is both an extension of (5) and its rigorous justification. When no restriction is made on the principal moments A, B, C, (7) is a transcendental equation in all three angular variables α, β, γ. In the trivial limiting case of complete dynamical symmetry, $A = B = C$, the basic property of direction cosines reduces the left member to a constant: independence of α, β, γ then shows that all orientations are possible permanent configurations. Intermediate degrees of dynamical symmetry evidently determine equilibrium zones of lesser extent, but nevertheless infinite in number of points (i.e. orientations) represented. Generalization to elliptic orbit in the first instance entails the loss of energy integral (7), but standard canonical transformation and perturbation procedures are then applicable and well suited to orbits of moderate eccentricity. It is intuitively clear that attendant reduction of equilibrium region increases with orbit eccentricity. Total degeneracy of the equilibrium zone into discrete points and corresponding discrete equilibrium orientations has been demonstrated by J. L. Synge by taking the fictitious limit of zero moment of inertia in the earth-pointing direction $C = 0$, as indicated in Ref. 1.

For the sake of completeness, the explicit forms of kinetic and potential energies and the relevant direction cosines are tabulated as follows:

$$2T = A(\dot{\alpha}\cos\beta\cos\gamma + \dot{\beta}\sin\gamma + a_1\Omega)^2 + B(-\dot{\alpha}\cos\beta\sin\gamma + \dot{\beta}\cos\gamma + b_1\Omega)^2 + C(\dot{\alpha}\sin\beta + \dot{\gamma} + c_1\Omega)^2 \qquad (9)$$

$$2U = 3\Omega^2[Aa_3^2 + Bb_3^2 + Cc_3^2] \qquad (10)$$

$$\begin{aligned} a_1 &= \cos\beta\cos\gamma & a_3 &= \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ b_1 &= -\cos\beta\sin\gamma & b_3 &= \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma \\ c_1 &= \sin\beta & c_3 &= \cos\alpha\cos\beta \end{aligned} \qquad (11)$$

Portions of the potential which are independent of angular orientation have been omitted from (10), which is seen to be a form of MacCullagh's expression for the potential of a distant attracting mass.

It is hardly necessary to emphasize that the result enunciated in Ref. 1 is correctly scrutinized by applying the more succinct principle (7) without approximation or special assumptions, as indicated previously. Readers who might still cling to the excessively differentiated system (1–3) should recognize that (4) demonstrates an invariance of the differential system with respect to displacement of origin of angular coordinates; this provides an alternative interpretation of the result (5) or (7).

The angle variables employed in the present discussion are seen to possess the same advantages of convenience secured in a different manner by Jeffreys.[2]

For more complete discussions of special integrals of energy, of permanent configurations and surfaces of zero velocity, of effects of gyroscopic terms, and of other concepts utilized in the preceding discussion, the reader should consult standard treatises such as Refs. 3 and 4.

Received April 10, 1964.

* Professor of Aerospace Engineering. Member AIAA.

† Equation numbers refer to and notation conforms with Ref. 1.

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[1] Michelson, I., "Equilibrium orientations of gravity-gradient satellites," AIAA J. 1, 493 (1963).

[2] Jeffreys, H., "The moon's principal librations in rectangular coordinates," Monthly Notices Roy. Astronom. Soc. 115, 189 (1955).

[3] Brouwer, D. and Clemence, G. M., *Methods of Celestial Mechanics* (Academic Press, Inc., New York, 1961), Chap. X.

[4] Moulton, F. R., *An Introduction to Celestial Mechanics* (Macmillan Company, New York, 1914), revised ed., Chap. VII.

Encl 3

Euler Angles for Libration Analysis

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INCREASINGLY stringent satellite attitude stability requirements have necessitated accurate dynamical descriptions of librational motion consisting of limited angular displacements about satellite centroid. Although Euler's angle convention has always been followed in treatises on dynamics of rotating bodies, it introduces a peculiar inconvenience in discussions of librations: small angular displacements of arbitrary direction cannot be represented by a set of small values of the Eulerian angles. Although quaternions or different angle conventions could be introduced which are not subject to this disadvantage, there is an understandable reluctance to adopt these, owing to the attendant loss of accessibility to the classic literature and methods which this would entail. It is shown below that sets of finite Euler angles can be found which possess all of the properties desired for treating general librational motions, thus obviating the necessity for introduction of unconventional analytical techniques.

The need for a convention for representing angular displacements is recalled to be a consequence of the fact that these are not vector quantities, since magnitude, direction, and also sequence determine the resultant of two or more rotations; i.e., angular displacements do not enjoy the commutative property of addition. Among the many ways of securing an arbitrary rotation by compounding rotations about a set of orthogonal directions, Euler's convention, being adequate, has long held universal acceptance. A rotation ψ about one axis X, followed by a rotation θ about a perpendicular axis Y, and a final rotation φ about the original axis bring the axes to positions x, y. The two right-handed triads X, Y, Z and x, y, z are then related by equations compactly expressed in matrix form[1]:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\psi\sin\theta & -\cos\psi\sin\theta \\ \sin\theta\sin\varphi & \cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi & \sin\psi\cos\varphi + \cos\psi\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & -\cos\psi\sin\varphi - \sin\psi\cos\theta\cos\varphi & -\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (1)$$

Although zero displacements Ψ, θ, φ clearly do reduce the transformation matrix to unit diagonal form, it is also seen directly that first-order small values of these angles do not correspond to a rotation of arbitrary direction. For this purpose, it is sufficient to compare (1) with the transformation that corresponds to infinitesimal rotations α, β, γ about axes X, Y, Z, as done in librations analysis[2]:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Note in particular that the matrix element $\sin\psi\sin\theta$ vanis[illegible] to the second order in the Eulerian transformation ([illegible] whereas the corresponding Z rotation γ in (2) is only first-[illegible]der small. Hence, small values of the Euler angles do [illegible] represent small rotations that have a Z component. T[illegible] geometrical reason for this is to be found in the unsymmetri[illegible] character of Euler's convention: two rotations about X, [illegible] about Y, and none about Z. The more symmetrical rotati[illegible] sequence that leads to the transformation (2), on the ot[illegible] hand, entails exactly one rotation about each of three a[illegible] and thus is of a different class from Euler's, the same being [illegible] of course when the angles α, β, γ are finite and arranged i[illegible] definite sequence.

The complete generality of Euler's rotations nonethe[illegible] assures that there must be values of ψ, θ, φ which corresp[illegible] to all rotations, including arbitrary infinitesimal rota[illegible] about three orthogonal directions. These are found by [illegible]sidering those sets of three Euler rotations which lea[illegible] triad invariant, the final configuration congruent to the i[illegible] One of these is $\psi = \pi/2$, $\theta = \pi/2$, $\varphi = 0$, leading to t[illegible] required infinitesimal rotation by means of the angles

$$\psi = (\pi/2) + \alpha \qquad \theta = (\pi/2) + \gamma \qquad \varphi = \beta$$

When the angles α, β, γ are small,

$$\alpha \ll 1 \qquad \beta \ll 1 \qquad \gamma \ll 1$$

the substitution of (3) into (1) recovers (2). The suitabili[illegible] angles (3) is also geometrically evident because the two [illegible]tions of approximate magnitude $\pi/2$ have the effect of sep[illegible]ing each rotation axis from the others by an angular dist[illegible] $\pi/2$. This is clearly necessary for the representation of a [illegible]sultant small rotation of arbitrary direction, but it is [illegible]cluded by the Euler sequence when ψ or θ is limited to [illegible] values, the remaining angle φ also being small.

The usefulness of a set of rotations such as (3) (other [illegible] may also be found which serve equally well) is that acc[illegible] calculation of libration amplitudes, mode coupling, sta[illegible] and related characteristics requires consideration of [illegible] infinitesimal and finite values of α, β, γ in unrestricted [illegible]binations; angles (3) permit the partially stabilized lib[illegible] to be joined to the completely stabilized libration in wh[illegible] $\alpha = \beta = \gamma = 0$. This is accomplished, moreover, wi[illegible] the orthodox framework of Euler rotations in a generality [illegible] possible with the small values of ψ, θ, φ, which correctly re[illegible]sent only the limiting state of zero libration.

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* Professor of Aerospace Engineering, Department of Mechanical and Aerospace Engineering. Member AIAA.

References

[1] MacMillan, W. D., *Dynamics of Rigid Bodies* (Dover Publi[illegible]tions, Inc., New York, 1960), pp. 105–108.

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Encl 4

Libration Damping and Lunar Mass

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GRAVITATIONALLY oriented satellites are of interest currently because of the possibility they afford to control orientation in space for long time durations without a continuing power requirement. A major design problem is the provision of adequate damping for rotational oscillations; the theory of physical librations of the moon is a useful guide to the underlying dynamics. Consideration of the interactions between librational and orbital motions leads to a new relationship between masses and orbital distances and reveals a condition on friction terms.

Energy dissipation within the moon is directly related to the couple-producing term in the gravitational potential, the sun's effect predominating over the earth's. If the energy of moon's orbital motion around earth is expressed as

$$k^2 M_e M_m / -2r$$

and its variation determined by dissipation, the relationship can be written as

$$\frac{M_e M_m}{r^2}\,\delta r + \frac{M_s M_m}{R^3}\,\delta\phi_1 = 0 \tag{1}$$

where the Gaussian constant k^2 has been suppressed as a common factor. Earth, moon, and sun masses are denoted M_e, M_m, M_s; earth-moon and sun-moon mean distances are represented by r and R; effects of lunar orientation toward sun and lunar mass asymmetries determine ϕ_1, the explicit form of which is given by MacCullagh's formula (see, e.g., Ref. 1).

Tidal dissipation within the earth due to the proximity of the lunar mass can likewise be related to variation of earth's orbit around sun, giving

$$\frac{M_s M_e}{R^2}\,\delta R + \frac{M_e M_m}{r^3}\,\delta\varphi_2 = 0 \tag{2}$$

where ϕ_2 likewise depends on the differences between earth's principal inertia moments and the earth-moon orientation.

Although the two Eqs. (1) and (2) do not form a simultaneous system, it is observed that the products of diagonal coefficients form an equality

$$(M_e M_m / M_s M_s) = (r/R)^5 \tag{3}$$

as is easily verified by direct substitution of known numerical values for each of the quantities contained in this equation. It should be noted that Eq. (3) has not been deduced from (1) and (2), so that a further relationship is implied between the differential coefficients which they contain:

$$\frac{\delta\phi_1}{\delta\phi_2} = \frac{\delta R}{\delta r} \tag{4}$$

The ratio on the right side of (4) can be evaluated on the basis of orbital considerations (e.g., by requiring the total moment of momentum of earth and moon orbital motions to remain constant). Then the terms on the left relate dissipation within the satellite (moon) to that within the primary. Reciprocity relationships are suggested by the form of (4).

Equation (3) recalls the formula given by Laplace for the radius of the "activity sphere" outside of which the sun's attraction is more important than the earth's on the motion of a third body. If the geometric mean of earth and moon masses is replaced by earth mass, in fact, Eq. (3) becomes identical to Laplace's formula (see, e.g., Ref. 2). That relationship is known to be devoid of precise significance, despite its utility; in contrast, Eq. (3) may be considered surprising in its precision, despite the largely heuristic development given here.

The striking simplicity of the relationship (3), moreover, leads one to expect that it could be found in treatises on celestial mechanics; this is not the case. Nor does the classic literature appear to provide a sound dynamical basis for the equation.

Received August 6, 1964; revision received August 28, 1964.
* Professor of Aerospace Engineering. Member AIAA.

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[1] Routh, E. J., *Advanced Dynamics of Rigid Bodies* (Dover Publications, Inc., New York, 1955), p. 340.
[2] Tisserand, F., *Mécanique Céleste* (Gauthier-Villars, Paris, 1896), Tome IV, p. 200.

Encl. 5

Reprinted from

THE ASTRONOMICAL JOURNAL VOLUME 70, NUMBER 2 MARCH 1965

Abstracts of Papers Presented at the 117th Meeting of the American Astronomical Society, held 28–31 December 1964 at Montreal, Canada

Theory of Lunar Librations Re-Examined. IRVING MICHELSON, *Illinois Institute of Technology.* —The principal features of lunar rotation were concisely summarized with a high degree of precision by Cassini in 1693 in the set of three empirical laws that are known by his name. Laplace one century later furnished a dynamical theory for the actual small departures of the motion from the idealized form enunciated by Cassini, and suggested measurements from which lunar mass distribution characteristics might be inferred. In modern times Commission 17 of the International Astronomical Union, recognizing the inconclusive character of all such subsequent efforts, has urged the application of a rigorous method for the entire process of data reduction and interpretation. A generalized formulation utilizing contemporary methods of analysis has been carried out which now permits discussion in fundamental dynamical terms of physical librations of the moon and also of artificial satellites in arbitrary elliptic orbits.

Among the features of the lunar motion which were inaccessible on the earlier theory is the dependence of the Cassini motion upon given orbital characteristics (eccentricity and inclination) and inclination of the lunar equator to the ecliptic; this limitation is now removed. Restrictions on small-amplitude librations are also relaxed, both in relation to libration mode coupling and the role of arbitrary principal moment of inertia differences. Finally the complete dynamical basis of Cassini motion is demonstrated with due regard for significant but earlier ignored effects of relative motion of lunar centroid.

Equilibrium Orientations of Gravity-Gradient Satellites

IRVING MICHELSON*
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THE gravitational torque that acts on an orbiting satellite represents an automatic mechanism for attitude control in space. Since configurations sufficiently elongated in the earth-vertical direction experience a positive restoring couple due to the gradient of gravitational attraction when displaced from equilibrium orientation, systems based on this principle can be of a passive character. Practical interest in gravitationally oriented satellites is conditioned by the additional amount of mechanical control which will be required to assure adequate precision of orientation. A calculation is indicated which demonstrates that there is an infinity of equilibrium satellite orientations, in which no control is therefore required. All are in the neighborhood of those discrete orientations for which the gravitational torque is zero.

An equilibrium orientation in space is defined by the directions of orbital angular velocity and instantaneous earth-vertical. If satellite principal inertia axes x_1 and x_3 occupy these directions, respectively, and x_2 is directed opposite to instantaneous linear orbital velocity, the axes form a right-handed triad, gravitational torque is zero, and orbital centrifugal forces are balanced by gravitational attraction. When small angular displacements α, β, γ are produced by rotating about x_1, x_2, x_3, respectively, the equilibrium is disturbed and the ensuing angular vibrations about satellite mass center are governed by an extended form of Euler's equations of rigid-body motion:

$$A\ddot{\alpha} + 3\Omega^2(B - C)\alpha = 0 \tag{1}$$

$$B\ddot{\beta} + 4\Omega^2(A - C)\beta + \Omega(A - B - C)\dot{\gamma} = 0 \tag{2}$$

$$C\ddot{\gamma} + \Omega^2(A - B)\gamma - \Omega(A - B - C)\dot{\beta} = 0 \tag{3}$$

Principal-axis moments of inertia are denoted by A, B, C and Ω represents orbital angular speed for an orbit assumed for simplicity to be circular; dots indicate differentiations with respect to time. Eqs. (1–3) differ from the forms given in treatises on dynamics, owing to two causes.[1] One is the inclusion on the left-hand sides of the equations of the components of gravitational torque, and the other is due to the complete form of the total moment of momentum for a body in motion about a point *not* fixed in space.

In order to obtain a useful integral of the system of Eqs. (1–3), it is convenient to rewrite them as

$$\ddot{\alpha} = \partial U_1/\partial\alpha \tag{1'}$$

$$\ddot{\beta} + 2a\dot{\gamma} = \partial U_2/\partial\beta \tag{2'}$$

$$\ddot{\gamma} - 2b\dot{\beta} = \partial U_3/\partial\gamma \tag{3'}$$

where clearly a and b are constants, and U_1, U_2, U_3 are functions only of α, β, γ, respectively, given by

$$a = \frac{\Omega(A - B - C)}{2B} \qquad b = \frac{\Omega(A - B - C)}{2C}$$

$$U_1(\alpha) = \frac{-3}{2}\Omega^2\frac{(B - C)}{A}\alpha^2 \qquad U_2(\beta) = -2\Omega^2\frac{(A - C)}{B}\beta^2$$

$$U_3(\gamma) = -\frac{1}{2}\Omega^2\frac{(A - B)}{C}\gamma^2$$

Multiplying Eqs. (1′, 2′, and 3′) in turn by $\Omega\dot{\alpha}$, $b\dot{\beta}$, and $a\dot{\gamma}$ and adding, one obtains by means of a single integration

$$(\Omega\dot{\alpha}^2 + b\dot{\beta}^2 + a\dot{\gamma}^2)/2 = \Omega U_1 + bU_2 + aU_3 + K' \tag{4}$$

where K' is a constant of integration. An equilibrium orientation is recognized by the simultaneous and persistent conditions $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$. Thus if each of the angular velocities $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ is put equal to zero in Eq. (4), the right-hand side furnishes the geometrical condition that is necessary to secure the corresponding equilibrium orientation. The method employed to obtain the result hardly differs from that used by Jacobi in his treatment of the problem of Lagrange's three particles.[2]

Writing now

$$U(\alpha,\beta,\gamma) = \Omega U_1 + bU_2 + aU_3$$

Eq. (4) reduces to

$$U(\alpha,\beta,\gamma) = -K' \tag{5}$$

Although the particular orientation given by $\alpha = \beta = \gamma = 0$ is seen from Eqs. (1–3) to represent one orientation of dynamical equilibrium (that one which corresponds to zero gravitational torque), Eq. (5) shows that there is an entire one-parameter family of points in α,β,γ space for which the same condition is true. For all of the orientations, in general, a net gravitational torque is present but is exactly balanced by the couple generated by the static gradient of centrifugal forces in the orbital motion.

A case of special interest from the practical viewpoint is that in which the satellite possesses axial symmetry about the earth-vertical direction, so that $A = B$. In this case it is easy to see that Eq. (5) reduces to

$$\alpha^2 = \tfrac{2}{3}\beta^2 + K$$

and evidently the constant K must vanish, since $\alpha = 0$, $\beta = 0$ are values corresponding to one solution.

In addition to demonstrating that many equilibrium orientations are to be found in the same neighborhood of attitudes, the present result may be useful in determining advantageous contours for communications satellite reflector disks.

Finally it is noted that the present calculation bridges the gap between two well-known results. Although a spherically symmetric mass distribution possesses a double infinity of equilibrium orientations (i.e., it has no preferred directions), and two point masses attached to a weightless rod possess only six discrete equilibrium orientations (two of which are stable[3]), the calculation shows that for a general distribution of mass there are a single infinity of orientations replacing each of the six discrete ones found by Synge.

Received by IAS October 23, 1962.

* Professor of Mechanical Engineering, and Visiting Professor at the Institut des Specialités Industrielles de Nancy, Universite de Nancy, France, during Spring Semester 1963. Member AIAA.

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[1] Michelson, I., "Liberational dynamic-response limits of gravity-gradient satellites," AIAA J. 1, 216 (1963).

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[3] Synge, J. L., "On the behaviour, according to Newtonian theory, of a plumb line or pendulum attached to an artificial satellite," Proc. Roy. Irish Acad. 60A, 1–10 (January 1959).

ENCL 7

PRELIMINARY DRAFT [redacted]

I. Michelson

N65-88286

LUNAR LIBRATIONS THEORY -- CRITIQUE AND REFORMULATION

SUMMARY

The standard theory of physical librations of the Moon as formulated by Laplace takes for its basis the equations of Euler for the rotational motion of a rigid body about a fixed point. An elementary development proceeding from first principles is given which facilitates scrutiny of approximations made by Laplace and also aids direct physical interpretation. It is shown that corrections are needed in a theory that can be used in the reduction of quantitative observational data, tantamount to a fresh formulation.

The traditional analysis is extended in the first instance by allowing for the nonuniform motion of lunar centroid relative to Earth-Moon barycenter. Whereas the conventional theory recognized only the action of Earth's gravitational couple on the Moon, this consideration demonstrates that it is a dynamic balance between gravitational and centrifugal couples which determines the librations. It is also shown that, in the absence of the inertial effect, free periods of the motion are wrongly predicted. Equations more accurate than Euler's in this regard are given by Hamilton's Principle with the additional advantage of avoiding perilous linearization approximations. This formulation immediately accommodates also to the inclusion of solar effects which are shown to have been omitted without justification in past.

An effect unexplained and altogether unexplainable on the Laplace theory is the role of central importance played by the ecliptic, as expressed by Cassini. Present formulation, by contrast, demonstrates the key importance of the actually dominant Solar tidal potential within the Moon.

By recognizing the autonomous character of lunar librations, an

integral of motion is obtained which serves particularly well to aid discussion of low-energy motion such as libration. When it is taken in conjunction with virial and stability statements, it furnishes a direct relationship between the lunar ellipticity parameter f and other constants which are more accurately known.

Lagrangian and Hamiltonian Functions for Rotating Moon

The motion of a particle of the Moon is referred to the center of mass of that body in the first instance. When the lunar centroid motion is referred to the Earth-Moon barycenter, and that point in turn referred to an inertial reference frame, then the laws of Newtonian mechanics are applicable. Hence denoting by $\bar{R}_o$ the vector extending from origin of inertial reference frame to barycenter of Earth-Moon system, by $\bar{R}_1$ the vector from Earth-Moon barycenter to lunar centroid, and by $\bar{r}$ the position vector of a fixed point P of the moon relative to lunar centroid, then the location of P in the inertial frame is given by the vector $\overline{OP}$

$$\overline{OP} = \bar{R}_o + \bar{R}_1 + \bar{r}.$$

By means of the unit vectors $\bar{e}_{R_o}$ and $\bar{e}_{R_1}$ directed along $\bar{R}_o$ and $\bar{R}_1$ it is possible to express the velocity of P in the inertial frame as

$$\bar{v} = \frac{d}{dt}(\overline{OP}) = \dot{\bar{R}}_o + \dot{\bar{R}}_1 + \dot{\bar{r}} = \dot{R}_o\bar{e}_{R_o} + \dot{R}_1\bar{e}_{R_1} + \bar{\Omega}_o \times \bar{R}_o + (\bar{\Omega}_o + \bar{\Omega}_1) \times \bar{R}_1$$

$$\bar{\Omega}_o = \frac{\bar{R}_o \times \dot{\bar{R}}_o}{R^2} \qquad \bar{\Omega}_1 = \frac{\bar{R}_1 \times \dot{\bar{R}}_1}{R_1^2} \qquad + (\bar{\Omega}_o + \bar{\Omega}_1 + \bar{\Omega}_2) \times \bar{r} \qquad (1)$$

where $\bar{\Omega}_o$ is the angular velocity vector, normal to plane determined by $\bar{R}_o$ and $\dot{\bar{R}}_o$, giving the velocity of the end-point of $\bar{R}_o$ normal to that vector. In the same way, $\bar{\Omega}_1$, $\bar{\Omega}_2$ give the relative angular velocities of the corresponding moving reference frames.

Although the total kinetic energy of the Moon is given by the integration over the parts, as

$$\frac{1}{2}\int \bar{v}.\bar{v}\, dm,$$

the rotational motion $\bar{\Omega}_2$ of the Moon relative to its centroid is studied by

means of the term in the kinetic energy alone which contains this quantity. When the integration is carried out using center-of-mass coordinates, the relevant part of the kinetic energy is given by the expression

$$T = \frac{1}{2} \int (\bar{\sigma} \times \bar{r}).(\bar{\sigma} \times \bar{r})\, dm$$

where brevity of notation is achieved by writing

$$\bar{\sigma} \equiv \bar{\Omega}_o + \bar{\Omega}_1 + \bar{\Omega}_2.$$

For further simplicity the principal axes x_1, x_2, x_3 are employed, the corresponding components of the absolute angular velocity being denoted σ_1, σ_2, σ_3, and principal inertia moments designated as A, B, C. It follows at once that

$$T = \frac{A}{2} \sigma_1^2 + \frac{B}{2} \sigma_2^2 + \frac{C}{2} \sigma_3^2. \tag{2}$$

By writing the kinetic energy in the form (2), the influence of rotational motion on the orbital motion is neglected: $\bar{R}_o$, $\bar{\Omega}_o$, $\bar{R}_1$, $\bar{\Omega}_1$ are considered to be independently determined -- an approximation justified by the vastly different energies of orbital and lunar rotational motions, as well as the near-sphericity of the lunar mass distribution. More accurate treatment, if desired, may be sought along lines indicated in the investigations of Kondurar (Ref.) or by exploiting the concept of gravitating center for non-centrobaric bodies, in the manner which has been applied to tidal theory by Nahas (Ref.).

The gravitational potential energy of the Moon consists of the parts representing mutual force attractions by Earth and Moon independent of the Moon's mass distribution, plus considerably smaller contributions, the most important of which are the tidal potentials due to Earth and Sun treated as distant mass points. These are written as the leading terms of MacCullagh's formula:

$$V = \frac{-k^2 M_e}{2R^3} (A + B + C - 3I_e) - \frac{k^2 M_s}{2\rho^3} (A + B + C - 3I_s) \qquad (3)$$

where k^2 is the universal gravitational constant, M_e and M_s are the masses of Earth and Sun, whereas R and ρ stand for the distances from lunar centroid to that of Earth and that of Sun -- and it is noted that R is not identified with the barycentric distance R_1 earlier introduced. The moment of inertia of the Moon about the line connecting its centroid to that of Earth is designated I_e, and analogous meaning is attributed to I_s. It is of interest to note that the second term contains a coefficient smaller than that of the first term, in the ratio

$$\frac{M_s}{\rho^3} \frac{R^3}{M_e} = 5.5 \times 10^{-3}$$

but the second term will be seen nonetheless to be of greater interest than this numerical value suggests. The reason is found in the small differences between principal inertia moments A, B, C, and the fact that lunar orientation is such that I_e differs very little from one of these, while I_s ranges essentially from one of these to another each week, so that the overall influence of the solar term ~~may be~~ is substantially enhanced.

The Lagrangian function is formed as the difference of kinetic and potential energies, from (2) and (3):

$$L = \frac{A}{2}\sigma_1^2 + \frac{B}{2}\sigma_2^2 + \frac{C}{2}\sigma_3^2 + \frac{k^2 M_e}{2R^3}(A + B + C - 3I_e) + \frac{k^2 M_s}{2\rho^3}(A + B + C - 3I_s), \qquad (4)$$

Potential and kinetic energies (2) and (3) are both not explicitly time-dependent, so that their sum furnishes the value of the Hamiltonian function. The explicit form of this function in terms of a set of coordinates and corresponding conjugate momenta naturally depends on the choice of coordinates, several distinct sets of which will be found to accord with alternative approximation and perturbation schemes. A set of reference axes and specific nomenclature of lunar principal inertia axes are found to be useful in analyzing basic features of the motion as well as for defining the coordinates later employed.

Coordinate axes and angles of rotation

Axes X_1, X_2, X_3 serve as a system of reference coordinates with origin located at Moon centroid such that X_1 is normal to Earth-orbit plane (direct sense); X_3 is along the line connecting Earth and Moon centroids, directed away from Earth; X_2 completes a right-handed triad. Principal inertia axes of the Moon are denoted x_1, x_2, x_3 and are located by an ordered sequence of rotations α, β, γ starting from reference axes X_1, X_2, X_3: α is a rotation in the positive sense about X_1; β is a rotation in positive sense about the displaced X_2 axis; γ is a rotation in positive sense about twice-displaced X_3 axis. The order of rotations is immaterial only when these rotations are infinitesimal; in general, therefore, the relationships between reference coordinates and principal axis coordinates are characterized by a matrix which is not skew-symmetric. The explicity transformation relationships are exhibited in Fig. 1, where direction cosines a_1, a_2, ..., c_3 are expressed in terms of the angles α, β, γ and additional useful relationships between these quantities are tabulated. Principal inertia moments about x_1, x_2, x_3 axes are respectively denoted A, B, C; the components of angular velocity $\bar{\Omega}_2$ relative to reference axes are denoted p, q, r and are linear functions of $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, i.e., of the time derivatives of the rotation angles:

$$\left.\begin{aligned} p &= a_1\dot{\alpha} + \sin\gamma\,\dot{\beta} \\ q &= b_1\dot{\alpha} + \cos\gamma\,\dot{\beta} \\ r &= c_1\dot{\alpha} + \dot{\gamma} \end{aligned}\right| \qquad (5)$$

Preference for the present rotation convention, as compared with Euler angles for example, is related to the occasional inconvenience of the latter set: an arbitrary small rotation is not describable by a set of small values of the Euler angles (this being a "1-2-1" convention, while α, β, γ form a

"1-2-3" set). The unsuitability of Euler angles for general motions has been noted earlier (see, e.g., Tisserand).

Libration in Two-Body Motion

The customary analysis of lunar librations, following Laplace, has been based on writing the classical rigid body rotation equations of Euler, the angular velocities being those measured relative to reference axes defined by Earth-Moon configuration and treating lunar centroid as the origin of a Newtonian reference frame. According to this view, ~~Moon~~ Earth and Sun enter only through the gravitational potential V_G, having no role in the inertia of the motion. Such an analysis may be termed a "one-body" rotational motion.

In the present formulation the inertial influences of Sun and Earth are indicated by the presence of terms $\bar{\Omega}_o$ and $\bar{\Omega}_1$ in the kinetic energy (2). The differential equations of motion obtained by employing the expression (2) in Hamilton's Principle reduce to the classical one-body rotational motion when both Ω_o and Ω_1 are suppressed (see Note A, for example). In view of the hitherto unquestioned acceptance of the suitability of one-body libration analysis, it is of interest to depart from it in easy stages, thus permitting the closest comparisons with the traditional development. Thus the retention of $\bar{\Omega}_1$ and neglect of $\bar{\Omega}_o$, effectively accounting for motion of Moon relative to Earth but not of Earth-Moon relative to Sun, can be expected to furnish a more complete insight to the actual motion without entering at once upon the full added complexity suggested by accounting for both $\bar{\Omega}_o$ and $\bar{\Omega}_1$. It will in fact be seen that features of the motion appear which are of fundamental importance and which are not present in earlier analyses. In order to distinguish from the most general discussion which can be based on the present formulation, namely, that in which Sun, Earth, and Moon are all

accounted, the calculations below wherein Earth and Moon are considered alone will be termed "two-body" librational analysis. To the extent that solar influences are later seen to be dominated by those of Earth, the former may be later accounted by application of perturbation techniques, and the two-body analysis serves to determine the canonical variables for the more complete calculation.

Neglecting non-uniform motion of Earth-Moon barycenter relative to a true inertial reference frame by omitting the term $\bar{\Omega}_o$, the kinetic energy of lunar rotational motion becomes, from (2) and (5)

$$T = \frac{A}{2}\{(\Omega + \dot{\alpha})a_1 + \sin\gamma\ \dot{\beta}\}^2 + \frac{B}{2}\{(\Omega + \dot{\alpha})b_1 + \cos\gamma\ \dot{\beta}\}^2 + \frac{C}{2}\{(\Omega + \dot{\alpha})c_1 + \dot{\gamma}\}^2 \qquad (6)$$

where the subscript on Ω_1 has been dropped; this can also be written as

$$T = \frac{A}{2}\{\Omega^2 + \Omega(\dot{\alpha} + \sin\beta\ \dot{\gamma}) + \dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 + 2\sin\beta\ \dot{\alpha}\dot{\gamma}\} - \frac{(A-B)}{2}\{-\cos\beta\ \sin\gamma\ (\Omega+\dot{\alpha}) + \cos\gamma\ \dot{\beta}\}^2 - \frac{(A-C)}{2}\{\sin\beta\ (\Omega+\dot{\alpha}) + \dot{\gamma}\}^2 \qquad (6')$$

where the role of dynamic asymmetries (A-B) and (A-C) is put in evidence; it is known that these terms are positive and small compared with any one of the principal inertia moments. Important features are: i) the fact that the kinetic energy is of second degree in $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ but not homogeneous as a function of these quantities; ii) the separation of kinetic energy into parts porportional to the moment of inertia A and parts which are several orders smaller owing to the smallness of (A-B) and (A-C); and iii) the representation of librational "relative" motion by terms carrying a dot above -- omission of all of these terms leaves the kinetic energy for configurations

of relative equilibrium, a quantity of decisive importance in stability discussions to follow.

It may be remarked at this point that the relative equilibrium corresponding to $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$ is of a form determined by choice of reference axes, and does not correspond to a condition of zero rotation, but to a rotation speed equal always to the orbital angular velocity. For an eccentric orbit, of course, this is without strict dynamical meaning, but is considered as a useful basic motion owing to the observed fact that these quantities are small enough only to affect the energies very slightly. A more accurate approximation is consistent with Cassini's Theorems and partially specifies $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$. Note also that approximations of this type, having direct energetic interpretations, are of a different character from the usual ones in librations analysis which assume that <u>displacements</u> differ little from the basic "Cassini Motion" which has an empirical basis rather than dynamical.

Decomposing the kinetic energy into parts proportional to zero, first and second powers of the angle rates $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, it may be written as

$$T = T_o + T_1 + T_2 \ , \tag{6''}$$

a form which will be useful in the discussion which follows.

The interpretation of the separate contributions to (6) or (6') is immediate: rotation $\bar{\Omega}$ associated with orbital motion is directed along axis X_1, while the components of relative motion $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ are referred to axes intermediate between the set X_1, X_2, X_3 and the instantaneous positions of principal inertia axes x_1, x_2, x_3 (only for infinitesimal angular displacements α, β, γ can each of the quantities $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ be identified with the angular velocities about the separate axes). The total energy shown by

(6) represents the sums of resolved components along each principal axis direction, both for the orbital and for the relative rotations.

Noting from Fig. 1 that the direction cosines a_1, b_1, c_1 in (5) are independent of the angle α, the same being true for the coefficients of $\dot{\beta}$ and $\dot{\gamma}$, it follows that

$$\frac{\partial T}{\partial \mathcal{Z}_\alpha} = 0$$

which will yield further useful interpretation below. The derivatives with respect to β and γ, for relative equilibrium $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$ may also be noted:

$$\left.\frac{\partial T}{\partial \beta}\right|_o = -\Omega^2\{(A - C)\sin\beta\cos\beta + (A - B)(-\cos\beta\sin\gamma)\sin\beta\sin\gamma\}$$

$$\left.\frac{\partial T}{\partial \gamma}\right|_o = -\Omega^2(A - B)\cos\beta\sin\gamma\cos\beta\cos\gamma = -\Omega^2(A - B)(-b_1a_1); \tag{7}$$

the interpretation of these terms as centrifugal couples may be anticipated. Turning to the potential energy due to Earth, the portion which depends on lunar mass asymmetry and orientation is given from (3) as

$$V_e = \frac{-k^2M}{2R^3}(A + B + C - 3I_e)$$

where again a subscript is omitted. The moment of inertia I_e is closely given by

$$I_e = A\frac{x_1^2}{R^2} + B\frac{x_2^2}{R^2} + C\frac{x_3^2}{R^2}$$

which is easily brought to the form

$$A + (B - A)b_3^2 + (C - A)c_3^2$$

by recalling that x_1, x_2, x_3 are the coordinates of Earth, located by the X_3

axis and its direction cosines.

Hence

$$V = \frac{-k^2M}{2R^3}\left[(A - B)(3b_3^2 - 1) + (A - C)(3c_3^2 - 1)\right] \tag{8}$$

and partial derivatives are noted for later reference:

$$\left\{\begin{aligned}
-\frac{\partial V}{\partial \alpha} &= \frac{3k^2M}{R^3}\left[(A - B)\, b_3 \frac{\partial b_3}{\partial \alpha} + (A - C)c_3 \frac{\partial c_3}{\partial \alpha}\right] \\
&= \frac{3k^2M}{R^3}\left[(A - B)\, b_3b_2 + (A - C)c_3c_2\right] \\
-\frac{\partial V}{\partial \beta} &= \frac{3k^2M}{R^3}\left[(A - B)\, b_3 \frac{\partial b_3}{\partial \beta} + (A - C)\, c_3 \frac{\partial c_3}{\partial \beta}\right] \\
&= \frac{3k^2M}{R^3}\left[(A - B)(\sin\alpha \cos\gamma + \cos\alpha \sin\beta \sin\gamma)\right. \\
&\quad \left.(\cos\alpha \cos\beta \sin\gamma) + (A - C)\cos\alpha \cos\beta\, (-\cos\alpha \sin\beta)\right] \\
-\frac{\partial V}{\partial \gamma} &= \frac{3k^2M}{R^3}(A - B)\, b_3 \frac{\partial b_3}{\partial \gamma} = \frac{3k^2M}{R^3}(A - B)\, b_3\, (-a_3),
\end{aligned}\right. \tag{8'}$$

That these derivatives represent components of gravitational couple is readily shown on the basis of the meaning of potential energy; it will also be demonstrated in a manner that facilitates physical interpretation of the derivatives (7)

Direct Calculation of Gravitational Couple

Denoting the Earth-Moon centroids distance by R, the distance from Earth centroid to a point P of the Moon is given by

$$\ell^2 = (R_3 + X_3)^2 + X_2^2 + X_1^2$$

while the unit vector from the same direction is

$$\bar{e}_\ell = \frac{X_1\bar{\imath}_1 + X_2\bar{\imath}_2 + (X_3 + R)\bar{\imath}_3}{\ell}$$

in terms of reference axes and their associated unit vectors. Consequently unit mass at point P is attracted toward Earth center by force

$$\bar{f} = \frac{-k^2M}{\ell^2}\,\bar{e}_\ell$$

and this force creates a moment relative to lunar centroid which is

$$\bar{r} \times \bar{f}.$$

$$\bar{r} = X_1\bar{\imath}_1 + X_2\bar{\imath}_2 + X_3\bar{\imath}_3$$

Total gravitational couple due to Earth is then obtained by integration over all mass particles:

$$\bar{Q} = \int(\bar{r} \times \bar{f})\, dm.$$

By means of approximations of the type

$$\frac{X_1}{\ell^3} \doteq \frac{X_1}{R^3}\{1 - \frac{3X_3}{R}\}$$

involving neglect of higher powers of ratios which are inferior in value to 10^{-2} in the present case, the components of gravitational couple parallel to reference axes identified by subscripts are given by

$$Q_I = \frac{3k^2M}{R^3}\{(A - C)\, c_2c_3 + (A - B)\, b_2\, b_3\}$$

$$Q_{II} = \frac{-3k^2M}{R^3}\{(A - C)\, c_1c_3 + (A - B)\, b_1b_3\} \qquad (9)$$

$$Q_{III} = 0$$

These components, resolved along principal axis direction x_1, give the first of the formulas

$$H_1 = a_1Q_I + a_2Q_{II} + a_3Q_{III} = \frac{-3k^2M}{R^3}(B - C)\, b_3c_3$$

$$H_2 = \frac{3k^2M}{R^3}(A - C)\, a_3c_3 \qquad (9')$$

$$H_3 = \frac{-3k^2M}{R^3}(A - B)\, a_3b_3$$

~~where~~ and H_2, H_3 give the components parallel to x_2 and x_3, respectively.

The identity between the third equation of set (9) and the corresponding equation of set (8) indicates the consistency of approximations made in the one case in the potential, in the other in the evaluation of force moments directly. Likewise the first of equations (8) and the first of equations (9) show that $-\frac{\partial V}{\partial \alpha}$ gives the gravitational couple about the X_1 axis. It is thus shown for the derivatives indicated in (8) that each has the precise meaning: the component of resultant gravitational couple referred to the direction in space with respect to which the corresponding rotation is made. These directions are not orthogonal, so that the sum of squares of (8) does not give the sum of squares of (9) or (9'). It is noted that for small values of the angles, the components (9') become, to the first order of small quantities

$$H_1 = \frac{-3k^2M}{R^3}\alpha \qquad H_2 = \frac{-3k^2M}{R^3}\beta \qquad H_3 = 0.$$

For completeness it is noted that

Centrifugal Couple

When an irregularly shaped body moves along a curved path each mass point which comprises it experiences a centripetal force. In the same manner that a resultant centrifugal force is associated with the body as a whole, acting through its centroid, a couple about that point is also defined which is proportional to principal inertia moment difference. As an aid to the identification of terms in the kinetic energy which represent this effect, a direct calculation is employed which is analogous to that for the gravitational couple already evaluated.

The relative motion of Moon about Earth is referred to the barycenter of the two, the distance of which from lunar centroid is denoted by S. Then point P in the Moon effectively experiences a centrifugal force per unit mass given by

$$d\bar{F} = \Omega^2\{X_2\bar{l}_2 + (S + X_3)\,\bar{l}_3\};$$

its moment relative to lunar centroid is

$$\bar{r} \times d\bar{F}$$

and integration over the entire mass gives the complete centrifugal couple as

$$\bar{K} = \Omega^2 \int \bar{r} \times \{X_2\bar{l}_2 + (S + X_3)\bar{l}_3\}\ dm$$

This integral is readily evaluated, without approximation, the components parallel to reference axes being found to have the values

$$K_I = 0, \quad K_{II} = -\Omega^2\{(A - C)c_1c_3 + (A - B)b_1b_3\}$$

$$K_{III} = \Omega^2\{(A - C)c_1c_2 + (A - B)b_1b_2\} \qquad (10)$$

Resolved along principal axes, the components of centrifugal couple are

$$G_1 = a_1K_I + a_2K_{II} + a_3K_{III} = \Omega^2\{(A - C)b_1c_1 - (A - B)b_1c_1\}$$

$$G_2 = b_1K_I + b_2K_{II} + b_3K_{III} = -\Omega^2(A - C)a_1c_1 \qquad (10')$$

$$G_3 = \qquad = \Omega^2(A - B)a_1b_1$$

Comparing with the kinetic energy for "relative equilibrium" $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$ shows that

$$\frac{\partial T}{\partial \gamma} = G_3$$

while the fact that $\frac{\partial T}{\partial \alpha} = 0$ conforms with the value $K_I = 0$. For completeness it is also noted that

$$K_{II} \cos\alpha + K_{III} \sin\alpha = \frac{\partial T}{\partial \beta} \quad ,$$

so that is is established that the (positive) partial derivatives of kinetic energy with respect to α, β, γ, give the components of centrifugal couple along the X_1, X_2', and x_3 axes respectively -- here X_2' denotes the axis having direction cosines $\{0, \cos\alpha, \sin\alpha\}$ with respect to the reference set X_1, X_2, X_3.

Energy Condition on Lunar Elongation Parameter f

Both kinetic and potential energies being expressed as functions of angular displacement coordinates α, β, γ and their time derivatives $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, with no explicity dependence on time t, it is known that an integral exists of the form

$$\dot{\alpha}\frac{\partial L}{\partial \dot{\alpha}} + \dot{\beta}\frac{\partial L}{\partial \dot{\beta}} + \dot{\gamma}\frac{\partial L}{\partial \dot{\gamma}} - L = \text{const.} \tag{11}$$

Implications and interpretations will be discussed for the case of relative equilibrium $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$, in order to demonstrate the manner in which the ratio of principal inertia moment difference f, given by

$$f = \frac{A - B}{A - C}$$

is partially determined. The kinetic energy (6) reduces to

$$T_o = \frac{\Omega^2}{2}\left[A - (A - C)c_1{}^2 - (A - B)b_1{}^2\right] \tag{6''}$$

so that (11) becomes

$$V - T_o = \text{const.} \tag{12}$$

for which direct physical interpretation is available. The constancy of the energy difference in (12) assures that this function of the variables α, β, γ has three vanishing partial derivates, so that

$$\begin{aligned} -\frac{\partial V}{\partial \alpha} + \frac{\partial T_o}{\partial \alpha} &= 0 \\ -\frac{\partial V}{\partial \beta} + \frac{\partial T_o}{\partial \beta} &= 0 \\ -\frac{\partial V}{\partial \gamma} + \frac{\partial T_o}{\partial \gamma} &= 0 . \end{aligned} \tag{13}$$

Each term appearing in these equations has already been interpreted as a couple component; the first equation, for example, expresses the fact that the X_1 component of Earth-gravitational couple annuls the corresponding component of centrifugal couple. The remaining equations express the same condition for other (non-orthogonal) components. By resolving along the three reference axes, it is seen that the resultant gravitational and centrifugal couples are in balance; principal axis components then also satisfy the same condition. Hence equation (12) has this meaning, and further analysis shows that when the quantity $V - T_o$, the stationary value of which is assured by the conditions (13), is an actual minimum, then <u>secular</u> stability, as contrasted with ordinary stability, is assured.

Substitution from (8) and (6") gives the explicity form of (12):

$$-\frac{3k^2M}{\rho^3\Omega^2} c_3^2 + c_1^2 + f\left(\frac{-3k^2M}{\rho^3\Omega^2} b_3^2 + b_1^2\right) = \text{const.} \tag{14}$$

where terms in T_o and V which are independent of α, β and γ have been removed to the right side of the equation. In the case of circular orbit, the quantity $\frac{k^2M}{\rho^3\Omega^2}$ is ~~equal to unity~~ constant, while for eccentric orbits it is expressible as a function of true anomaly. Denoting this quantity by λ, the equations (12) and (13) take the form

$$\begin{aligned}
&-3\lambda c_3^2 + c_1^2 + f(-3\lambda b_3^2 + b_1^2) = \text{const.} \\
&-3\lambda c_3\frac{\partial c_3}{\partial\alpha} + c_1\frac{\partial c_1}{\partial\alpha} + f\left(-3\lambda b_3\frac{\partial b_3}{\partial\alpha} + b_1\frac{\partial b_1}{\partial\alpha}\right) = 0 \\
&-3\lambda c_3\frac{\partial c_3}{\partial\beta} + c_1\frac{\partial c_1}{\partial\beta} f\left(-3\lambda b_3\frac{\partial b_3}{\partial\beta} + b_1\frac{\partial b_1}{\partial\beta}\right) = 0 \\
&-3\lambda c_3\frac{\partial c_3}{\partial\gamma} + c_1\frac{\partial c_1}{\partial\gamma} + f\left(-3\lambda b_3\frac{\partial b_3}{\partial\gamma} + b_1\frac{\partial b_1}{\partial\gamma}\right) = 0
\end{aligned} \tag{15}$$

Sets of equations of this form have been studied in relation to the problem

of motion of a rigid body about a fixed point of the body (Ref. Golubov) Their fundamental interest in the study of physical librations is that they impose conditions on the parameter f, which quantity has heretofore only been estimated by arduous and indirect procedures depending on the most delicate observations of the physical librations themselves (the usual procedure employed being based on a theoretical analysis which has altogether neglected the centrifugal couple terms b_1 and c_1).

Equations (13) furnish a clear mode of appraisal of the error committed by neglect of centrifugal couple terms: the physical interpretation of a balance of centrifugal and gravitational couples is reduced (when b_1, c_1, are neglected) to the statement that gravitational couple itself must vanish, which is meaningless for a dynamically asymmetric Moon that experiences any physical libration motion at all. Even for small angles it is seen that b_1 and c_1 are of the same order as b_3, so that their neglect is not justifiable.

Recalling that centrifugal terms b_1 and c_1 relating to Earth-motion relative to Moon are represented by $\bar{\Omega}_1$ in (1), the foregoing discussion points to the importance of including also the effects of Sun-motion relative to Moon, represented by $\bar{\Omega}_0$. Although such terms have been seen to carry a smaller distance factor, it is now clear that their influence will be that of additional centrifugal and gravitational couples, these being much less restricted by small values of α, β, γ than the Earth couples. Looking ahead to the discussion of motion as described by Cassini, moreover, the role of the ecliptic which it involves can also be linked to inclusion of solar effects. Hence it is evident that equations (15) cannot be expected to furnish a quantitatively accurate theory, but are of chief interest in revealing qualitative features of the motion, relationships between observed constants and unknown constants, and for examining the stability of motion.

Kinetic Energy of General Librational Motion

By writing the kinetic energy (6') as

$$T = \frac{A}{2}\{\Omega^2 + 2\Omega(a_1p + b_1q + c_1r) + p^2 + q^2 + r^2\}$$

$$-\frac{(A - C)}{2}\{(c_1\Omega + r)^2 + f(b_1\Omega + q)^2\}$$

where p, q, r are given by (5), it is possible to establish an ordering of terms in two different ways. In the first place with regard to the generalized velocities $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ it is clear that T is not a homogeneous function of these quantities, but rather can be written as the sum of three functions of different degrees which separately have this property:

$$T = T_o + T_1 + T_2$$

$$T_o = \frac{A\Omega^2}{2} - \frac{(A - C)\Omega^2}{2}[c_1{}^2 + fb_1{}^2]$$

$$T_1 = A\Omega(a_1p + b_1q + c_1r) - (A - C)\Omega\,[c_1r + fb_1q]$$

$$T_2 = \frac{A}{2}(p^2 + q^2 + r^2) - \frac{A - C}{2}[r^2 + fq^2]$$

Here T_o is homogeneous of degree zero in $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, T_1 is homogeneous of degree unity in the same variables, and T_2 is homogeneous of degree two. Dynamical systems possessing kinetic energy expressions of this form are sometimes called "unnatural" and require separate analysis in many general respects (cf. Whittaker's Analytical Dynamics). The term T_o represents the kinetic energy of a configuration in "relative equilibrium" in the moving reference frame X_1, X_2, X_3; it is seen to depend on the configuration through the direction cosines b_1, c_1 which, incidentally are observed to involve only two of the coordinates α, β, γ. That dependence is restricted by the principal

moment differences represented by A - C and f, and since the first of these is of order 10^{-4} for the Moon, that term is evidently much smaller than the one that precedes it. The disproportion is further accentuated by the fact that c_1 and b_1 are themselves small as evidenced by the empirical fact that physical librations are small (but note here that the usual assumption is followed, to assume that the first radius of the Moon and the North pole of its equator may be identified with principal inertia axes).

The term T_1, homogeneous of first order in $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ also contains two parts, the second of which is again much smaller than the first; the term a_1p is the main part of T_1, owing to the fact that it is of order unity when b_1, c_1 are small. T_2 is the energy of pure relative motion, of second degree in $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, and again composed of one part which is much smaller than the other. Hence in addition to the decomposition of T into parts of distinct homogeneous degree indices, a further hierarchy is observed related to the smallness of (A - C). In this manner it appears that the next stage of precision beyond the relative equilibrium already formulated involves retention of the first term in T_1; in this manner both the sought parameter f and the approximate libration can be accounted while terms clearly much smaller than each retained term can be neglected.

Note A

Identification with the theory employed by Koziel

If the motion of the Earth-Moon barycenter is either ignored or treated as uniform, relative to an inertial reference frame, then $\bar{\Omega}_o$ is discarded in considering the motion of a particle within the Moon. When the same treatment is accorded to the motion of lunar centroid relative to Earth-Moon barycenter, $\bar{\Omega}_1$, is deleted also. The motion then appears as $\bar{\Omega}_2$, representing the angular velocity relative to a frame of reference which moves with lunar centroid. Let the components of this velocity along principal axes x_1, x_2, x_3, be denoted p, q, r. Then in place of kinetic energy (2), there is obtained

$$T = \frac{A}{2} p^2 + \frac{B}{2} q^2 + \frac{C}{2} r^2$$

while the angular momentum is simply

$$Ap\bar{\imath}_1 + Bq\bar{\imath}_2 + Cr\bar{\imath}_3$$

where principal-axis unit vectors have been introduced.

The time of rate of change of angular momentum relative to lunar centroid is obtained from the foregoing expression by noting that the unit vectors are time dependent. Since

$$\frac{d\bar{\imath}_1}{dt} = r\bar{\imath}_2 - q\bar{\imath}_3$$

and two analogous equations give the variations of $\bar{\imath}_2$, $\bar{\imath}_3$, it is seen that the rate of change of angular momentum is given as

$$\{A\dot{p} + (C-B)qr\}\bar{\imath}_1 + \{B\dot{q} + (A-C)rp\}\bar{\imath}_2 + \{C\dot{r} + (B-A)pq\}\bar{\imath}_3 \qquad \text{(A1)}$$

where time derivatives are indicated by a dot.

Considering the potential energy due to the Earth and the asymmetric mass distribution within the Moon, the relevant part of (3) is

$$V_e \equiv \frac{-k^2 M_e}{2R^3}(A + B + C - 3I_e) \tag{A2}$$

which is reduced by writing I_e in the form

$$I_e = A\frac{x_1^2}{R^2} + B\frac{x_2^2}{R^2} + C\frac{x_3^2}{R^2}$$

where x_1, x_2, x_3 are the principal axis coordinates of Earth. In order to express the components of gravitational couple as derivatives of V_e with respect to α, β, γ (which is valid when these are small quantities), it is sufficient to express x_1, x_2, x_3 in terms of these angles treated as small. Since

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = \begin{vmatrix} 1 & \gamma & -\beta \\ -\gamma & 1 & \alpha \\ \beta & -\alpha & 1 \end{vmatrix} \cdot \begin{vmatrix} X_1 \\ X_2 \\ X_3 \end{vmatrix}$$

in the stated limit, it is readily seen that

$$R^2\frac{\partial I_e}{\partial \alpha} = A2X_1 0 + B2X_1X_3 + C2X_3(-X_2) = 2(B-C)X_2X_3 = 2(B-C)x_2x_3$$

and $$R^2\frac{\partial I_e}{\partial \beta} = 2(C-A)x_3x_1 \qquad R^2\frac{\partial I_e}{\partial \gamma} = 2(A-B)x_1x_2$$

within the order of present approximation. Hence the gravitational couple is

$$-\frac{\partial V_e}{\partial \alpha}\bar{i}_1 - \frac{\partial V_e}{\partial \beta}\bar{i}_2 - \frac{\partial V_e}{\partial \gamma}\bar{i}_3 = -\frac{3k^2M_e}{R^5}\{(B-C)x_2x_3\bar{i}_1 + (C-A)x_3x_1\bar{i}_2 + (A-B)x_1x_2\bar{i}_3\} \tag{A3}$$

The result (A3), obtained here from the potential V_e, is also found more directly in the limit of small angles α, β, γ, by integration of the forces acting, multiplied by coordinates.

Equating respective components of couple and angular momentum derivative

from (A3) and (A1),

$$\left.\begin{aligned}
\frac{dp}{dt} + \frac{C-B}{A}\,qr &= \frac{3k^2M_e}{R^5}\frac{(C-B)}{A}\,x_2x_3 \\
\frac{dq}{dt} + \frac{A-C}{B}\,rp &= \frac{3k^2M_e}{R^5}\frac{(A-C)}{B}\,x_3x_1 \\
\frac{dr}{dt} + \frac{B-A}{C}\,pq &= \frac{3k^2M_e}{R^5}\frac{(B-A)}{C}\,x_1x_2
\end{aligned}\right\} \quad (A4)$$

To adapt to Koziel's notation, the corresponding quantities are seen to be related as shown:

Present Notation	Koziel Notation
x_1, A, p	z, C, r
x_2, B, q	y, B, q
x_3, C, r	-x, A,-p

which bring the set of equations to the form

$$-\frac{dp}{dt} + \frac{B-C}{A}\,rq = \frac{3k^2M_e}{R^5}\,\frac{B-C}{A}\,zy$$

$$\frac{dq}{dt} + \frac{C-A}{B}\,(-p)(r) = \frac{3k^2M_e}{R^5}\,\frac{C-A}{B}\,(-x)z$$

$$\frac{dr}{dt} + \frac{A-B}{C}\,q(-p) = \frac{3k^2M_e}{R^5}\,\frac{A-B}{C}\,(-x)y$$

Finally putting $\alpha \equiv \frac{C-B}{A}$, $\beta \equiv \frac{C-A}{B}$, $\gamma \equiv \frac{B-A}{C}$ and writing M in place of k^2M_e, gives exactly the equations (1·03) of Koziel (Acta Astronomica Ser. A. Vol. 4, p. 66; 1948)

$$\begin{aligned}
\frac{dp}{dt} + \alpha qr &= 3M\alpha R^{-5}yz \\
\frac{dq}{dt} - \beta pr &= -3M\beta R^{-5}xz \qquad (A5) \\
\frac{dr}{dt} + \gamma pq &= 3M\gamma R^{-5}\,xy
\end{aligned}$$

Summarizing the special assumptions which permit the reduction of present formulation to that of Koziel:

i) Ignore non-uniform motion of Earth-Moon barycenter and also that of lunar centroid

ii) Assume all angular displacements to be infinitely small, hence neglect squares and products of these quantities

iii) Neglect the gravitational couple exerted by the Sun.

Equations (A2) thus obtained are the traditional point of departure for analysis of physical librations of the Moon. In the usual subsequent development, it may be further remarked, no use is made of the fact that an autonomous dynamical system is being treated (a fact which affords a more advantageous formulation), and other approximations are introduced which impair the accuracy of later results.

Most significant among consequences of assumptions listed above is the fact that no dynamical mechanism is retained to counter-balance the Earth gravitational couple (more complete analysis shows that centrifugal couples related to $\bar{\Omega}_1$ and $\bar{\Omega}_2$ play a leading role in this respect). Stability analysis employing known integrals of autonomous system, in addition, provide new relationships between lunar flattening parameter f and orbital parameters that are accurately known.

In evaluating the gravitational couple terms on the right side of equations (A5) with the aid of MacCullagh's formula, an advantage in directness is offset by the fact that the approximations which it introduces are not exhibited as clearly as is done in more elementary derivations. In view of the form of couple components, moreover, in which principal axis coordinates of attracting body appear but the rotation angles are absent which served in the intermediate calculations, it appears that an advantage of clarity is gained by a return to first principles.

Newton's law of gravitation

$$dF = \frac{+k^2 M\ dm}{\rho^2}$$

expresses the force of attraction experienced by particle dm under the influence of mass M at distance ρ from the particle. Regarding Earth as a mass point located by lunar principal axis coordinates x_1, x_2, x_3, particle dm being situate at (ξ_1, ξ_2, ξ_3), the complete couple is evaluated by an integration over all mass points dm. In vector form this becomes

$$+ k^2 M \int \frac{1}{\rho^3} \begin{vmatrix} \bar{\imath}_1 & \bar{\imath}_2 & \bar{\imath}_3 \\ \xi_1 & \xi_2 & \xi_3 \\ x_1-\xi_1 & x_2-\xi_2 & x_3-\xi_3 \end{vmatrix} dm \qquad \text{(A6)}$$

where the integrand is seen to represent the vector product of the position vector of dm

$$\bar{r} \equiv \xi_1 \bar{\imath}_1 + \xi_2 \bar{\imath}_2 + \xi_3 \bar{\imath}_3$$

and the elementary force dF written in vector form. Noting that

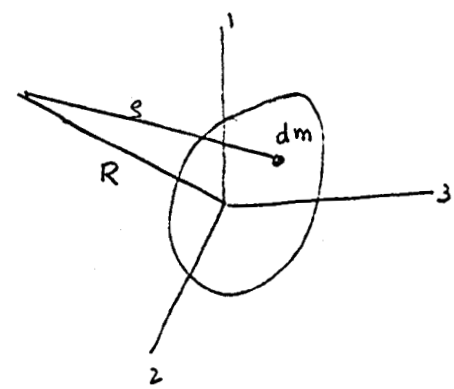
$$\rho^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2$$

and that integration variables ξ_1, ξ_2, ξ_3 are limited to values smaller than R, the distance from lunar centroid to M,

$$R^2 = x_1^2 + x_2^2 + x_3^2$$

it is clear that the integration (A6) can be effected in a series of terms proportional to negative integral powers of R. The x_1-component, for example, is the coefficient of the corresponding unit vector $\bar{i}_1$

$$k^2M \int \frac{(\xi_2 x_3 - \xi_3 x_2)\ dm}{\{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2\}^{\frac{3}{2}}} \qquad \text{(A7)}$$



Since

$$\rho^3 = \{R^2 + r^2 - 2(x_1\xi_1 + x_2\xi_2 + x_3\xi_3)\}^{\frac{3}{2}}$$

we have

$$\rho^{-3} = R^{-3}\ \{1 + \frac{3(x_1\xi_1 + x_2\xi_2 + x_3\xi_3)}{R^2} - \frac{3}{2}\frac{(\xi_1^2 + \xi_2^2 + \xi_3^2)}{R^2}$$

$$+ O([\frac{x_1\xi_1 + x_2\xi_2 + x_3\xi_3}{R^2}]^2)\}$$

When it is recalled that integrals of terms such as $\xi_1\xi_2$ vanish because of the defining property of principal inertia axes, it follows that the integral (A7) has the value

$$k^2M \frac{3}{R^5} \cdot x_3x_2 \int (\xi_2{}^2 - \xi_3{}^2)\, dm = \frac{3(C-B)\, x_2x_3}{R^5} k^2M \qquad (A8)$$

which is the x_1 component of gravitational couple in agreement with (A4); the x_2 and x_3 components are obtained by permuting inertia parameters and indices. Terms neglected are those having coefficients R^{-7} and smaller, the omission of which is evidently justified when R is large compared with the maximum dimension of the body -- when the dimensions of the Moon are compared with Moon-Earth distance R, it is seen that the first neglected term carries a coefficient which is about 10^{-4} times smaller than the preceding term which is retained. The actual magnitudes, of course, are also affected by the factor $(C - B)x_2x_3$ for the retained term and the factor which appears with R^{-7}; the smallness of inertia moment differences and Earth coordinates must thus be taken into account.

Concerning the coordinates x_1, x_2, x_3 which appear by product pairs in the separate components of gravitational couple, it is clear that although no assumptions have been made on their values, there are large differences among them which must be recognized. The empirical fact that the $-x_3$ axis is directed very nearly Earthward (and points exactly toward Earth each month as the Moon crosses the ecliptic, if T. Mayer's first meridian may truly be taken as a principal axis) implies that x_2 and x_3 are by comparison each very small. It follows that the x_3-component of couple, which is proportional to product of x_1 and x_2, is on the whole very much smaller than the other two components. This characteristic is seen to contrast with the couple exerted by the Sun, leading to a conclusion about the importance of that effect.

The conventional form of the equations of physical libration of the Moon exhibit Earth's gravitational couple as the terms on the right side of equations (A5), this being the only agency for altering the angular momentum terms on the left side. By what has been shown for the couple components in the case of Earth, it is clear that the analogous effect of Sun can be accounted by an additional term in each case. Thus for the x_1 component of <u>combined</u> Earth and Sun gravitational couples, (A8) is replaced by

$$-3(B - C)k^2 \left[\frac{M_e}{R_e^5} x_2^e x_3^e + \frac{M_s}{R_s^5} x_2^s x_3^s \right] \qquad (A9)$$

where subscripts and superscripts are introduced to distinguish the separate effects; the x_2- and x_3- components are of the same form involving only notation permutations.

The customary neglect of Sun's couple in comparison with Earth's is based on its greater distance:

$$\frac{R_s}{R_e} \doteq 0.389 \times 10^3.$$

That the distance ratio implied in (A9) more than offsets the greater mass of Sun is seen by noting that

$$\frac{M_s}{M_e} \left(\frac{R_e}{R_s}\right)^3 \doteq 5.6 \times 10^{-3} \; ; \qquad (A10)$$

hence the coefficient of the product of Sun's coordinates $\frac{x_2^s x_3^s}{R_s^2}$ is this much smaller than the corresponding term for Earth. But when account is taken of the fact already noted that x_2^e is small compared with x_3^e, or more appropriately, that

$$\frac{x_2^e}{R_e} \ll 1 \qquad\qquad \frac{x_3^e}{R_e} \doteq 1,$$

then it is seen that the Sun's couple is smaller only when $\frac{x_2^s x_3^s}{R_s^2}$ is also small, which is certainly not always true. Although a strict comparison of Earth and Sun couples depends on a knowledge of x_1^e, x_2^e, x_3^e, which requires the description of librational motion that is not available until the present problem is solved, it is still possible to obtain useful rough comparisons in the following manner.

Observing that all three components of Earth couple would be zero if both x_1^e and x_2^e were zero, in which case the surviving Solar couple terms would dominate despite the coefficient handicap (A10), consider a range of small but non-zero values of Earth coordinates and determine the portions of the Moon's orbit about Earth wherein the Sun's couple is greater than Earth's. Evidently these regions are furthest from syzygyes and quadratures.

... to be continued

Note B

On the Inadequacy of Euler's Rigid Body Equations for Libration Analysis

It is well established that the natural frequencies have unequal values for pitching and rolling motions of a stably oriented satellite. The pitching mode (motion in the orbital plane) frequency is $\sqrt{3}\,\Omega$ where Ω is the orbital angular velocity which is constant for circular orbit, whereas the rolling mode (motion in plane normal to orbit, containing Earth) is 2Ω. The calculation of these values for a satellite idealized as a plumb-line was given by J.L. Synge in 1958 (Proc. Roy. Irish Acad., Vol. 60, Section A), and generalized for arbitrary mass distribution by I. Michelson in 1962 (ARS Journal, Vol. 29, p.1735). The different values are explained by nature of the centrifugal couple which appears when relative motion of satellite is properly accounted: resultant gravitational couple is equated to rate of change of absolute angular momentum. The equations of Laplace which are fundamental to the analysis of Koziel and others are readily seen to be inconsistent with this fact.

Take the equations in the form (A4) and observe that each of the products qr, rp, and pq is small to the second order when rotation angles α, β, γ are each first-order small. Hence these may be neglected as compared with the other (first order) terms on the left sides of the equations. Then the identical numerical coefficient of each right side determines natural frequency values $\sqrt{3}\Omega$ for each mode. The conclusion is that the Euler equations (A4), which are properly derived for the motion of a body with one point fixed, are inadequate for the study of the inherent dynamic features of satellite librations.

When the kinetic energy is written as in equation (6), the distinct natural frequencies mentioned above are recovered. Lunar librations should

therefore be studied not only by accounting for Moon motion relative to Earth through Ω, but also including the Ω_0 effect of annual motion there omitted.

The above criticism of Laplace's formulation does not apply to the formulation which was given by Lagrange in 1780: natural periods given by his equations (Oeuvres, Tome Cinquième, p. 83) are entirely consistent with those of Michelson already indicated.

Note C

Secular Stability Criterion and Lunar Ellipticity Parameter f

Stability of equilibrium in a dynamical system is characterized by a minimum value of the potential energy in the most elementary examples, but a more delicate condition is established when the reference frame in which equilibrium is observed is itself in non-uniform motion. Criteria given by H. Lamb are extended to include effects of slowly varying rotation; this corresponds to the orbital eccentricity interaction with physical librations. The virial theorem taken in conjunction with any specification of libration (such as Cassini's, for example) yields a relationship involving the parameter f and other quantities which are more accurately known. It appears that the value of f is not a sensitive function of the specified libration, so that even in the absence of accurate corrections to Cassini's motion, a value of f can be deduced which should not involve large uncertainty.

The specific form of the fundamental energy quantity in these discussions is readily contrasted with the simpler and older results in the following manner: in place of $V - T_o$ in the theory given by Lamb, it is found that $V - T_o + T_2$ is minimuzed for a secularly stable configuration of relative equilibrium. Here T_o and T_2 are those parts of the total kinetic energy which are homogeneous functions of degree zero and two respectively in the generalized velocities; V is the potential energy.

Note D

Alternate Formulation Employing Angular Velocities as Fundamental Variables

Euler's rigid body rotation equations (A4) as derived above in two different conventional ways exhibit the angular velocities as dependent variables. Regardless of whether Euler rotation angles are employed or an unconventional set such as α, β, γ, the expression of angular velocity components as linear functions of the angle time-rates yields a sixth order differential system. The procedure introduced by Hess and widely followed by contemporary Russian dynamicists is different in this respect, that no ordered sequence of rotations is required, the angular velocities being treated as fundamental quantities. An additional set of three equations is employed, the underlying symmetries of which are exploited to give new and deep insights of the motion of a rigid body about one of its fixed points. The motion of present interest is of course not in this category, but the Hessian analysis with alterations holds considerable appeal nonetheless, owing to the remaining though imperfect analogy.

Encl. 8

Abstract of Paper to be Presented at the XVIth International Astronautical Congress, Athens, Greece, September 18, 1965.

DEDUCTIONS FROM NONLINEAR ANALYSIS OF SATELLITE PHYSICAL LIBRATIONS

I. Michelson

Passive control of satellite orientation in space is hindered by the inherent low-frequency character of free oscillations. In view of the poor efficiency of damping devices under these conditions, it is desirable to examine in the fullest detail the dynamic features of rigid satellites of general configuration in eccentric orbits. Nonlinear Analysis of Physical Librations, reported at the XVth IAF Congress, elucidated the manner in which orientations or relative equilibrium are extended by the inclusion of small compound fictitious forces of relative motion. Further consequences of the autonomous nature of the motion have been established, notably to relate the lowest energy libration modes near equilibrium to the inertia parameters of the satellite and to orbital characteristics.

Accounting for inertia effects in the distinctive manner of Lagrange's study of physical librations, a basic class of motions is defined by special conditions for secular stability of relative equilibrium in a non-uniformly rotating system. By assuming that "Cassini" motion conforms to these stability requirements, the orientations and principal inertia moment difference ratios are inter-related. The fundamental dynamical basis of such motion in the case of the Moon is thereby clarified, furnishing a theoretical basis for the observed inclination of its equator and permitting analogous deductions with regard to artificial satellites. The formulation considers three finite degrees of freedom of physical libration, the order of the system of equations being reduced by ignoring solar pressure and other explicitly time-dependent effects.

An important consequence of the generality of the motion considered is the demonstration that equilibrium attitude is strongly affected by orbit eccentricity and dynamic asymmetry of satellite. These same parameters also play a major role in determining the nature and stability of periodic motions in the vicinity of equilibrium orientations. Canonical formulation is discussed, in which these two quantities define a perturbation procedure.

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