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ON THE DETERMINATION OF THE SECULAR AND CRITICAL EFFECTS IN THE MOTION OF SATELLITES USING A NON-SINGULAR SET OF VECTORIAL ELEMENTS

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Abstract

Under the influence of the Sun and the Moon a highly eccentric orbit of a satellite can become a nearly circular one or a nearly circular orbit might become eccentric. In light of this the author wishes to suggest the combination of the Gibbsian rotation vector and the Hamiltonian vector as a non-singular set of elements to be used for all eccentricities and all inclinations. The method of Halphen is suggested for the computation of secular effects and the method of Liouville for the computation of the near resonance effects caused by the commersurability of mean motions of the disturbed and the disturbing body.

The combination of these two methods can give information about the general behavior of the orbit over an interval of thousands of years in the case of a minor planet and over an interval of a number of years in the case of an artificial satellite moving in cislunar space.

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NOTATIONS

$\underset{\sim}{r}$ - the position vector of the disturbed body,

$r = |\underset{\sim}{r}|$.

ξ, η - the coordinates of the disturbed body in an ideal system of coordinates.

M, ω, Ω, i, e, n - the standard elliptic elements of the disturbed body.

The initial values of the elements are denoted by a zero subscript.

L - the orbital mean longitude.

ΔL - the perturbations in the mean orbital longitude.

$c^2 = p = a(1-e^2)$.

$\underset{\sim}{P}$, $\underset{\sim}{Q}$, $\underset{\sim}{R}$ - the Gibbsian vectorial elements of the disturbed body.

The elements of the disturbing body will be designated by the corresponding symbol with a prime.

χ - the true orbital longitude reckoned from the departure point.

$u = -e \sin \chi/\sqrt{p}$

$v = e \cos \chi/\sqrt{p}$

E = the eccentric anomoly of the disturbed body.

$\lambda = E + \chi$ - the eccentric orbital longitude.

$\underset{\sim}{h}$ - the vector appearing in the Hamiltonian integral of the two body problem.

$\underset{\sim}{g}$ - the Gibbsian rotation vector.

$\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ - the unit vectors along the axes of an ideal system rigidly connected to the osculating orbit plane.

$\underset{\sim}{i_o}$, $\underset{\sim}{j_o}$, $\underset{\sim}{k_o}$ - the initial values of $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$.

$\underset{\sim}{F}$ - the disturbing force lying in the orbital plane.

$\underset{\sim}{F_o}$ - the disturbing force $\underset{\sim}{F}$ averaged over the orbit of the disturbing body.

Z - the component of the disturbing force normal to the orbit plane.

Z_o - the component Z averaged over the orbit of the disturbing body.

X - the component of the disturbing force in the direction of $\underset{\sim}{i}$.

Y - the component of the disturbing force in the direction of $\underset{\sim}{j}$.

S - the component of the disturbing force in the direction of $\underset{\sim}{r}$.

T - the component of the disturbing force in the direction of $\underset{\sim}{k} \times \underset{\sim}{r}$.

The gravitational constant is put equal to one.

In his previous work the author has emphasized the importance of long period effects for the estimation of the life time of satellites moving in highly eccentric orbits (Musen, 1963). In his work a collection of formulae was given for the computation of the long period lunar effects in the motion of artificial satellites using Halphen's method (Halphen, 1888).

Originally the author suggested the use of the classical elliptic elements, which are quite convenient if the eccentricity does not become small. The system of equations giving the variations of the elliptic elements contains the eccentricity and the inclination as "small divisors" thus making the computation very difficult if the orbit becomes a nearly circular one.

Unfortunately, this is exactly what may happen to originally highly eccentric orbits in cislunar space. The results obtained by Shute (1962) on the basis of Halphen's method show that under the influence of the Sun and the Moon a highly eccentric orbit can become a nearly circular one or a nearly circular orbit might become eccentric. Thus, the demarkation line between the two extreme types of orbits is not clearly defined.

In light of the accumulated information we wish to reconsider the problem and to suggest a non-singular set of elements to be used for all eccentricities and all inclinations.

This system consists of the vector appearing in the Hamiltonian expression for the velocity in the two body problem and of the Gibbsian rotation vector (Gibbs, 1901). The Gibbsian vector determines the rotation of the orbital plane from its initial position to the position at a given time. The author made use of the Hamiltonian vector (1954) in his vectorial theory

of the perturbations of elements. The Gibbsian vector was used by the author (1961) in an article on the generalization of Stromgren's method of special perturbations (Stromgren, 1929).

The combination of these two vectors represents a non-singular system of elements because no "small divisors" appear in the equations for their variations. Two sources of difficulty associated with the determination of perturbations for nearly circular orbits can be recognized. The first is our tendency to obtain the unit vector directed toward the perigee. This can be avoided by representing the position vector in terms of the Hamiltonian vector directly. The second source is the use of the eccentric and the mean anomalies. Both sources cause the presence of e as a small divisor.

The difficulty can be resolved by combining the perturbations in the perigee position with the mean and the eccentric anomaly, thus introducing the eccentric orbital longitude instead of the eccentric anomaly. Kepler's equations should be re-written in terms of the mean and the eccentric orbital longitudes. This fact was recognized by Herget (1962), who was the first to obtain a proper form of Kepler's equation for small eccentricity.

The Hamiltonian vector $\underset{\sim}{h}$ is defined as

$$\underset{\sim}{h} = \frac{e\,\underset{\sim}{Q}}{\sqrt{p}} = u\underset{\sim}{i} + v\underset{\sim}{j} \tag{1}$$

where

$$u = -\frac{e \sin \chi}{\sqrt{p}} \quad , \quad v = +\frac{e \cos \chi}{\sqrt{p}} \tag{2}$$

Introducing the eccentric orbital longitude

$$\lambda = E + \chi$$

and the vector

$$\underset{\sim}{\rho} = \underset{\sim}{i} \cos \lambda + \underset{\sim}{j} \sin \lambda \tag{3}$$

the position vector of the satellite can be written in a form depending directly upon the vector $\underset{\sim}{h}$:

$$\underset{\sim}{r} = a \underset{\sim}{\rho} - \frac{c^2}{\sqrt{a} + c} \underset{\sim}{h} \; \underset{\sim}{h} \cdot \underset{\sim}{\rho} + a \; c \; \underset{\sim}{k} \times \underset{\sim}{h} \tag{4}$$

$$= \xi \underset{\sim}{i} + \eta \underset{\sim}{j}$$

For the determination of p we have

$$\frac{1}{p} = \frac{1}{a} + h^2 \tag{5}$$

The semi-major axis remains invariant under secular action.

Eliminating E, M, and e from

$$\frac{r}{a} = 1 - e \cos E$$

$$E - e \sin E = M$$

in favor of λ, L and the components of $\underset{\sim}{h}$, we deduce

$$\frac{r}{a} = 1 - cv \cos \lambda + cu \sin \lambda \tag{6}$$

$$\lambda - cv \sin \lambda - cu \cos \lambda = L \tag{7}$$

In addition to the vectors $\mathbf{r}$ and $\mathbf{h}$ we will also make use of the vector $\mathbf{r}_o$ and $\mathbf{h}_o$ obtained from $\mathbf{r}$ and $\mathbf{h}$ by the subsitution of $(\mathbf{i}_o, \mathbf{j}_o, \mathbf{k}_o)$ for $(\mathbf{i}, \mathbf{j}, \mathbf{k})$

$$\mathbf{r}_o = \xi\, \mathbf{i}_o + \eta\, \mathbf{j}_o\,, \quad \mathbf{h} = u\mathbf{i}_o + v\mathbf{j}_o$$

Decomposing $\mathbf{h}$ along the axes rigidly connected with the osculating orbit plane, we have (Musen, 1954)

$$\frac{d\mathbf{h}}{dt} = \left(1 + \frac{r}{p}\right)\mathbf{F} - \frac{\mathbf{r}_o\,\mathbf{r}_o \cdot \mathbf{F}_o}{rp} \tag{8}$$

Designating by $\mathbf{Fo}$ the disturbing force averaged along the orbit of the disturbing body, we have for the secular effect in $\mathbf{h}$:

$$\frac{d\mathbf{h}}{dt} = \frac{1}{2\pi} \int_{-u_e}^{2\pi - u_e} \left[\left(1 + \frac{r}{p}\right) \mathbf{Fo} - \frac{\mathbf{r}_o\,\mathbf{r}_o \cdot \mathbf{Fo}}{rp} \right] dL \tag{9}$$

Eliminating dL in favor of $d\lambda$ by means of (7) we obtain

$$\frac{d\mathbf{h}}{dt} = \frac{1}{2\pi} \int_{0}^{2\pi} \left[\left(1 + \frac{r}{p}\right) \mathbf{Fo} - \frac{\mathbf{r}_o\,\mathbf{r}_o \cdot \mathbf{Fo}}{\tau\, p} \right] \frac{r}{a}\, d\lambda \tag{10}$$

We have

$$\mathbf{Fo} = -\,2m'\,(\mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j}) \cdot A \cdot \mathbf{r} \tag{11}$$

$$\mathbf{r} \cdot \mathbf{Fo} = -2m'\,\mathbf{r} \cdot A \cdot \mathbf{r} \tag{12}$$

where A is the fundamental dyadic which was used by Halphen to determine the components of the averaged disburbing force $\underset{\sim}{Fo}$ in the directions of $\underset{\sim}{P'}$, $\underset{\sim}{Q'}$, $\underset{\sim}{R'}$. No attempt is made in this article to modify the method of computing A. Designating the components of $\underset{\sim}{Fo}$ in the directions of $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{r}$ and $\underset{\sim}{k}$ by X_o, Y_o, S_o, Z_o respectively, we have

$$X_o = - 2m' \, i . A . \underset{\sim}{r}, \tag{13}$$

$$Y_o = -2m' \, \underset{\sim}{j} . A . \underset{\sim}{r}, \tag{14}$$

$$S_o = - 2m' \, \underset{\sim}{r} . A . \underset{\sim}{r}/r, \tag{15}$$

$$Z_o = - 2m' \, \underset{\sim}{k} . A . \underset{\sim}{r} \tag{16}$$

We deduce from (10) taking (13) - (15) into account the following two scalar equations for the actual computation of the secular perturbations in $\underset{\sim}{h}$:

$$\frac{du}{dt} = \frac{1}{2\pi} \int_0^{2\pi} \left[\left(1 + \frac{r}{p}\right) X_o - \frac{\xi . S_o}{p} \right] \frac{r}{a} \, d\lambda, \tag{17}$$

$$\frac{dv}{dt} = \frac{1}{2\pi} \int_0^{2\pi} \left[\left(1 + \frac{r}{p}\right) Y_o - \frac{\eta \, S_o}{p} \right] \frac{r}{a} \, d\lambda. \tag{18}$$

Introducing the Gibbsian vector $\underset{\sim}{g}$, we can express the vectors ($\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$) of the system rigidly connected with the orbit plane in terms of their initial values (i_o, j_o, k_o) and in terms of $\underset{\sim}{g}$. We have (Gibbs, 1901)

$$\underset{\sim}{U} = \underset{\sim}{U_o} + \frac{2}{1 + g^2} \left[\underset{\sim}{g} \times \underset{\sim}{U_o} + \underset{\sim}{g} \times (\underset{\sim}{g} \times \underset{\sim}{U_o}) \right] \tag{19}$$

where $\underset{\sim}{U}$ and $\underset{\sim}{U_o}$ can take the following values:

$$\underset{\sim}{U} = \underset{\sim}{i}, \ \underset{\sim}{j}, \ \underset{\sim}{k}, \ \underset{\sim}{r}, \ \underset{\sim}{h},$$

$$\underset{\sim}{U_o} = \underset{\sim}{i_o}, \ \underset{\sim}{j_o}, \ \underset{\sim}{k_o}, \ \underset{\sim}{r_o}, \ \underset{\sim}{h_o}.$$

The differential equation for the variation of $\underset{\sim}{g}$ was established in the author's previous work (Musen, 1961). It is

$$\frac{d\underset{\sim}{g}}{dt} = \frac{1 + g^2}{4\sqrt{p}} \ Z \ (\underset{\sim}{r} + \underset{\sim}{r_o}) \tag{20}$$

From this equation we deduce for the secular perturbations in $\underset{\sim}{g}$

$$\frac{d\underset{\sim}{g}}{dt} = \frac{1}{2\pi} \ \frac{1 + g^2}{4\sqrt{p}} \int_0^{2\pi} Zo \ (\underset{\sim}{r} + \underset{\sim}{r_o}) \ \frac{r}{a} \ d\lambda \tag{21}$$

Taking into account

$$\underset{\sim}{r} = \underset{\sim}{r_o} + \frac{2}{1 + g^2} \ [\underset{\sim}{g} \times \underset{\sim}{r_o} + \underset{\sim}{g} \times (\underset{\sim}{g} \times \underset{\sim}{r_o})] \tag{22}$$

and decomposing $\underset{\sim}{g}$ along the axes $(\underset{\sim}{i_o}, \ \underset{\sim}{j_o}, \ \underset{\sim}{k_o})$,

$$\underset{\sim}{g} = g_1 \ \underset{\sim}{i_o} + g_2 \ \underset{\sim}{j_o} + g_3 \ \underset{\sim}{k_o} \tag{23}$$

we deduce the following three symmetrical scalar equations for the actual computation of the secular effects in $\underset{\sim}{g}$:

$$\frac{dg_1}{dt} = \frac{1}{2\pi} \int_0^{2\pi} \frac{Zo}{2\sqrt{p}} \left[(1 + g_1^2)\xi + (g_1 g_2 - g_3)\eta \right] \frac{r}{a} d\lambda, \quad (24)$$

$$\frac{dg_2}{dt} = \frac{1}{2\pi} \int_0^{2\pi} \frac{Zo}{2\sqrt{p}} \left[(g_1 g_2 + g_3)\xi + (1 + g_2^2)\eta \right] \frac{r}{a} d\lambda, \quad (25)$$

$$\frac{dg_3}{dt} = \frac{1}{2\pi} \int_0^{2\pi} \frac{Zo}{2\sqrt{p}} \left[(g_1 g_3 - g_2)\xi + (g_2 g_3 + g_1)\eta \right] \frac{r}{a} d\lambda \quad (26)$$

The components of the Hamiltonian vector with respect to the inertial system are determined using equation (19), thus

$$\underset{\sim}{h} = \underset{\sim}{h_o} + \frac{2}{1 + g2} \left[\underset{\sim}{g} \times \underset{\sim}{h_o} + \underset{\sim}{g} \times (\underset{\sim}{g} \times \underset{\sim}{h_o}) \right].$$

The Laplacian vector $\underset{\sim}{e} = e \underset{\sim}{P}$ can be obtained from the relation

$$\underset{\sim}{e} = c \underset{\sim}{h} \times \underset{\sim}{k}. \quad (27)$$

If we desire to have the standard elements Ω and i, we can easily deduce them from the vectorial equation

$$\underset{\sim}{e_1} \sin\Omega \sin i - \underset{\sim}{e_2} \cos\Omega \sin i + \underset{\sim}{e_3} \cos i = \underset{\sim}{k_o} + \frac{2}{1 + g^2} \left[\underset{\sim}{g} \times \underset{\sim}{k_o} + \underset{\sim}{g} \times (\underset{\sim}{g} \times \underset{\sim}{k_o}) \right].$$

For the secular perturbations in the orbital mean longitude we have

$$\frac{dL}{dt} = - \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 S_o}{a\sqrt{a}} d\lambda + \frac{p\sqrt{a}}{\sqrt{a} + \sqrt{p}} \left(u \frac{dv}{dt} - v \frac{du}{dt} \right).$$

In addition to secular effects the resonance caused by the commensurability of mean motions of the disturbed and the disturbing body is of great significance in the determination of the orbital stability over a long interval of time.

If the process of integration is performed analytically, a small divisor will appear in the denominator as a result of integration of the critical trigonometrical terms.

However, such resonance effects can be determined analytically only under the restriction that the eccentricity and the inclination are small and that only one critical argument is present in the trigonometrical development of the disturbing function. In addition, in the existing analytical theory the secular effect and the resonance effect are treated separately.

In the real problems we might have more than one critical argument and the mutual effect of the secular and the critical terms might not be negligible.

No restrictions on the eccentricity or the inclination can be imposed in the problem of motion of a satellite in the cislunar space. For all these reasons the author would like to suggest a method of special perturbations for treating the combined secular and resonance effects instead of an analytical development.

Let us assume that

$$\frac{n'}{n} \backsim \frac{p}{p'} \quad (p, p' > o \text{ and are relative primes}). \tag{29}$$

The arguments in the development of the right hand sides of the equations for the variations of elements are of the form

$$i\,L - i'\,M'.$$

Using the Liouville substitution (Liouville, 1836)

$$\theta = p L - p' M',$$

$$L = p' \varphi, \tag{30}$$

we bring the arguments to the form

$$(i p' - i' p) \varphi + \frac{i'}{p'} \theta$$

Evidently an argument will be a critical one if

$$i = kp, \quad i' = kp', \text{ (k is a positive integer).} \tag{31}$$

If the last condition is not satisfied, the term is a short periodic one. This leads to the conclusion that the resonance terms in the equations for variation of h and g are obtained by the process of averaging with respect to φ.

Putting

$$\Psi^{(u)}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left[\left(1 + \frac{r}{p} \right) X - \frac{\xi S}{p} \right] d\varphi, \tag{32}$$

$$\Psi^{(v)}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \left[\left(1 + \frac{r}{p} \right) Y - \frac{\eta S}{p} \right] d\varphi, \tag{33}$$

$$\psi^{(g_1)}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Z}{2\sqrt{p}} \left[(1 + g_1^2)\, \xi + (g_1 g_2 - g_3)\, \eta \right] d\varphi, \tag{34}$$

$$\psi^{(g_2)}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Z}{2\sqrt{p}} \left[(g_1 g_2 + g_3)\, \xi + (1 + g_2^2)\, \eta \right] d\varphi, \tag{35}$$

$$\psi^{(g_3)}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Z}{2\sqrt{p}} \left[(g_1 g_3 - g_2)\, \xi + (g_2 + g_3 g_1)\, \eta \right] d\varphi, \tag{36}$$

$$\psi^{(n)}(\theta) = - \frac{1}{2\pi} \int_0^{2\pi} \frac{3}{\sqrt{a}} \left(\frac{T}{\sqrt{p}} + \underset{\sim}{F} \cdot \underset{\sim}{g} \right) d\varphi, \tag{37}$$

$$\psi^{(L)}(\theta) = - \frac{1}{2\pi} \int_0 \frac{2\, r\, S}{\sqrt{a}}\, d\varphi + \frac{p\sqrt{a}}{\sqrt{a} + \sqrt{p}} \left[u\, \psi^{(v)} - v\, \psi^{(u)} \right], \tag{38}$$

$$A_0^{(x)} = \frac{1}{2\pi} \int_0^{2\pi} \psi^{(x)}(\theta)\, d\theta, \tag{39}$$

we deduce for the effect of the critical terms a set of equations of the form

$$\frac{dx}{dt} = \psi^{(x)}(\theta) - A_0^{(x)} = A_1 \cos\theta + B_1 \sin\theta + A_2 \cos 2\theta + B_2 \sin 2\theta + \cdots \tag{40}$$

In these integrals $d\varphi$ can be replaced by $\frac{1}{p'} \frac{r}{a} d\lambda$. The upper limit becomes $2 p' \pi$.

The computation of the position vector of the disturbing body is then performed using the formulae

$$\underset{\sim}{r'} = \underset{\sim}{P'} a' (\cos E' - e') + \underset{\sim}{Q'} a' \sqrt{1-e'^2} \sin E' \tag{41}$$

$$E' - e' \sin E' = \frac{p}{p'} (\lambda - cv \sin \lambda - cu \cos \lambda) - \frac{1}{p'} \theta \tag{42}$$

The effect of the critical terms decreases for higher multiples of the argument θ. Assuming that the effect of the critical terms with the arguments 3θ, 4θ, ... is negligible, we deduce the following expressions for the coefficients of four dominant critical terms

$$A_1 = \frac{\psi(o) - \psi(\pi)}{2} \tag{43}$$

$$B_1 = \frac{\psi(\pi/2) - \psi(3\pi/2)}{2} \tag{44}$$

$$A_2 = \frac{\psi(o) - \psi(\pi/2) + \psi(\pi) - \psi(3\pi/2)}{4} \tag{45}$$

$$B_2 = \psi(\pi/4) - \frac{\sqrt{2}+1}{4} [\psi(o) + \psi(\pi/2)] + \frac{\sqrt{2}-1}{4} [\psi(\pi) + (3\pi/2)] \tag{46}$$

Thus the problem of determining the coefficients of the first four dominant terms is reduced to the computation of four single integrals for each scalar element. There is no necessity to perform a double harmonic

analysis in full. This method of reducing the problem to the computation of single integrals was suggested by Liouville (1836), but it was not used before, because the numerical evaluation of all these integrals requires a large number of subdivision points and the computation of a large number of special values of X, Y, Z, S, T. Liouville's approach can be used conveniently only now with the advent of modern electronic digital computers.

After the coefficients A_1, B_1, A_2, B_2, ... are determined the equations (40) are combined together with the corresponding equations giving the secular derivatives of the elements. The combined equations are then integrated numerically. The value of the critical argument to be substituted into (40) is given by the equation

$$\theta = i\,[L_0 + \Delta L + n_0\,(t - t_0)] - i'\,[M_0 + \Delta M_0 + n_0'\,(t - t_0)]$$

$$+ \int_{t_0}^{t} dt \int_{t_0}^{t} \left(i\,\frac{dn}{dt} - i'\,\frac{dn'}{dt}\right) dt$$

Conclusion

The combined effect of the secular and the critical terms is decisive in the determination of the orbital stability over a long interval of time.

The combination of Halphen's and Liouville's methods is able to provide an answer valid at least over an interval of thousands of years in the case of a minor planet and for a number of years in the case of a satellite moving in the cislunar space. Our application of the method of secular effects lead to some conclusions which were not anticipated beforehand. It remains to be seen what interesting conclusions will follow from the superposition of the secular and the critical terms and from the computation of their direct actions as well as their interactions.

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