NASr-21(07)

MEMORANDUM
RM-4775-NASA
OCTOBER 1965

FACILITY FORM 602
N66 11757
(ACCESSION NUMBER)
35
(PAGES)
CR 67953
(NASA CR OR TMX OR AD NUMBER)
(THRU)
1
(CODE)
23
(CATEGORY)

COMPUTATIONAL RESULTS FOR DIFFUSE TRANSMISSION AND REFLECTION FOR HOMOGENEOUS FINITE SLABS WITH ISOTROPIC SCATTERING

GPO PRICE $ _____
CFSTI PRICE(S) $ _____
Hard copy (HC) _____
Microfiche (MF) _____
ff 653 July 65

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PREPARED FOR:
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

The RAND Corporation
SANTA MONICA • CALIFORNIA

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This research is sponsored by the National Aeronautics and Space Administration under Contract No. NASr-21. This report does not necessarily represent the views of the National Aeronautics and Space Administration.

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PREFACE

A basic problem in the theory of planetary and stellar atmospheres is the determination of the diffuse reflection and transmission patterns. In this Memorandum, prepared for the National Aeronautics and Space Administration, tables and graphs are developed from the results of equations for the diffuse reflection and transmission coefficients. These graphs are useful aids for developing an intuitive understanding of how the reflection and transmission coefficients are influenced by changes in the albedo for single scattering, the optical thickness of the atmosphere, and the angle of incidence.

This work is part of a continuing study of numerical methods of solution of radiative transfer processes. It is an important first step toward solving realistic inverse problems in which the authors infer the vertical stratification of an atmosphere based on satellite measurements of diffusely reflected radiation.

SUMMARY

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Diffuse reflection and transmission coefficients are computed for radiative transfer in homogeneous slabs of finite thickness with isotropic scattering. The results are summarized in graphs of these coefficients for a complete range of albedos and thicknesses. Representative tables of transmission coefficients are also included.

Author

CONTENTS

PREFACE iii

SUMMARY v

LIST OF FIGURES ix

Section

I. INTRODUCTION 1

II. SOME SELECTED TABLES OF TRANSMITTED INTENSITIES 5

III. GRAPHIC SURVEY OF TRANSMITTED INTENSITIES 8

IV. GRAPHIC SURVEY OF REFLECTED INTENSITIES 20

REFERENCES 28

LIST OF FIGURES

1. Transmitted intensities t per unit incident flux albedo $\lambda = 0.1$ 9

2. Transmitted intensities t per unit incident flux albedo $\lambda = 0.2$ 10

3. Transmitted intensities t per unit incident flux albedo $\lambda = 0.3$ 11

4. Transmitted intensities t per unit incident flux albedo $\lambda = 0.4$ 12

5. Transmitted intensities t per unit incident flux albedo $\lambda = 0.5$ 13

6. Transmitted intensities t per unit incident flux albedo $\lambda = 0.6$ 14

7. Transmitted intensities t per unit incident flux albedo $\lambda = 0.7$ 15

8. Transmitted intensities t per unit incident flux albedo $\lambda = 0.8$ 16

9. Transmitted intensities t per unit incident flux albedo $\lambda = 0.9$ 17

10. Transmitted intensities t per unit incident flux albedo $\lambda = 0.95$ 18

11. Transmitted intensities t per unit incident flux albedo $\lambda = 1.0$ 19

12. Reflected intensities t per unit incident flux albedo $\lambda = 0.1$ 21

13. Reflected intensities t per unit incident flux albedo $\lambda = 0.2$ 21

14. Reflected intensities t per unit incident flux albedo $\lambda = 0.3$ 22

15. Reflected intensities t per unit incident flux albedo $\lambda = 0.4$ 22

16. Reflected intensities t per unit incident flux albedo $\lambda = 0.5$ 23

17. Reflected intensities t per unit incident flux
albedo λ = 0.6 23

18. Reflected intensities t per unit incident flux
albedo λ = 0.7 24

19. Reflected intensities t per unit incident flux
albedo λ = 0.8 24

20. Reflected intensities t per unit incident flux
albedo λ = 0.9 25

21. Reflected intensities t per unit incident flux
albedo λ = 0.95 26

22. Reflected intensities t per unit incident flux
albedo λ = 1.0 27

I. INTRODUCTION

A complete set of transmission and reflection functions has been computed for the problem of radiative transfer in finite homogeneous slabs with isotropic scattering. We give the basic equations, and present selected tables of transmission functions, as well as a graphical survey of the transmission and reflection functions for a full range of albedos and thicknesses.

Consider a plane-parallel slab of finite optical thickness x, whose upper surface is uniformly illuminated by parallel rays of net flux π per unit normal area. Let u be the direction cosine of the incident radiation, and v be that of the emergent diffusely-scattered radiation. The slab is assumed to be homogeneous and isotropic, with constant albedo for single scattering λ. Let

$t(x,v,u)$ = the intensity of the transmitted radiation having direction cosine v, with direction cosine of incident radiation u, and thickness of the slab x. (1)

Let us introduce the quantity $T(x,v,u)$ by means of the formula,

$$t(x,v,u) = \frac{T(x,v,u)}{4v} \quad . \tag{2}$$

Similarly, we define

$r(x,v,u)$ = the intensity of the reflected radiation having direction cosine v, with direction cosine of incident radiation u, and thickness of the slab x, (3)

and we introduce the function S,

$$r(x,v,u) = \frac{S(x,v,u)}{4v} \quad . \tag{4}$$

The T and S functions correspond to the transmission and reflection functions of Chandrasekhar.[1]

It is well known that the S and T functions satisfy the equations,[1,2,3]

$$\begin{aligned} \frac{\partial T}{\partial x} = - \frac{1}{v} T + \lambda \left\{ e^{-x/u} + \frac{1}{2} \int_0^1 T(x,v',u) \frac{dv'}{v'} \right\} \\ \cdot \left\{ 1 + \frac{1}{2} \int_0^1 S(x,v,u') \frac{du'}{u'} \right\} \quad , \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{\partial S}{\partial x} = - \left(\frac{1}{u} + \frac{1}{v}\right) S + \lambda \left\{ 1 + \frac{1}{2} \int_0^1 S(x,v',u) \frac{dv'}{v'} \right\} \\ \cdot \left\{ 1 + \frac{1}{2} \int_0^1 S(x,v,u') \frac{du'}{u'} \right\} \quad . \end{aligned} \tag{6}$$

By allowing the thickness of the slab to approach zero, we obtain the initial conditions

$$T(0,v,u) = 0 \ , \qquad S(0,v,u) = 0 \quad . \tag{7}$$

Note that the T and S functions are symmetric in u and v.

The integrals which appear in the basic Eqs. (5) and (6) are approximated by Gaussian quadrature. Let N be the order of the quadrature formula, let v_i, $i = 1,2,\ldots,N$ be the appropriate abscissas, and let w_i, $i = 1,2,\ldots,N$ be the Christoffel weights (Ref. 3). We

introduce the functions $T_{ij}(x)$ and $S_{ij}(x)$, $x \geq 0$, as solutions of the system of ordinary differential equations with initial conditions at $x = 0$,

$$\dot{T}_{ij} = -\frac{1}{v_i} T_{ij} + \lambda \left\{ e^{-\frac{x}{v_j}} + 0.5 \sum_{k=1}^{N} T_{kj} \frac{w_k}{v_k} \right\} \cdot \left\{ 1 + 0.5 \sum_{k=1}^{N} S_{ik} \frac{w_k}{v_k} \right\} ,$$

$$T_{ij}(0) = 0 \quad ,$$

$$\dot{S}_{ij} = -\left(\frac{1}{v_i} + \frac{1}{v_j}\right) S_{ij} + \lambda \left\{ 1 + 0.5 \sum_{k=1}^{N} S_{kj} \frac{w_k}{v_k} \right\} \cdot \left\{ 1 + 0.5 \sum_{k=1}^{N} S_{ik} \frac{w_k}{v_k} \right\} , \tag{8}$$

$$S_{ij}(0) = 0 \quad ,$$

$$i = 1,2,\ldots,N \ , \qquad j = 1,2,\ldots,N \ .$$

Reflected and transmitted intensities are produced using the formulas,

$$r_{ij}(x) = \frac{S_{ij}(x)}{4v_i} \quad ,$$

$$t_{ij}(x) = \frac{T_{ij}(x)}{4v_i} \quad . \tag{9}$$

We integrate numerically the differential equations of Eq. (8) using a quadrature formula of order N = 7, a step size of $\Delta x = 0.005$, an Adams-Moulton integration method, and an IBM 7044 computer. Reflected intensities have be tabulated in a previous work.[2] In the next section we present selected tables of the transmitted intensities, and in the third and fourth sections we give graphs of the transmitted and reflected intensities. The complete survey requires three-quarters of an hour of computing.

Alternatively, S and T functions may be calculated from the X and Y functions of Chandrasekhar.[1] We produce S and T functions of from the X and Y functions computed earlier.[4] Complete agreement to at least four places is found. These results are consistent with current knowledge of source functions[5,6] and internal intensities.[7]

The methods used here, together with the quasilinearization technique,[8] enable us to treat various inverse problems.[9]

II. SOME SELECTED TABLES OF TRANSMITTED INTENSITIES

Transmitted intensities are given in Tables 1 through 4 for four different slabs. There are seven angles of incidence and seven angles of transmission. The seven angles, which are arc cosines of the abscissas $\{v_i\}$, are

$$\begin{aligned} &\text{Angle } 1 \cong 88.54 \text{ degrees} , \\ &\text{Angle } 2 \cong 82.57 \text{ degrees} , \\ &\text{Angle } 3 \cong 72.72 \text{ degrees} , \\ &\text{Angle } 4 \cong 60.00 \text{ degrees} , \\ &\text{Angle } 5 \cong 45.34 \text{ degrees} , \\ &\text{Angle } 6 \cong 29.45 \text{ degrees} , \\ &\text{Angle } 7 \cong 12.95 \text{ degrees} . \end{aligned} \tag{10}$$

The incident angle is constant in each row; the transmitted angle is constant in each column. Thus, for example, the intensity in the direction 45.34 degrees from the normal is 6.984×10^{-3} when the incident angle is 60.0 degrees, the albedo is 0.1, and the thickness is 1. (This is found in the fourth row, fifth column of Table 1.)

Table 1

TRANSMITTED INTENSITIES FOR A SLAB WITH ALBEDO $\lambda = 0.1$, THICKNESS $x = 1$

Incident angle ↓ Transmitted angle →

	1	2	3	4	5	6	7
1	8.456E-06	1.326E-05	9.615E-05	1.991E-04	2.439E-04	2.555E-04	2.564E-04
2	6.734E-05	1.533E-04	7.556E-04	1.288E-03	1.465E-03	1.484E-03	1.468E-03
3	1.123E-03	1.737E-03	3.199E-03	3.996E-03	4.068E-03	3.927E-03	3.806E-03
4	3.913E-03	4.982E-03	6.725E-03	7.278E-03	6.984E-03	6.562E-03	6.286E-03
5	6.739E-03	7.967E-03	9.625E-03	9.818E-03	9.175E-03	8.513E-03	8.110E-03
6	8.742E-03	9.997E-03	1.151E-02	1.143E-02	1.055E-02	9.726E-03	9.241E-03
7	9.821E-03	1.107E-02	1.249E-02	1.225E-02	1.124E-02	1.034E-02	9.815E-03

Table 2

TRANSMITTED INTENSITIES FOR A SLAB WITH ALBEDO $\lambda = 0.5$, THICKNESS $x = 3$

Incident angle ↓ Transmitted angle →

	1	2	3	4	5	6	7
1	3.529E-05	4.209E-05	5.672E-05	1.011E-04	1.863E-04	2.686E-04	3.179E-04
2	2.138E-04	2.551E-04	3.460E-04	6.264E-04	1.137E-03	1.610E-03	1.886E-03
3	6.621E-04	7.954E-04	1.116E-03	2.050E-03	3.518E-03	4.769E-03	5.468E-03
4	1.987E-03	2.424E-03	3.450E-03	5.722E-03	8.626E-03	1.086E-02	1.204E-02
5	5.147E-03	6.182E-03	8.325E-03	1.213E-02	1.633E-02	1.931E-02	2.082E-02
6	9.193E-03	1.084E-02	1.398E-02	1.891E-02	2.392E-02	2.731E-02	2.896E-02
7	1.218E-02	1.422E-02	1.794E-02	2.347E-02	2.887E-02	3.241E-02	3.411E-02

Table 3

TRANSMITTED INTENSITIES FOR A SLAB WITH ALBEDO λ = 1.0, THICKNESS x = 3

Transmitted angle →

Incident angle ↓	1	2	3	4	5	6	7
1	1.672E-03	2.035E-03	2.552E-03	3.163E-03	3.733E-03	4.118E-03	4.307E-03
2	1.034E-02	1.258E-02	1.578E-02	1.958E-02	2.309E-02	2.540E-02	2.651E-02
3	2.979E-02	3.627E-02	4.560E-02	5.672E-02	6.657E-02	7.274E-02	7.560E-02
4	6.214E-02	7.577E-02	9.547E-02	1.182E-01	1.369E-01	1.478E-01	1.526E-01
5	1.031E-01	1.256E-01	1.575E-01	1.924E-01	2.192E-01	2.339E-01	2.400E-01
6	1.409E-01	1.711E-01	2.132E-01	2.574E-01	2.898E-01	3.067E-01	3.133E-01
7	1.650E-01	1.999E-01	2.480E-01	2.974E-01	3.327E-01	3.506E-01	3.574E-01

Table 4

TRANSMITTED INTENSITIES FOR A SLAB WITH ALBEDO λ = 1.0, THICKNESS x = 30

Transmitted angle →

Incident angle ↓	1	2	3	4	5	6	7
1	2.344E-04	2.350E-04	3.566E-04	4.385E-04	5.183E-04	5.835E-04	6.236E-04
2	1.448E-03	1.760E-03	2.202E-03	2.708E-03	3.201E-03	3.604E-03	3.852E-03
3	4.163E-03	5.063E-03	6.334E-03	7.788E-03	9.206E-03	1.036E-02	1.108E-02
4	8.615E-03	1.048E-02	1.311E-02	1.612E-02	1.905E-02	2.145E-02	2.292E-02
5	1.432E-02	1.741E-02	2.178E-02	2.678E-02	3.166E-02	3.564E-02	3.809E-02
6	1.997E-02	2.428E-02	3.038E-02	3.735E-02	4.415E-02	4.971E-02	5.313E-02
7	2.388E-02	2.904E-02	3.634E-02	4.468E-02	5.281E-02	5.946E-02	6.355E-02

III. GRAPHIC SURVEY OF TRANSMITTED INTENSITIES

The transmitted intensities are plotted for λ = 0.1, 0.2, ... 0.9, 0.95, 1.0 in Figs. 1 through 11, respectively. In each figure there are three separate graphs, one for each of three incident angles approximately 88.5, 45.3, and 13.0 degrees. The abscissas in each graph are the transmitted angles. The ordinates are the intensities on a logarithmic scale. Many curves are shown on each graph; these correspond to various optical thicknesses. Note that as the slabs are made increasingly thicker, limiting shapes of the intensity curves are obtained for all albedos.

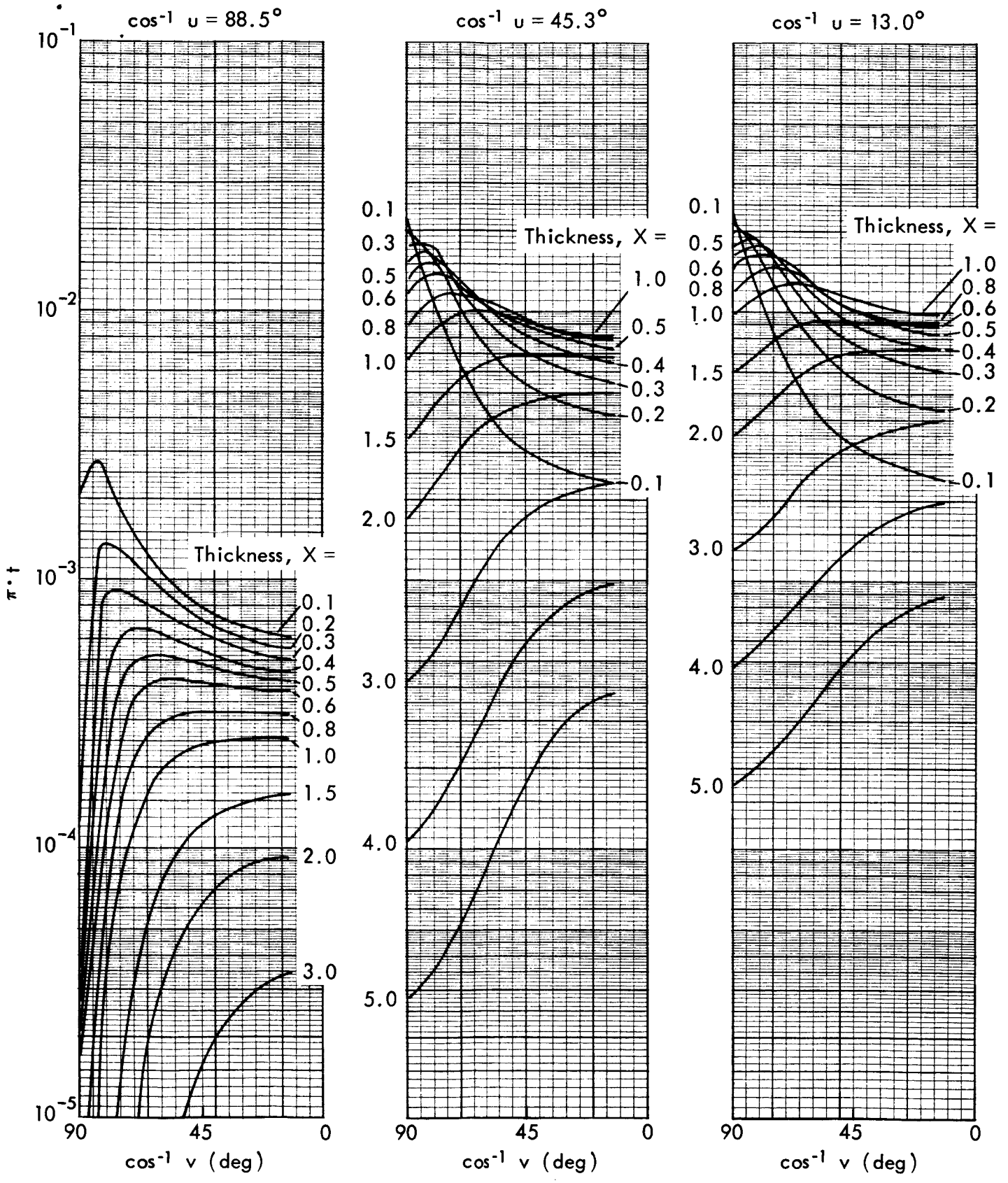

Fig. 1—Transmitted intensities t per unit incident flux albedo λ = 0.1

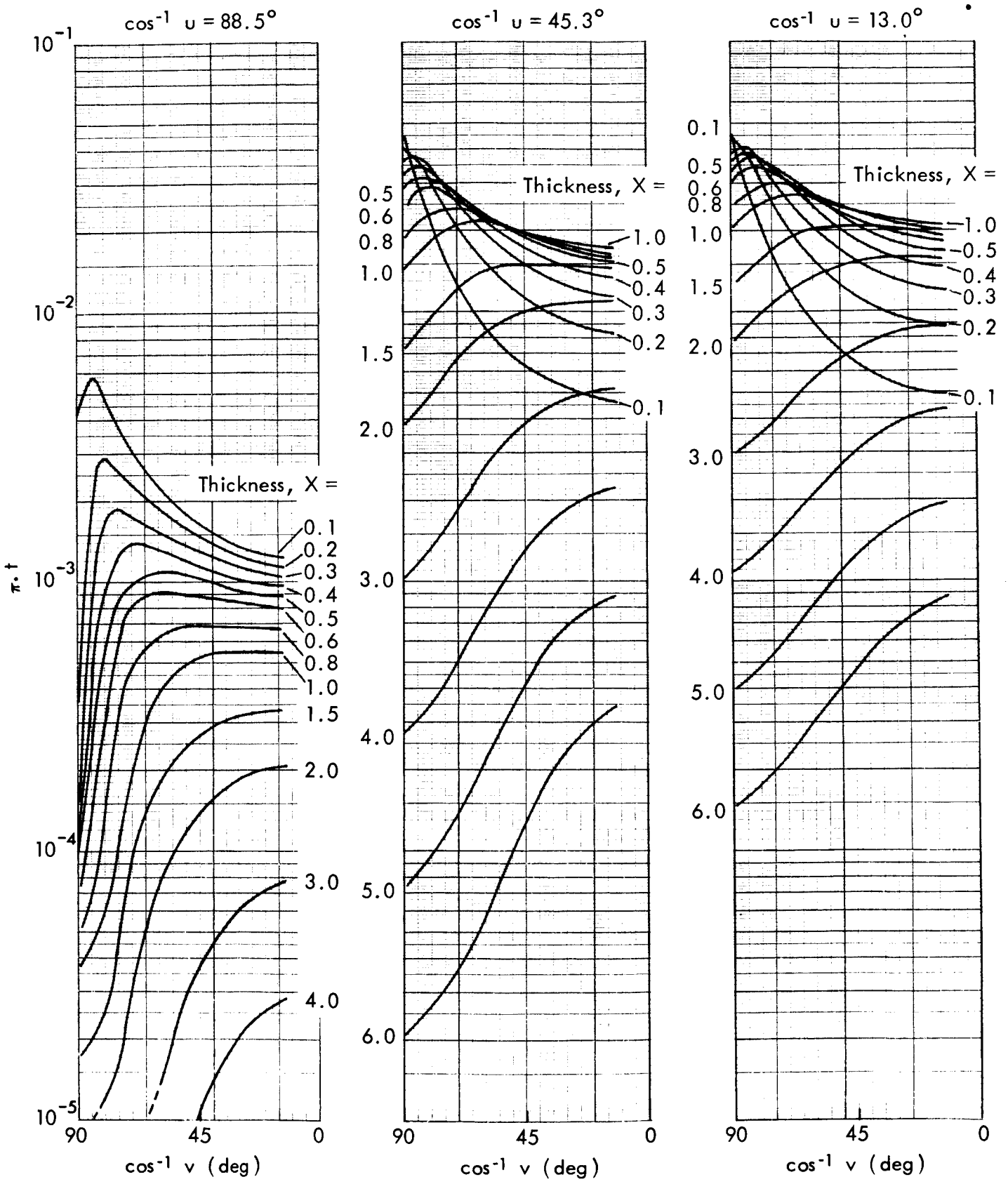

Fig. 2—Transmitted intensities t per unit incident flux albedo λ = 0.2

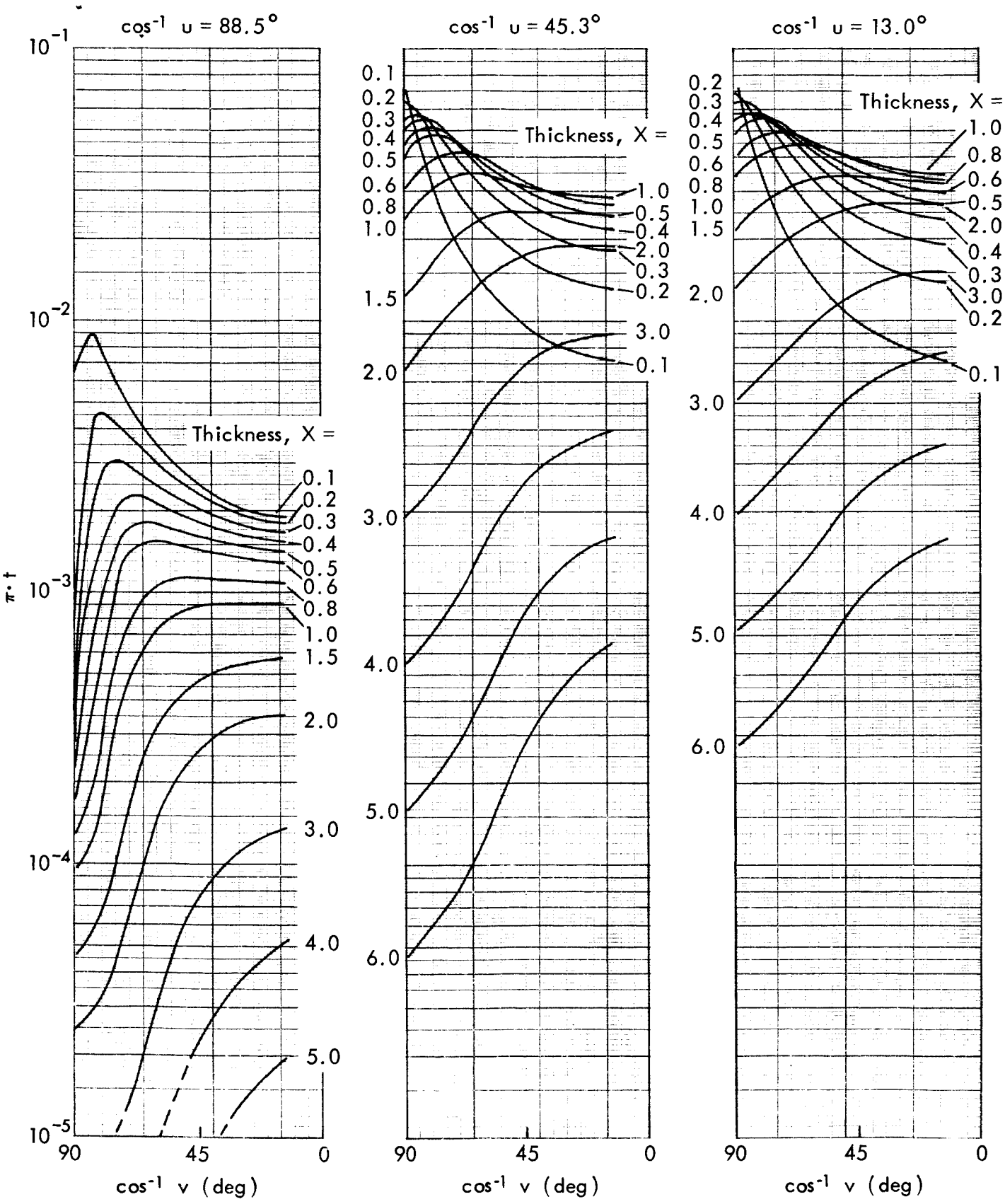

Fig. 3—Transmitted intensities t per unit incident flux
albedo λ = 0.3

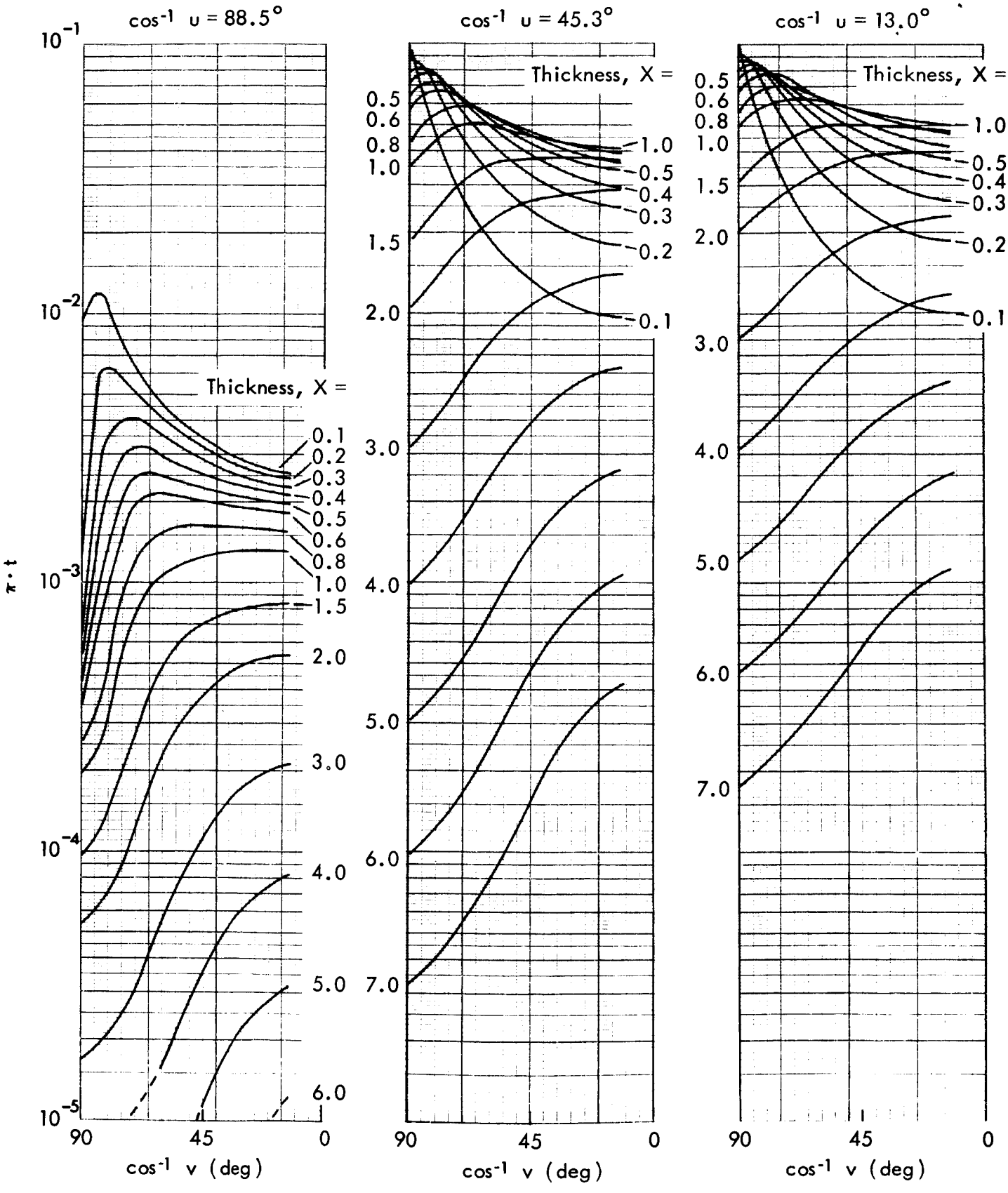

Fig. 4—Transmitted intensities t per unit incident flux albedo $\lambda = 0.4$

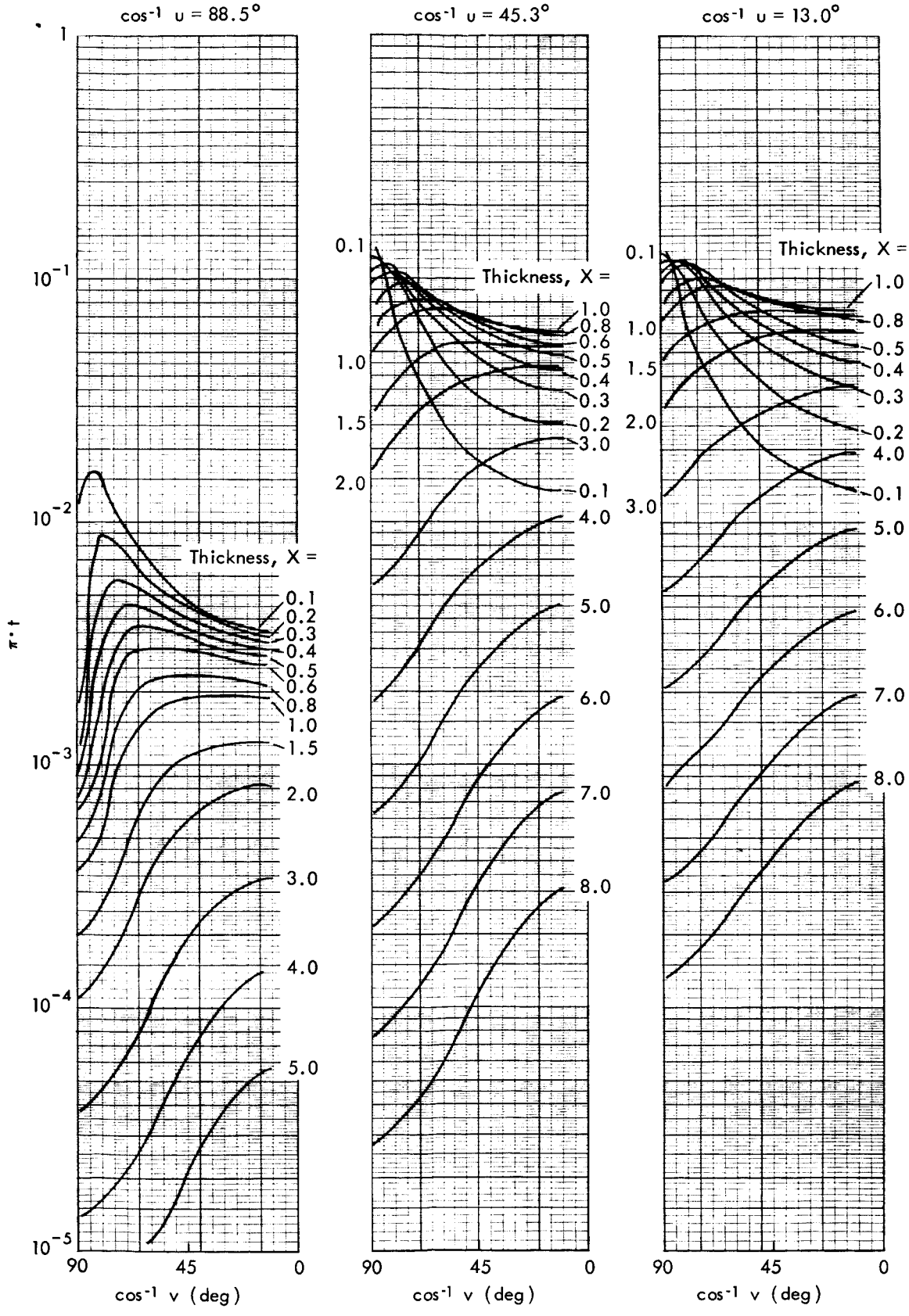

Fig.5—Transmitted intensities t per unit incident flux
albedo λ = 0.5

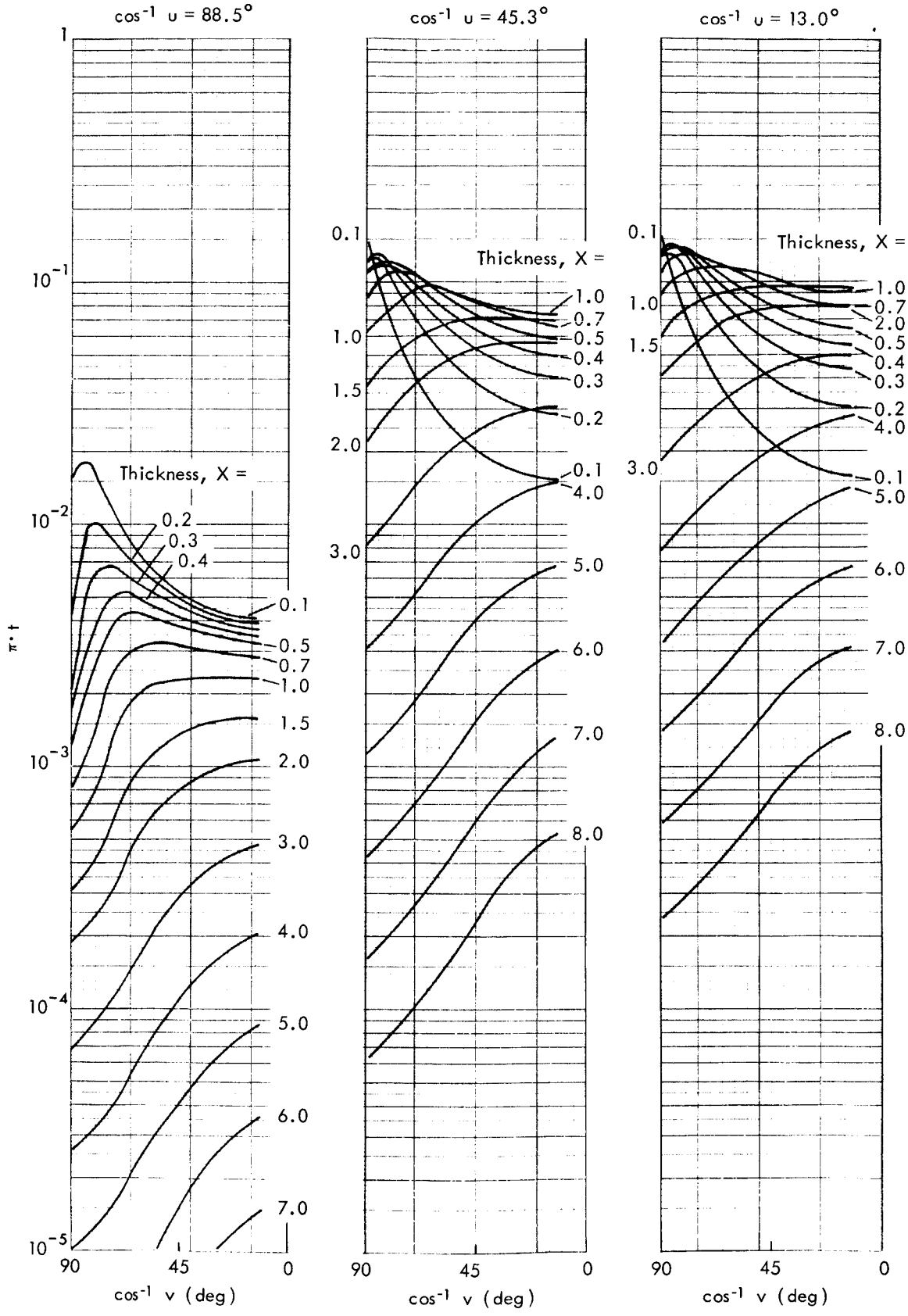

Fig. 6—Transmitted intensities t per unit incident flux
albedo λ = 0.6

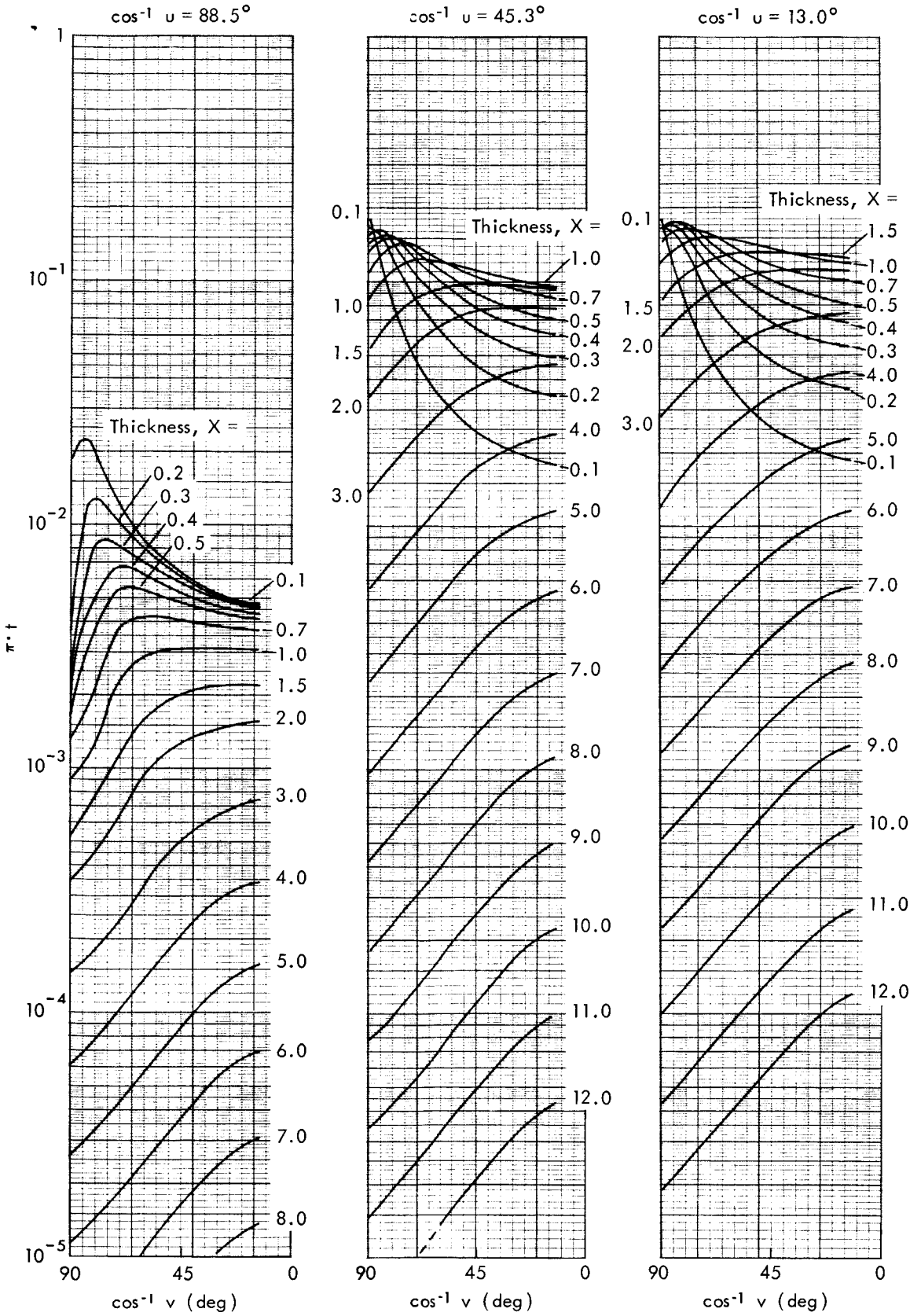

Fig. 7—Transmitted intensities t per unit incident flux
albedo $\lambda = 0.7$

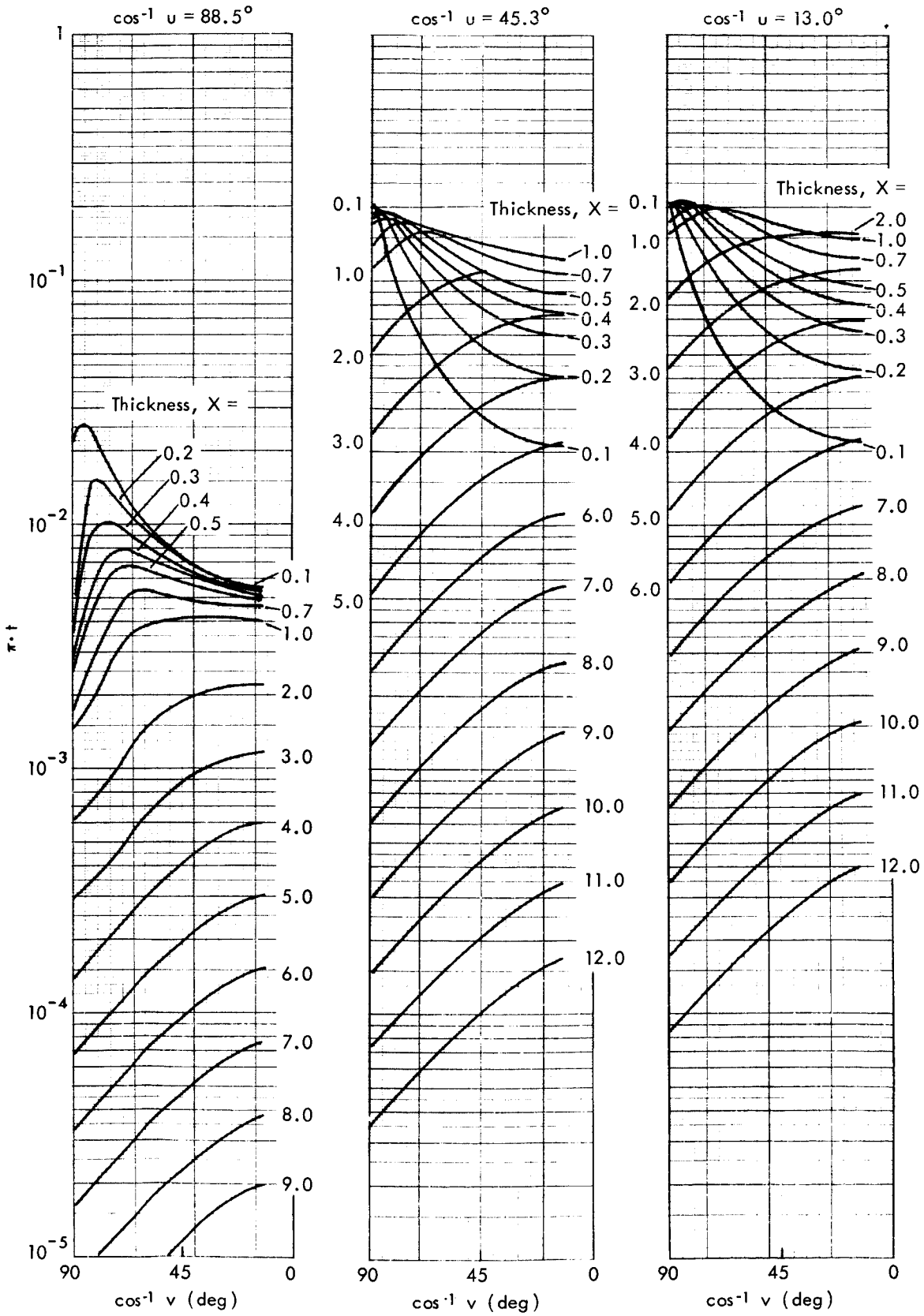

Fig. 8—Transmitted intensities t per unit incident flux
albedo λ = 0.8

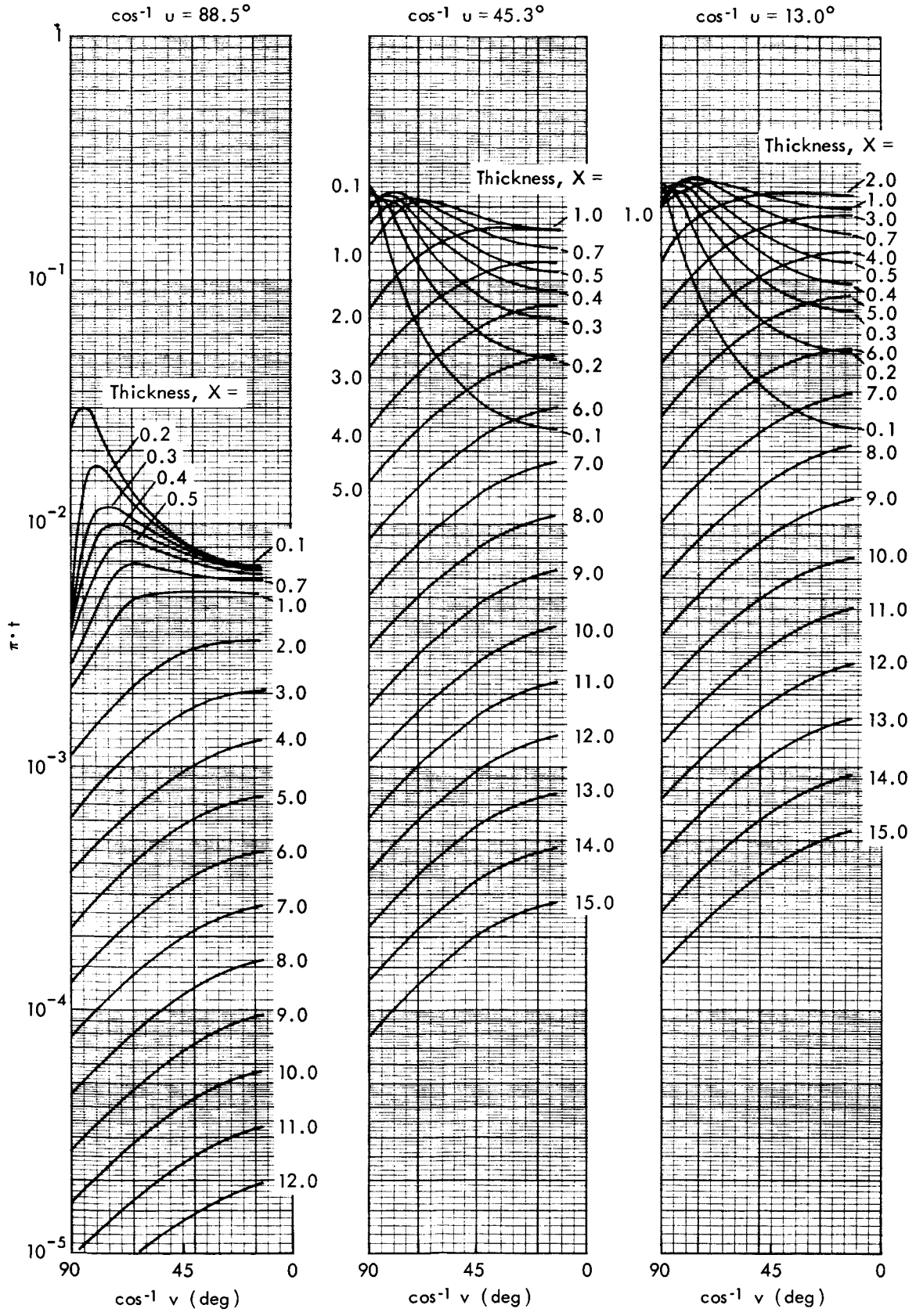

Fig. 9—Transmitted intensities t per unit incident flux
albedo $\lambda = 0.9$

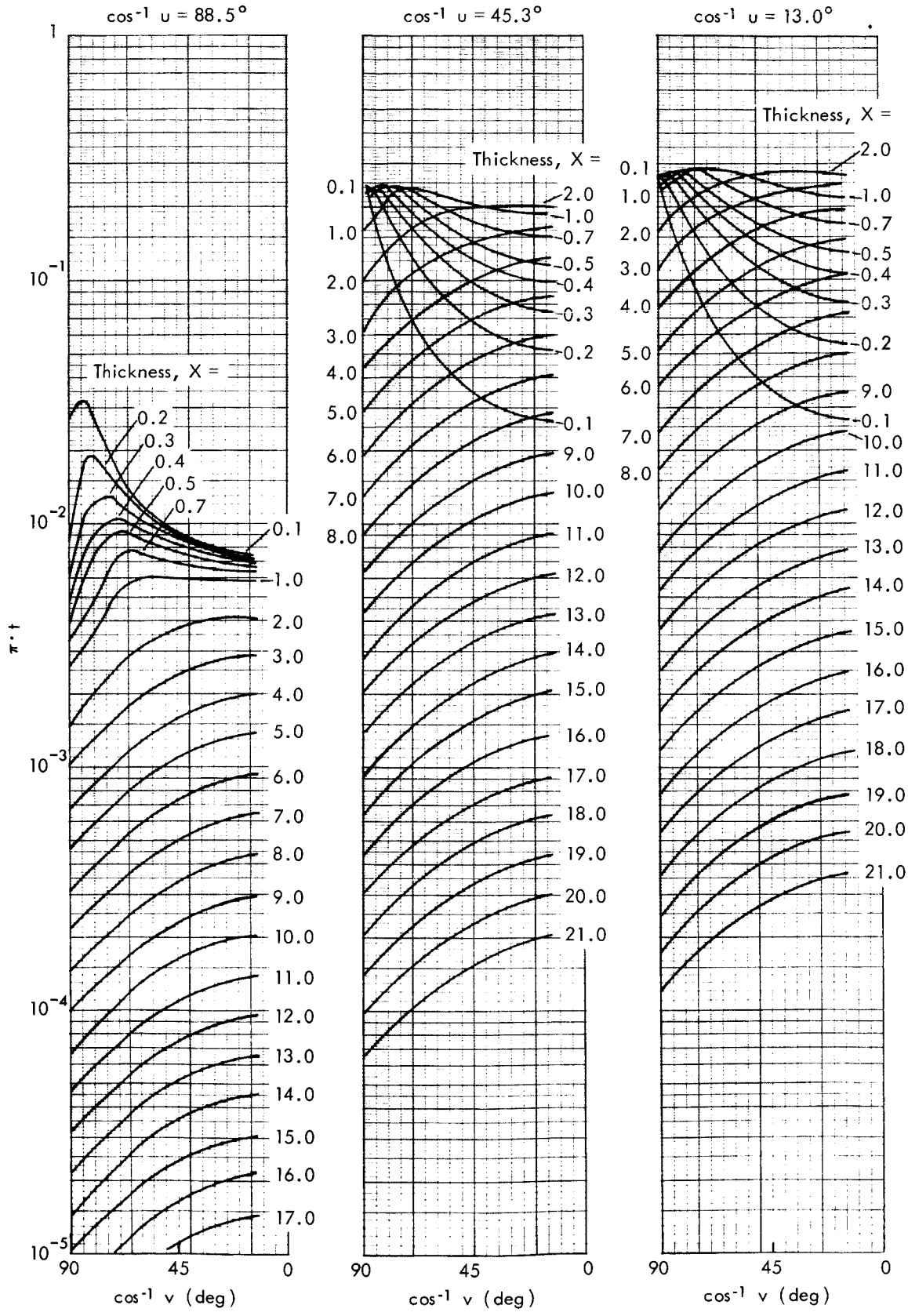

Fig. 10—Transmitted intensities t per unit incident flux
albedo λ = 0.95

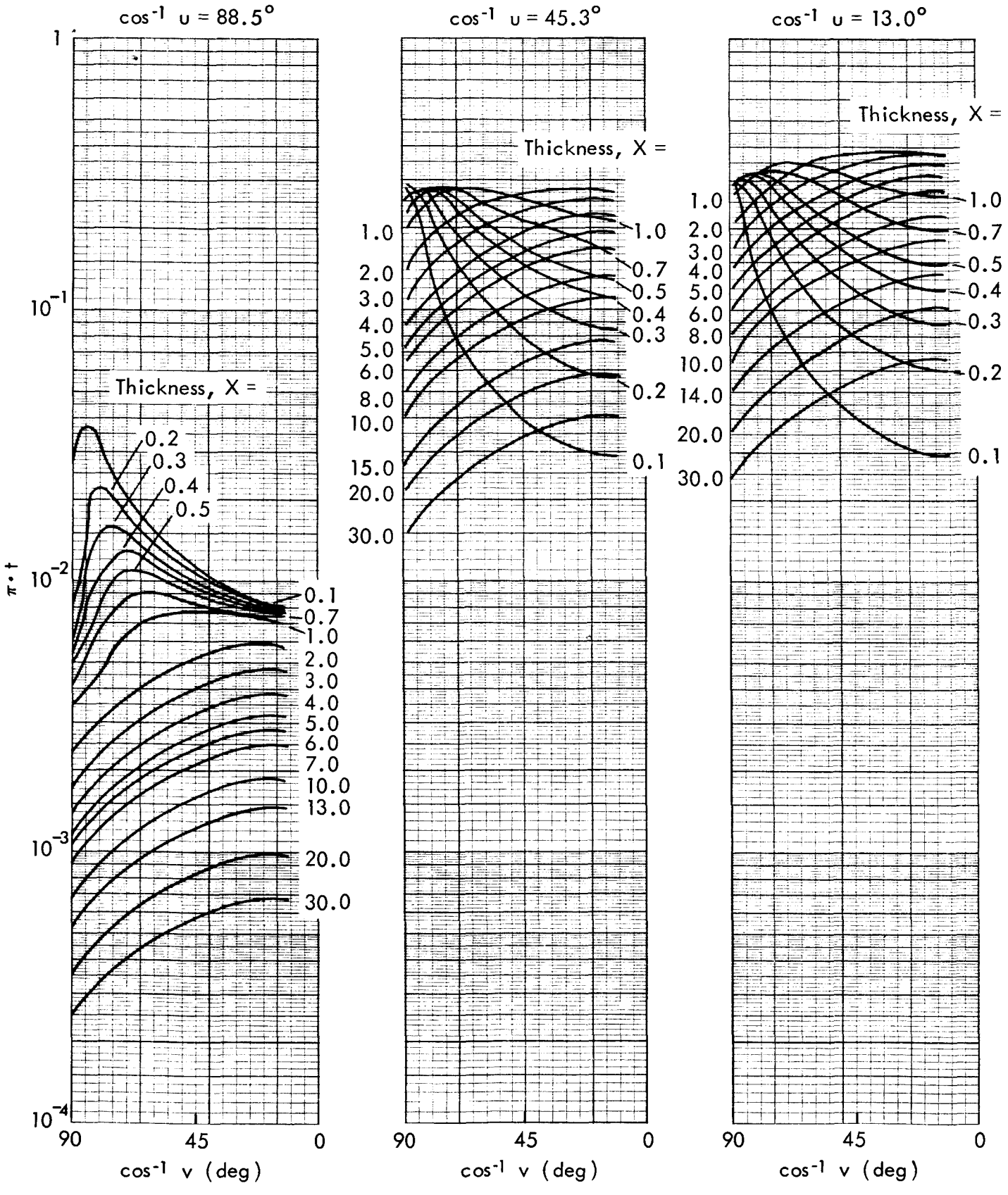

Fig. 11—Transmitted intensities t per unit incident flux albedo $\lambda = 1.0$

IV. GRAPHIC SURVEY OF REFLECTED INTENSITIES

Figures 12 through 22 are graphs of the reflected intensities plotted against the emergent angle. The three incident angles are about 88.5, 45.3, and 13.0 degrees. The ordinates are intensities on a linear scale. For all albedos but one, the intensity curves reach limiting values corresponding to the semi-infinite slab as the thickness is increased. For the case of conservative scattering ($\lambda = 1$), Fig. 22 shows the intensity patterns for the semi-infinite slab by means of dashed lines.

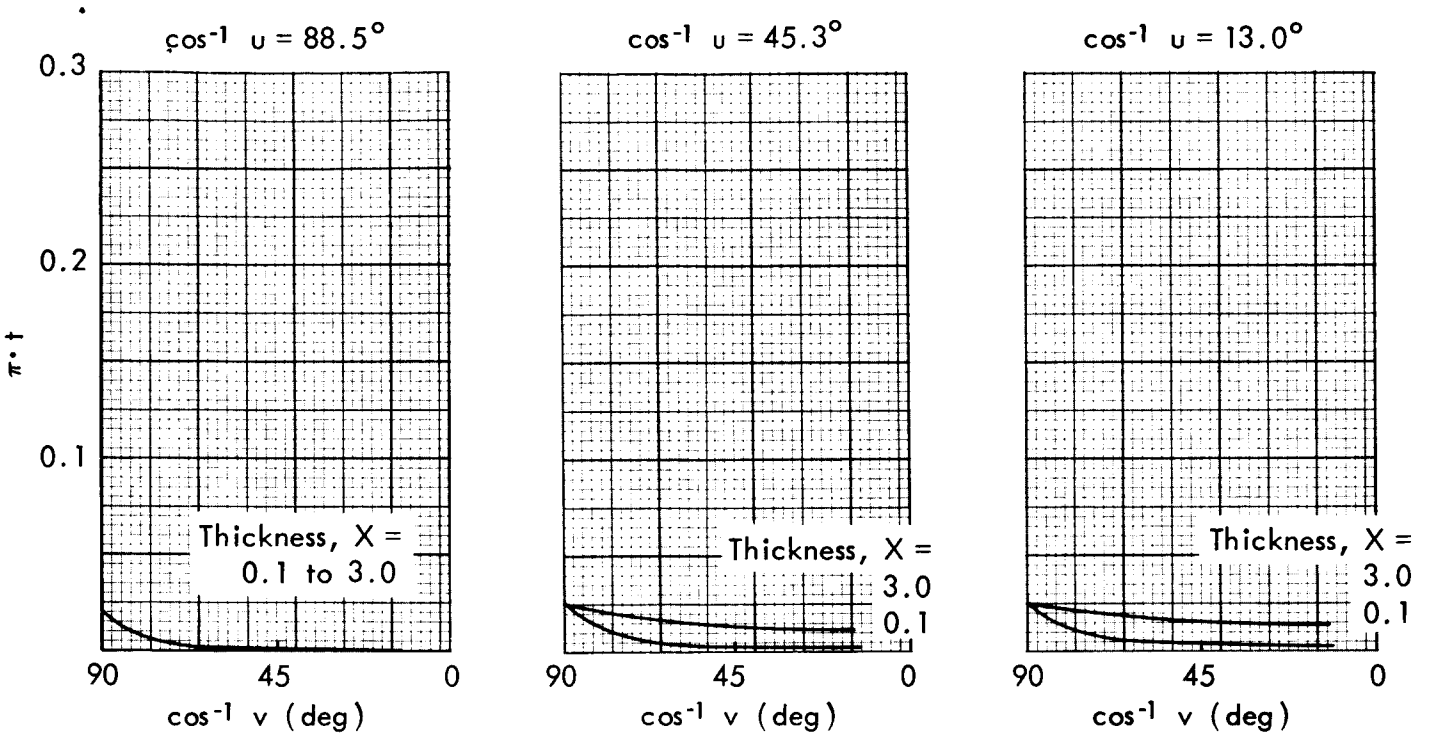

Fig. 12—Reflected intensities t per unit incident flux
albedo λ = 0.1

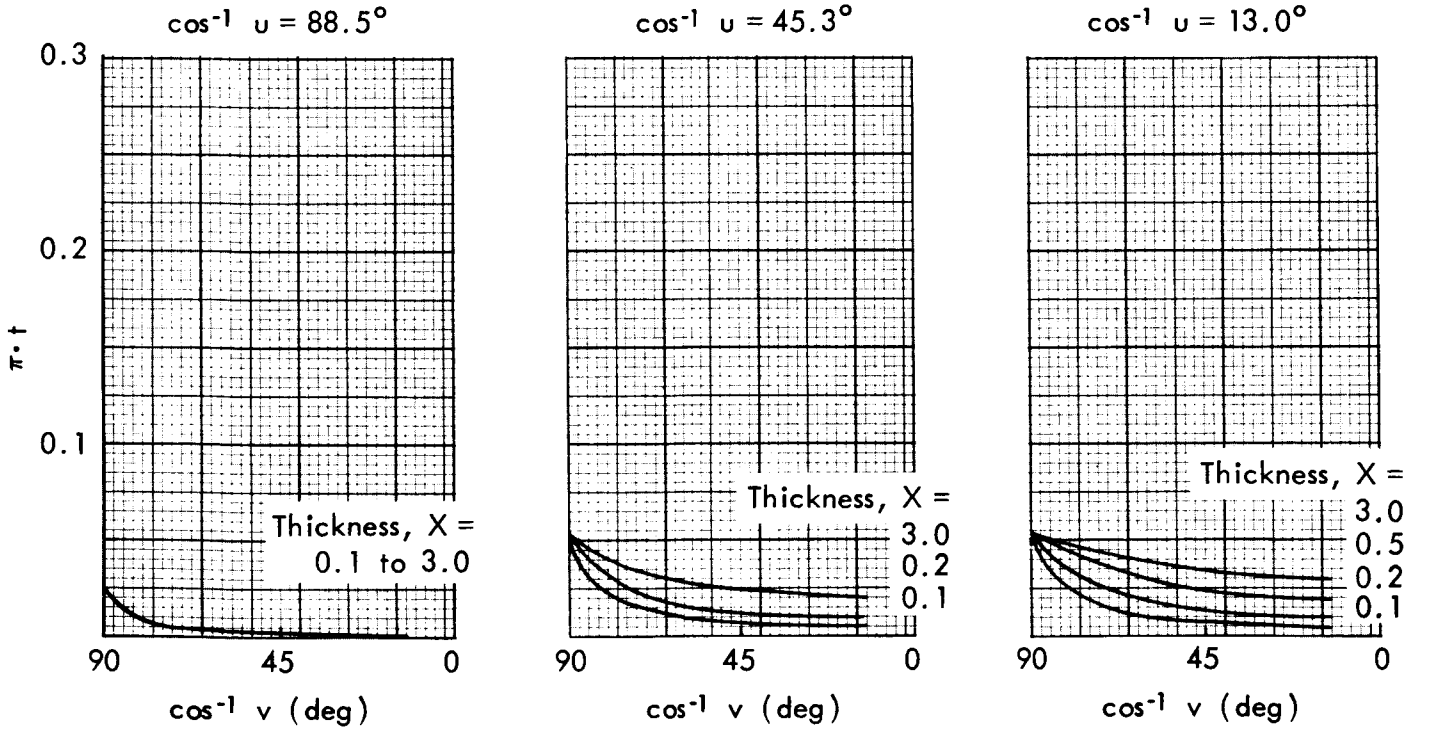

Fig. 13—Reflected intensities t per unit incident flux
albedo λ = 0.2

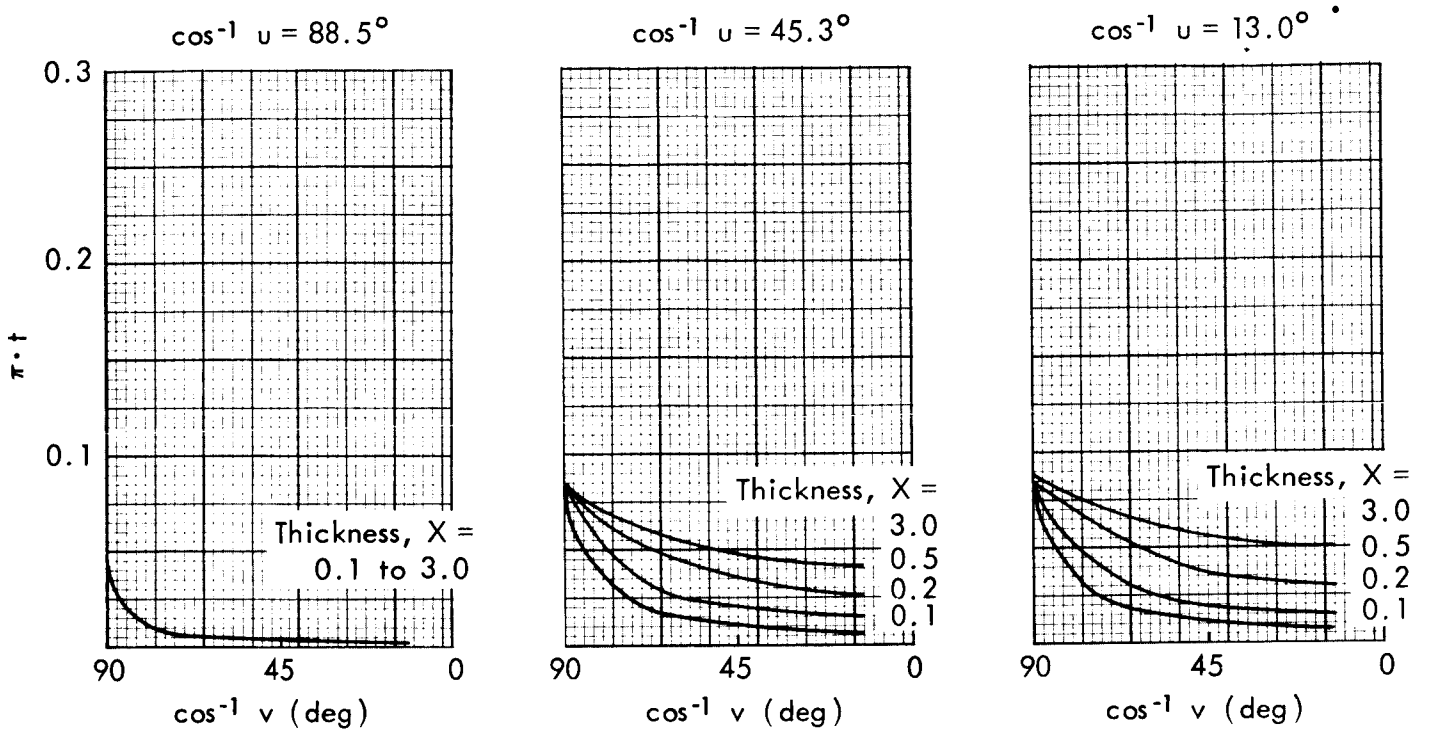

Fig. 14—Reflected intensities t per unit incident flux
albedo λ = 0.3

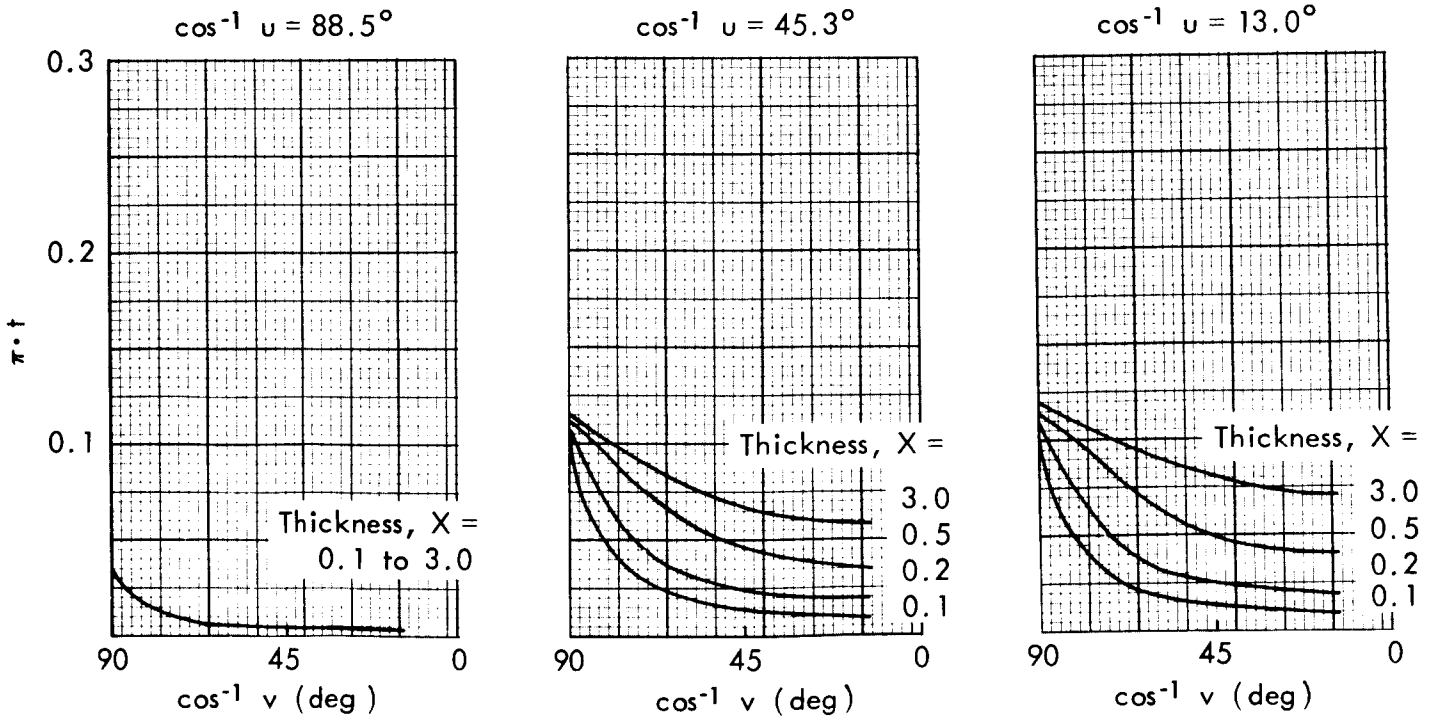

Fig. 15—Reflected intensities t per unit incident flux
albedo λ = 0.4

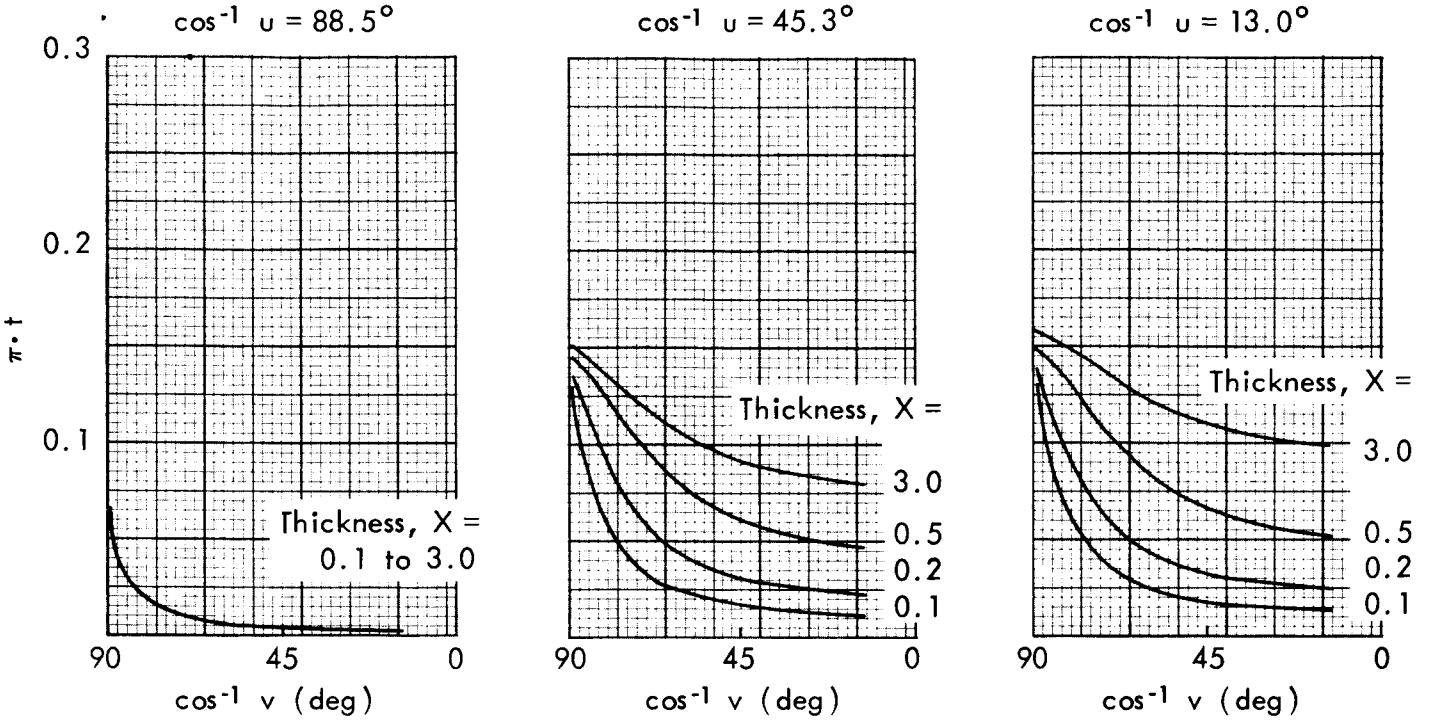

Fig. 16—Reflected intensities t per unit incident flux
albedo λ = 0.5

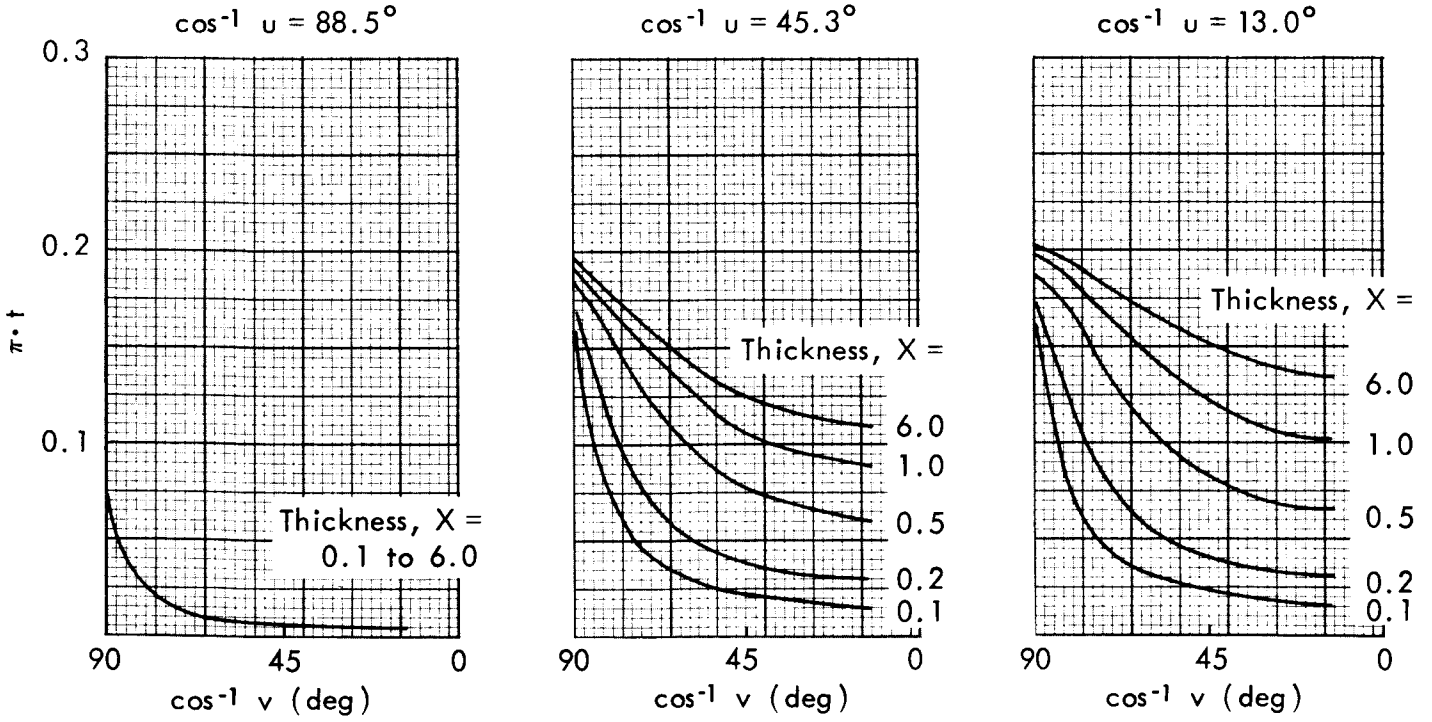

Fig. 17—Reflected intensities t per unit incident flux
albedo λ = 0.6

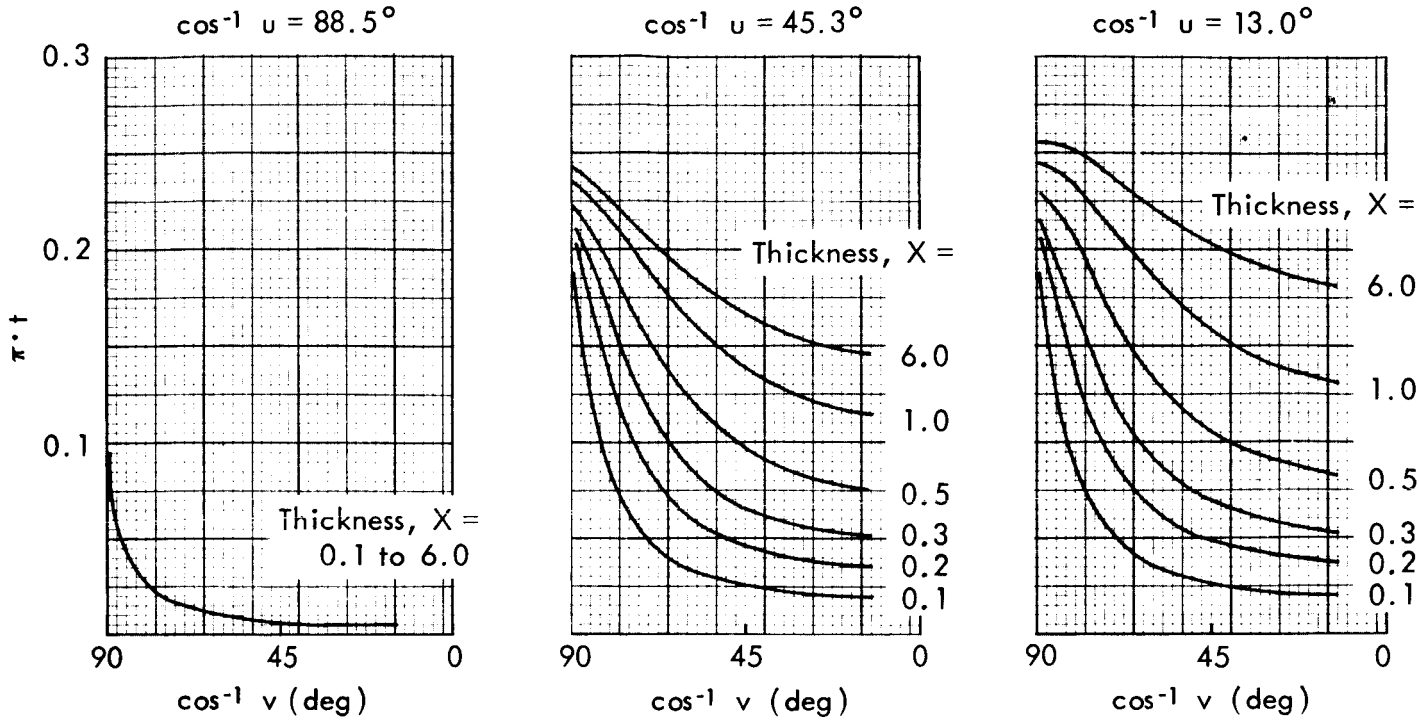

Fig. 18—Reflected intensities t per unit incident flux
albedo λ = 0.7

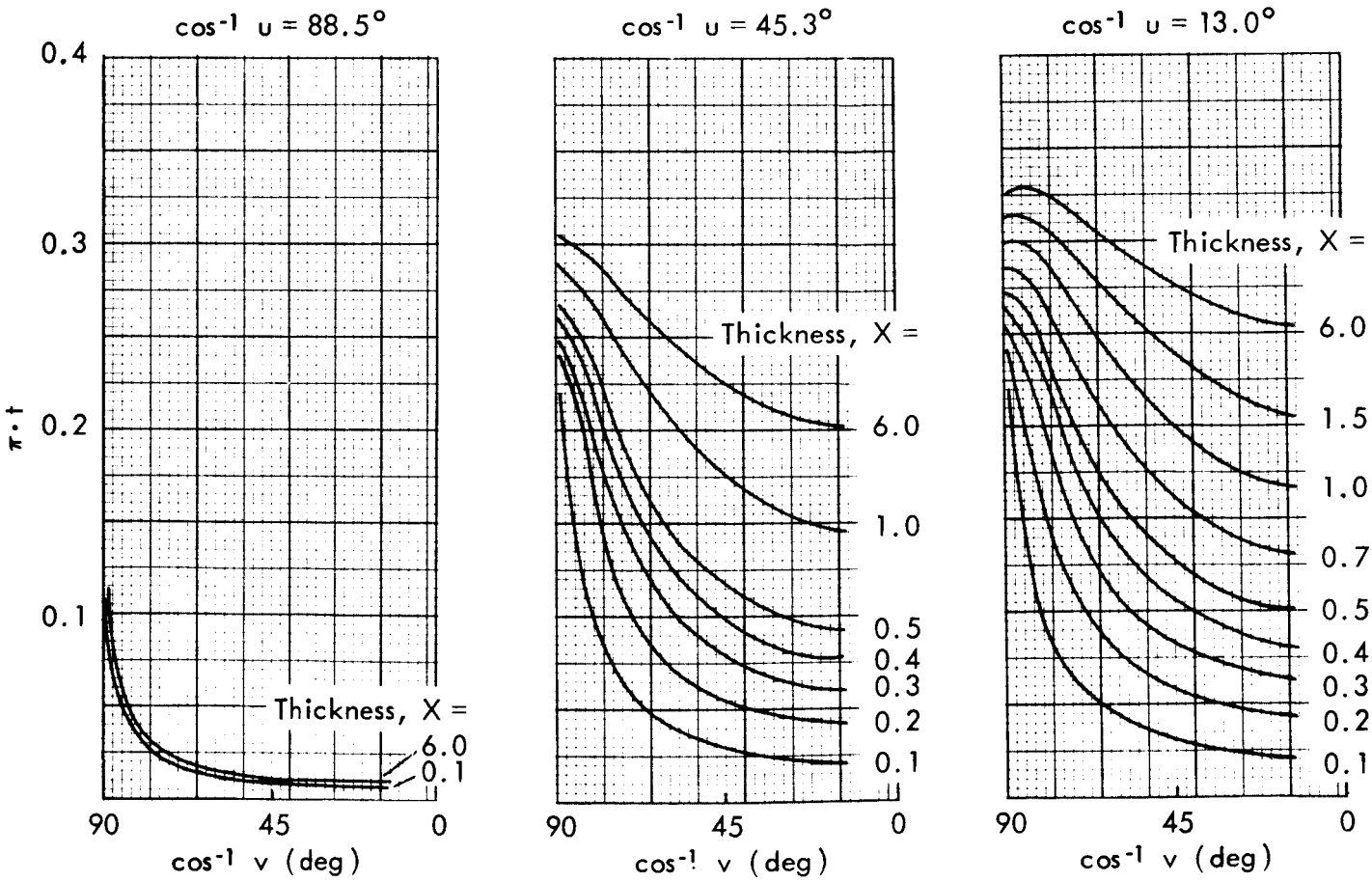

Fig. 19—Reflected intensities t per unit incident flux
albedo λ = 0.8

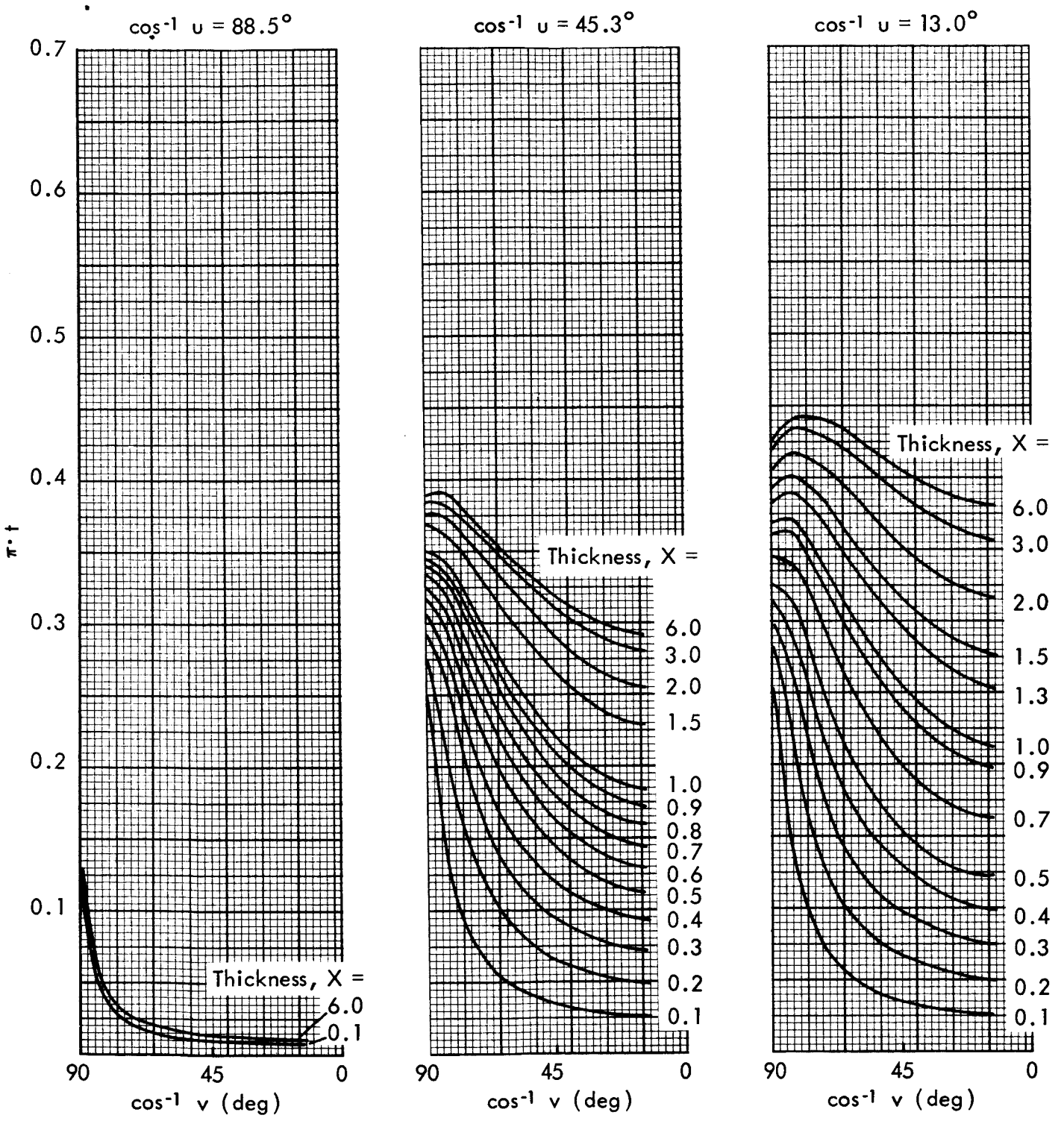

Fig. 20—Reflected intensities t per unit incident flux
albedo $\lambda = 0.9$

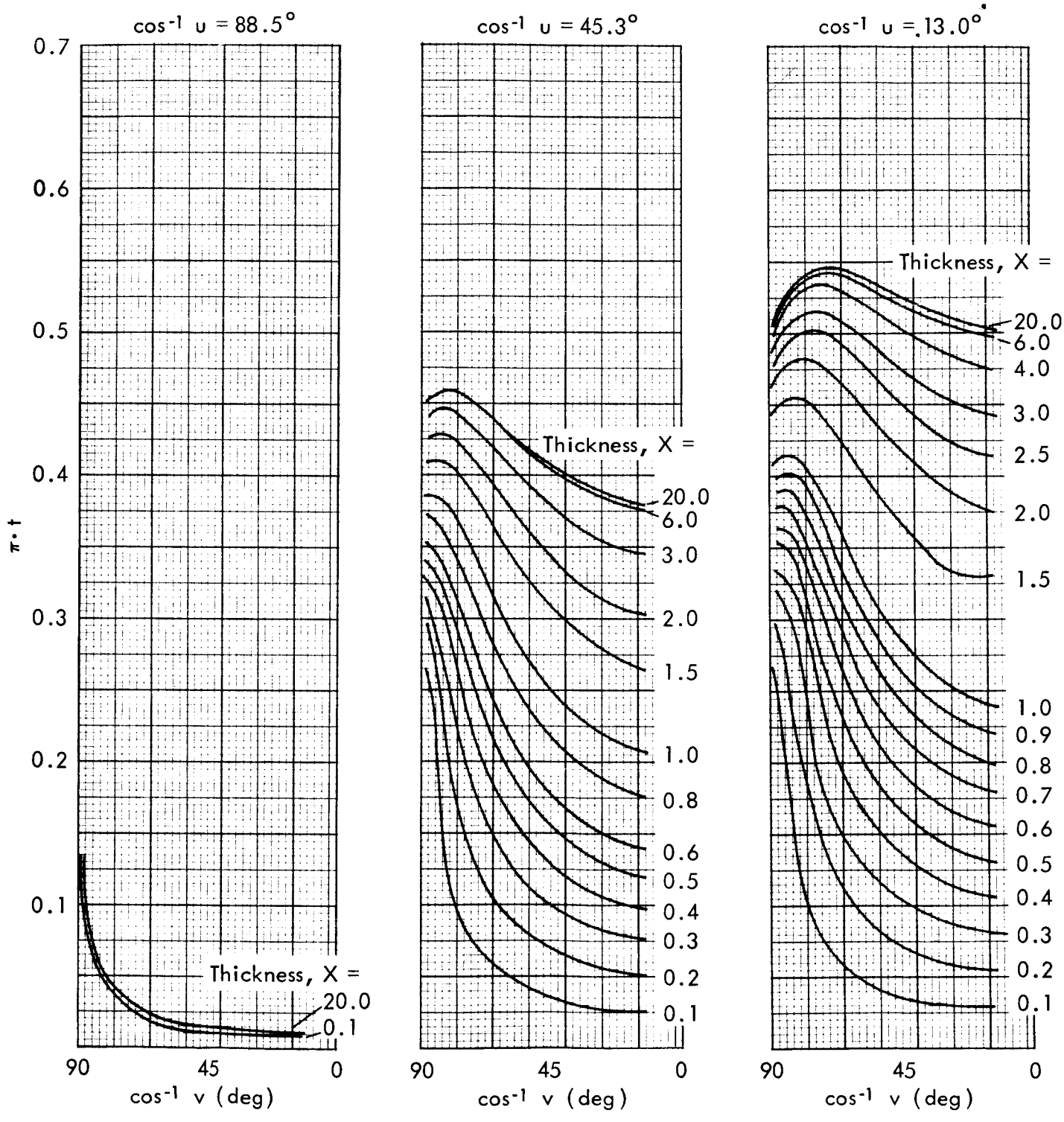

Fig. 21—Reflected intensities t per unit incident flux
albedo $\lambda = 0.95$

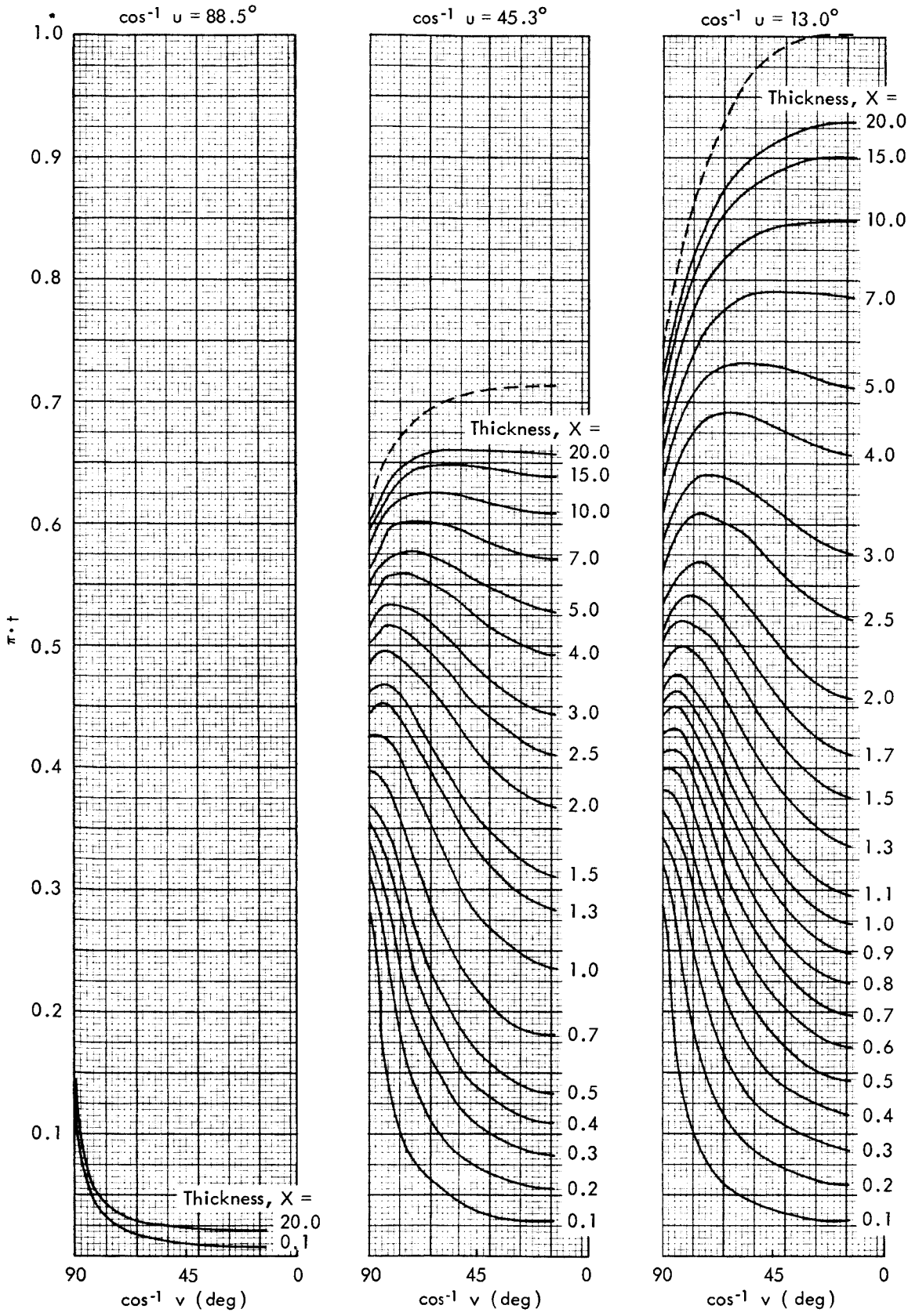

Fig. 22—Reflected intensities t per unit incident flux
albedo λ = 1.0

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