GPO PRICE $ _____

CFSTI PRICE(S) $ _____

Hard copy (HC) 7.77

Microfiche (MF) 2.25

ff 653 July 65

NASA CR-62026 – Vol. I
R-6152

FACILITY FORM 602

N66 26546 (ACCESSION NUMBER)

(THRU)

477 (PAGES)

1 (CODE)

CR-62026 (NASA CR OR TMX OR AD NUMBER)

23 (CATEGORY)

THE DESIGN AND ANALYSIS OF SENSITIVITY EXPERIMENTS

VOLUME I

By D. Rothman, M. J. Alexander,
and J. M. Zimmerman

May 1965

Distribution of this report is provided in the interest of information exchange. Responsibility for the contents resides in the author or organization that prepared it.

Prepared under Contract No. NAS 8-11061 by
ROCKETDYNE
Canoga Park, Calif.

for George C. Marshall Space Flight Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

FOREWORD

This report was prepared by personnel of the Stochastic Processes and Operations Research Unit of the Rocketdyne Research Department under Contract NAS 8-11061, "A Study of Methods for the Design and Analysis of Sensitivity Experiments", for the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration. The work was administered under the technical direction of the Propulsion and Vehicle Engineering Division, Engineering Materials Branch of the George C. Marshall Space Flight Center with Dr. John B. Gayle acting as project manager.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the support of the National Aeronautics and Space Administration and in particular the guidance of Dr. John B. Gayle in carrying out the work described in this report and in making available the computing facilities of the George C. Marshall Space Flight Center.

The authors profited from discussions at many government and industrial laboratories too numerous to mention. Their assistance was of great value in selecting topics to be included in this report.

Several of the authors' colleagues at Rocketdyne participated in this effort. In particular, Phyllis M. Hait and Bernyce J. Byars performed the coding of the simulation programs and the preparation of all necessary input data.

Finally, we are grateful to Naomi Monatt and Agnes Bishop who typed the entire occasionally illegible manuscript in admirable fashion.

TABLE OF CONTENTS

VOLUME I: Sections I through V-B; VOLUME II: Appendix V-C.

Section I - Introduction 1

Section II - Types of Sensitivity Experiments
Part A - General Considerations 8

II-A1. Assumptions About the Nature of the Response Function 9

II-A2. Costs of Sensitivity Experiments 12

II-A3. Information, Loss Functions, and Optimum Experimental Designs 15

II-A4. Experiments with More Than One Stimulus Variable 17

Section II - Types of Sensitivity Experiments
Part B - Specific Classes of Problems 19

II-B1. Inverse Response Problems 20

II-B2. Discriminating Between Response Functions 26

II-B3. Estimating and Ranking Response Functions 29

II-B4. Testing for Monotonicity 32

II-B5. Finding and Testing Response Function Models 34

II-B6. Safety and Reliability Problems 37

Section III - Designs for Sensitivity Experiments
Part A - General Remarks 40

III-A1. Sequential, Block-Sequential, or Pre-Selected Designs: Open Ended or Fixed-Sample Designs 41

III-A2. Designs Restricted to Certain Test Levels 44

III-A3. Setting Errors and Measurement Errors 46

III-A4. Stochastic Approximation 48

III-A5. Efficiency of Designs for the Inverse Response Problem in the Distribution-Free Case 52

Section III - Designs for Sensitivity Experiments
Part B - Specific Designs 54

III-B1. Up-and-Down Design 55

III-B2. Bartlett Design 59

III-B3. 20 ZIL Design 60

III-B4. Run Down Design 62

III-B5. Robbins-Monro Design and Variations 64

III-B6. Alexander Design 73

III-B7. The Naval Powder Factory (NPF and NPF Inverted) Designs 82

III-B8. Block-Sequential Design for the 50% Inverse Response Problem when the Response Function is Cumulative Normal 84

III-B9. Multivariate Sensitivity Design 88

III-B10. Nonparametric Design for Discriminating Between Two Response Functions 91

III-B11. Design for Discriminating Between Two Cumulative Normal Response Functions 97

III-B12. Langlie Design 101

III-B13. Split-the-Difference Design 104

III-B14. Rothman Design 107

III-B15. Derman Design 111

III-B16. Safety Design for Exponential Stress and Normal cdf Response 113

III-B17. Probit Design 115

Section IV - Methods for Analyzing Sensitivity Experiments
Part A - General Remarks 117

IV-A1. The Relationship Between the Design and the Analysis of a Sensitivity Experiment 118

IV-A2. Use of Prior Information 119

IV-A3. Notion of "Percent Defective" 121

IV-A4. Maximum Likelihood Estimates 122

IV-A5. Transformations 123

IV-A6. Tests of Hypothesis 125

Section IV - Methods for Analyzing Sensitivity Experiments
Part B - Specific Analysis Methods 127

IV-B1. Maximum Likelihood Estimates of
Cumulative Normal Response Functions 128

IV-B2. The Minimum Overlapping Subset (MOS) 137

IV-B3. Two-Subset Method 143

IV-B4. Dixon-Mood Analysis 147

IV-B5. Comparing One Sample with a Standard Response
Function, or Two or More Samples with Each Other 153

IV-B6. Method of Reversals 157

IV-B7. Analysis of Multivariate Sensitivity Design with
Non-Interacting Stimuli 165

IV-B8. Probit Analysis 170

IV-B9. A Likelihood Ratio Test for Monotonicity 174

IV-B10. A Likelihood Ratio Goodness-of-Fit Test 176

IV-B11. Test for the Existence of an Unknown Stimulus Variable 178

References 180

Appendix - V-A. Computer Program Listings 183

V-A1. Input Data for "MLE": A FØRTRAN Program to Compute Maximum-
Likelihood Estimates of the Parameters of a Cumulative
Normal Response Function 184

V-A2. Listings and Sample Outputs 186

Appendix - V-B. Mathematical Details 211

V-B1. Growth of Information for the 50% Inverse Response
Problem when the Response Function is Cumulative Normal 212

V-B2. Maximum Likelihood Estimates of Partially Ordered Parameters 221

V-B3. Minimum Overlapping Subset 225

V-B4. Approximate Number of Tests in Alexander Design for Small α 230

V-B5. Derivation of Discrimination Designs 232

V-B6. The Inverse Response Problem for the Cumulative Normal Response Function 236

V-B7. Response Distribution at a Specified Combination of Stimulus Variables in the Absence of Interaction 241

V-B8. Distribution of Responses Due to an Unknown Stimulus Variable 243

V-B9. Models for Multivariate Experiments with Interacting Stimuli 249

Bibliography 257

SECTION I

INTRODUCTION

INTRODUCTION

In this report we shall be concerned with that broad field of experimentation, generally referred to as sensitivity testing, in which each test is characterized by a quantal response. To some sample specimen or realization of a system one or more stimuli are applied and the result is either a "response" or a "non-response", depending on whether some critical physical threshold was or was not exceeded for that particular sample. The aim of most sensitivity experiments is to describe one or several aspects of the distribution of these critical levels for a population of specimens or systems as a function of the stimulus variables.

Sensitivity testing has been widely employed for many years in connection with the development and evaluation of explosives, propellants, detonation devices, and armor-piercing projectiles. Perhaps its earliest implementation was in biological studies of dosage mortality and response to drugs. More recently, sensitivity experiments have been employed in the evaluation of new materials subject to stress in various environments, and in delineation of unstable combustion regions in chemical propulsion systems. Although the applications have been diverse, quantal response experiments generally have many characteristics in common. Two basic problems are invariably encountered: (1) the construction of suitable test apparatus which provides a convenient yet also realistic framework for the evaluation of the response of a specimen or system to representative stimuli; and (2) the experimental planning or design of actual test programs, and the statistical analysis of the resulting quantal response data. Even though the first of these questions arises in most practical situations, the nature of

the solution depends significantly on the details of the particular application, and we shall have nothing more to say in these pages in this connection.

Fortunately, there is much that can be said about the planning and analysis of sensitivity experiments which has general applicability in many different situations, and it is to this question that we shall direct our full attention.

The statistical design and analysis of quantal response tests is in many respects quite different from the analogous treatment of classical factorial experiments. In the latter, the responses at a given treatment combination are generally continuously distributed, and often the effects of a great many independent variables are simultaneously evaluated. But in sensitivity testing, the results of each test fall into one or the other of two qualitative categories, and attention is usually restricted (until recently almost exclusively) to studying the effects of a single independent variable (stimulus or stress) at a time. Despite these apparent limitations, sensitivity experiments possess several redeeming features, not the least of which is the ease with which they permit the utilization of sequential planning techniques. Sequential planning, that is, establishing the precise conditions under which a particular test is to be conducted on the basis of previous test results, is attractive to most experimenters and permits the implementation of relatively efficient statistical methods. Generally speaking, the mathematical tools which in recent years have been used to develop new designs and analyses for sensitivity problems have been developed in the field of stochastic processes, often with these problems directly in mind. However, these newer more efficient methods have been implemented only to a very limited degree; sensitivity testing is still conducted using familiar experimental design and analysis techniques which are, often unknown to the experimenter, in many cases relatively inefficient.

This report is an attempt to remedy this situation by making a variety of newer and/or more efficient planning and analysis procedures available to the sensitivity experimenter in a form which is hopefully suitable for direct implementation. In order to accomplish this, we have separated experimental designs and analysis procedures into different sections. Since in many cases several methods of analysis can be used in conjunction with a single design, and vice versa, this separation should permit the user to take full advantage of the many combinations at his disposal. The procedures themselves are presented in the style of a manual. All the rules and limitations are given in a complete but brief form, with only a few comments concerning mathematical justification. For the latter, the reader is referred either to specific references in the open literature, or in the case of the newer methods, to appendices included in this report.

It has been the authors' experience that one of the reasons many experimenters have not taken advantage of recent developments in the design and analysis of sensitivity experiments has been the lack of quantitative information on which to base a selection of a particular method for a given problem. In most cases mathematical comparisons of experimental designs and analysis procedures are very difficult, and one is forced to resort to simulation or Monte Carlo techniques. Several limited studies of the latter type have previously appeared in the literature, and their results have been partially summarized in the present report. In addition, an extensive simulation has recently been completed at Rocketdyne which permits a satisfactory comparison of many of the newer methods with those already in use. The general conclusions that can be drawn from all these simulations are reported below in the discussions of individual design

and analysis methods. Details of the general Rocketdyne simulation are given in an appendix.

It goes without saying that the selection of an appropriate procedure for designing or analyzing a sensitivity experiment depends on the experimenter's ability to provide a clear formulation of his objectives and on the assumptions and constraints which underlie the testing. In order to assist in this process we have included a section called "Types of Sensitivity Problems" in which the major classes of sensitivity experiments are reviewed. Included are discussions of the inverse response problem, programs for the discrimination between response functions, selection of a response function, estimation of the parameters in a response function, ranking response functions, testing for monotonicity of the response function, and, finally, safety and reliability evaluation programs.

Although a claim of completeness cannot be made, we have attempted to include (1) all those recently developed design and analysis procedures which appear to have merit for general applications, and (2) those methods which have already found wide use in the field.

Many of the procedures in the latter category are based on a specific assumption concerning the form of the response function (e.g., cumulative normal), and are relatively efficient only when the goal of the tests is to estimate the behavior of this function near the median or mean. Recently, much attention has been given to the problem of estimating portions of the "tails" of these response functions, that is, stress levels with either very small or very large response probabilities. Several methods of this type have been included in this report.

In addition, there has been a growing recognition of the difficulties of making proper assumptions about the type of response function being sampled, and more significantly, the serious effect incorrect assumptions of this type can have on the efficiency of the design and analysis. This has led in recent years to the development of several distribution-free design and analysis procedures which assume generally only that the response function is nondecreasing with increasing stimulus. These are given considerable attention in the text.

Although methods for handling sensitivity experiments with several stimuli are discussed, we shall consider situations in which there is only a single observable quantal response. There do not appear to be methods available for handling the multi-response situation, except in very special cases (see Ref. 29).

Many of the methods we describe for the analysis of quantal response data require only a minimum of calculation on a slide rule or desk calculator. In a few cases, however, (e.g., general maximum-likelihood procedures) the use of a large electronic computer is almost mandatory. A few FORTRAN computer codes which have been prepared to handle these situations are described and listed in an appendix. Copies of these program decks will be made available to interested users. A brief description of the general Rocketdyne simulation program is also provided.

Finally, it must be pointed out that, although the authors have taken considerable pains to present methods for the design and analysis of a great variety of sensitivity experiments, it will be clear to the user of this manual that the basic orientation is towards those situations generally encountered in engineering research and development programs. This bias seemed unavoidable, due to the

limitations of the experience of the authors and the necessarily limited size of this report. It is hoped that readers of this manual will find procedures which in their present form are suitable for implementation to the problem at hand. If not, the reader is encouraged to construct his own designs, supplementing the ideas and techniques given below with his own.

SECTION II

TYPES OF SENSITIVITY EXPERIMENTS

PART A - GENERAL CONSIDERATIONS

II-A1. ASSUMPTIONS ABOUT THE NATURE OF THE RESPONSE FUNCTION

The selection of a procedure for conducting a sensitivity experiment and analyzing the test results depends significantly on the amount of information the experimenter possesses about the type of response function from which he is sampling. Generally speaking, if the experimenter has had extensive previous experience with the test apparatus and specimens he is employing,* it should be possible to select a specific type of cumulative distribution for the response function (e.g., normal, log normal, uniform). It may be necessary in this connection to transform the stimulus variable(s) before simple distributional forms can be assumed (see Section IV-A5). (In some cases it may even be possible to establish good a priori estimates of the parameters of the assumed distribution, such as the mean and standard deviation in the normal case.) When distributional assumptions are justified, the improvement in the quality of the design may be significant (see Section II-B1).

Unlike some other types of experimental designs (e.g., factorial) and analyses (e.g., regression), sensitivity procedures depend critically on the accuracy of the distributional assumptions. Thus, for example, if it is assumed that a response function is cumulative normal when it is really, say, uniform, then the design and analysis chosen will be satisfactory only when the median response (that is, the 50 percent stimulus level) is of primary interest. In other regions, corresponding to small or large response probabilities, highly inefficient designs and inaccurate analyses will result. Thus, generally speaking, a parametric approach should be selected only after an extensive amount of previous experience and completed data analyses are available.

*or if he is able to find a model for his process, as in Ling and Hess (Ref. 31).

Since this is often not the case, and moreover, since many sets of sensitivity data clearly are not consistent with the classical response function forms, it becomes necessary to employ distribution-free (or nonparametric) methods. Most generally, one assumes only that the fraction response distribution at a given stimulus level is binomial and that the true response probabilities are monotone nondecreasing with increasing stimulus. (The question of monotonicity is further discussed in Section II-B4.) Although distribution-free designs generally require more tests and permit less powerful analyses, there are several which are surprisingly efficient - even in some cases closely rivaling parametric procedures (see, for example, Section III-B6). The relative efficiency of the two approaches depends among other things on which portion of the response function is of particular interest. Near the 50 percent level, both are generally effective. In the tail regions below 0.01 and above 0.99, inaccurate assumptions about the form of the response function generally have a deleterious effect. Recently developed nonparametric methods can be quite effective in these regions and are to be desired in all cases where substantial information about the response function is not available.

For very low (< .01) or very large (> .99) response probabilities, the sample requirements of a purely distribution-free approach may be forbidding, yet extreme sensitivity to the selection of the form of the overall response function virtually prohibits the use of completely distributional methods except in certain special circumstances. For example, the existence of even a small percentage of defective items, or "duds," in a population which otherwise has a cumulative normal response function, may lead to badly biased estimates in the tails. To handle such problems, which occur more and more frequently of late in connection with safety or reliability studies, a hybrid "tailwise parametric" approach has recently been formulated: assumptions concerning the form of the response

function are made only in some probability interval which contains the very small (or very large) response probability of interest. Nothing is assumed about the behavior of the response function outside of this range. Unfortunately, such techniques have not been explored in practice to any extent, nor to our knowledge, have they been evaluated by simulation. Consequently, it is possible only to outline these tailwise parametric procedures which is done in Section II-B1 below.

Another approach to the study of the tails of the response function is to implement the techniques of accelerated or overstress testing by incorporating one or more additional stimulus variables. If, for example, one is interested in the value of a stimulus variable (stimulus "A") corresponding to a very low response probability (e.g., .001) one can conduct a series of experiments in which in addition to A a second stimulus ("B") is employed at levels which will assure at least a few responses in a reasonable number of tests. Stimulus B may or may not be a realistic physical excitation for the item being tested, but the responses it helps produce through interaction with the stimulus A must be of the type of interest to the experimenter. If this multivariate sensitivity experiment (see Section III-B9) can be efficiently designed, and in particular if the interaction between the stimuli can be characterized, then the originally required information concerning the stimulus A can be recovered by appropriate analysis in relatively fewer tests than if a multivariate experiment were conducted.

II-A2. COSTS OF SENSITIVITY EXPERIMENTS

In this section we shall briefly discuss a few of the major cost factors that generally arise in the selection of a plan for sensitivity experimentation. Some of these are self-evident and quite commonly considered; others are occasionally overlooked. In principle, all should be carefully evaluated before selecting the design and analysis procedures.

a) Cost of specimen or item being tested and expense of running the test. This is perhaps the most obvious of all costs and naturally provides a limiting factor on the total allowable sample size.

b) Cost of data analysis. The majority of distributional and essentially all the distribution-free design and analysis procedures do not require extensive computation expenditures. The major exception is the maximum-likelihood analysis method which requires a large digital computer for practical implementation. Some procedures are more readily carried out than others, however, and in a few cases computation time may be a factor. The effect is primarily through the time delay between successive tests required to carry out the more complex techniques. This increases the total elapsed test time which may itself introduce additional costs.

c) Cost of having a response occur. When the items being tested are expensive or their response may cause local damage (e.g., an explosion), it may be desirable to select an experimental design which will lead to fewer responses even if this compromises the quality of the data analysis.

d) Cost of having a nonresponse occur. In some situations it turns out to be more expensive to cope with a nonresponse than with a response. For example, in tests of electrical initiators, the specimen must be rendered harmless after a nonresponse, and this may take considerably more time than disposing of an item which has already been activated. Here an experimental design which limits nonresponses may be preferable.

e) Costs dependent on the level of the stimulus variable(s). In many sensitivity experiments, the cost of a particular test depends significantly on the level of the stimulus (or stimuli) applied. If, for example, this cost increases significantly with increasing stimulus, then an experimental design should be selected which limits the number of high stimulus level tests and which makes sure that those performed are as efficient as possible. If the cost dependence on the level of the stimulus is known explicitly, it may be possible to use this information in deriving an "optimum" design (see Section II-A3).

f) Effects of batch size on specimen preparation costs. In certain sensitivity experiments, one or more of the stimulus variables are inherent in the specimen itself (e.g., configuration factors, composition) rather than in the set-up of the test apparatus. In designing sequential tests with such items, therefore, preparation of the specimens must be accomplished also in a sequential manner as stimulus levels for subsequent tests are established. A program of this type can be very costly, and it may be necessary to prepare at least several specimens at a time (batch preparation); similar economies may be possible through simultaneous testing (batch testing). Information which relates the cost per item as a function of the batch size will assist in the preparation of an experimental design in such cases (see Section III-A1).

g) Costs arising from the number of stimulus levels which are to be permitted during testing. From the experimental design point of view, it is desirable to be able to test at any level of the stimulus variable in some specified interval. This may be very expensive, however, and require substantially more sophisticated test apparatus. Consequently, experimental designs which limit testing to a few specified stimulus levels may often be desirable.

II-A3. INFORMATION, LOSS FUNCTIONS, AND OPTIMUM EXPERIMENTAL DESIGNS

In this section we describe briefly a macroscopic approach to the design of sensitivity experiments based on cost considerations. The method is directed towards the planning of large scale complex sensitivity experiments (e.g., those with several stimulus variables) and results in the establishment of broad design characteristics which the optimum plan must satisfy. As long as they are consistent with such requirements, detailed experimental plans can then be implemented. Our purpose here is simply to mention the macroscopic viewpoint and point out some of its virtues. All our attention in this report will be directed towards specific detailed designs and analysis procedures. Those interested in implementing the cost optimization viewpoint may find further details given in Ref. 1.

Broadly speaking, the experimenter's objective is to find a program plan which balances the costs described in the previous section against the cost quality of the ultimate test results. In order to relate these two factors quantitatively it is necessary to introduce the notion of information and attempt to evaluate the costs of "misinformation".

The usual approach is to define information as the inverse of some appropriate average variance of all the estimates of response function characteristics which are of interest. This variance, and thus the information, depends on the experimental design. Next the major cost factors' (see Section II-A2) dependence on the experimental design is represented in terms of appropriate cost functions. These would include, for example, the explicit effect of stimulus level on cost (if one exists) and the costs of responses and nonresponses (if these are different). The latter would have to be included as expected costs (depending on the design), since the true response probabilities at various stimulus levels are generally not known in advance.

Finding the cost of misinformation (that is, a cost which is directly proportional to the variance of the estimates of interest) is generally a difficult task, but partial answers are sometimes possible. For example, in the "safety" evaluation of current excited detonators by sensitivity testing one seeks an estimate of that current level below which a detonation is, with very high probability, unlikely to occur. Errors in the estimates of this level will be carried over to the system design and will result either in more frequent accidental detonations and damage to the system, or in expensive overdesign to provide unnecessary safety.

A "loss function" is then constituted which contains two terms: the cost of experimentation and the cost equivalent of the information to be generated, both as a function of the experimental design. Finally, the loss function is studied to find that experimental plan which minimizes the total cost. Generally this optimization cannot be obtained in closed mathematical form and computer search procedures may be necessary. Often this entire procedure is feasible only if large sample theory is used to express the cost of misinformation. The resulting design is then said to be "asymptotically optimal".

It is interesting to note that in many cases only very broad characteristics of the optimum design are specified by this procedure (e.g. the mean and variance of the stimulus levels at which testing is to be conducted, or perhaps just one or two response probabilities whose corresponding stress levels should be converged to in the course of the experiment).

II-A4. EXPERIMENTS WITH MORE THAN ONE STIMULUS VARIABLE

Until very recently, available methods for the design and analysis of sensitivity experiments have been limited to the case in which there is only a single stimulus variable. For a great many situations univariate methods are quite satisfactory, since experimental conditions often can be so controlled as to limit the effects of all variables except one selected for study.

In other cases, however, this may not be possible, or what is more important may not be desirable, and an experimental design and analysis involving several stress variables is required. There are essentially two types of multivariate experiments:

1. Those with a single stress variable and one or more environmental variables.
2. Those with several stimulus variables (and perhaps one or more environmental variables).

In the former case, information about the effect of the single stimulus variable on the probability of response is desired under a variety of combinations of the environmental variables. Problems of this type can be handled by combining univariate sensitivity procedures with ordinary factorial design and analysis methods. An example of this approach is given in Ref. 2.

If there are several stimuli whose effect is of interest, then a multivariate sensitivity approach is indicated. Recently developed methods for handling such problems are given in Section III-B9 and IV-B7.

Alternatively, truly multivariate sensitivity experiments may be handled (although perhaps less efficiently) as in problems of type 1 by combining factorial designs with univariate sensitivity methods. One singles out one of the stimuli (for reasons of cost, ultimate program objective, etc.) and considers the others as "environmental" variables. Examples of this approach are given in Refs. 1 and 2 and will

not be considered further in this report.

The experimenter should be alert to the possibility of additional "hidden" stimulus variables in apparently univariate sensitivity experiments. A method of testing for such variables when their levels are uncontrolled and uncorrelated with the known stimulus variable is given in Section IV-B11.

SECTION II

TYPES OF SENSITIVITY EXPERIMENTS

PART B - SPECIFIC CLASSES OF PROBLEMS

II-B1. INVERSE RESPONSE PROBLEMS

a. Introduction

The most commonly encountered type of sensitivity problem is that of finding at what level of the stimulus variable a given percent response will occur. For example, in biological assay it is often necessary to determine the dose (called LD 50 or ED 50) which is effective half the time, or in testing explosives, it is often of interest to find the stress that results in a detonation, say, 95% of the time. In each of these situations we are concerned with inverting the relationship which gives the probability of a response as a function of the stimulus; thus the terminology (probably due to J. W. Tukey) of the "Inverse Response Problem".

The general problem can be stated more precisely as follows: let x be a stimulus variable and let the probability of a response for a given x be $M(x)$. Then, for a given probability $0 < \alpha < 1$, we wish to determine that value of x, x_α, such that

$$M(x_\alpha) = \alpha \quad .$$

The proper choice of design and analysis procedures for inverse response problems depends on two factors: (1) whether assumptions concerning the form of $M(x)$ can be made, or whether a distribution-free approach is required, and (2) the value of α. We first consider the distributional question.

b. Form of M(x) Known

When distributional assumptions are appropriate, many experimental design and analysis procedures are available which lead to estimates* of the distribution

*Throughout this report we shall use the notation "$\hat{x}$" to denote an estimate of the quantity "x".

parameters. Thus, theoretically, the entire distribution can be described and the solution of the inverse response problem is available for any α. In practice, however, such results are generally useful only when α is close to .5, and become increasingly less satisfactory for very small and very large values of α, unless the distributional form and prior estimates of its parameters can be completely relied on. In the latter case, Chernoff has derived "asymptotically optimal designs" (Ref. 4) when the response function is normal, for all α (see Section III-A4). However, the efficiency of these designs for small samples and small α has not been extensively evaluated.

Of course even for α near .5, the distribution of the estimates of the response function parameters depends on the experimental design and those detailed specifications (e.g., level of first test, step size in up-and-down procedures) which must be selected in advance. The optimum choice of these design parameters depends on the assumed distribution. (For example, the step size for the Bruceton up-and-down design should be close to σ, the standard deviation, if a normal distribution is assumed, and the initial test should be as close to μ as possible.) Unless the experimenter has some previous results on which to base his choices of design parameters, they will generally not be close to those values leading to the most efficient experimental design. Therefore, a good design should be self-corrective, i.e., capable of overcoming poor choices of the preselected design parameters.

One good procedure which can be used with a distributional analysis for α near .5 is the "delayed" Robbins-Monro design (Section III-B5); this design compensates for a poor choice of initial test level, and the choice of the other design parameter turns out not to be critical. Further, both the design and analysis are quite simple. The up-and-down designs are generally useful also in this connection, again for α near .5. In particular, the Bruceton or up-and-down procedure has simple design rules and a straightforward analysis method (Section III-B1). But, while the Bruceton procedure compensates for a poor choice of initial test level, the choice of the fixed step size can significantly affect the quality of the estimate of x_α. When the response function can be assumed to be normal, it is possible to select asymptotically optimal designs by means of the theory of Chernoff (Ref. 4). These procedures are summarized in Section III-A4.

c. The Distribution-Free Case

Without assumptions on the form of the response function to relate the response probability at various stimulus levels, information obtained by testing at one level gives little information about the response probability at any other. For example, knowing $x_{.5}$ tells us nothing about the value of $x_{.05}$; if we assume that the response function is monotone nondecreasing with increasing stimulus (see Section II-A1), then we may conclude that $x_{.05} < x_{.5}$, but that is all. While monotonicity, generally, only "loosely" relates the response fractions at different stimulus levels, it can be used effectively if a great many tests are concentrated in a small interval. Thus, for distribution-free designs a major objective is to concentrate the test levels as near to x_α as possible. How easily this can be done depends on the value of α. We distinguish three cases:

Distribution-Free Designs When $\alpha \sim .5$

When $\alpha \sim .5$, it is not difficult to verify that most tests are in a neighborhood of x_α, since approximately equal numbers of responses and nonresponses will occur in such a region. Thus, there are many distribution-free designs which are reasonably efficient in this case. Since in such designs the initial test level must be chosen in advance, good designs should, as before, compensate for a poor choice. Suggested designs for this problem are the delayed Robbins-Monro Design (Section III-B5), the Alexander Design (Section III-B6), the Rothman Design (Section III-B14), and the Split-the-Difference Design (Section III-B13). All except the Rothman Design have simple design rules and straightforward analysis procedures.

Distribution-Free Designs for $\alpha \sim .05$ or $\sim .95$

As the value of α tends to zero or one, it becomes more difficult not only to estimate x_α but even to determine some interval or region containing it. Without distributional assumptions, it is necessary to perform a great many tests in this region to obtain a reasonable estimate of x_α. Thus, a suitable design for this case must not only correct for a poor selection of the first test level, but also continually force the successive test levels to remain in a neighborhood of x_α. The Alexander Extreme Value Design and the Rothman Design have been formulated explicitly to handle this problem and appear to be very efficient (Sections III-B6 and III-B14).

Distribution-Free Designs for $\alpha \sim .01$ or $\sim .99$

For such extreme values of α, a great many tests are required before the experimenter can have any assurance that he is even close to x_α. If a large

number of tests can be tolerated, the Alexander or Rothman Designs can be used. If, as is more generally the case, cost and time considerations limit the sample size, then a procedure based on a tailwise parametric assumption (see below) should be used. Since the Alexander Design, and to a lesser extent a stochastic approximation design for the $\alpha = .05$ case, provide a great many tests at levels below $x_{.05}$ (see example at the end of Section III-B6), using one of these designs and a tailwise parametric analysis may be satisfactory.

d. The Tailwise Parametric Case

We next consider the problem of estimating x_α for very small α (e.g., $\alpha < 0.01$) or very large α (e.g., > 0.99), by means of a tailwise parametric approach (see Section II-A1). Our discussion will be preliminary and brief, since this method has not been fully developed, and more importantly, has not been applied in practice nor evaluated by simulation.

We consider the case of small values of α; completely analogous remarks can be made in the complementary situation when α is near unity. First we establish a response probability interval $(0,\beta)$ in which a form for the response probability as a function of the stimulus is to be assumed. Choice of the upper limit β depends on α and in how wide an interval the experimenter can rely on his assumption; for $\alpha = 0.01$, for example, β might be taken as 0.2; for $\alpha = 0.001$, β might be 0.1. We say nothing about the behavior of the response function in the interval $(\beta,1)$. Next we assume a model for the response function over the domain of response probabilities less than β. Polynomial, exponential, or perhaps doubly exponential models can be considered, in addition to those usually assumed.

The experimenter now chooses a design which would place most of the data in this region of the response function (e.g., Section III-A4). When testing is complete, any data which seem to be outside of this interval are eliminated; finally, the remaining data are analyzed by conventional means.

There are several open questions that arise in applying this tailwise parametric approach. First, of course, is the selection of the model. This may not be as critical as it seems; however, the quality of the estimate of x_α should not depend significantly on the form chosen because only a <u>local</u> model is being employed. Secondly, the quantitative influence of the choice β needs to be evaluated. As β increases one is forced to apply a model over a wider and wider interval ($\beta = 1$ corresponds to the fully parametric case), which is generally undesirable. On the other hand, for β too small, estimation of x_α in a reasonable number of tests may be difficult. A procedure for establishing the optimum choice of β which balances these effects awaits formulation.

II-B2. DISCRIMINATING BETWEEN RESPONSE FUNCTIONS

In many types of sensitivity experiments it is desired to compare the response functions of two or more materials or devices. In quality control testing, for example, production batches are sampled to see if the response function of the current lot matches some standard response function. This is roughly the technique for armor plate production, where projectile penetration ("ballistic limit") tests are used for quality control (e.g., the requirement might be of the form, 0 penetrations in 5 tests under $x = 2000$ ft/sec). A similar situation is found in materials testing (e.g., 0 responses out of 20 trials at some specified height of the impactor) and in squib production (1 Amp, no-fire; 2 Amp, fire).

Another class of problems in which it is desired to discriminate between response functions is in the development of new materials or devices, where each new version is tested to see if it has been improved sufficiently to meet certain specified standards. Such a problem occurs in sensitivity testing of newly developed explosives, where the explosive must be able to withstand the ordinary stress of handling and transportation, no matter what other properties it has.

This type of experimental situation often has been treated as simply a binomial comparison problem; that is, the test item is compared to the reference at only one stress level. Generally this is not the most efficient way to test against a standard. A better design would be one in which tests were made at those levels which, based on previous data in the test sample, seemed to have the most promise for demonstrating a difference between the test material and the standard. Such designs are given in Sections III-B10 and III-B11, but have not yet been evaluated by simulation.

Another important problem encountered in sensitivity testing is to determine whether several samples of data are consistent with a single response function. For example, we may ask whether data from different lots of a production device, different manufacturers of a device, or different formulations of a material, or even from identical materials on different impact testers, are consistent with each other. This situation arises when there is no commonly accepted standard or when there is a danger that the reference has been obtained under environmental conditions which are different from those of the experimenter.

One example of this type of discrimination problem has occurred in programs to develop liquid rocket fuels with less tendency towards combustion instability (here the stress variable is the amplitude of an intentional pressure disturbance). Other examples are those programs in which chemical additives are incorporated in a basic explosive formulation which is then tested to see if any are promising in reducing sensitivity.

At any step of such a test program, the material to be tested might be thought of as one population, and all other materials which have not yet been shown to be different from it might constitute a second population. Within this two-sample framework, designs are given in Sections III-B10 and III-B11 (as above, these designs have not yet been evaluated by simulation).

Tests of hypothesis corresponding to all of these problems are given in Section IV-B5.

Sometimes the problem is not to find any difference which may exist between the response functions, but rather to test the null hypothesis that, for a

certain α, x_α is the same for both response functions (i.e., they intersect at x_α). In this case one would test each population as for the α-inverse response problem, and then compare the two estimates of x_α as in Phase V of the analysis procedure given in Section IV-B1.

II-B3. ESTIMATING AND RANKING RESPONSE FUNCTIONS

In some sensitivity problems the aim is to find estimates of the unknown parameters of the response function $M(x)$, or perhaps functions of these parameters. For example, if the response function is assumed to be cumulative normal with parameters μ and σ, then an objective of the experiment might be to estimate some function $f(\mu,\sigma)$ such as $\mu + k\sigma$, with k specified. It should be emphasized in this connection that it is in a certain sense not enough to have as an experimental objective the independent estimation of μ and σ. Such problems are not "well-posed" in the sense that the optimum procedure, or the choice of any desirable design and analysis method for that matter, depends on the relative importance of the estimates of μ and σ in the experimenters objective.

Such estimation problems may occur if there is a response function parameter which is of particular interest. For example, the standard deviation σ in the case above, or a scale parameter in general, may be a measure of repeatability or control, and therefore may have independent interest apart from the response probabilities themselves. Or, as in the model given by Ling and Hess (Ref. 31), one of the parameters may correspond to a threshold or "zero activation energy" which is of particular physical significance. Asymptotically optimal designs (in the sense of the variance of the estimates) for such problems generally resemble designs for an inverse response problem. For example, if the response function is cumulative normal and an estimate of σ is desired, then an asymptotically optimal design is one which is asymptotically optimal for the α inverse response problem, $x_\alpha = \mu + k\sigma$, as α approaches 1 (i.e., as k becomes infinite).

To our knowledge, the "threshold" problem has been solved only in the special case of a step response function; the first few tests of optimal designs for one-, two-, three-, and infinite-sample experiments are given by Marks (Ref. 18), and these

results are approximated in the "split-the-difference" design (Section III-B13).

Often the experimenter has no interest in the actual values of the stress function, perhaps because the stress variables and/or environmental conditions are not those encountered in the field. In such cases the test objective is only to rank several materials or devices according their sensitivity to the laboratory stress variables, and hope that this ranking also applies under field conditions. Even when such hopes are justified an additional difficulty in this approach is that a material relatively more sensitive at one stress level may be relatively less sensitive at another if our assumptions permit the response functions to cross. Yet it is certainly arbitrary to rank materials according to their estimated response fractions at a specified stress, or according to the stress levels estimated to correspond to a specified response probability.

One way out of this dilemma is to assume that all response functions (for the materials in question) are identical except for a location parameter,* in addition to whatever assumption, if any, is made about the form of the response function. For example, if the ith response function is assumed cumulative normal with parameters μ_i and σ_i, then this additional assumption would give $\sigma_i = \sigma$, for all values of i, and the response functions could not cross.

Asymptotically optimal designs for these composite problems are generally made up of individual designs for each response function which are themselves each asymptotically optimal for an appropriate inverse response problem (see Section II-B1). If the response function is cumulative normal or logistic, Hodges and Lehmann (Ref. 13)

*If $M_1(x) = M_2(x-\mu)$, then M_1 and M_2 are said to differ only by a location parameter, μ.

have shown that the 50% inverse response problem is the best. Thus the designs for ranking problems would not differ greatly from those now used in most laboratories.

However, analysis of several sets of data, under the assumption that only the location parameters are different, does not appear to have been previously carried out in practical applications. Indeed, such an analysis is very difficult in the distribution-free case (which would correspond to an extension of the method of Section IV-B6) and generally not feasible. But in the distributional case, maximum-likelihood procedures for handling a single set of data (as for the cumulative normal response function in Section IV-B1) may be easily extended to the multi-sample ranking problem.

II-B4. TESTING FOR MONOTONICITY

In most physical situations, a natural assumption in testing responses to stress is that the probability of a response increases monotonically with increasing stimulus level.* Of course, in experimental situations the sample response fractions do not necessarily follow this rule. In small samples, a decrease in response fraction with increasing stimulus is a problem only in distribution-free designs, although there are corrective procedures, such as the "method of reversals," which can be used to obtain monotone estimates (see Section IV-B6).

Generally speaking, non-monotonicity observed in the response function is a serious question when either a large number of tests has been performed at the stress levels indicating the reversal, or when a sharp decrease in the response over successive stimulus levels is found. Even in these cases it is first necessary to determine whether these larger discrepancies can still be attributed to experimental error or actually indicate a non-monotone function (see Section IV-B11 where such tests are discussed).

If such a test fails on a reasonably high confidence level, some further explanation for the observed non-monotonicity must be found. One possibility is to hypothesize the existence of an additional unknown stimulus, which operates independently of the known stimulus variable. This situation is discussed in Section V-B . Assuming that such a "hidden" stimulus exists yields the interesting result that the variance of the response distribution at any given stress

*Well known exceptions to this are projectile penetration studies, in which the probability of penetration of some standard armor plate may not always increase as the velocity of the projectile is increased.

level of the known stimulus is actually reduced. Thus, this approach is better suited to explaining apparent mild non-monotonicities in large samples, rather than when a sharp decrease in the response fraction is observed with increasing stimulus. In the latter case, if all experimental procedures have been carefully checked, the phenomenon of non-monotonicity may actually exist.

II-B5. FINDING AND TESTING RESPONSE FUNCTION MODELS

Response function models have traditionally been chosen from those cumulative distributions commonly used in other branches of statistics, such as the normal,

$$M(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(y-\mu)^2/2\sigma^2} dy ,$$

logistic,

$$M(x) = \frac{1}{1+e^{-\pi x/\sqrt{3}\sigma}} ,$$

uniform

$$M(x) = \begin{cases} 0 , & x < A \\ \frac{x-A}{B-A} , & A \le x \le B \end{cases}$$

or transformations of these (see Section IV-A5). Sometimes the experimenter has a "candidate" for a response function which is based on physical considerations, such as the Ling and Hess (Ref. 31) model,

$$M(x) = \begin{cases} 1-e^{-\alpha(x-X_o)^n e^{-X^*/x}} & , x \ge X_o \\ 0 & , x < X_o \end{cases}$$, where α, n, X_o, and X^* are parameters,

or his candidate may be one for which design and analysis procedures are well known, such as the normal.

If testing and conclusions are confined to a region near the 50% point, $x_{.5}$, then the experimenter can hardly go wrong with any model he uses. Thus the normal distribution is commonly used in these cases without any concern for verification of the model.

There are some problems, however, for which sample sizes are prohibitive unless the experimenter can rely heavily on a response model. If such a situation is met repeatedly by an experimenter in a certain type of sensitivity testing, he may wish to choose a model for that type of data once and for all by taking an extremely large sample of data for one standard material or device. Such data is generally taken in the "tails" of $M(x)$, since it is in these regions where differences between response function are most noticable. He then tries to fit different distributional forms to the data until one of these passes a "goodness-of-fit" test (e.g., Section IV-B10); subsequently that form is assumed without question for similar sensitivity tests.

The difficulty with such exploratory experiments, however, is in posing them properly. For example, suppose that the experimenter is generally interested in the $\alpha = .001$ inverse response problem, and would like to know if his response functions are cumulative normal. If he concludes they are really normal, he will be able in the future to take advantage of his conclusion by employing an efficient design consisting of a mixture of tests aimed at $\mu + 1.575\sigma$ (see Section III-A4).

A design which has been used for exploratory problems of this type is the Bartlett design (Section III-B2). This design gives a decreasing sequence of tests, and the experimenter ends the sequence when he estimates that his next test will be below the level $x_\alpha = \mu - 3.09\sigma$,* or when he runs out of tests (cf Hammer, Ref. 32).

But careful attention to the use to be made of the conclusion drawn from the exploratory program suggests a design which may be better. The experimenter can mix an asymptotically optimal distribution-free design for the $\alpha = .001$ inverse response problem (a stochastic approximation, as in Section III-A4) with an

*$K = -3.09$ corresponds to $\alpha = .001$

asymptotically optimal distributional design (a mixture of stochastic approximations of $\mu + 1.575\sigma$). The relative sample sizes of the two component designs can be chosen such that, under the null hypothesis of normality, the estimates of $x_{.001}$ given by each set of data would have equal variance. Thus the distribution-free component design would require about 5.6 times as many tests as the distributional component design.

If there are two candidate response functions, the experimenter would mix in equal proportions two designs, one for each, of the type given above in the normal case.

One point must be made strongly: if the experimenter does not carefully define the use he intends to make of his response function, it is difficult to properly pose his exploration problem.

II-B6. SAFETY AND RELIABILITY PROBLEMS

There are many situations in which sensitivity experimentation is employed to estimate the safety or reliability of a device or system. The former case is commonly associated with finding levels of the stress variable which correspond to very small fraction responses. In reliability problems one is generally interested in stress levels associated with very high fraction responses. Safety and reliability problems can be treated as inverse response problems (see Section II-B1), if it is assumed that the appropriate stress density functions (see below) are "delta functions" at x_α (i.e., densities with all their mass concentrated at the solution of the inverse response problem). The only virtue of such an approach is that it generally gives conservative estimates for safety and reliability. In this section we formulate the safety and reliability problems in a more realistic manner.

If $M(x)$ denotes a response function, and if $f_1(x)$ denotes the density of the maximum accidental stresses encountered by a device in a "lifetime" of handling, shipping, etc., then it is natural to define the safety S, by

$$S = 1 - \int_{-\infty}^{\infty} M(x) f_1(x) dx \ .$$

Similary, if $f_2(x)$ denotes the density of actuating stresses, then the reliability, R can be represented by

$$R = \int_{-\infty}^{\infty} M(x) f_2(x) dx \ .$$

A realistic problem for a sensitivity experiment would be to estimate the safety (reliability) or demonstrate that it meets a certain specification, given $f_1(x)(f_2(x))$.

For example, if $M(x)$ is assumed to be a normal cdf* and $f_1(x) = Ae^{-Bx}$ for $x > \mu - 4\sigma$, then

$$S \simeq 1 - \frac{A}{B} e^{-B(\mu - B\sigma^2/2)}.$$

A design for this safety problem is outlined in Section III-B16.

For the extreme response probability values usually associated with safety and reliability problems, it is hard to see how one could dispense with a distributional assumption on $M(x)$ in selecting design and analysis procedures. However, this problem has not yet been fully investigated.

If the function $f_1(x)$ is completely unknown, then strictly speaking there is no way of estimating safety. In this case, if there are two or more devices whose safeties are to be evaluated, the only way of ensuring that one device is safer than another is to demonstrate that one response function is always less than another; that is, the response functions do not cross. One way to approach this is to assume that the response functions are all identical except for their location parameters, and treat the problem as one of ranking response functions (see Section III-B3).

If the function $f_1(x)$ is imperfectly known, standard methods of error analysis may be used to find the marginal variance in the estimate of the safety due to errors in $f_1(x)$. For example, in the exponential model given above, suppose σ_A^2 and σ_B^2 are the estimated variances in our estimates of A and B, respectively. Then the variance σ_S^2 in the estimate of the safety is given by

*cdf = cumulative distribution function

$$\left[\sigma_S/(1-S)\right]^2 \cong \left[\sigma_A/A\right]^2 + \left[\sigma_B(1/B + \mu - B\sigma^2)\right]^2 .$$

Further error terms would be added to this expression in the usual fashion to account for imperfect estimates of μ and σ.

SECTION III

DESIGNS FOR SENSITIVITY EXPERIMENTS

PART A - GENERAL REMARKS

III-A1. SEQUENTIAL, BLOCK-SEQUENTIAL, OR PRE-SELECTED DESIGNS; OPEN ENDED OR FIXED-SAMPLE DESIGNS

In choosing a particular design one must be guided by both the experimental costs (or limitations in the preparation of test specimens) and the efficiency of the design. The relative importance of the two will indicate which types of design should be used. In this section we discuss some basic characteristics of designs for sensitivity experiments which bear on establishing a balance between these factors.

a. Pre-Selected Designs

In this type of design, also known as one-block or non-sequential, all stress levels are planned in advance. Since the levels are determined before any information is available, these designs are usually inefficient and require a large number of tests. Even a two-block design is generally much more efficient (see below). However, if considerations such as long lead times, economies of fabrication or dosage preparation, or unusual delays in observing responses (as in bioassay) are of greater importance, this type of design may be necessary. An example is the usual type of probit design (Section III-B17).

b. Block-Sequential Designs

If several tests at a time, but not all, must be planned in advance, then a block-sequential design may be used. In this type of design the stress levels for a given block are not determined until the preceding block has been completed and all of the results observed. In some cases, both the number of blocks and the allocation of tests to the different blocks ("staging") are determined before the start of the experiment, as in the block-sequential and run-down designs (see below). These choices are made in order to produce the

greatest gain in information about the parameter to be estimated. It is also possible to determine the number of tests in the next block, as well as the stress levels, from the results of the previous blocks.

The tests in any block may or may not be at the same stress level; however, early in an experiment there is generally a definite advantage in allowing different stress levels within a block.

c. Sequential

If the stress level for each test is not determined until the result of the previous test has been observed (and perhaps also analyzed) the design is said to be sequential. The stress level chosen may depend only on the previous test and its result, as in the up-and-down design and in the Robbins-Monro process; in this case the design is a "Markov design". In other designs the stress level is determined from some or all previous tests, as in the Alexander designs. The latter type is more efficient, and a Markov design should generally not be used unless the experimenter does not have the opportunity for a more thorough analysis between tests.

If the experimenter is restricted in advance to a certain sample size, the design is said to be a fixed sample size design, or close-ended. A more desirable policy is to continue testing until the precision of the estimate(s) meets certain requirements (thus the sample size is a function of the results). In this case the design is said to be open-ended. While open-ended designs will give better results, considerations such as cost and availability of materials often necessitate a bound on the sample size. Most of the designs (stochastic approximation, Alexander, etc.) in this Manual are given in open-ended form. They can be used as

close-ended designs, however, without any modification, simply by terminating the experiment at the specified sample size. Strictly speaking, in the close-ended case there should be a different version of each design for each sample size, but it is believed that the improvements thereby obtained would generally be rather small, even for small sample sizes.

III-A2. DESIGNS RESTRICTED TO CERTAIN TEST LEVELS

There are several reasons why an experimenter might want to restrict his test levels to certain discrete values. In some cases these are the only levels at which he can test at all (e.g., the card gap test in which there must be an integral number of cards). In other cases it is convenient to construct test apparatus which allows for rapid and accurate stress level settings at a few selected values (e.g., notches every half-inch for drop testing). Sometimes the experimenter wants to be able to perform a simple analysis predicated on such a fixed level design (e.g., the Dixon-Mood analysis, Section IV-B4, for the Bruceton design, Section III-B1). Often there is an economy involved in "tooling" for fabrication of only a few types of items, when these items have their "stress" levels built in (e.g., solid propellant critical diameter studies).

All indications are that these restrictions really have very little effect on most experiments, although there is almost certain to be a noticeable deleterious effect on efficiency for very large sample sizes. At any rate, the experimenter may modify any of the designs given in this Manual to conform with such a design requirement as follows:

Suppose the design calls for the next test at stress level s. Let $\{s_j\}$ denote the stress levels actually available for experimentation, where $s_j < s_{j+1}$, $j = 1, 2, \cdots$.

Case 1: $s = s_j$, for some j

Simply test at $L = s_j$.

Case 2: $s_j < s < s_{j+1}$, for some j

Choose a random number γ uniformly distributed in $[0,1]$. If $\gamma \leq (s-s_j)/(s_{j+1}-s_j)$, test at $L = s_{j+1}$, otherwise test at $L = s_j$.

This procedure avoids the convergence of the test levels to a single fixed level, which could occur if the above rules were replaced by the selection of the fixed level s_j nearest to s.

III-A3. SETTING ERRORS AND MEASUREMENT ERRORS

In many types of experiments there are errors made in controlling the level of a stress variable or in measuring and interpreting the resulting responses. The first of these is referred to as a "setting error", and is common, for example, in projectile penetration studies where the velocity of the projectile (the stimulus variable) is only specifiable with a standard deviation, say, of 15 ft/sec. The effect of this error may be described by means of a method given at the end of Appendix V-B1.

Apart from whatever setting errors may exist there are always measurement errors in determining what the stimulus level(s) actually was during a test. In almost all cases, however, these errors are very small and can be ignored both in performing the tests and analyzing the results. In a few cases this may not be the case (for example, again consider projectile penetration studies where measurement errors are not small although they are generally smaller than the setting error). The effect of measurement errors may be treated by the methods of Appendix V-B1 by thinking of them as setting errors and assuming that all measurements are made perfectly. Alternatively when the measurement variance is known it is possible to modify the analysis (e.g., maximum-likelihood estimates) in order to correct for measurement errors, although the additional calculations are extremely tedious.

Of greater importance is the second problem mentioned. In fact, one of the real difficulties in sensitivity experimentation is in the interpretation of the test results. When is a result a response? Did that projectile actually penetrate the armor plate? Was that sound a detonation or a rapid deflagration? In drop testing, how should a "second strike" detonation be counted? In lethal dosage experiments, how should one interpret a severely incapacitated subject?

One solution in those cases where such questions are not easily answered is to allow the experimenter, within certain guide lines, to grade the response subjectively between 0 and 1 for all doubtful situations. Although we shall not do so, almost all of the methods given in the Manual may be modified in a straightforward manner to allow for such graded responses.

III-A4. STOCHASTIC APPROXIMATION

Stochastic approximation is of interest in sensitivity testing because it is concerned with experimental procedures which converge to a desired value when the observations being made involve error. Some of these schemes can be applied to the inverse response problem, which can be formulated as follows: If for a given stimulus x the probability of a response is $M(x)$, then we wish to find that value x_α, of x such that

$$M(x_\alpha) - \alpha = 0.$$

This differs from the analogous classical mathematical problem of finding roots of equations in that $M(x)$ is unknown and must be estimated on the basis of random observations with mean value $M(x)$.

Assuming that a unique root exists, Robbins and Monro (Ref. 6) devised an iterative procedure which converges to x_α under general conditions which include the inverse response problem in a special case. It is necessary to assume that $M(x)$ is a monotone function, continuous in the neighborhood of x_α.

Since the publication of the original Robbins-Monro paper a great deal of work has been done on stochastic approximation processes. Most of this deals with such mathematical topics as convergence, optimal choice of the process parameters, asymptotic properties of estimators, etc. Very few application of the method have been made to experimental design problems. More recently, there has been a greater interest in applying the procedure to experiments with quantal responses. Several empirical studies have been made and computer simulation evaluations have been used in order to critically examine the value of using stochastic approximation strategies under various conditions (see Appendix V-C).

Stochastic approximation designs for the inverse response problem generally depend on two parameters which must be chosen before testing begins. In the non-parametric case, when no assumptions are made concerning the response function, these are L_1, the first test level, and a certain constant, c. When parametric assumptions can be made, these can be related to the parameters of the assumed distribution (for example, under normal assumptions $L_1 = \hat{\mu} - K_\alpha \hat{\sigma}$ (where K_α is unit standard deviate) and $c = 1.253\hat{\sigma}$ where $\hat{\mu}$ and $\hat{\sigma}$ are 'guesses' of the mean μ and standard deviation σ). In stochastic approximation processes the estimate, $\hat{x}_\alpha$, is simply the next test level, and it can be shown that this is a sufficient statistic converging to x_α not only in probability but with probability 1. However, for application to the design of inverse response experiments we are more interested in the estimates for small samples which will depend on the choice of the design parameters. The more critical of these is L_1 (or $\hat{\mu}$ in the parametric case) which (as in most designs for the inverse response problem) is at some a priori 'best guess' of x_α.

A poor guess for L_1 results in the acquisition of a great many tests before a reasonable approximation to x_α is found. All of the variations of the original Robbins-Monro procedure are concerned with overcoming this deficiency. The Kesten (Ref. 7) and Odell (Ref. 8) variations have also the more general design objective of always testing in a neighborhood of x_α. These questions are discussed in detail in Section III-B5. A comprehensive review of stochastic approximation methods is given in Ref. 15.

If the response function is known to be cumulative normal, Chernoff (Ref. 4) has shown that asymptotically optimal designs for the inverse response problem are divided into two classes (see Section V-B6 for details). We consider the inverse

response problem for $x_\alpha = \mu + k\sigma$.

Case 1: $|K| \leq 1.5750360$

It may be shown that an asymptotically optimal design is any stochastic approximation of $x_\alpha = \mu + k\sigma$; the variance of the resulting estimate is asymptotic to

$$H = \alpha(1-\alpha)/[M'(x_\alpha)]^2 N,$$

where N denotes the sample size. For $K = 0$, $H = \pi\sigma^2/2N$.

Case 2: $|K| > 1.5750360$

Here an asymptotically optimal design is a mixture, in the proportions $.5+.787518/K$ and $.5-.787518/K$, of stochastic approximations of $\mu + 1.5750360\sigma$ and $\mu - 1.5750360\sigma$, respectively. The variance of the resulting estimate is asymptotic to

$$H = 1.6436K^2\sigma^2/N .$$

Thus when Case 1 obtains, the distributional assumption does not influence the asymptotically optimal design, since the latter is always a stochastic approximation to $x_\alpha = \mu + k\sigma$ even in the distribution-free case.* If Case 2 obtains, however, the distributional assumption significantly influences the asymptotically optimum design since a distribution-free design of this type is still a stochastic approximation of $\mu + k\sigma$, with the same variance expression as in Case 1, whereas an optimal parametric design is the mixture given in Case 2. We consider an extreme example to show how much better the distributional design can be in the latter case. Consider the nonparametric inverse response problem for $\sigma = .000,000,287$ which

*Any stochastic approximation of x_α is asymptotically efficient in the distributional case, whereas it is difficult to find stochastic approximations which are asymptotically optimal in the distribution-free case.

leads to an asymptotic variance of $1.3 \times 10^5 \sigma^2/N$, when the response function is (unknown to the experimenter) cumulative normal. But if the experimenter can completely rely on the assumption of normality, he can proceed as in Case 2 for $\mu - 5\sigma$, and his asymptotic variance will then be "only" $41\sigma^2/N$.

III-A5. EFFICIENCY OF DESIGNS FOR THE INVERSE RESPONSE PROBLEM IN THE DISTRIBUTION-FREE CASE

In evaluating and comparing designs for the inverse response problem, we would like to find that one which produces the best estimates in the sense of minimum variance for a specified sample size. For most designs, an empirical (simulation) study is required to estimate the variance for various sample sizes, and such studies are very useful in comparing designs under various conditions. However, in evaluating the results of simulation, it is desirable to use a standard against which each design can be measured.

For a class of stochastic approximation designs (which includes the Robbins-Monro), Hodges and Lehmann obtained an expression for the asymptotic normal variance. This variance, when asymptotically optimum design parameters are used (see Section III-B5), is given by

$$H_n(\hat{x}_\alpha) = \frac{\alpha(1-\alpha)}{[M'(x_\alpha)]^2 N} \quad , \quad M'(x_\alpha) = \left.\frac{dM(x)}{dx}\right|_{x=x_\alpha}$$

where N is the sample size, $M'(x)$ is assumed to be continuous at x_α, and $0 < M'(x_\alpha) < \infty$. This is the best presently available theoretical standard for comparing designs and is extremely useful in the following cases:

1. If for small N, the variance of a design is close to the standard, the design can be considered efficient.
2. If for a reasonably large N the variance of a particular design is, say, $5H_N(\hat{x}_\alpha)$ the design would be abandoned whereas if it were, say, $1.2H_N(\hat{x}_\alpha)$, the design and possible modifications should be investigated further.

In using this asymptotic variance standard, the following limitations should be kept in mind:

1. It is not known in general how large N must be before this expression can be implemented in practical situations.
2. Until further work is completed, an asymptotic variance of $H_N(\hat{x}_\alpha)$ does not imply asymptotic optimality of a design, although this is presently believed to be the case for all distribution-free inverse response problems and for certain parametric inverse response problems.

SECTION III

DESIGNS FOR SENSITIVITY EXPERIMENTS

PART B - SPECIFIC DESIGNS

III-B1. UP-AND-DOWN DESIGN

a. Introduction

This most widely known sensitivity design was first implemented at the Explosives Research Laboratory at Bruceton, Pennsylvania. It is properly used in conjunction with distributional assumptions for the inverse response problem with α near 0.5. For normal response curves, Dixon and Mood developed a simple approximate maximum likelihood method of analysis (described in Section IV-B4), along with analysis methods for small samples. General maximum-likelihood estimation procedures (Section IV-A4) can be used when the response function is assumed to be other than cumulative normal. Because of the extensive literature and experience with the up-and-down design we shall give only a brief description here. Furthermore we shall describe only a few of the many variations which have appeared. (A related design is discussed in Section III-B15.)

One of the advantages of the up-and-down design is the simplicity of its rules. These are given below assuming that the stress variable has been appropriately transformed (see Section IV-A5).

b. Design Rules

1. Choose a step-size, d, which will be fixed throughout the design.
2. The first test is at L_1, a best a priori guess of $x_{.5}$.
3. If this results in a response, the next test is at L_1-d; if it is a nonresponse the second test is at L_1+d.
4. If the nth test is at L_n and the result of that test is y_n, then

$$L_{n+1} = \begin{cases} L_n + d & \text{if } y_n = 0 \\ L_n - d & \text{if } y_n = 1. \end{cases}$$

5. The design generally can be considered open-ended; special termination rules such as "stop after K changes of response" are often used in practice.

The two parameters, d and L_1, of the design must be chosen in advance. The conventional choices are as follows: for normal response curves, $L_1 = \mu = x_{.5}$ and $d = \sigma$; for logistic response curves, $L_1 = x_{.5}$ and $1.5 \leq d \leq 2$. A poor choice of L_1 results in a greater number of tests. (The design will, in this case, have an initial series of all responses or all nonresponses and these tests should generally not be used in the analysis (see Section IV-B4).). The asymptotic properties of the design depend heavily on d, and the variance of the estimate of $x_{.5}$ increases sharply as d becomes large. However, if d is very small, then the choice of L_1 becomes more critical, introducing appreciable bias if it is far from $x_{.5}$.

It should be emphasized that this design is effective for the inverse response problem only for α near .5.

c. Variations

The following two variations have been introduced to overcome poor choices of L_1 and d. They are both improvements of the original design but their relative merit is unknown.

Two-stage Design (Cochran and Davis, Ref. 9)

1. Perform an up-and-down design with $d = 1.25\sigma$ until a change of response occurs.
2. Let L_1 be the midpoint between the two last levels. With L_1 as the first test level, use either with a Robbins-Monro design with $c = 1.25\sigma$ (see Section III-B5) or an up-and-down with step size σ.

Wetherill Design, (Ref. 5)

1. Use an up-and-down test with step-size σ until 5 changes of response have occurred.
2. Change to an up-and-down design with step-size $1/2\sigma$.

d. Up-and-Down Design for Delayed Observations

If there is a delay before the result of a test is known, a saving in time can be obtained by running several small independent up-and-down sequences simultaneously rather than one long design. A small sample analysis method such as that of Brownlee et al. (Section IV-B4) can be used in this connection. There appears to be no appreciable loss of efficiency if, for example, a 40-test design is broken into four series of 10 tests, provided that a good guess for μ is made at the outset.

e. Evaluation of the Up-and-Down Basic Design for Inverse Response Problem

When a step-size of $d = \sigma$ is used, as is generally recommended, the variance of the estimator for the inverse response problem for $x_\alpha = \mu + K\sigma$ is given asymptotically (Ref. 12) by

$$\mathrm{Var}\,(\hat{\mu} + K\hat{\sigma}) \simeq (2.02 + 3.75K^2)\sigma^2/N \ .$$

The relative asymptotic efficiency of this basic up-and-down design for various inverse response problems, relative to the optimum results of Chernoff summarized in Section IIIA-4, is tabulated below:

K (in $\mu + K\sigma$)	$e = \frac{Var_{min}(\text{Chernoff})}{Var_{\text{up-and-down with } d = \sigma}}$
0	78%
.5	58%
1.0	40%
1.5	36%
2.0	39%
3.0	41%
∞*	44%

*The limit of the inverse response problem as $K \rightarrow \infty$ is equivalent to the estimation of the parameter σ.

III-B2. BARTLETT DESIGN, (Ref. 10)

a. Introduction

This simple distribution-free design is intended for the exploration of the lower tail of a response function for the purpose of selecting a type of distribution to use in analyzing subsequent smaller samples of data taken in similar test situations. It may also be used for the inverse response problem for small values of α. A similar design holds for the upper tail and large values of α.

b. Design

The experimenter selects a set of test levels equally spaced with increment Δ. The first test is at a level where responses are expected roughly 50% of the time. Testing continues at this level until the second response; then testing begins at the next lower level. Testing continues at this lower level until the second response occurs, etc.

c. Comments

If the response function is known to be normal, a step of $\Delta = .5\sigma$ is better than $\Delta = \sigma$, at least for the 1% and .1% inverse response problems and for sample sizes over 300; the larger step size is better for sample sizes under 80 (see simulation results in Appendix V-C). In general, this is typical of the way that Δ depends on the intended sample size. Almost any parametric or nonparametric method of analysis may be used on data collected by the Bartlett design.

III-B3. 20 ZIL DESIGN

a. Introduction

This is an extremely simple distribution-free design for estimating the "zero-threshold" stimulus level, that is, the highest stress level at which no responses are possible. Even in those situations where it is not possible to assume the existence of a threshold, the design is still of use for the inverse response problem if $.01 < \alpha < .10$. The design will be described for this range of α values; a similar design may be used for the case $.90 < \alpha < .99$.

b. Design

A fixed set of test levels with equal spacing Δ is chosen. The first test is made at a level where the experimenter believes that the response probability is about 50%. The quantity Δ is chosen so that the initial test level is three levels above the stress level believed to be the threshold, or, for the inverse response problem, three levels above x_α.

Testing continues at a level until the first response occurs, or until 20 nonresponses have been observed. If the first response occurs within the first five tests at a level, the next test is taken two levels below; if the first response is on the sixth to twentieth test at a level, the next test is taken on level below; if there are twenty nonresponses at a level, the sequence ends.

More than one sequence may be used; in each case the new sequence starts three levels above the level at which the previous sequence terminated.

c. Comments

In the conventional implementation of this design three sequences are run, and then the level below the lowest level at which a response is observed is taken as a

nonparametric estimator of the threshold.

If the 20 ZIL design is intended for the inverse response problem for small α, a parametric analysis would be preferred, because the simple conventional estimator has a bias which changes drastically with step size. For example, if $\alpha = 0.05$ and the response function can be assumed to be cumulative normal, maximum-likelihood estimates of μ and σ would be used to compute an estimate of $\mu - 1.645\sigma$ (the experimental objective when $\alpha = .05$). Although the 20 ZIL design is not nearly efficient for this inverse response problem, the danger in the assumption of normality is greatly reduced relative to, say, the Bruceton design, because no extrapolation is involved. For example, if $\Delta = \sigma$, the estimate of $x_{.05} = \mu - 1.645\sigma$ will be at a stress level in the middle of the data.

III-B4. RUN DOWN DESIGN

a. Introduction

This design is used for the inverse response problem when the response function can be assumed to be cumulative normal and when 200 tests can be made. Tests are concentrated near the 35% and 10% response levels, and the design is satisfactory for inverse response problems from $\alpha = 25\%$ to $\alpha = 10\%$. It is not recommended for the inverse response problem when $\alpha \leq 5\%$.

An analogous procedure exists when $75\% \leq \alpha \leq 90\%$. This design was originally intended to be used in conjunction with an elementary final analysis based on a log logistic assumption for the response function. The analysis employs the results of the last 180 tests, which are at only two levels. The Run Down design is discussed in (Ref. 11).

b. Design

Phase I: Initialization

A Bruceton design (Section III-B1) is used to conduct the first twenty tests. Let $m = \hat{\mu}$ and $s = \hat{\sigma}$ denote the estimates of μ and σ obtained by the Dixon-Mood (Ref. 12) or maximum-likelihood methods of analysis (Sections IV-B4, IV-B1).

Phase II: The 35% Response Level

The next fifty items are tested at $m - .4s$

Phase III: The 10% Response Level

The next 130 tests are tested at $m - 1.3s$, except in one situation. If the fifty tests in Phase II gave five or less responses, then eighty more tests are taken at $m - .4s$, followed by fifty at $m - .2s$.

c. Comments

For the completed sample, maximum likelihood estimates of μ and σ are suggested, (see Section IV-B1).

This design cannot really claim to be very efficient, since in general it is entirely established on the basis of the analysis performed at the end of Phase I. The design is useful when there is cost or time saving in having 130 tests at the same level, or when there is relatively little time for analysis between tests. When these conditions do not prevail, one of the many sequential designs available, such as the Alexander Design or the Rothman Design (Sections III-B6, III-B14) is recommended.

III-B5. ROBBINS-MONRO DESIGN AND VARIATIONS

a. Introduction

Several designs for sensitivity experiments have been derived from the general stochastic approximation scheme of Robbins-Monro which was formulated in 1951 (Ref. 6). These designs, generally considered open-ended, can be used either with or without distributional assumptions on the response function. The rules for these designs assume that it is possible to test at any stress level.

b. Basic Design Rules

1. First test at L_1 some a priori best guess of x_α.

2. If the response at L_1 is given by $y_1 = 1$, then the second test is at stress level

$$L_2 = L_1 - a_1(1-\alpha)$$

where a_1 is the first term of a sequence $\{a_n\}$ given below. If $y_1 = 0$ then the second test is at stress level

$$L_2 = L_1 + a_1\alpha \quad .$$

3. Let the nth test be at some stress level L_n and the response for that test be y_n (0 or 1); then the next test will be at stress level L_{n+1} given by

$$L_{n+1} = L_n + a_n(\alpha - y_n)$$

where $\{a_n\}$ is a specified sequence of numbers which must satisfy

$$\sum_{1}^{\infty} a_n = \infty ,$$

$$\sum_{1}^{\infty} a_n^2 < \infty .$$

If it is assumed that $M(x)$ is locally linear around x_α then the asymptotically best values for the sequence are given by

$$a_n = c/n .$$

The optimum value of c which asymptotically minimizes the variance of the estimate of x_α is

$$c = \frac{1}{M'(x_\alpha)} \qquad M'(x_\alpha) = \left.\frac{dM(x)}{dx}\right|_{x=x_\alpha}$$

Since c must be chosen in advance, this relationship is of little use when $M(x)$ is unknown. When distributional assumptions for $M(x)$ can be made, c can be expressed in terms of the distribution parameters (through the relation $c = 1/(M'(x_\alpha))$), which in turn generally must be guessed at before the program begins. In either case, the choice of c is not critical as long as it is larger than the optimum (see Refs. 9,13). When assumptions or prior information concerning distributional parameters is possible, the selection of L_1, the best guess of x_α, also is made in terms of these parameters. The selection of L_1 and c when the distribution is normal is discussed below.

4. There is a theoretical expression for n, the required total number of tests, based on the criterion that the asymptotic variance of the estimate be

less than a specified value (Ref. 14). Since this condition depends on parameters which are difficult to estimate (e.g., $M'(x_\alpha)$) with satisfactory precision, the testing is generally stopped after a predetermined number of tests.

5. The estimate of x_α after n tests is L_{n+1}.

c. Discussion and Variations

As can be seen from the design rules, the value of L_1 must be guessed and the optimum c is given in terms of the derivative of an unknown function; thus in general c must also be guessed before testing begins. Since all of the variations of the design involve both of these parameters, we shall postpone the discussion of how the estimates are affected by them until these designs have been described.

Delayed Process (Cochran-Davis) for $\alpha = 1/2$, (Ref. 9)

From the equation for determining successive test levels

$$L_{n+1} = L_n + a_n(\alpha - y_n)$$

we see that if a series of all responses or nonresponses occurs (so that $\alpha - y_n$ remains constant), the step sizes $L_{n+1} - L_n$ continually decrease because $\{a_n\}$ is a decreasing sequence. If L_1 is far from x_α, then these decreasing steps would require a large number of tests before we are in a neighborhood of x_α.

If we are testing in a region around $x_{.5}$ we would expect in any efficient design approximately the same number of responses and nonresponses. Thus if the first few tests result in all responses or nonresponses, we would be led to

believe that our guess of L_1 was not a good one. In the delayed process version of the Robbins-Monro design this poor guess is counteracted by not allowing the step sizes to decrease until a change of response occurs. The rules can be stated as follows:

1. First test at L_1 some a priori best guess of x_α.
2. If y_1 (the response at L_1) is 0, then

$$L_2 = L_1 + \alpha a_1 \quad .$$

Our step size will remain αa_1 until a response of 1 occurs. That is, if $y_2 = y_3 = \cdots = y_k = 0$ then

$$L_{n+1} = L_n + \alpha a_1 \qquad (n = 1, 2, \cdots, k) \quad ,$$

but if the response at L_{k+1} is 1, then our step size will decrease as in the Robbins-Monro design. Similarly, if $y_2 = y_3 = \cdots = y_k = 1$ our step size will be constant throughout these tests, that is

$$L_{n+1} = L_n - a_1(1-\alpha) \quad \text{for} \quad n = 1, 2, \cdots, k \quad .$$

3. When the first change of response occurs we introduce the second term of our sequence $\{a_k\}$ and continually decrease the step size from then on. If $y_1 = y_2 = \cdots = y_k = 0$ but $y_{k+1} = 1$ then

$$L_{k+2} = L_{k+1} - a_2(1-\alpha).$$

Similarly for $y_1 = y_2 = \cdots = y_k = 1$ but $y_{k+1} = 0$, we have

$$L_{k+2} = L_{k+1} + a_2\alpha$$

4. For all $n > k$ the stress level is given by

$$L_{n+1} = L_n + a_{n-k+1}(\alpha - y_n) \quad .$$

5. The sequence $\{a_n\}$ and the stopping rule are the same as for the Robbins-Monro design.

If L_1 is close to $x_{.5}$ this variation differs only slightly from the Robbins-Monro design, and estimates of $x_{.5}$ are essentially the same for the two. The variance of the estimates in this variation will increase as L_1 moves away from $x_{.5}$, but much less drastically than for the Robbins-Monro design. This improvement is obtained without introducing any complications in the calculations.

Variations in Sequences ($\alpha = 1/2$)

Changes in the selection of the sequence $\{c/n\}$ can also be considered, again to overcome a poor choice of L_1. Alternate choices for $\{a_n\}$ are $\{2c/(n+1)\}$ and $\{3c/(n+2)\}$. When L_1 is close to $x_{.5}$ the asymptotically optimum sequence $\{c/n\}$ used in the Robbins-Monro design gives better estimates as would be expected. However, use of the alternate sequences yields improved estimates when L_1 is not close to $x_{.5}$. In all cases the delayed response design gives better estimates than the Robbins-Monro design either with or without sequence variations.

Kesten Variation, (Ref. 7)

In this design the objective is to obtain more rapid convergence of the test levels to x_α by using decreasing step size only when testing in the region close to x_α. Frequent changes in sign of $L_{n+1} - L_n$ is the criterion used to indicate that the testing is in this region. This is equivalent to a change

of response, i.e., $y_{n-1} \neq y_n$. The rules for the Kesten variation are:

Rules 1 and 2 are the same as in the Robbins-Monro design.

3. Use the sequence $\{a_n\} = \{c/n\}$ which is the same as for the Robbins-Monro design. Then if the nth test is at stress level L_n with response y_n, the $(n+1)^{st}$ test is at

$$L_{n+1} = L_n + c_n(\alpha - y_n)$$

where

$$c_1 = a_1$$

$$c_2 = a_2$$

$$c_j = a_{t(j)}$$

with

$$t(j) = 2 + \sum_{i=3}^{j} \delta[(L_i - L_{i-1})(L_{i-1} - L_{i-2})]$$

and

$$\delta(x) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases} .$$

The function $t(j)$ is constructed so that the step size remains the same when the responses at two levels L_{i-2} and L_{i-1} are the same, but decreases when the response at L_{i-2} is different from the response at L_{i-1}.

4. The remaining rules are the same as for the Robbins-Monro design.

For all values of α the Kesten variation gives better estimates than those obtained with the original Robbins-Monro design.

Odell Modification, (Ref. 8)

This is a modification of the Kesten design. As in the latter, the step size is only decreased when testing is in the region of x_α (based on the same criterion); however, in this region the test levels are determined by linear interpolation. In the Odell modification more than one test is performed at each test level.

The rules can be stated as follows:

1. Choose k, the number of tests to be performed at the first level and Δk, the increment to be added to k with increasing trials (see step 5).

2. Perform k tests at L_1, the best a priori guess of x_α. Let y_{1i} $(i=1,2,\ldots k)$ be observed responses, then the average response at L_1 is given by

$$\bar{y}_1 = \frac{1}{k} \sum_{i=1}^{k} y_{1i} .$$

3. The next test level, L_2, is given by

$$L_2 = L_1 + a_1(\alpha - \bar{y}_1) .$$

4. If the nth test level is at L_n with average response $\bar{y}_n$, the $(n+1)^{st}$ test level is given by

$$L_{n+1} = L_n + c_n(\alpha - \bar{y}_n),$$

where $c_1 = a_1$

$c_2 = a_2$

and if $c_{j-1} = a_\ell$ for $\ell \geq 2$, then

$$c_j = \begin{cases} a_\ell & \text{when} \quad \alpha \notin (\bar{y}_{j-1}, \bar{y}_j) \\ \dfrac{(L_j - L_{j-1})}{\bar{y}_j - \bar{y}_{j-1}} & \text{when} \quad \alpha \in (\bar{y}_{j-1}, \bar{y}_j) \end{cases}$$

$$c_{j+1} = \begin{cases} a_\ell & \text{when} \quad c_j = a_\ell \quad \text{and} \quad \alpha \notin (y_j, \bar{y}_{j+1}) \\ \dfrac{L_{j+1} - L_j}{\bar{y}_{j+1} - \bar{y}_j} & \text{when} \quad \alpha \in (y_j, \bar{y}_{j+1}) \\ a_{\ell+1} & \text{when} \quad \alpha \notin (\bar{y}_j, \bar{y}_{j+1}) \\ & \text{and} \quad c_j = \dfrac{L_j - L_{j-1}}{\bar{y}_j - \bar{y}_{j-1}} \end{cases}$$

where

$$\alpha \in (\bar{y}_j, \bar{y}_{j+1}) \quad \text{if} \quad \bar{y}_j \le \alpha \le \bar{y}_{j+1} .$$

5. If $(L_j - L_{j-1})(L_{j-1} - L_{j-2}) < 0$, Δk tests are added to the sample size k.

Discussion

Stochastic approximation designs give unbiased estimates only when $\alpha = 1/2$. This can be explained most graphically on the basis of an example given by Wetherill (Ref. 5) as follows: assume $\alpha = .75$ and the first test is at $x_{.75}$ but results in a nonresponse. Then the second test would be at $x_{.75} + .75c$, where c is defined as before, and at least 31 tests would be required to return to $x_{.75}$ using the Robbins-Monro design. This number would be reduced to eight tests for the Kesten variation.

But in all cases there will be a bias in the estimate of x_α if $\alpha \neq .5$, irrespective of the choice of x. Of course, the estimates are asymptotically unbiased, by the definition of stochastic approximation.

The Robbins-Monro design and its variations are primarily intended for use in the distribution-free case, although they may be used in conjunction with distributional assumptions as indicated earlier in this section. In both cases, the accuracy of the estimate, $\hat{x}_\alpha$, will depend on the choice of these design parameters: for example when the response function is assumed to be cumulative normal, the Robbins-Monro design gives satisfactory precision of the estimate only when either μ or σ is well known in advance. The delayed response variation overcomes a poor choice of L_1 particularly if the estimate of σ is between $2\sigma/3$ and $3\sigma/2$.

In the distribution-free case the choice of the parameters c and L_1 will affect the efficiency of the design. In general the estimates of x_α are not sensitive to errors in c, provided c is not less than its optimum. The choice of L_1 is much more critical, particularly when the Robbins-Monro design is used, due to the continually decreasing step sizes in this case. By slowing this decrease all of the variations are less adversely affected by the choice of L_1. The delayed reaction design does this with no increase in computational requirements, so on the basis of simplicity it is preferred.

The results of the various simulations which have been made using these stochastic approximation designs are described in Appendix V-C.

III-B6. ALEXANDER DESIGN

a. Introduction

In sequential designs new levels for testing are determined from previous test results, and this may be accomplished in many different ways. For the up-and-down designs which have predetermined step size, new levels are determined from the results of the previous test only. Stochastic approximation designs use the same information, but in addition the variable step size depends on the total number of tests already performed. In the Alexander design the step size is constant, but new test levels depend on all previous test results.

The design, which is intended for the inverse response problem, uses alternately increasing and decreasing sequences to bound the sought-for stimulus level x_α. The testing ends when x_α is, with a specified probability, located within an interval of length not more than 2Δ (Δ = step size). From this interval estimate, an estimate of x_α is found by linear interpolation.

The initiation and termination rules for the sequences are defined in terms of monotone estimates of the response probabilities at the test levels. In the original design, which should be used for α around .5, maximum-likelihood estimates are used. However, for extreme values of α, it is more efficient to use both maximum-likelihood estimates and an estimate based on confidence bounds. How these are found is indicated in the design for extreme values of α.

When $\alpha \leq .01$ or $\alpha \geq .99$ and only a limited number of tests is available, the .05 design can be used in conjunction with a tailwise parametric assumption (see Section II-A1).

Simulations of both .5 and .05 designs have been carried out (see Appendix V-C). The design is generally more efficient than other available nonparametric designs, and is asymptotically as efficient as the best parametric stochastic approximation when distributional assumptions are valid.

b. The General Design

1. The first test is at L_1, the a priori best guess of x_α.

2. By the method of reversals monotone estimates are evaluated at all test levels after each test (see Section IV-B6).

3. Testing will be performed by alternately increasing and decreasing sequences of test levels.

4. The first test of an increasing (decreasing) sequence is at either the highest (lowest) level, strictly above (below) the last test level, at which the estimate is less (greater) than or equal to α or, if there is no such level, at the level above (below) the last test level.

5. An increasing (decreasing) sequence will be terminated at the first level at which the estimate after a test at that level is strictly greater (less) than α.

6. The rules for ending the design depend on the value of α and are given explicitly below.

c. The .5 Design

For $\alpha = .5$ the estimates used in following the design rules are the maximum-likelihood estimates given by the method of reversals. When the testing is finished we wish to have an interval I such that $\text{Prob}(x_\alpha \epsilon I) > P$, where P is some prescribed probability. The length of I depends on the particular experiment; it is never more than twice Δ, the step size, but in most cases

it is Δ. The occurrence of an interval of length 2Δ corresponds to the situation when $\hat{x}_\alpha$ is at the center of I. Testing will be stopped when either of the following conditions is satisfied: (a) there are three adjacent test levels $L_0 < L_1 < L_2$ such that the response estimate at L_1 is .5 and the response estimates $\hat{p}_0$ and $\hat{p}_2$ at levels L_0 and L_2, respectively, lead to the confidence statements

$$\text{Prob}\{p_0 > .5\} < (1-P)/2$$

$$\text{Prob}\{p_2 < .5\} < (1-P)/2$$

or (b) there are two adjacent levels $L_0 < L_2$ for which the above confidence statements can be made.

When P is .5 (as was used in the computer simulation study), then the conditions for L_0 and L_2 are given by the following table:

A	B
0	2
1	3
2	4
3	6
4	7
5	8
6	9
7	10
8	12
9	13
10	14

In this table, A denotes the number of responses at L_0 and B denotes the minimum number of nonresponses which must be observed at that level for the condition in (a) to be satisfied. Similarly, if A is the number of nonresponses at L_2 then at least B responses at L_2 are required for termination.

In the following simulated example, the Alexander design is used for $\alpha = .5$, $\Delta = .5$, with a cumulative normal response function, $\mu = 0$, $\sigma = 1$:

Test Number	Stress	Response
1	1.3	1
2	.8	1
3	.3	0
4	.8	1
5	.3	1
6	-.2	1
7	-.7	0
8	-.2	0
9	.3	1
10	-.2	1
11	-.7	0
12	-.2	1

Since the response fraction at -.2 is 3/4, while at .3 it is 2/3, the method of reversals must be used, giving 5/7 at -.2 and at .3. Linear interpolation between 5/7 at -.2 and 0/2 at -.7 gives as final estimate $\hat{x}_{.5} = -.35$.

d. The Alexander Design for Extreme Values of α

A consistent nonparametric design for the inverse response problem has stress levels converging to x_α. Therefore, when $\alpha \sim .05$ we would expect approximately 19 nonresponses for every response. Thus in this case a good design forces some testing in the stimulus region below the lowest level at which a

response has been observed. (We refer to this as the "zero region".) Since the maximum-likelihood estimates in the zero region are all zero, a new "estimate", $\tilde{p}$, will be introduced to insure testing in the neighborhood of x_α. This estimate is actually used only to determine when to terminate a decreasing sequence. In all other cases, maximum-likelihood estimates are still employed.

The definition of $\tilde{p}_1$ is given as follows: let the interval Z_1 be that part of the zero-region $\geq L_1$ and let N be the number of nonresponses observed in Z_1. Then $\tilde{p}_1$ is given implicitly by

$$(1-\tilde{p}_1)^N = 1-P$$

where P is some specified probability.

The objective in using the new type of "estimate" is to be reasonably sure that decreasing sequences end below x_α. From the rules for the design, a decreasing sequence will terminate at level L_1 where $\tilde{p}_1 < \alpha$. From the definition of $\tilde{p}_1$ the following confidence statement can be made:

$$\text{Prob}\{p_1 < \alpha | \text{observed responses}\} > P \quad ,$$

i.e., on the basis of the observed responses the probability that L_1 is below x_α is greater than P.

A uniform set of rules for the design can now be given:

1. The first test of an increasing sequence is at the level below that at which a response has been observed. If the result of this test is a response the sequence ends; otherwise, one more test (at the next higher level) is performed.

2. The first test of a decreasing sequence is at the level below that at which a response has been observed. The sequence ends at level L_1 such that in the corresponding interval Z_1 (the zero region $\geq L_1$) at least N nonresponses have been observed. Values for N can be found from

$$(1-\alpha)^n = 1-P \ , \quad N = [n] + 1 \quad .$$

The following table gives values of N for P = .5

α	N
.10	7
.09	8
.08	9
.07	10
.06	12
.05	14
.04	17
.03	23
.02	35
.01	69

3. Testing ends when there are three adjacent levels L_1, L_2, and L_3 such that at least one response has been observed at L_3 (and none at a lower level) and a total of at least N nonresponses have been observed at L_2 and L_3. The value of N is given in the preceding table.

As α becomes smaller the number of tests necessary to terminate a decreasing sequence and, therefore, the total number of tests in the design, becomes quite

large. Because of this, it is recommended that for very small α (say, $\leq .02$), tailwise parametric assumptions should be used in conjunction with a .05 design (see Section II-B1).

4. The estimate, $\hat{x}_\alpha$, is found by linear interpolation between the response probabilities estimated at L_2 and L_3 by the method of reversals.

e. Discussion

These designs have the advantage that once the rules are understood, the actual procedure is fairly straightforward and the intermediate calculations between tests are extremely simple. The parameter x_α may be estimated by the method of reversals and linear interpolation (see Section IV-B6). Of course, as with any nonparametric design, distributional or tailwise parametric assumptions can be used to estimate x_α. For data taken by this design, the risk in making these assumptions is much less than that involved in using Bruceton data and the normal assumption to estimate $x_{.01} = \mu - 2.326\sigma$, for example.

In Appendix V-C complete results of simulations of the Alexander designs for $\alpha = .5$ and .05 are given. It appears that this design, particularly for extreme values of α, is much more efficient than other nonparametric designs which are not based on analysis of all previous results at each stage. Nonparametric estimates from the Alexander design are asymptotically as efficient as those from parametric stochastic approximations of x_α.

One of the advantages of these designs is the small number of tests required. In Appendix V-B4, a formula is given for the upper bound of the expected sample size for extreme values of α. When $\alpha = .05$, the bound is 76 tests. In our simulations, the average number of tests was about 45 for $\alpha = .05$; for $\alpha = .5$ the average number of tests was about 11.

The step size, or increment between test levels, denoted by Δ, should be small if the first test is close to x_α, and larger if the first test is further away from x_α. A suitable value for Δ based on the results of extensive simulations of the Alexander design is $\Delta = W/7$ for the 50% problem and $\Delta = W/40$ for the 5% problem, where W is the "prior sigma," that is, the standard deviation of the first test around x_α (cf. Sections III-B13, III-B14). This recommendation is intended for the case of a very poorly known response function. The experimenter can do somewhat better if he has an estimate of the value of $M'(x_\alpha)$ (cf. Section III-B5).

Example

The following data were simulated using a normal response function with $\mu = 0$, $\sigma = 1$, so that for $\alpha = .05$, $x_\alpha = -1.645$. The first test was at -3σ and the step size chosen was $.25\sigma$. (The X's and O's indicate responses and non-responses, respectively.)

Stimulus Level	Test Results									
-3.00	O									
-2.75	O									
-2.50	O	O				O				
-2.25	O	O				O		O		
-2.00	O	O		O		O		O		O
-1.75	O	O		O		O	O	O	O	O
-1.50	O	O		O		X	O		O	
-1.25	O	O		O		O				
-1.00	O	O	O	O	O	O				
- .75	O	X	O		O					
- .50	X									
	I_1	D_1	I_2	D_2	I_3	D_3	I_4	D_4	I_5	D_5

In the above table, the columns indicate sequences (I for increasing, D for decreasing). The final estimate is obtained by linear interpolation which yields $\hat{x}_\alpha = -1.675$. Note that a total of 44 tests was required.

III-B7. THE NAVAL POWDER FACTORY (NPF AND NPF INVERTED) DESIGNS

a. Introduction

This type of design is intended for finding "threshold" stress levels or for the inverse response problem. Thus it is similar to the 20 ZIL design (Section III-B3), except that it would be used for less extreme values of α (e.g., from 10% to 30% for the version given below).

For large α, a complementary design to the one given below may be obtained by interchanging "responses" with "nonresponses" and "-" with "+" in the design rules.

b. NPF Inverted Design (for Lower Tail)

Phase I: Initialization

A level S and step size Δ are chosen so that $M(S)$ is believed to be .5, and so that $S-2\Delta$ is believed to be the desired stress level (x_α or threshold).

The first test is at $L_1 = S - 2\Delta$

Phase II: Increasing Sequence

If the first test is a nonresponse, the second test is at $L_2 = L_1 + \Delta$. In general, $L_{i+1} = L_i + \Delta$ until the ith test is a response. Then $L_{i+1} = L_i - \Delta$, and a decreasing sequence is started.

Phase III: Decreasing Sequence

Testing continues at a level until either a response is observed, in which case testing proceeds to the next lower level, or three nonresponses are observed, in which case the design is terminated.

c. Comments

This design is sometimes repeated five times, starting each subsequent replication one level above the final level of the previous replication. Further details on this design are given in Ref. 28.

d. Example: $\Delta = 0.1$, $S = 7.1$

i	L_i	R_i
1	6.9	0
2	7.0	1
3	6.9	1
4	6.8	1
5	6.7	0
6	6.7	0
7	6.7	0

III-B8. BLOCK-SEQUENTIAL DESIGN FOR THE 50% INVERSE RESPONSE PROBLEM WHEN THE RESPONSE FUNCTION IS CUMULATIVE NORMAL

This design is for determining the stress level at which the response is 0.5 (inverse response problem for $\alpha = .5$) when it is assumed that the response function is cumulative normal. Further, it is assumed that from considerations of specimen preparation it is necessary to complete the testing in a limited number of "blocks". All of the stimulus levels for a given block must be determined before the tests are performed. In such cases the design problem breaks into two parts: (1) the determination of the number of tests for each block, given the total number available for all blocks, and (2) the explicit calculation of the stress levels at which these tests should be performed.

We shall give a procedure for handling this design problem in which the selection of block sample sizes is handled independently of the specification of the individual stress levels within each block. The approach is based on a general asymptotic estimation of the growth of information for sensitivity experiments (see Appendix V-B1) and leads to the approximate optimization of the number of tests in each block. The general treatment of the design for any number of blocks is indicated in Appendix V-B1. We shall limit attention in this section to two-block experiments.

a. Suppose that the assumed cumulative normal response function has parameters μ and σ, both of which are unknown, and that the experimenter's original guess at μ, denoted by $\tilde{\mu}_o$, is normally distributed with mean μ and (known) standard deviation s_o.

The amount of prior information (see Appendix V-B1) is expressed in the form $E_o = (\pi/2)\sigma^2/s_o^2$, which gives the number of asymptotically efficient tests (at μ) providing an asymptotically equivalent amount of information.

b. Approximate optimum sample sizes for two-block experiments for two values of E_o are given below:

Sample Size of First Block		1	2	3	4	5	6	7	8
Total Sample Sizes for Which the Given First Block is Optimum	$E_o = .02$	2-3	4-7	8-11	12-15	16-19	20-24	25-29	30-34
	$E_o = 2.00$	2-3	4-6	7-10	11-14	15-19	20-25	26-31	32-34

It should be noted that the effect of E_o on the optimum staging of the two blocks is slight. It may be shown furthermore that the loss of expected information relative to the optimum is small if non-optimal sample sizes are used, even to the extent of selecting equal sample sizes in the two blocks. For example, if 32 tests are available and $E_o = .02$, then the (8,24) design gives about 17.36 units of information, whereas the (16,16) design gives about 14.77 units (in terms of the equivalent number of asymptotically efficient tests). Either way, the two-block designs are significantly less efficient than the three-block design (3,5,24), which would give about 23.11 units.

At any rate, we can summarize the results given in this table by the empirical formula

$$n_1/N \cong (8N)^{-1/4}, \quad .02 \le E_o \le 2.00, \quad \begin{array}{l} n_1 = \text{size of first block;} \\ N = \text{total sample size} \end{array}.$$

c. Given the (approximately) optimum allocation of the number of tests to each block, we next consider the specification of the test levels within each block. The general idea of this design is to use s_o, and numbers derived from it, to determine the spacing of the tests in the first, and subsequent, blocks. The location of each block will be determined by the latest estimates and by the response "imbalance" in the previous block.

The experimental design will be defined inductively. Let

$n_j \equiv$ the number of tests allocated to the jth block ,

$m_o \equiv 0$

$$m_j \equiv \sum_{i=1}^{j} n_j \ ,$$

$$s_j \equiv s_{j-1}\sqrt{\frac{m_{j-1} + .5}{m_j + .5}} \ ,$$

where m_j denotes the cumulative sample size up to the jth block, and s_j is an estimate of the standard deviation of $\tilde{\mu}_j$, the latter being an estimate of μ made after the jth block. (The choice of estimators is discussed below.)

Suppose we have just finished the jth block, and wish to determine the stress levels for the $(j+1)^{st}$ block. We use the cumulative normal function $g(x;\tilde{\mu}_j, s_j)$ with parameters $\tilde{\mu}_j$, s_j. Let r_j be the fraction of responses in the jth block. Then the n_{j+1} stress levels for the $(j+1)^{st}$ block are the values x_k $(k = 1, 2, \cdots, n_{j+1})$ determined by

$$g(x_k;\tilde{\mu}_j, s_j) = \frac{-.01-.98r_j+k}{n_{j+1}} \qquad (k = 1, 2, \cdots, n_{j+1}) \quad .$$

If r_j is .5, the stress levels for the $(j+1)^{st}$ block are "equi-spaced in probability" and centered around $\tilde{\mu}_j$ using this procedure. If r_j is near 1.0, the stress levels are moved downward; if near 0.0, they are moved upward.

d. As a first estimate $\hat{\mu}_j$ of μ we may use the maximum likelihood estimate (see IV-B1) or the estimate given by the MOS method (see IV-B2). If neither applies, then all tests through the jth block resulted either in all responses

or all nonresponses. If there are only responses, say, then $\hat{\mu}_j = S_{min} - s_j/\sqrt{m_j}$, where S_{min} denotes the lowest stress level tested.

Once $\hat{\mu}_j$ is obtained, we incorporate the prior information concerning knowledge of μ by using the estimate

$$\tilde{\mu}_j = \alpha\tilde{\mu}_o + (1-\alpha)\hat{\mu}_j \quad ,$$

where

$$\alpha = .2/(m_j+1) \quad .$$

If the prior information were .02, the correct value of α would be $.02/(m_j+.02)$. If the prior information were as high as 2, then the correct value of α would be $2/(m_j+2)$. Since the value of E_o is not known, but generally in this range, the value $.2/(m_j+1)$ has been chosen as a compromise.

III-B9. MULTIVARIATE SENSITIVITY DESIGN

a. Introduction

Little attention has been given to the design of multivariate sensitivity experiments, that is, experiments with several stimulus variables. In the field of biological assay, the emphasis has been on determining suitable distributional assumptions for the response function and on methods of analysis, rather than on the design of the experiments (see, for example, Ref. 16). Sometimes it is assumed that only one of the variables is a "sensitivity" variable, so that one of the univariate sensitivity designs can be combined with a factorial or perhaps rotatable design for the other variables (see Refs. 1 and 2).

As described in Section II-A4, the problems encountered here are largely analogous to those for univariate sensitivity experiments. If there are k stimulus variables, the finding and testing of response function models involve the consideration of a $(k+1)$-dimensional response surface. The inverse response problem becomes the determination of a k-dimensional equal-probability contour.

The selection of a multivariate design depends on whether the stimulus variables "interact". Various definitions have been used for this term; in what follows we shall use a definition based on the assumption that, if a failure does not occur, the object tested returns to its original condition. (This is not an acceptable assumption for biological experiments where doses are generally additive.) Then, if the response, when the stimuli are applied simultaneously, is the same as when they are applied consecutively, the stimuli will be referred to as "noninteractive". An assumption of noninteraction can be tested by a χ^2-test (see Section IV-B7).

When the stimuli do not interact, the following simple nonparametric design can be used to determine estimates of the probability of response at each level combination at which a test is performed. It is assumed that the response function is monotone nondecreasing in each stimulus variable, for fixed values of the other variables, and the estimates will be constrained to satisfy this condition. There do not appear to be any good designs available for multivariate sensitivity experiments with interaction, although some preliminary work which describes the equal-probability contours in this case has been accomplished (see Appendix V-B9).

b. Multivariate Design Without Interaction

We first treat explicitly the bivariate case for which the definition of non-interaction can be expressed mathematically as follows: if the probability of a nonresponse when stimulus x is at level x_i is q_{x_i} and when stimulus y is at level y_j is q_{y_j}, then the probability of a nonresponse when both are applied at the level combination (x_i, y_j) is $Q(x_i, y_j)$, where

$$Q(x_i, y_j) = q_{x_i} q_{y_j} \quad .$$

Using this relationship and properties of Poisson's generalized binomial distribution, estimates $\hat{Q}(x_i, y_j)$ of $Q(x_i, y_j)$ are easily determined when the following design is used (see also Appendix V-B7 and Section IV-B7).

Bivariate Design

Consider the $n_1 x n_2$ points on the rectangular grid given by (x_i, y_j) $[i = 1, 2, \cdots, n_1; j = 1, 2, \cdots, n_2]$. Perform ℓ tests at each of these $n_1 x n_2$ points.

This simple design derives its usefulness from the method of analysis associated with it (see Section IV-B7). However, this analysis depends heavily on the assumption of noninteraction, which should be verified by the test given in that section.

It is difficult to make general statements concerning the best choices of n_1, n_2, and ℓ, since these depend on one's prior knowledge of the response surface. From preliminary investigations it appears that n_1 and n_2 should be taken equal and ℓ as small as possible, consistent with the total number of tests available and other restrictions on the design.

III-B10. NONPARAMETRIC DESIGN FOR DISCRIMINATING BETWEEN TWO RESPONSE FUNCTIONS

a. Introduction

In this section we give a design for discriminating between two monotone non-decreasing response functions which are otherwise both unspecified (see the following section for discrimination between cumulative normal response functions). The design is derived on the basis of (1) maximizing (roughly) the probability of concluding that the two response functions are not identical, when in fact they are not, and (2) ensuring that, as long as the functions are unequal over some interval(s), this probability approaches one as the sample size becomes infinite (see Appendix V-B5 for discussion). In the basic design given, it is assumed that, for each test, the experimenter can sample from either population; variations are given for experiments in which only one of the two populations can be sampled. The rules for the design do not include criteria for terminating the experiment; these may be developed separately using the techniques of Section IV-B5, which gives an asymptotic analysis for use in the discrimination experiments discussed in this section.

b. Design

Phase I: Initialization

The first phase of testing is to conduct an up-and-down test series on each population (see Section III-B1), with common step size Δ, starting at the same stress level for each series. This initial level should be half-way between the 50% response levels expected for the two functions, and Δ should be the average of what would be used if the two functions were tested separately. This phase ends when at least two responses and two nonresponses have been recorded for each population, or even sooner if the sum of the absolute values

of the differences between estimates of the two response functions at corresponding stress levels is strictly greater than 1; in the latter case the methods of Phase II are more efficient.

Phase II: Discrimination

At each step of Phase II, both sets of data, and the combined set of data, are analyzed by the "method of reversals" (see Section IV-B6). For each (tested) stress level s, the resulting estimates of the response probabilities for population 1, population 2, and (under the null hypothesis of equality) the combined populations are denoted by $\hat{p}_1(s)$, $\hat{p}_2(s)$, and $\hat{p}_3(s)$, respectively. For each and every (tested) stress level s and for each population i, $i = 1, 2$, the "marginal discriminator"

$$D_i(s) = \hat{p}_i(s)\ln[\hat{p}_i(s)/\hat{p}_3(s)] + \hat{q}_i(s)\ln[\hat{q}_i(s)/\hat{q}_3(s)]$$

is computed, where $\hat{q}_j(s) = 1-\hat{p}_j(s)$, and $x[\ln(x)]$ is defined as usual to be zero if x is zero. Let I and S be values of i and s for which $D_i(s)$ is maximized. Then the next test should be from population I and at stress level S. If the (I,S) pair is not unique, one of the pairs should be chosen at random from those pairs which maximize $D_i(s)$, unless there are other design considerations. (For example, the normal analysis of the next section, even though it may not be strictly appropriate, can be used to break ties by selecting that pair which most closely resembles the optimum computed under normal assumptions.)

Each time this discrimination phase is performed, testing continues until the total sample size is twice the sample size at the point when Phase II was last entered, or until $D_i(s)$ is everywhere zero, whichever occurs first. The experiment then proceeds to Phase III. The doubling rule for terminating Phase

II insures that, under the null hypothesis of a difference between populations, the probability of being in Phase II approaches one as the sample size becomes infinite; this in turn insures the asymptotic efficiency of the design.

Phase III: Densification

The purpose of the third phase of testing is to continue testing if Phase II leads nowhere, and to insure that the test levels for both populations approach dense subsets of the space of possible test levels as the sample size goes to infinity (a dense subset of the real line has infinitely many points on any interval). Thus, no matter how small the interval(s) on which there is a difference, this interval will be found eventually. The probability of testing in this interval will then approach one as the sample size becomes infinite, which is the same as saying that the probability of spotting the difference will approach one.

For each population, one test is taken half-way between every pair of successive stress levels at which tests have previously been made, one test is taken Δ above the highest stress previously tested, and one test is taken Δ below the lowest stress previously tested. The experiment then returns to Phase II.

c. Discussion and Variations

If there is not much time between tests for computing the quantities $D_i(s)$ for all values of s and i, the experimenter may choose to recompute only those two or three values which seem most promising; after the experiment is well into Phase II, it is generally satisfactory to call for blocks of five or ten tests at the same stress level.

If at some stage early in the experiment, the value $D_i(s+\Delta)$ is only slightly less than the maximum value $D_i(s)$, then parabolic interpolation between the values $D_i(s-\Delta)$, $D_i(s)$, and $D_i(s+\Delta)$ may be used to find a new stress level x, $s < x < s+\Delta$, for the next test.

It is easy to modify the above design if one of the populations has already been evaluated in a different experiment and may no longer be studied. In some instances it may be thought of as a "standard" population against which the second is to be tested, in which case it may be represented by a table of standard or perfectly known values, say, $p_2(s_j)$, $j = 1, \cdots, M$. Then testing on the other population would be restricted to these levels, s_j, $j = 1, \cdots, M$, and $p_3(s_j)$ would be set equal to $p_2(s_j)$ for all values of j.

If costs differ according to a function $C_i(s)$ of population and stress level, then the function $D_i(s)/C_i(s)$ may be used in Phase II, rather than $D_i(s)$. If tests from population 2 are much more expensive than those from population 1, then Phase I may end as soon as $p_1(s) \neq p_2(s)$, for some s. This permits earlier entry into Phase II, where use of the function $D_i(s)/C_i(s)$ inhibits testing from population 2. In Phase III, tests from population 2 may simply be deleted in this case.

d. <u>Examples</u>

Example I: Two Test Populations; Basic Design

Stimulus Level	Response Fractions		
	Population 1	Population 2	Combined
10	1/1		1/1
9	1/2	2/2	3/4
8	0/1	1/2	1/3
7		0/1	0/1

Now suppose that we are in Phase II. We compute

$$D_1(10) = D_2(10) = 0,\ D_1(9) = .5\ln(2/3) + .5\ln(2) = .14,$$

$$D_2(9) = \ln(4/3) = .29,\ D_1(8) = \ln(3/2) = .41,$$

$$D_2(8) = .5\ln(3/2) + .5\ln(3/4) = .06,$$

$$D_1(7) = D_2(7) = 0.$$

Thus the next test in Phase II would be at level 8 from population 1. If there is no response from this test, the subsequent test in Phase II should either be at the same level and population or at level 9, population 2. If there is a response, the subsequent test would be at level 9, population 2.

Example II: Comparison with a Reference

Stimulus Level	Response Fractions		
	Test Population (1)	Reference Population (2)	Combined
10	3/4	43/44	46/48
9	2/4	37/44	39/48
8	1/4	22/44	23/48

To find the next test in Phase II, we compute

$$D_1(10) = .75\ln(18/23) + .25\ln(6) = .2641$$

$$D_1(9) = .5\ln(8/13) + .5\ln(8/3) = .2477$$

$$D_1(8) = .25\ln(12/23) + .75\ln(36/25) = .1108 \quad .$$

Therefore the next test would be at $s = 10$ (parabolic interpolation gives $s = 9.64$).

Example III: Data at the End of Phase I

Stimulus Level	Response Fractions					
	Population 1		Population 2		Combined	
	Data	Estimates	Data	Estimates	Data	Estimates
9	1/1	1.00	1/2	0.50	2/3	0.67
8	1/2	0.67*	0/2	0.00	1/4 *	0.40
7	1/1	0.67		(0.00)	1/1	0.40

Phase I has just ended because of the apparent differences in response fractions. Phase II would now begin with a test at $s_2 = 8$ or $s_2 = 7$ (the maximum $D_1(s)$ is $D_2(8) = D_2(7) = \ln(5/3)$), probably with $s_2 = 8$.

Suppose that this test is a response; then $D_1(9) = \ln(3/2)$ would be the maximum, and the subsequent test, from population 1, would be at level 9.

*These estimates were obtained by the method of reversals, Section IV-B6.

III-B11. DESIGN FOR DISCRIMINATING BETWEEN TWO CUMULATIVE NORMAL RESPONSE FUNCTIONS

a. Introduction

In this section we give a design for discriminating between two response functions which are cumulative normal distributions with possibly different (and unknown) parameters (see the previous section for a non-parametric discrimination design). This parametric design is derived on the basis of maximizing the probability of concluding that the two response functions are not identical, when in fact they are not, and on ensuring that this probability approaches one as the sample size becomes infinite. Rules for terminating the experiment are not given, but may be developed separately using the techniques of Section IV-B5.

b. Design

Phase I: Initialization

Each population is tested until there is at least one response at a lower level than some non-response. This may take about twenty tests. Estimates of μ and σ can then be made by the Minimum Overlapping Subset method (see Section IV-B2) or, if enough time is available, by the method of maximum likelihood (see Section IV-B1).

An excellent design for this phase is a version of the Robbins-Monro design (see Section III-B5). The test levels are defined recursively as follows:

$$L_{n+1} = L_n + A(R_n - .5)/n,$$

where L_n denotes the stress level for the nth test, R_n denotes the response to the nth test, (0 or 1).

The parameter A should be between 2 and 2.5 times the value guessed for the standard deviation σ, and L_1 should be at the experimenter's initial estimate

of μ. The up-and-down method, with conventional step size, may also be used in Phase I.

Phase II: Discrimination

Denote the estimates of the normal parameters obtained in Phase I by $\hat{\mu}_1$, $\hat{\sigma}_1$ for the first set, $\hat{\mu}_2$, $\hat{\sigma}_2$ for the second set, and $\hat{\mu}_3$, $\hat{\sigma}_3$ for the combined data. Suppose that less data have been taken from Population 1 than from Population 2. Then the next test will be from this population at level

$$S_1 = .792\hat{\mu}_1 + .208\hat{\mu}_3 + \delta,$$

where

$$\delta = 2.752\left[\hat{\sigma}_3^2/(\hat{\mu}_3 - \hat{\mu}_1)\right] \ln(\hat{\sigma}_3/\hat{\sigma}_1).$$

Because of convergence problems in the derivation of the above formula, the absolute value of δ should be subject to the constraint

$$|\delta| \leq 1.575\hat{\sigma}_1 ;$$

if the constraint is violated, use instead

$$S_1 = .792\hat{\mu}_1 + .208\hat{\mu}_3 + 1.575\hat{\sigma}_1 \operatorname{sgn}(\delta)$$

to compute the next test level. These results are discussed further in Appendix V-B5.

After the new result is observed, the data from the population tested and the combined set of data are both re-analyzed, and another test is designed by the same rules.

c. Variations

If the two response functions appear to be quite "obviously" different (e.g., if the sum of the absolute value of the differences between the nonparametric estimates

of the two response functions at corresponding stress levels is strictly greater than two) before Phase I is concluded, the experimenter is advised to switch immediately to the nonparametric design of Section III-B10. Also, the test levels given by the above formula begin to converge as the sample size gets larger, and larger and larger blocks of tests at a single stress level may be planned without repeating the analysis after each test. For example, after twenty tests, ten from each population, it may be feasible to plan for two blocks of tests, ten from each population, at the same stress level.

d. Examples

Example 1:

Suppose the following data has already been collected:

Stimulus Level	Response Fractions		
	Test Population (1)	Population (2) Reference	Combined
10	3/4	43/44	46/48
9	2/4	37/44	39/48
8	1/4	22/44	23/48

The maximum-likelihood estimates are roughly $\hat{\mu} = 9.00$, $\hat{\sigma} = 1.48$, $\hat{\mu}_2 = 8.00$, $\hat{\sigma}_2 = 1.00$, $\hat{\mu}_3 = 8.06$, $\hat{\sigma}_3 = 1.10$, and it follows that $S_1 = 9.86$. In this case it would not be very inefficient if the experimenter planned for five tests at $S = 9.86$ without bothering to re-analyze the data after each test.

Example 2: Typical Data from Phase I

Stimulus Level	Response Fractions		
	Population 1	Population 2	Combined
9.50		1/1	1/1
9.17		1/1	1/1
9.00		0/1	0/1
8.92		1/1	1/1
8.08	0/1		0/1
8.00	1/1	0/1	1/2
7.83	0/1		0/1
7.50	0/1		0/1
7.00	0/1		0/1
MOS Estimates			
$\hat{\mu}$	8.04	8.96	8.655
$\hat{\sigma}$	.11	.11	.742

Since the sample size is equal for both populations, we choose to test from population 1 which has a shortage of responses. But

$$\delta = 2.752\left[(.742)^2/.615\right]\ln(6.75) > 1.575\hat{\sigma}_1$$

Thus the next test would be at

$$S_1 = .792(8.04) + .208(8.655) + 1.575(.11) = 8.34 .$$

(If the Robbins-Monro process had simply been continued for population 1, the next test level would have been

$$L_6 = 8.08 + .20 = 8.28.)$$

III-B12. LANGLIE DESIGN (Ref. 17)

a. Introduction

This open-ended design is essentially distribution-free, although it is most often used in conjunction with the assumption that the underlying response function is cumulative normal (say, with parameters μ, σ). It requires that all stress levels in some a priori specified interval be available for testing. The estimates $\hat{\mu}$, $\hat{\sigma}$ of the distribution parameters μ, σ can be found by standard methods of analysis (see, for example, maximum likelihood estimation, Section IV-B1).

b. Design

The rules for determining successive levels for tests are as follows:

1. Choose an interval (A,B) such that at stress level A almost no responses are expected, and at stress level B almost all responses are expected.
2. The first test is at level $L_1 = (A+B)/2$.
3. If there was a response at L_1 then $L_2 = (A+L_1)/2$.
   If there was a nonresponse at L_1 then $L_2 = (B+L_1)/2$.
4. Assume tests have been made at levels $L_1, L_2, \cdots, L_p, L_{p+1}, \cdots, L_n$. Starting at L_n and working back, find the first level, L_p, such that there are as many responses as nonresponses in the pth through n^{th} tests. Then

$$L_{n+1} = (L_p+L_n)/2 \quad .$$

5. If there is no level L_p then

$$L_{n+1} = \begin{cases} (A+L_n)/2 \ , & \text{if a response was observed at } L_n \\ (B+L_n)/2 \ , & \text{if a nonresponse was observed at } L_n \end{cases} \quad .$$

c. Discussion

When $\sigma = 0$, all of the nonresponses will occur at lower stimulus levels than all the responses. This is called a "degenerate" outcome. In the Langlie design degenerate sets of data occur frequently for small samples even when $\sigma \neq 0$. This would indicate that the level sequencing rules do not insure that a sufficient number of tests is performed close to μ. (A degenerate outcome does not indicate that μ is in an interval between the highest nonresponse and lowest response, since, as Langlie proved, for any point of the interval (A,B) and any n there exists an outcome with a degenerate interval containing that point.

Although it is not necessary to have a priori knowledge of σ in using the Langlie design, the initial interval (A,B) must be chosen in advance. In the simulation carried out by Langlie (see Ref. 17), a great many more degenerate outcomes occurred when the large interval $(-12\sigma, 8\sigma)$ was selected than for the smaller interval $(-4\sigma, 4\sigma)$. In the former case more than half (53%) the sets of data were degenerate for $n = 11$, while only 11% were degenerate in the latter case. However, even with the smaller interval, a degenerate outcome was still encountered when $n = 25$.

If the stress levels converge to either end point of the interval (A,B), then one must conclude that the interval was inappropriately chosen. If the levels converge to some interior point, then the response function is actually a step function (i.e., $\sigma = 0$). In other cases the levels "wander" over the entire interval. (Indeed, this is one of the objectives of the design.)

The Langlie design is about as effective for the inverse response problem as the Bruceton design when large samples are available. More precisely, Langlie

has shown as a result of his simulations that using maximum likelihood estimates

$$\mathrm{Var}(\hat{\mu}+k\hat{\sigma}) \approx (2.5+3.2k^2)\sigma^2/n$$

which is very close to the variance obtained from the Bruceton design for commonly used step sizes. The relative asymptotic efficiency of the Langlie design for various cumulative normal inverse response problems (based on the results of Chernoff summarized in Section III-A4) is tabulated below:

k (in $\mu+k\sigma$)	$e = \mathrm{Var}_{min}/\mathrm{Var}_{Langlie}$
0	63%
.5	52%
1.0	40%
1.5	38%
2.0	43%
3.0	47%
∞*	51%

Consequently, the Langlie design for large samples is to be considered only for estimation of x_α for α in a neighborhood of 50 percent, and even in this case other designs (Sections III-B1, III-B5, etc.) are more efficient. For small samples the Langlie design is generally inefficient because of the persistence of degenerate outcomes.

*The limit of the inverse response problem as $k \to \infty$ is equivalent to the estimation of the parameter σ.

III-B13. SPLIT-THE-DIFFERENCE DESIGN

a. Introduction

In many cases, during preliminary phases of experimentation, the standard deviation, w, of the experimenter's estimate of x_α for the α inverse response problem is believed to be much larger than the length of the stimulus interval in which the response function goes from 0 to 1. For example, if the response function is cumulative normal with unknown mean μ and known standard deviation σ, and if the standard deviation of the initial estimate of μ is $w = 100\sigma$, then the above situation holds because the response function goes from .001 to .999 in an interval of length 6.180σ.

In this section we give a design for experimenting in the above situation, which is based on the idealization that the response function $M(x)$ is a step with unknown point of discontinuity x_o:

$$M(x) = \begin{cases} 1 \ , & x > X_o \\ M_o \ , & x = X_o, \\ 0 \ , & x < X_o \end{cases} \quad \text{where} \quad 0 \le M_o \le 1 \ .$$

If in fact the response function is a step, the design presented is approximately optimal, for all sample sizes. It has one defect, however: if the level X_o should change in the course of the experiment, there is no way to detect this chang When applying the design to the preliminary problem stated above, the experimenter should use it only as long as the standard deviation of the experimenter's latest guess of x_α is still greater than the length of the stimulus interval over which the response function goes from 0 to 1. Otherwise the estimates obtained will begin to converge to the wrong value. In the above

example, it can be shown that the split-the-difference design should be used for no more than the first six tests.

b. Design

Let the experimenter's initial guess, y_o, of X_o be normally distributed with mean X_o (unknown) and standard deviation w (known). The first test is at

$$L_1 = y_o \quad .$$

The second test is at

$$L_2 = L_1 + 1.167w$$

if the first test is a nonresponse, and at

$$L_2 = L_1 - 1.167w$$

if the first test is a response.

The general rules for planning the $(r+1)^{st}$ test are:

1. If the first r tests have been responses, let

$$L_{r+1} = L_r - 1.167w/\sqrt{r} \quad .$$

2. If the first r tests have been nonresponses, let

$$L_{r+1} = L_r + 1.167w/\sqrt{r} \quad .$$

3. If the first r tests include both responses and nonresponses, and if the last result was a response, let s_r denote the highest level previously tested

below L_r. That is,

$$S_r = \max(L_j | j = 1, \cdots, r-1, L_j < L_r) \quad .$$

Note that S_r is not necessarily L_{r-1}.

If the last result was a nonresponse, let S_r denote the lowest level previously tested above L_r. That is,

$$S_r = \min(L_j | j = 1, \cdots, r-1, L_j > L_r) \quad .$$

In both cases, the interval between L_r and S_r is the "interval of uncertainty" in which the point of discontinuity X_o may lie, and the next test is at the midpoint of this interval,

$$L_{r+1} = (L_r + S_r)/2 \quad .$$

The design should be terminated after $2+\log_2(w/\delta)$ tests, where δ denotes the length of the interval in which the response function is believed to go from 0 to 1.

c. Discussion

The quantities $A_r = 1.167w/\sqrt{r}$ in steps 1 and 2 have been chosen so that $\sum_{i=1}^{r} A_i$, $r = 1, 2, 3$, match the values 1.17w, 1.99w, and 2.657w, which are shown to be optimal by Marks (Ref. 18) for step response functions and infinite sample sizes. For example, if $y_o = 0$, then two nonresponses and a response would lead to the successive test levels 0, 1.17w, 1.99w, and 1.537w according to Marks, and to 0, 1.167w, 1.992w, and 1.580w according to our convenient approximation.

III-B14. ROTHMAN DESIGN

a. Introduction

This design is a type of stochastic approximation for the nonparametric inverse response problem which makes use of the "method of reversals" analysis (see Section IV-B6) after each test to plan the next test level. For small samples the design resembles the "split-the-difference" design (see Section III-B13), but for large samples the test levels and the estimates both converge to the true value, x_α, unlike the "split-the-difference" design.

b. Design

Let the experimenter's initial guess, y_o, of x_α be normally distributed with mean x_α (unknown) and standard deviation w (known). The first test is at

$$L_1 = y_o + 1.167w(\alpha-.5).$$

The second test is at

$$L_2 = L_1 + 1.167w$$

if the first result is a nonresponse, and at

$$L_2 = L_1 - 1.167w$$

if the first result is a response. The general rules for planning the $(r + 1)^{st}$ test are:

1. After the rth test, all of the data are analyzed by the method of reversals.
2. The value

$$\Delta = .6745\sqrt{\alpha(1-\alpha)Y_r/r}, \quad Y_r = \sum_{i=1}^{r}(1/i) \text{ is computed.}$$

3. If there are any stress levels at which the estimated response is greater than or equal to $\min(\alpha + \Delta, 1)$, let S_1 denote the lowest of these. If there

are any stress levels at which the estimated response is less than or equal to $\max(\alpha - \Delta, 0)$, let S_2 denote the highest of these.

4. If both S_1 and S_2 exist, as is generally the case for large sample sizes, let

$$L_{r+1} = (S_1+S_2)/2 + 1.167w(\alpha - \alpha_r),^*$$

where α_r is the fraction of responses in the first r tests.

If S_1 exists, but not S_2, let

$$L_{r+1} = (S_1+L_r)/2 - 1.167w/\sqrt{r}$$

If S_2 exists, but not S_1, let

$$L_{r+1} = (L_r+S_2)/2 + 1.167w/\sqrt{r}$$

If neither S_1 nor S_2 exists, let

$$L_{r+1} = L_1$$

c. Discussion

The quantity Y_r in Step 2 is approximated sufficiently well by the first two terms of the infinite series

$$Y_r = \gamma + \ln(r+.5) + 1/24(r+.5)^2 - \ldots$$

where γ is Euler's constant given by

$$\gamma = .5772\ldots$$

This quantity Y_r is actually the expected number of plateaus when the method of reversals is applied to r points sampled from a constant response function, and therefore r/Y_r is the approximate number of data points in each plateau

*For extremely large sample sizes, this second term should be replaced by $(\alpha-\alpha_r)/d_r$, where d_r is an estimate of $M'(x_\alpha)$ from the r data points. Possibly $d_r = 2\Delta/(S_1-S_2)$.

near x_α (see Section IV-B6). Thus $(S_1+S_2)/2$ represents the center of an approximate 50% confidence interval on x_α.

This distribution-free design, combined with the estimator $\hat{x}_\alpha = (S_1+S_2)/2$, is about as good (asymptotically) as any parametric design which converges to x_α, although the latter is not necessarily an asymptotically optimal parametric design for the inverse response problem. For example, a parametric stochastic approximation of $\mu - 4\sigma$ in the cumulative normal case is definitely not asymptotically optimal for the .0032% inverse response problem; the correct procedure is a mixture of parametric stochastic approximations of $\mu - 1.575\sigma$ and $\mu + 1.575\sigma$ (see Ref. 4).

If the initial estimate, y_o, is uniformly distributed in an interval of width D (rather than normally distributed), the above design may be used with $w = .22D$.

d. <u>A Simulated Example</u>

Let $\alpha = .05$, $w = 5$, and the response function be cumulative normal with $\mu = 0$, $\sigma = 1$. Then $x_\alpha = -1.645$. Stress levels are given below to the nearest .1.

We make one observation on a random variable normally distributed with mean -1.645 and standard deviation 5. We find $y_o = -.2$, and therefore test first at

$$L_1 = -.2-2.6 = -2.8; \text{ suppose } R_1 = 0.$$

Here R_r denotes the response to the rth stress. At this point we have $S_2 = -2.8$, S_1 doesn't exist, and

$$L_2 = (-2.8 -2.8)/2 + 5.8 = 3.0; \text{ suppose } R_2 = 1.$$

Now $S_1 = 3.0$, $S_2 = -2.8$. $\alpha_2 = .5$,

$$L_3 = (3.0 - 2.8)/2 -2.6 = -2.5; \text{ suppose } R_3 = 0.$$

Now $S_1 = 3.0$, $S_2 = -2.5$, $\alpha_3 = .333$,

$L_4 = (3.0-2.5)/2 - 1.65 = -1.4$; suppose $R_4 = 0$.

Now $S_1 = 3.0$, $S_2 = -1.4$, $\alpha_4 = .25$,

$L_5 = (3.0-1.4)/2 - 1.2 = -.4$; suppose $R_5 = 1$.

Now $S_1 = -.4$, $S_2 = -1.4$, $\alpha_5 = .40$,

$L_6 = (-.4-1.4)/2 - 2.04 \cong -3.0$; etc.

The following table gives the rest of the simulated example:

r	L_r	R_r	r	L_r	R_r	r	L_r	R_r
6	-3.0	0	26	-1.5	0	46	-1.9	0
7	-2.6	0	27	-1.5	0	47	-1.9	0
8	-2.3	0	28	-1.5	0	48	-1.9	1
9	-2.1	0	29	-1.5	0	49	-2.1	0
10	-1.9	0	30	-1.5	0	50	-2.1	0
11	-1.8	0	31	-1.4	0	51	-2.1	0
12	-1.7	0	32	-1.4	0	52	-2.0	0
13	-1.6	0	33	-1.4	1	53	-2.0	0
14	-1.5	0	34	-1.8	0	54	-2.0	0
15	-1.4	0	35	-1.8	0	55	-2.0	0
16	-1.4	0	36	-1.8	0	56	-2.0	0
17	-1.3	0	37	-1.8	0	57	-2.0	0
18	-1.3	0	38	-1.7	1	58	-2.0	0
19	-1.2	0	39	-2.0	0	59	-2.0	0
20	-1.1	1	40	-2.0	0	60	-2.0	0
21	-1.7	0	41	-2.0	0	61	-1.9	0
22	-1.7	0	42	-2.0	0	62	-1.9	0
23	-1.6	0	43	-2.0	0	63	-1.9	0
24	-1.6	0	44	-2.0	0	64	-1.9	0
25	-1.6	0	45	-1.9				

After 64 tests, $S_1 = 1.4$, $S_2 = -1.9$, and $\hat{x}_\alpha = -1.65$ (which is much better than one should expect in general using this design).

III-B15. DERMAN DESIGN, (Ref. 24)

a. Introduction

This open-ended design can be used for the inverse response problem. It is distribution-free and is based only on the assumption that the response function is strictly monotone increasing. Tests are performed at specified, equally spaced fixed levels. The design is given here for $\frac{1}{2} \leq \alpha < 1$ but by suitable modification can be used for $0 < \alpha < \frac{1}{2}$.

b. Rules of design

1) The first test level L_1 is at some a priori best guess of x_α

2) Let Δ be the step size between fixed levels and let the nth test be at L_n with outcome y_n; then

$$L_{n+1} = \begin{cases} L_n - \Delta \; ; & \text{with probability } 1/2\alpha, \text{ if } y_n = 1 \\ L_n + \Delta \; ; & \text{with probability } 1-1/2\alpha, \text{ if } y_n = 1 \\ L_n + \Delta \; ; & \text{if } y_n = 0 \end{cases}$$

Tables of random numbers can be used to carry out these rules as follows: When a response occurs a random two digit number is compared to $\frac{100}{2\alpha}$. If the random number is smaller, the next level is a step lower; if it is larger, the next level is a step higher. If the two are equal a new random number is used.

c. Discussion

If $\alpha = .5$, the Derman and Bruceton designs are identical. The behavior for other values of α is unknown, although as Wetherill points out (Ref. 5) the use of random numbers in the design rules leads to inefficiencies for small samples. Convergence properties of the design are discussed by Derman in Ref. 24.

The estimate of x_α is that stress level most frequently tested. If several levels have the same number of tests, then $\hat{x}_\alpha$ is the average of these levels. Use of this design should be limited to the case where only a specified set of stimulus levels are available for testing. Even in this situation other distribution-free procedures (e.g., Section III-B6) are considerably more efficient.

III-B16. SAFETY DESIGN FOR EXPONENTIAL STRESS AND NORMAL cdf RESPONSE

a. Introduction

If the response function $M(x)$ of a system or device is assumed to be cumulative normal with (unknown) parameters μ and σ, and if the maximum stress density $f_1(x)$ that the system is exposed to is of the form Ae^{-Bx} for the range of interest (e.g., $x > \mu - 4\sigma$), then the safety in using the system (see Section II-B6) is given by

$$S = 1 - \int_{-\infty}^{\infty} M(x) f_1(x)\, dx \cong 1 - \frac{A}{B} e^{-B(\mu - B\sigma^2/2)}$$

The following design is intended for estimating S. It has not yet been simulated nor used in laboratory testing.

b. Design

Phase I

Let $K = -B\sigma$, where σ is unknown. Let σ_0 denote the initial estimate of σ. Let N denote the intended sample size. Using a step size of $\Delta = 1.575\sigma_0$, perform a Bruceton design (see Section III-B1). Continue until the sample has "overlap" (e.g., when the lowest level at which there is a response is below the highest level at which there is a nonresponse), or until N/4 tests have been performed, whichever comes first. Let σ_1 be the estimate of σ at this point (obtained by the methods of Section IV-B4).

Phase II

Let $K_1 = -B\sigma_1$. Depending on the value of $|K_1|$, denote the remainder of the sample to an inverse response design for $x_\alpha = \mu + k_1\sigma$ of the type given in Section III-A4. The parameter σ, and therefore K, should be reestimated after each test, if possible, and the design reoriented accordingly.

c. Comments

Data collected from this design may be analyzed by maximum-likelihood techniques (Section IV-B1), and the results used to get a maximum-likelihood estimate of S.

A similar design may be used for reliability estimation if the density of actuating stresses (see Section II-B6) is given by

$$f_2(x) = Ae^{Bx}, \quad x < \mu + 4\sigma .$$

III-B17. PROBIT* DESIGN (Ref. 30)

a. Introduction

This design, the oldest and one of the best-known, has been applied mostly to biological problems, in which connection it was originally developed. Two of the most common applications have been to the determination of the relative potencies of two or more drug preparations, or to the description of the entire response curve for doses of a single preparation (see Section II-B5). The Probit analysis (see Section IV-B8) has generally been used in connection with this design, although other maximum likelihood techniques (see Section IV-B1) can be employed. The Probit design is generally used in conjunction with distributional assumptions (mainly normal and more recently also logistic), and thus it may be necessary to first transform the stimulus variable (see Section IV-A5).

b. Design Rules

In the simplest version of this design, k stimulus levels are chosen and n tests are conducted at each level. The choice of k, n, and the spacing between stimulus levels depend significantly on the particular application, the number of specimens available, the nature of the distributional assumptions (if any), and many other factors.

c. Comments

This simple form of the Probit design is entirely nonsequential (see Section III-A1), and as a consequence is relatively inefficient compared with any even

*The term "probit", introduced by Bliss, is a contraction of the words, "probability unit".

partially sequential procedure. Furthermore, since the design requires a great many tests, it would appear that its use is limited to situations in which all test levels must be planned in advance of any testing, and cost or specimen availability considerations are not critical.

There is virtually an unlimited number of variations of the Probit design. These include: (1) a variable number of tests at the k stimulus levels, either specified before testing begins or during the program; (2) selecting only a few of the k levels before testing begins, the rest being established on the basis of the analysis of the data from these few levels; and (3) testing at decreasing levels until all nonresponses are observed, and/or testing at increasing levels until all responses are observed. Most of these variations essentially represent attempts to sequentialize the Probit design, and in each case more efficient procedures exist. Therefore, we will make no attempt to further describe or discuss this class of designs.

SECTION IV

METHODS FOR ANALYZING SENSITIVITY EXPERIMENTS

PART A - GENERAL REMARKS

IV-A1. THE RELATIONSHIP BETWEEN THE DESIGN AND THE ANALYSIS OF A SENSITIVITY EXPERIMENT

In all but a few cases, any method for analyzing sensitivity data can, in principle, be used in conjunction with any design. The exceptions pertain primarily to those situations in which an experimental design has been devised so as to permit a particularly simple analysis. The most common examples of the latter case are all of the variations of the Dixon-Mood method (Section IV-B4) which is intended for use only on data collected from the Bruceton design or one of its variations.

In handling multivariate sensitivity data the experimenter can use the particular design and corresponding analysis methods given in this manual (Sections III-B9 and IV-B7), although other methods of analysis may be possible.

Although sensitivity data can often be analyzed by any one of several procedures, all methods may not be equally efficient. For example, the MOS method (Section IV-B2) requires data reasonably well balanced between responses and nonresponses in order to be effective.

Statements of limitations and flexibility accompany each of the analysis methods described in Section IV of this manual.

IV-A2. USE OF PRIOR INFORMATION

Almost all methods of analyzing sensitivity data deal only with the data in the current sample, and not with information about points or parameters of the response function available before testing begins. It is generally assumed that such prior information has been fully used already in designing the current sample through proper selection of the design parameters (e.g., level of first test, step size). An example of the effect of using such information is a version of the Robbins-Monro process due to Dvoretzky (Ref. 19) in which the small sample variance is actually better than the theoretical minimum given in Section III-A5 which is based on zero prior information.

Information may be given concerning those aspects of the response function in which we are interested, in the form of bounds (as in Dvoretzky) or a density (as we generally use) on the initial estimates ("prior density"). If such information is quoted in units of equivalent asymptotically efficient tests (i.e. the number of asymptotically efficient tests which would provide the same information under large sample conditions), and if prior information E_o is a significant fraction of the sample size (e.g., E_o=.8, N=5), then a special effort should be devised to incorporate the prior information in the experimental analysis. The difficulty in carrying this out is that, in order to express E_o in the proper units, two quantities must be known (cf Section V-B1):

1. A scale parameter of the experimenter's prior density, and
2. A scale parameter of the true response function.

   The first of these is by definition known, but the second is generally unknown and must be estimated from the sample at hand, which may be very small.

Once E_o is estimated, the maximum likelihood estimate of the quantities of interest is averaged with the initial guess using N and E_o as the respective weights. An example of the implementation of prior information is given in Section III-B8.

IV-A3. NOTION OF "PERCENT DEFECTIVE"

In some types of sensitivity testing, the population of test items is "contaminated" with a certain fraction, d, of "defective" items which will not respond to any stimulus. If the response function would have been $M(x)$ without these defectives, then it is $(1-d)M(x)$ with them.

One obvious consequence of such a situation is that it would be meaningless to work on the inverse response problem for $\alpha > 1-d$; a stochastic approximation intended for x_α would not converge or even be bounded.

In distributional problems, the parameter d may be known in advance or estimated from the sample (by maximum likelihood). For $M(x)$ normal, both cases have been covered by Eissner (Ref. 20), but the modification of the Chernoff optimal design theory (Ref. 4) to allow for contamination has not yet been worked out.

In some sensitivity testing, it is standard practice to make a second test at an extremely high stress level, on those items which have not yet responded. If they do not respond on this second test, it is assumed that they are "duds", and are not counted in the data analysis. In this way the problem of estimating the percent defective is avoided.

IV-A4. MAXIMUM LIKELIHOOD ESTIMATES

Maximum likelihood estimates (MLE's) of points or parameters of a response function are those values which maximize the sample likelihood function,

$$L = \prod_{i=1}^{K} [M(\ell_i)]^{f_i} [1-M(\ell_i)]^{n_i-f_i} ,$$

where the data is given by response fractions f_i/n_i at the stress levels ℓ_i, $i=1...K$. Such estimates may be distributional (e.g., Section IV-B1) or distribution-free (e.g., Section IV-B6) depending on assumptions about the response function $M(x)$. Under suitable regularity condition on $M(x)$ maximum likelihood estimates are known to be asymptotically efficient, asymptotically unbiased, and asymptotically normally distributed.

These estimates are "design-independent" in the sense that the order in which the data were acquired in no way affects the calculation. Consequently, ordinary MLE's may not be very efficient in certain cases relative to other analysis procedures. Moreover, when used in the straightforward manner indicated above, MLE's do not possess the faculty of incorporating prior information (see Section IV-A2) which may be a serious defect for small samples.

Calculation of MLE's has been programmed for automatic computation in two important cases: (1) $M(x)$ is normal, and (2) $M(x)$ is monotone nondecreasing, but otherwise unspecified. These computer programs are described in Appendix V-A.

IV-A5. TRANSFORMATIONS

a. Transformations of the Stress Variables

Most stimulus variables possess an inherent or natural experimental measure. For example, in drop tests the height of the impactor, in inches or centimeters, is the obvious quantification of the stimulus variable. In many applications, however, distributional assumptions are appropriate and convenient in terms of a transformation of the stimulus variable. Thus in a particular drop test, we may be able to assume that the response is cumulative normal as a function of the logarithm of the drop height. Some design and analysis procedures, e.g., the Bruceton design (Section III-B1) and Dixon-Mood analysis (Section IV-B4), require that the stimulus levels be transformed to a new scale on which the response function is cumulative normal.

More generally, if it is desired to employ a distribution which has some non-zero probability associated with every possible value of the independent variable (such as the normal), and if in fact the smallest stimulus value is "0", then a logarithmic transformation is often very convenient. A variety of transformations has been used in practice, most common being the aforementioned log function and the log log function.

The main point to emphasize in connection with the implementation of the designs and analyses in this report is that all procedures, tests of hypotheses, analyses, etc., are to be performed in terms of the transformed variable. The inverse transformation should be applied only to the final estimate(s) of interest. For example, in an inverse response problem, if the 1% level is desired and the response function is normal, $\hat{x}_{.01} = \hat{\mu}-2.326\hat{\sigma}$ should be computed in the transformed space, and only then should the result be transformed to the original scale.

b. Transformations of Responses

The analysis of sensitivity data based on distributional assumptions can be extremely tedious. For example, the calculation of exact maximum-likelihood estimates (see Section IV-B1) requires the iterative solution of simultaneous equations and is difficult without the assistance of a digital computer. One way to minimize this difficulty is by using approximate methods which sacrifice little accuracy, but permit reasonably rapid calculations. Perhaps the best example of this is the widely used Dixon-Mood analysis for normal sensitivity data. Their estimators are approximations to the exact maximum-likelihood estimators for μ and σ which are quite satisfactory when the experiment is designed and analyzed as directed; computations may be performed easily with the assistance of a desk calculator.

Another approach to simplifying distributional analyses, which has found wide use in biological applications, is to find a transformation of the observed responses for which the transformed response curve has a simpler or standard form (e.g., linear). The transformation and curve parameters are obtained by an iterative procedure which is greatly facilitated by the use of certain readily available tables. The inverse transformation then gives estimates of the original distributional parameters and any percentage point of the original response curve.

Perhaps the most common transformation of this type is that performed as part of the Probit analysis of normal response data (see Sections III-B17 and IV-B8). Here the transformed response variate is linear, and the iterative procedures required to estimate the two linear parameters are not difficult to apply using the many computational aids (e.g., tables) that have been developed. Other examples are the logit and angular transformations. These are given explicitly in Section IV-B8.

IV-A6. TESTS OF HYPOTHESIS

In the course of analyzing sensitivity experiments, and often as a primary objective of these experiments, tests of hypothesis about response functions are performed. Examples are: (1) tests for monotonicity of a response function, which should be the first step in the analysis of any set of data (e.g., Section IV-B11); (2) tests for goodness-of-fit of a particular response function model, which should be applied only if monotonicity is not rejected (e.g., Section IV-B12); (3) tests for the existence of an unknown stimulus variable, if monotonicity has been rejected or is doubtful (Section IV-B13); and (4) discrimination tests of several samples against each other or against a standard (Section IV-B5).

Most such tests are constructed by setting up a null hypothesis and a class of alternatives which includes the null hypothesis as a special case. For example, in testing goodness-of-fit, the null hypothesis might be that the response function is cumulative normal, and the alternatives might be all types of monotone nondecreasing functions.

In likelihood ratio tests of hypothesis, the likelihoods L_o and L of the sample(s) are computed under the null hypothesis and alternative hypothesis, respectively. The alternative hypothesis usually requires more parameters, and gives a higher likelihood, than the null hypothesis. If f denotes the excess number of parameters, and if λ denotes the likelihood ratio L_o/L, then $X^2 = -2\,\ell n(\lambda)$ is asymptotically distributed as X^2 with f degrees of freedom (asymptotic in the sense of requiring large enough sample sizes at the stress levels tested, even for a small number of these levels).

The experimenter now specifies his "Type I" error, δ which is the probability of rejecting the null hypothesis when it is really true. If the X^2 value corresponding to (X^2, f) exceeds $1-\delta$, he will reject the null hypothesis. Values generally used for δ are .05 or .01; too low a value will make it difficult to reject the null hypothesis when it in fact should be rejected.

SECTION IV

METHODS FOR ANALYZING SENSITIVITY EXPERIMENTS

PART B - SPECIFIC ANALYSIS METHODS

IV-B1. MAXIMUM LIKELIHOOD ESTIMATES OF CUMULATIVE NORMAL RESPONSE FUNCTIONS

a. Introduction

Suppose that a response function is assumed cumulative normal with parameters μ and σ, and that it is desired to calculate the maximum likelihood estimates (MLE's) of these parameters. An iterative (Newton-Raphson) procedure is generally necessary, and to insure convergence and reasonable calculation time, it is of prime importance to obtain good "first guesses". The estimates are then calculated to the desired accuracy, and it must be verified that the solution is at a maximum of the likelihood function. Finally, the covariance matrix of the MLE's may itself be estimated by the usual asymptotic theory. A FORTRAN computer program which carries out these calculations is listed in Appendix V-A.

b. Calculation of Estimates

Phase I: First Guesses

Excellent first guesses for the MLE's are given by the formulae

$$\mu_o = (1-\alpha)\mu_1 + \alpha\mu_2$$

$$\sigma_o = (1-\alpha)\sigma_1 + \alpha\sigma_2 \quad ,$$

where μ_o, σ_o denote first guesses,

μ_1, σ_1 denote estimates from the MOS method (see Section IV-B2),

μ_2, σ_2 denote estimates from the two subset method (see Section IV-B3),

$\alpha = |f-.5|$, and

f denotes the fraction of responses in the entire set of data.

For well centered data we have $f \cong .5$, and the initial estimates are given by the MOS method. For badly centered data we have $f \cong 0$ or $f \cong 1$, and the initial estimates are averages of both the MOS and two subset procedures, neither of which is reliable in this case.

If the test data are taken at just a few stimulus levels, the response fractions may be plotted on normal probability paper, a straight line curve-fit, and the results used as first guesses for μ and σ. This technique may actually be necessary if the other procedures given above fail to provide first guesses within the domain of convergence of the Newton-Raphson iteration; however, it is not generally recommended because there are often stress levels at which the response fractions are 0 or 1, and therefore may not be plotted on normal probability paper.

Phase II: Iteration

The normal equations that the maximum likelihood estimates satisfy are

$$\frac{\partial L}{\partial \mu} = \frac{1}{\sigma}\left[\sum_j \frac{Z_j}{q_j} - \sum_i \frac{Z_i}{p_i}\right] = 0 ,$$

$$\frac{\partial L}{\partial \sigma} = \frac{1}{\sigma}\left[\sum_j \frac{t_j Z_j}{q_j} - \sum_i \frac{t_i Z_i}{p_i}\right] = 0 ,$$

where L denotes the natural logarithm of the likelihood function

$(L = \sum_j \ln q_j + \sum_i \ln p_i)$,

h_r denotes the rth stimulus level,

$t_r = (h_r - \mu)/\sigma$,

$Z_r = \frac{1}{\sqrt{2\pi}} \exp[-t_r^2/2]$,

$$p_r = \int_{-\infty}^{t_r} Z\, dt,$$

$$q_r = 1-p_r, \quad \text{and}$$

the indices i and j denote responses and non-responses, respectively.

For computation on an electronic digital computer (for example, an IBM 7094), it is safe to set $Z_r = 10^{-6}$ for all $|t_r| > 9$. For $t_r > 5$, we use $p_r = .9999999$; for $t_r < -5$, we use $p_r = 10^{-7}$; for all other values of t_r, we use

$$p_r = \begin{cases} v, \text{ if } t_r \geq 0 \\ 1-v, \text{ if } t_r < 0, \end{cases}$$

where $$v = 1+Z_r\eta(-1.33027443\eta^4 + 1.82125598\eta^3 - 1.78147794\eta^2 + .3565637826\eta - .31938153),$$

$$\eta = 1/(1 + .231641888|t_r|).$$

The Newton-Raphson iterative procedure for solving the two normal equations is summarized by the matrix equation

$$-\begin{pmatrix} \partial^2 L/\partial^2\mu & \partial^2 L/\partial\mu\partial\sigma \\ \partial^2 L/\partial\mu\partial\sigma & \partial^2 L/\partial^2\sigma \end{pmatrix} \begin{pmatrix} \Delta\mu \\ \Delta\sigma \end{pmatrix} = \begin{pmatrix} \partial L/\partial\mu \\ \partial L/\partial\sigma \end{pmatrix},$$

where $\Delta\mu$ and $\Delta\sigma$ denote the increment in the values of μ and σ, respectively, and

$$\partial^2 L/\partial\mu^2 = \frac{1}{\sigma^2}\left[\sum_j \frac{Z_j}{q_j}(t_j - Z_j/q_j) - \sum_i \frac{Z_i}{p_i}(t_i + Z_i/p_i)\right]$$

$$\partial^2 L/\partial\mu\partial\sigma = \frac{1}{\sigma^2}\left[\sum_j \frac{Z_j}{q_j}(t_j^2 - t_j Z_j/q_j - 1) - \sum_i \frac{Z_i}{p_i}(t_i^2 + t_i Z_i/p_i - 1)\right],$$

$$\partial^2 L/\partial\sigma^2 = \frac{1}{\sigma^2}\left[\sum_j \frac{t_j Z_j}{q_j}(t_j^2 - t_j Z_j/q_j - 2) - \sum_i \frac{t_i Z_i}{p_i}(t_i^2 + t_i Z_i/p_i - 2)\right] .$$

If M denotes $\max(|\mu|, |\sigma|)$, the iterative procedure should be terminated when

$$|\Delta\mu| < 10^{-5} M$$

and

$$|\Delta\sigma| < 10^{-5} M \quad .$$

Fifteen iterations are generally sufficient even in the wildest examples. The solution is at a maximum if the determinant of the above matrix is positive and if $\partial^2 L/\partial\mu^2$ is negative. It is at a minimum if both the determinant and $\partial^2 L/\partial\mu^2$ are positive.

Phase III: Covariance Matrix

The covariance matrix for the estimates is given asymptotically for large total sample sizes by

$$\begin{pmatrix} -E[\partial^2 L/\partial\mu^2] & -E[\partial^2 L/\partial\mu\partial\sigma] \\ -E[\partial^2 L/\partial\mu\partial\sigma] & -E[\partial^2 L/\partial\sigma^2] \end{pmatrix}^{-1} ,$$

where

$$E[\partial^2 L/\partial\mu^2] = -\frac{1}{\sigma^2}\sum_r Z_r^2/p_r q_r ,$$

$$E[\partial^2 L/\partial\mu\partial\sigma] = -\frac{1}{\sigma^2}\sum_r t_r Z_r^2/p_r q_r ,$$

$$E[\partial^2 L/\partial\sigma^2] = -\frac{1}{\sigma^2}\sum_r t_r^2 Z_r^2/p_r q_r ,$$

and the index r runs over all tests. If $t_r > 5$, the factor Z_r/q_r should be replaced by t_r; if $t_r < -5$, the factor Z_r/p_r should be replaced by $-t_r$.

Phase IV: Confidence Bands on Estimates

Suppose we have now computed the maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$, along with the asymptotic covariance matrix

$$V = \begin{pmatrix} A & B \\ B & C \end{pmatrix}, \qquad A,\ C > 0 \quad .$$

If we wanted to estimate $x_\alpha = \mu + k\sigma$, the estimate would be $\hat{x}_\alpha = \hat{\mu} + k\hat{\sigma}$, because the maximum likelihood estimate of a function of some parameters is the function of the maximum likelihood estimates of those parameters. The variable $\hat{x}_\alpha$ is asymptotically normal with mean x_α and variance

$$\begin{aligned} \mathrm{Var}(\hat{x}_\alpha) &\cong (1 \quad k)\ V(1 \quad k)^T \\ &= A + 2kB + k^2C \quad . \end{aligned}$$

Then

$$\hat{\sigma}_{\hat{x}_\alpha} \cong (A + 2kB + k^2C)^{\frac{1}{2}} \quad .$$

For small sample sizes the variate $(\hat{x}_\alpha - x_\alpha)/\hat{\sigma}_{\hat{x}_\alpha}$ is markedly non-normal, and the γ confidence bands on $\hat{x}_\alpha$ (e.g., $\gamma = .95$, $\gamma = .99$) are given approximately by

$$\hat{x}_\alpha \pm t_\delta(f)\hat{\sigma}_{\hat{x}_\alpha} \quad , \qquad \delta = (1 \pm \gamma)/2 \quad ,$$

where $t_\delta(f)$ denotes a value of the t-variate for f degrees of freedom. The value of f is computed very roughly as follows:

$$f = (C_{min}/C)N-2$$

which, from the results of Section III-A4, equals

$$1.6436\sigma^2/C \ - 2 \quad .$$

This formula is heuristic. The term $(C_{min}/C)N$ is the equivalent number of tests at $\mu \pm 1.575\sigma$ (asymptotically optimal for estimating σ), and the subtractive term accounts for the two parameters (μ and σ) being estimated.

This approximation may be examined for the Langlie design (Section III-B12), for which C is asymptotic to $3.2\sigma^2/N$, so that our approximation gives $f = .514N-2$. Values of f are compared to Langlie's values of n (Ref. 17, p. A-28).

N	f	n
10	3.1	3.4
15	5.7	5.4
30	13.4	12.5
50	23.7	23.6
100	49.4	54.3

Phase V: Comparing x_α Estimates for Two Independent Sets of Data

If $\hat{x}_{\alpha,1}$ and $\hat{x}_{\alpha,2}$ are estimates of x_α from two independent sets of data, then the null hypothesis $x_{\alpha,1} = x_{\alpha,2}$ may be tested as follows:

Let $Var(\hat{x}_{\alpha,1})$ and $Var(\hat{x}_{\alpha,2})$ denote the estimator variances computed in Phase IV, and let f_1 and f_2 denote the corresponding degrees of freedom. Then

$$S^2 \equiv Var(\hat{x}_{\alpha,1}-\hat{x}_{\alpha,2}) = Var(\hat{x}_{\alpha,1}) + Var(\hat{x}_{\alpha,2}) \quad ,$$

$$f \equiv S^2/[Var(\hat{x}_{\alpha,1})/f_1 + Var(\hat{x}_{\alpha,2})/f_2] \quad ,$$

and $(\hat{x}_{\alpha,1}-\hat{x}_{\alpha,2})^2/S^2$ is approximately F-distributed with $1,f$ degrees of freedom.

c. Examples

Example I:

Stress Level	Response Fraction
4	1/1
3	2/3
2	1/3
1	0/1

The estimates from the MOS method are $\mu_1 = 2.5$, $\sigma_1 = .829$. The estimates from the two-subset method are $\mu_2 = 2.5$, $\sigma_2 = 1.112$. Since $f = 4/8$, $\alpha = 0$, we have as first guesses $\mu_o = 2.5$, $\sigma_o = .829$. The final maximum likelihood estimates are $\hat{\mu} = 2.5$, $\hat{\sigma} = .856$. Four iterations were necessary to calculate these estimates. The covariance matrix is estimated as

$$\begin{pmatrix} .195 & 0 \\ 0 & .314 \end{pmatrix} .$$

Example II: (Golub and Grubbs, Ref. 21)

Stress Level	Response Fraction
2453	1/1
2433	0/1
2423	1/1
2415	0/2

Since $\mu_1 = 2428$, $\sigma_1 = 13.1$, $\mu_2 = 2443$, $\sigma_2 = 58.8$, $f = 2/5$, and $\alpha = .1$, we arrive at $\mu_o = 2429.5$, $\sigma_o = 17.7$. The final maximum likelihood estimates are $\hat{\mu} = 2431.6$, $\hat{\sigma} = 14.9$. Five iterations were necessary to calculate these

estimates. The covariance matrix is estimated as

$$\begin{pmatrix} 112 & 47 \\ 47 & 144 \end{pmatrix} .$$

Example III:

Stress Level	Responses/Trials
0	1/2
-1	1/7
-2	0/11

Since $\mu_1 = -.5$, $\sigma_1 = .525$, $\mu_2 = 0$, $\sigma_2 = 1.011$, $f = 2/20$, and $\alpha = .4$, we arrive at $\mu_o = -.3$, $\sigma_o = .72$ as first guesses. The final maximum likelihood estimates are $\hat{\mu} = -.105$, $\hat{\sigma} = .748$. The covariance matrix is estimated as

$$\begin{pmatrix} .306 & .184 \\ .184 & .191 \end{pmatrix} .$$

Example IV: It should be pointed out that the first guess procedure given above and used in Examples I, II, and III does not always work. Consider the following data

Stimulus Level	Responses/Trials	Stimulus Level	Responses/Trials
18.5	1/1	5.5	1/1
9.05	1/1	3.73	0/1
8.6	1/1	3.72	1/2
6.3	1/1		

First guesses computed by the above procedure do not work. In this case, the final estimates turned out to be $\hat{\mu} = 3.94$, $\hat{\sigma} = .50$, and covariance matrix

$$\begin{pmatrix} .40 & .64 \\ .64 & 1.58 \end{pmatrix}$$

IV-B2. THE MINIMUM OVERLAPPING SUBSET (MOS)

a. Introduction

The minimum overlapping subset (MOS) technique is a simple method of estimating the parameters μ and σ of a cumulative normal response function. The estimates are excellent for data from a wide range of experimental designs, as long as the number of responses is roughly between 20% and 80% of the sample size. The method takes some effort to master, but, once learned, it provides quick estimates for sample sizes of even 50 or 100. No tables are necessary, and even a slide rule is sufficiently accurate for the calculations.

In an experiment, the set of data which contains the most information is that composed of the responses which occur at stress levels below μ and the nonresponses which occur at stress levels above μ. From these information-rich points, which comprise a subset of perhaps a quarter of the total data, very good estimates of μ and σ can be found.

One attempts to collect these points, which in a probabilistic sense can be thought of as being on the "wrong" side of μ, by first taking the average of the lowest level at which there is a response, and the highest level at which there is a nonresponse. Then the original complete sample is examined to see if there are any points on the "wrong" side of this average. If there are any, the "worst" such response (that is, the one furthest below the average), and the "worst" such nonresponse (that is, the one furthest above the average), are set aside in a subset.

Next we see if any other points are on the "wrong" side of the mean of this subset. If there are none, the process is finished and the MOS consists of

only the two points already placed in the subset. Otherwise, the worst such wrong response point (if any) and the worst such wrong nonresponse point (if any) are added to the subset, and a new mean is calculated for the subset (which now consists of three or four elements). The process continues until all of the wrong points are in the subset and all of the right points are still in the original sample. The mean and standard error of the subset are then used to construct estimates of μ and σ. The formal rules for this procedure are given below.

b. Calculation of Estimates

We construct the minimum overlapping subset, T, by constructing a series of associated sets, $\{T_r\}$, as follows. Let $x_1 \leq x_2 \leq x_3 \leq \cdots$ denote the stress levels corresponding to each and every individual response, and let $y_1 \geq y_2 \geq y_3 \geq \cdots$ denote the stress levels corresponding to each and every nonresponse. The initial subset is empty, or null:

$$T_o = \emptyset \quad .$$

The initial guess at the mean of T is given by

$$\bar{x}_o = (x_1+y_1)/2 \quad .$$

The indices i_r and j_r are used to count the number of responses and nonresponses, respectively, in the subset T_r at the rth stage of the process. Therefore

$$i_o = j_o = 0 \quad .$$

We now define the subsets T_r recursively. Suppose we have arrived at the subset

$$T_r = (x_1, x_2, \cdots, x_{i_r}, y_1, y_2, \cdots, y_{j_r})$$

at the rth stage of the process. Let

$$\bar{x}_r = \frac{\sum_{i=1}^{i_r} x_i + \sum_{j=1}^{j_r} y_j}{i_r + j_r} \quad .$$

Then T_{r+1} is defined by adjoining to T_r the point x_{i_r+1}, if $x_{i_r+1} \le \bar{x}_r$, and/or the point y_{j_r+1}, if $y_{j_r+1} \ge \bar{x}_r$. The process continues until, at the Rth stage, no points satisfy these conditions (i.e., $T_{R+1} = T_R$). Then the minimum overlapping subset (MOS) is defined as

$$T = T_R \quad .$$

Now let $\bar{X}_T$ and S_T denote the mean and root mean square deviation of the stress levels of all the points in T, N denote sample size, N_T denote the number of points in the MOS, and $v = N_T/N$. Then our final approximate estimates of μ and σ are

$$\hat{\sigma} = \begin{cases} 3.32\, S_T \sqrt{v} & , \text{ for } v \le .25 \\ .83\, S_T/\sqrt{.5-v}, & \text{for } v > .25 \end{cases} ,$$

$$\hat{\mu} = \bar{X}_T - 1.253\hat{\sigma}(F_1+F_2-1) \quad ,$$

where F_1 denotes the response fraction within the points of the MOS, and F_2 denotes the response fraction within the set of points not in the MOS but still within $.6745\hat{\sigma}$ of $\bar{X}_T$. If the latter set is empty, then $F_2 = .5$.

The estimate $\hat{\sigma}$ may be justified intuitively as follows: suppose the design consists of alternating points $\mu+k\sigma$, $\mu-k\sigma$, $\mu+k\sigma$, $\mu-k\sigma$, etc. For an infinite sample, $S_T = k\sigma$. Now as k approaches 0, v approaches .5, but as k becomes very large, v approaches 0. Thus the factors $\sqrt{v}$ and $1/\sqrt{.5-v}$ are intended to compensate for the spacing used in the design. In most cases $.06 \leq v \leq .4$.

A crude estimate of the variances of the estimators is given by

$$\mathrm{Var}(\hat{\mu}) \approx \pi\hat{\sigma}^2/4N_T \quad ,$$

$$\mathrm{Var}(\hat{\sigma}) \approx .727\hat{\sigma}^2/NYZ \quad ,$$

where

$$Y = 1 - |2F_3-1| \quad ,$$

F_3 denotes the response fraction in the entire set of data, and

$$Z = \begin{cases} .5-v \ , & \text{if } v \geq .058 \\ 7.68v, & \text{if } v < .058 \end{cases} \quad .$$

In each case, the formula for the variance is asymptotically correct for designs asymptotically optimal for estimating the parameter in question.

c. Examples

Example I:

Stress	Responses/Trials	Responses	Nonresponses
4	1/1	$x_1 = 2$	$y_1 = 3$
3	2/3	$x_2 = 3$	$y_2 = 2$
2	1/3	$x_3 = 3$	$y_3 = 2$
1	0/1	$x_4 = 4$	$y_4 = 1$

In order, the calculated quantities are $T_o = \emptyset$, $\bar{x}_o = 2.5$, $i_o = j_o = 0$, and $Z_o = (2,3) = (x_1, y_1)$. The first subset to be tried is $T_1 = (2,3)$. Then $\bar{x}_1 = 2.5$, $i_1 = j_1 = 1$, $Z_1 = \emptyset$, and the MOS is $T = T_1 = (2,3)$. Now $\bar{x}_T = 2.5$, $S_T = .5$, $N = 8$, $N_T = 2$, $v = .25$, $\hat{\sigma} = .83$, $F_1 = .5$, $F_2 = .5$, $\hat{\mu} = 2.5$, $Var(\hat{\mu}) \approx .27$, $F_3 = .5$, $Y = 1$, $Z = .25$, and $Var(\hat{\sigma}) \approx .25$. This canonical set of data is analyzed in several other sections of this manual; the answers given by the MOS are in excellent agreement and require the least effort.

Example II: (Golub and Grubbs, Ref. 21)

Stress	Responses/Trials
2453	1/1
2433	0/1
2423	1/1
2415	0/2

Responses	Nonresponses
$x_1 = 2423$	$y_1 = 2433$
$x_2 = 2453$	$y_2 = 2415$
	$y_3 = 2415$

$T = (2423, 2433)$, $N = 5$, $N_T = 2$, $v = .4$, $\bar{x}_T = 2428$, $S_T = 5$, $\hat{\sigma} = 13.1$, $F_1 = F_2 = .5$, $\hat{\mu} = 2428$, $Var(\hat{\mu}) \approx 67.5$, $F_3 = .4$, $Y = .8$, $Z = .1$, $Var(\hat{\sigma}) \approx 312$.

Example III:

Stress	Responses/Trials
0	1/2
-1	1/7
-2	0/11

Responses	Nonresponses
$x_1 = -1$	$y_1 = 0$
$x_2 = 0$	$y_2 = -1$
	$y_3 = -1$
	$y_4 = -1$
	$y_5 = -1$
	$y_6 = -1$
	$y_7 = -1$
	$y_8 = \cdots = y_{18} = -2$

$T = (-1,0)$, $N = 20$, $N_T = 2$, $v = .1$, $\bar{x}_T = -.5$, $S_T = .5$, $\hat{\sigma} = .525$, $F_1 = .5$, $F_2 = .5$, $\hat{\mu} = -.5$, $\mathrm{Var}(\hat{\mu}) \approx .11$, $F_3 = .1$, $Y = .2$, $Z = .4$, $\mathrm{Var}(\hat{\sigma}) \approx .125$. It is reasonably apparent that the estimate of μ is rather poor, and the two-subset method (Section IV-B3) works much better for this set of data.

IV-B3. TWO-SUBSET METHOD

a. Introduction

The two-subset method is a very simple technique for estimating the parameters of a response function. Although the estimates are generally not good (sometimes they do not even exist), the method has the advantage of being suitable for desk calculation, and the results can be used as a first approximation for some more refined method of estimation. While it is described here for a two parameter response function (normal, uniform), the basic principle is easily extended to response functions with three or more parameters.

b. Procedure

1. Let α and β be the parameters of the response function $p = M(x;\alpha,\beta)$.

2. Let x be any stress level; let $n_1(x)$ be the number of tests at all levels less than x and $f_1(x)$ the fraction of responses in these $n_1(x)$ tests. Similarly $n_2(x)$ and $f_2(x)$ are the corresponding quantities for the data at the stress levels greater than x.

3. Any value of x which minimizes the quantity $|f_1(x)[1-f_1(x)]n_1(x)-f_2(x)[1-f_2(x)]n_2(x)|$ may be used to split the data into a "high" subset and a "low" subset.* This partition is not always unique; but if there are two such partitions, one would carry out the calculations for both cases and average the results.

4. Let S_1 and S_2 be the mean stress levels for the "low" and "high" subsets, respectively, and let $M^{-1}(p;\alpha,\beta)$ be the inverse of the response function.

*For data taken on the response-poor tail of the response function, for example, this criterion tends to provide subsets with the same total number of responses.

Then $\hat{\alpha}$ and $\hat{\beta}$, the estimates of the parameters α and β, are given by the simultaneous equations:

$$S_1 = M^{-1}(f_1;\hat{\alpha},\hat{\beta})$$

$$S_2 = M^{-1}(f_2;\hat{\alpha},\hat{\beta}) \quad .$$

If α and β are location and scale parameters, the above equations will be linear in $\hat{\alpha}$ and $\hat{\beta}$.

c. Examples

Example I: If our response function is cumulative normal our parameters will be μ and σ. Let the data be represented in the following fashion:

Stress Level	Response Fraction
4	1/1
3	2/3
2	1/3
1	0/1

The stress values in the two subsets are found to be (1,2,2,2) and (3,3,3,4), for which $n_1 = n_2 = 4$, $f_1 = 1/4$, $f_2 = 3/4$, $S_1 = 1.75$, $S_2 = 3.25$. The two simultaneous equations are

$$1.75 = \hat{\mu} - .67449\hat{\sigma}$$

$$3.25 = \hat{\mu} + .67449\hat{\sigma} \quad ,$$

and our estimates would be

$$\hat{\mu} = 2.5 \ ,$$

$$\hat{\sigma} = 1.112 \quad .$$

Example II: Cumulative Normal Response Function

Stress Level	Response Fraction	Stress Level	Response Fraction
.35	1/1	-1.9	0/7
.1	0/1	-2.15	0/6
-.15	0/1	-2.4	0/4
-.4	0/1	-2.65	0/3
-.65	0/1	-2.9	0/1
-.9	0/1	-3.15	0/1
-1.15	0/1	-3.4	0/1
-1.4	1/7	-3.65	0/1
-1.65	0/7		

(This set of data was generated from a population for which $\mu = 0$ and $\sigma = 1$. The maximum likelihood estimates are $\hat{\mu} = .58$ and $\hat{\sigma} = 1.10$.) The stress values in the two subsets are found to be (.35, .1, -.15, -.4, -.65, -.9, -1.15) and (-1.4, 7 times; -1.65, 7 times; -1.9, 7 times; -2.15, 6 times; -2.4, 4 times; -2.65, 3 times , 2.9, -3.15, -3.4, -3.65), for which $n_1 = 7$, $n_2 = 38$, $f_1 = 1/7$, $f_2 = 1/38$, $s_1 = -.4$, $s_2 = -2.058$. The two simultaneous equations are

$$-.4 = \hat{\mu} - 1.068\hat{\sigma}$$
$$-2.058 = \hat{\mu} - 1.938\hat{\sigma} \quad ,$$

and our estimates would therefore be

$$\hat{\mu} = 1.63$$
$$\hat{\sigma} = 1.90 \quad .$$

These are obviously much worse than the MLE's for the same example, but could be employed as "first guesses" for the Newton-Raphson iterative procedure used to calculate the MLE's.

Example III:

Stress Level	Response Fraction
0	1/2
-1	1/7
-2	0/11

The stress values in the two subsets are found to be (0,0), and (-1, 7 times; -2, 11 times), for which $n_1 = 2$, $n_2 = 18$, $f_1 = 1/2$, $f_2 = 1/18$, $s_1 = 0$, and $s_2 = -1.611$. The two equations are

$$0 = \hat{\mu}$$

$$-1.611 = \hat{\mu} - 1.593\sigma \quad ,$$

and the estimates are

$$\hat{\mu} = 0 \quad ,$$

$$\hat{\sigma} = 1.011$$

IV-B4. DIXON-MOOD ANALYSIS

a. Introduction

This analysis is used exclusively in conjunction with the Bruceton or up-and-down design (see Section III-B1) and is based on the assumption that the response function is cumulative normal. Since the procedure has found such wide application and has been described extensively in the literature (Refs. 12, 22), we shall give only a summary here. The estimates of the mean μ and standard derivation, σ of the normal response function are based on the moments of the observed response function and are approximations to the exact maximum likelihood estimates which are generally obtainable only from extensive iterative calculations (see Section IV-B1). These approximations are obtained by taking advantage of certain aspects of the Bruceton design and are quite good when the analysis is used as indicated below. The crucial restriction is that the fixed step size, d, used in the Bruceton design must be less than twice σ. Since this design is generally employed with $d = \sigma$ or even less, the Dixon-Mood approximation to the maximum likelihood estimators is quite satisfactory. It should be emphasized that this analysis and the corresponding Bruceton design may have to be carried out with a transformed stress variable (see Section IV-A5). The step size d is fixed in regard to this transformed variate. That is, the levels of the stress variable are chosen so that their transformed values are equally spaced with step d.

b. Details of Method

1. Because of the nature of the Bruceton design the total numbers of responses and nonresponses should be about equal. Let N be the smaller of these. For convenience, we will refer to the responses or nonresponses, whichever variable corresponds to N, as "events".

2. Let y_i $(i=0,1,2,\ldots k)$ be the stress levels (possibly transformed) at which tests were performed, starting at y_o, the lowest level at which an event occurred.

3. Let $n_i (i=0,1,2,\ldots k)$ be the number of events at each of the levels y_i. Then $n_o > 0$ and $\sum_{i=o}^{k} n_i = N$.

4. Let

$$A = \sum_{i=0}^{k} i\, n_i$$

$$B = \sum_{i=0}^{k} i^2 n_i$$

then an estimate, m, of the mean is given by

$$m = y_o + d\left(\frac{A}{N} \pm \frac{1}{2}\right)$$

where the plus sign is taken if the events being analyzed are nonresponses and minus if they are responses. An estimate, s, of σ is given by

$$s = 1.620d\left(\frac{NB-A^2}{N^2} + .029\right).$$

This formula should not be used when $\frac{NB-A^2}{N^2} < 0.3$; when this occurs or if $d > 2\sigma$ an alternate procedure described in Ref. 12 should be employed.

c. <u>Variations for Small Samples</u>

Brownlee, Hodges and Rosenblatt, (Ref. 23)

This method, which gives only an estimate for μ, can be used for samples of size 5 to 10. It is also useful when it is desired to conduct several Bruceton designs simultaneously and subsequently analyse all the observations. The design is described in Section III-B1.

Variation I

1. Let $y_i = (1,2,\ldots n)$ be the levels with equal spacing d at which tests have been performed. Let y_{n+1} be the level at which the $(n+1)^{st}$ test would be run if testing were continued.

2. Let $C = \sum_{i=2}^{n+1} y_i$; then the estimate of μ is given by $\hat{\mu} = \frac{C}{n}$. This analysis gives good estimates if the first test y_1 is within 2d of $x_{.5}$. If it is not the following additional variation can be used.

Variation II

If at level y_1 a response (nonresponse) occurs, let y_k bet the first level at which a nonresponse (response) occurs. Then let

$$C' = \sum_{i=k}^{n+1} y_i$$

and $\mu' = \frac{C'}{n+1-k}$.

The estimate μ' is more efficient when y_1 is more than 2d from $x_{.5}$, and also has less bias. However, if the initial guess of $x_{.5}$ is fairly accurate, Variation I is more efficient.

Dixon's New Small Sample Method*

In the Dixon-Mood estimation procedure, it is assumed that about an equal number of responses and nonresponses occur. This will generally be valid for large samples even if y_1 is far from $x_{.5}$.

For small samples this is not likely to hold, particularly when y_1 is far from $x_{.5}$, in which case an initial string of all responses or all nonresponses is likely to occur. The following analysis (Ref. 24) determines maximum-likelihood estimates by correcting for a string of constant responses and also for unequal numbers of responses and nonresponses during the remainder of the testing.

*We are indebted to Professor Dixon for permission to present a summary of these recent results and the two tables which follow prior to their publication in the Journal of the American Statistical Association.

Let N (the "nominal" test sequence length) be the number of tests, counting from the test before the first change of response. For $N \leq 6$, the estimate is $x_f + kd$ where x_f is the final test level, d the step-size and k is found from Table 1. In the latter, O's and X's signify nonresponse and response, respectively.

For $N > 6$, the estimate is

$$\frac{\sum X_i}{N} + \frac{d}{N}(A + C)$$

where the X_i are the levels at which the nominal tests were performed, d is the step size, and A and C are obtained from Table 2. In this table n_o and n_x are the number of O's and X's in the nominal test sequence.

This method of analysis is effective for small samples, being fairly independent of initial test level and step size.

Example

Log Dose	Results of Test
1.204	X
.903	O X X
.602	O O
.301	
0	

For the series OXXOXO, we have $x_f = .602$ and from Table 1, $k = .831$, so that the final estimate is $.602 + .831(.301) = .852$.

Table 1

Values of k for estimating LD_{50} from up-and-down sequence of trials of nominal length N. The estimate of LD_{50} is $x_f + kd$ where x_f is the final test level and d is the interval between dose levels. If the table is entered from the foot, the sign k is to be reversed.

N	Second Part of Series	k for Test Series Whose First Part is					Standard Error of LD_{50}
		O	OO	OOO	OOOO		
2	X	-.500	-.388	-.378	-.377	O	.90 σ
3	XO	.842	.890	.894	.894	OX	.78 σ
	XX	-.178	.000	.026	.028	OO	
4	XOO	.299	.314	.315	.315	OXX	.69 σ
	XOX	-.500	-.439	-.432	-.432	OXO	
	XXO	1.000	1.122	1.139	1.140	OOX	
	XXX	.194	.449	.500	.506	OOO	
5	XOOO	-.157	-.154	-.154	-.154	OXXX	.62 σ
	XOOX	-.878	-.861	-.860	-.860	OXXO	
	XOXO	.701	.737	.741	.741	OXOX	
	XOXX	.084	.169	.181	.182	OXOO	
	XXOO	.305	.372	.380	.381	OOXX	
	XXOX	-.305	-.169	-.144	-.142	OOXO	
	XXXO	1.288	1.500	1.544	1.549	OOOX	
	XXXX	.555	.897	.985	1.000^{+1}	OOOO	
6	XOOOO	-.547	-.547	-.547	-.547	OXXXX	.56 σ
	XOOOX	-1.250	-1.247	-1.246	-1.246	OXXXO	
	XOOXO	.372	.380	.381	.381	OXXOX	
	XOOXX	-.169	-.144	-.142	-.142	OXXOO	
	XOXOO	.022	.039	.040	.040	OXOXX	
	XOXOX	-.500	-.458	-.453	-.453	OXOXO	
	XOXXO	1.169	1.237	1.247	1.248	OXOOX	
	XOXXX	.611	.732	.756	.758	OXOOO	
	XXOOO	-.296	-.266	-.263	-.263	OOXXX	
	XXOOX	-.831	-.763	-.753	-.752	OOXXO	
	XXOXO	.831	.935	.952	.954	OOXOX	
	XXOXX	.296	.463	.500	$.504^{+1}$	OOXOO	
	XXXOO	.500	.648	.678	.681	OOOXX	
	XXXOX	-.043	.187	.244	$.252^{+1}$	OOOXO	
	XXXXO	1.603	1.917	2.000	2.014^{+1}	OOOOX	
	XXXXX	.893	1.329	1.465	1.496^{+4}	OOOOO	
		X	XX	XXX	XXXX	Second Part of Series	
		-k for Series Whose First Part is					

Table 2

Values of A and C for Approximate Estimate of LD_{50} for N>6

$n_o - n_x$	A	C for Test Series Whose First Part is			
		OO	OOO	OOOO	OOOOO
5	10.8	0	0	0	0
4	7.72	0	0	0	0
3	5.22	.03	.03	.03	.03
2	3.20	.10	.10	.10	.10
1	1.53	.16	.17	.17	.17
0	0	.44	.48	.48	.48
-1	-1.55	.55	.65	.65	.65
-2	-3.30	1.14	1.36	1.38	1.38
-3	-5.22	1.77	2.16	2.22	2.22
-4	-7.55	2.48	3.36	3.52	3.56
-5	-10.3	3.5	4.8	5.2	5.3
$n_x - n_o$	-A	XX	XXX	XXXX	XXXXX
		-C for Test Series Whose First Part is			

IV-B5. COMPARING ONE SAMPLE WITH A STANDARD RESPONSE FUNCTION, OR TWO OR MORE SAMPLES WITH EACH OTHER

a. Introduction

A frequent question in sensitivity experiments is to decide whether or not several samples of sensitivity data "come from" the same response function. Another question is to decide whether or not a set of sensitivity data agrees sufficiently well with a standard or previously specified response function. Experimental designs for these "discrimination" problems have been given earlier in Sections III-B10 and III-B11; we give below some likelihood ratio tests for the distributional and distribution-free discrimination problems.

b. Discrimination Between Several Sets of Sensitivity Data

Phase I: Analysis of Individual Sets of Data

Each of n sets of data is analyzed to obtain maximum likelihood estimates of the response function parameters (in the distributional case) and/or the response probabilities at the various test levels (in the distribution-free case). The maximized value of each sample likelihood is then computed. If for a particular set, the estimated response probability at ℓ_i is denoted by $\hat{p}_i$, $i = 1, \cdots, k$, and if the sample response fraction at ℓ_i is denoted by f_i/n_i, then the sample likelihood for this set is given by

$$L = \prod_{i=1}^{k} \hat{p}_i^{f_i} (1-\hat{p}_i)^{n_i-f_i} \quad .$$

Let L_j denote the maximized sample likelihood for the jth set, $j = 1, \cdots, n$, and let π_j denote the number of parameters needed to analyze these data. These parameters are needed to establish the degrees of freedom for the likelihood ratio test. In the distribution-free case, the analysis is performed by means

of the method of reversals (see Section IV-B6), and π_j is given by the number of plateaus.* In the distributional case, π_j is the minimum of the number of unknown distributional parameters and the number of different values among the $\hat{p}_i$, $i = 1, \cdots, k$. For example, in the normal case there are generally two unknown parameters, μ and σ. But for the data $f_1/n_1 = 1/1$, $f_2/n_2 = 0/1$, we have $\hat{p}_1 = \hat{p}_2 = .5$, and therefore $\pi = 1$, $L = 1/4$. For the data $f_1/n_1 = 0/1$, $f_2/n_2 = 1/1$, we have $\hat{p}_1 = 0$, $\hat{p}_2 = 1$, and therefore $\pi = 2$, $L = 1$.

Phase II: Analysis of Lumped Data

The next step is to take all of the data together and perform a similar analysis; let the results here be denoted by L_o and π_o.

Phase III: Likelihood Ratio

Let λ denote the likelihood ratio,

$$\lambda = L_o / \prod_{j=1}^{n} L_j \quad .$$

Then, if the sets of data are all from the same population, $X^2 = -2 \ln\lambda$ is approximately χ^2 distributed with

$$f = \sum_{j=1}^{n} \pi_j - \pi_o$$

degrees of freedom. As a test of the null hypothesis of equality of the n

*This is not exactly correct, but the true value of π_j for our purpose is not known in this case. For a strictly increasing response function, the number of plateaus approaches the number of stress levels as the sample size at every stress level becomes infinite; therefore, it is at any rate asymptotically correct to count plateaus.

response functions, the value corresponding to X^2 is found from a χ^2 table for max(f,1) degrees of freedom. A value over 95% or certainly over 99% would be considered significant in most cases.

Example of a Distribution-Free Discrimination:

Stress Level	Response Fractions and Estimates Given by the Method of Reversals					
	Set 1		Set 2		Combined	
3	2/2	1.0	0/2] 1/3	.33	2/4	.5
2	0/1	0.0	1/1]	.33	1/2	.5
1	0/2	0.0	0/1	.00	0/3	.0
	$\pi_1 = 2$		$\pi_2 = 2$		$\pi_o = 2$	
	$L_1 = 1$		$L_2 = 4/27$		$L_o = 1/64$	

Then $\lambda = 27/256$, $X^2 = 4.5$, $f = 2$, and the corresponding χ^2 value is about 90%. Therefore a difference between the response functions has not been demonstrated.

c. Testing a Set of Sensitivity Data Against a Standard

Let L denote the maximized sample likelihood when the data are analyzed (parametrically or nonparametrically) without reference to the standard or specified response function, and let π be the number of parameters used. Let L_o denote the likelihood of the sample under the assumed standard, or null hypothesis. Then $X^2 = -2\ \ell n(L_o/L)$ is approximately χ^2 distributed with π degrees of freedom.

Nonparametric Example: Suppose the response probabilities for the standard are .2 and .5 at stimuli 1 and 2, respectively, and the observed response fractions for the new set are 1/2 at 1 and 0/1 at 2. Then $\hat{p}_1 = \hat{p}_2 = 1/3$ by the method of reversals, $\pi = 1$, and $L = 4/27$. But according to the standard,

$$L_o = (.2)(.8)(.5) = .08 \quad ;$$

therefore

$$X^2 = -2 \ln(.54) = 1.23 \quad ,$$

which corresponds to about only 73%.

Parametric Example: Cumulative Normal Response Function

Stress Level	Response Fraction	Maximum Likelihood Estimate of Response Probability	Standard
2453	1/1	.924	.5
2433	0/1	.537	.5
2423	1/1	.282	.5
2415	0/2	.133	.5
		$\pi = 2$	
		$L = .0907$	$L_o = .03125$

$$\lambda = .345 \quad ,$$

$$X^2 = 2.13 \quad ,$$

which corresponds to 65% for two degrees of freedom.

IV-B6. METHOD OF REVERSALS

a. Introduction

The method of reversals is a maximum-likelihood procedure for obtaining distribution-free estimates of a monotone nondecreasing response function. The assumption of monotonicity implies that the estimates at adjacent stress levels are constrained to satisfy certain inequalities. (This type of problem is discussed in Ref. 27.)

In a univariate experiment, let the test levels be ℓ_i $(i = 1, 2, \cdots, k)$, and suppose they are ordered with increasing stress, $\ell_1 < \ell_2 < \cdots < \ell_k$. If $\hat{p}_i$ is to be the estimate of the probability of a response at ℓ_i, and if we assume that the response function is monotone nondecreasing, then necessarily

$$\hat{p}_1 \leq \hat{p}_2 \leq \cdots \leq \hat{p}_k \quad ,$$

and the response probabilities are said to be completely or linearly ordered. A simple algorithm, called the method of reversals, can then be used to find the estimates.

In multivariate experiments where the response function is assumed monotone in each variable, some pairs of response probabilities can be ordered. For example, if $\ell_1 = (x_1, y_1)$ and $\ell_2 = (x_2, y_2)$, then $x_1 \leq x_2$ and $y_1 \leq y_2$ imply $M(\ell_1) \leq M(\ell_2)$. However, there are pairs of stress levels for which a natural order does not exist (for example, $x_1 < x_2$ but $y_1 > y_2$). In multivariate experiments, therefore, the response probabilities are only partially ordered, and a more complicated algorithm must be used to find a complete ordering. Then the method of reversals can be applied.

In this section we describe both univariate and multivariate procedures.

b. Method of Reversals for Univariate Sensitivity Data

1. Let ℓ_i $(i = 1, 2, \cdots, k)$ be the k stress levels at which data have been collected, where $\ell_1 < \ell_2 < \cdots < \ell_k$. We wish to find estimates, $\hat{p}_i$, of the values $p_i = M(\ell_i)$, the response probabilities at the levels ℓ_i, which satisfy the constraints

$$\hat{p}_i \le \hat{p}_{i+1} \qquad (i = 1, 2, \cdots, k-1) \quad .$$

2. Let n_i $(i = 1, 2, \cdots, k)$ be the number of tests at ℓ_i and f_i the number of responses observed in the n_i tests. Consider the sequence

$$\frac{f_1}{n_1}, \frac{f_2}{n_2}, \cdots, \frac{f_k}{n_k} \quad .$$

If this sequence is nondecreasing, then the estimates $\hat{p}_i$ are given simply by

$$\hat{p}_i = \frac{f_i}{n_i} \quad .$$

3. If for some i, $f_i/n_i > f_{i+1}/n_{i+1}$, replace both by

$$\frac{F_{i,i+1}}{N_{i,i+1}} = \frac{f_i + f_{i+1}}{n_i + n_{i+1}} \quad .$$

The new sequence becomes

$$\frac{f_1}{n_1}, \frac{f_2}{n_2}, \cdots, \frac{f_{i-1}}{n_{i-1}}, \frac{F_{i,i+1}}{N_{i,i+1}}, \frac{f_{i+2}}{n_{i+2}}, \cdots, \frac{f_k}{n_k} \quad .$$

If this sequence still contains a "reversal" (i.e., a pair of consecutive fractions for which the first is greater than the second), then replace this pair

by a single term as above. This process is continued until one obtains a nondecreasing sequence:

$$\frac{\phi_1}{\eta_1}, \frac{\phi_2}{\eta_2}, \frac{\phi_3}{\eta_3}, \text{ etc.,}$$

where each of the ratios has the form

$$\frac{\phi_j}{\eta_j} = \frac{f_i + \cdots + f_{i+s}}{n_i + \cdots + n_{i+s}}$$

for appropriate i and s.

4. The final estimates are then given by

$$\hat{p}_i = \cdots = \hat{p}_{i+s} = \frac{\phi_j}{\eta_j} \quad .$$

If it is desired to obtain estimates of the response function at stress levels other than those at which tests have been conducted, then linear interpolation using these values can be performed.

It is convenient for some applications to introduce at this point the notion of a "plateau", which is defined as a maximum stress interval over which the estimated response is constant.

Example: Let $k = 8$ and the response fractions f_i/n_i at the k levels $\ell_1, \cdots, \ell_k$ be 1/3, 1/4, 1/3, 1/5, 1/2, 2/5, 3/5, 1/2, respectively. Then a nondecreasing sequence is determined as follows:

	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8
Step One	1/3	1/4	1/3	1/5	1/2	2/5	3/5	1/2
Step Two	2/7		2/8		3/7		4/7	
Final Step	4/15				3/7		4/7	

,

where the "reversals" are indicated. The final nondecreasing estimates are given by

$$\hat{p}_1 = \hat{p}_2 = \hat{p}_3 = \hat{p}_4 = 4/15 \quad ,$$

$$\hat{p}_5 = \hat{p}_6 = 3/7 \quad ,$$

$$\hat{p}_7 = \hat{p}_8 = 4/7 \quad .$$

Thus, over the plateau from ℓ_1 to ℓ_4 we use the estimate 4/15; for the plateau $[\ell_5, \ell_6]$, 3/7; and for the plateau $[\ell_7, \ell_8]$, 4/7. To obtain estimates in the interval (ℓ_4, ℓ_5) linear interpolation may be used. For example, at $(\ell_4 + \ell_5)/2$ the estimate would be $(4/15 + 3/7)/2 = 73/210$.

c. Monotonic Estimates for Multivariate Sensitivity Data

As was indicated in the introductory paragraphs of this section, the stimulus levels for multivariate experiments in general cannot be completely ordered. In this section we give an algorithm for estimating a complete ordering of the stimulus levels which is based on the known partial ordering and the observed response fractions. Then the method of reversals is applied using this complete ordering as in the univariate case. It can be shown that the resulting multivariate response surface estimates are maximum likelihood under the assumption that this surface is monotone nondecreasing in each stress variable.

The algorithm is initiated by considering any order which is consistent with the partial order naturally exhibited by the stimulus level combinations. Then the observed responses at the stimulus level combinations are used to determine a new order (which is still consistent with the original partial ordering). This process is continued until an ordering is found which satisfies other

requirements on the estimates (described later). The resulting complete ordering is not unique, but any one arrived at by this process will give identical estimates when the method of reversals is subsequently applied.

Details of Algorithm

Assume that there are t stimulus variables, and tests have been performed at n distinct level combinations $Q_i = Q(x_1^{(i)}, x_2^{(i)}, \cdots, x_t^{(i)})$, $i = 1, 2, \cdots, n$. Here $x_j^{(i)}$ is the level of the jth variable for the ith "point" (combination of stimulus variables). Note that the $x_j^{(i)}$ are not necessarily distinct.

If Q_1 and Q_2 are such that for all j $(j = 1, 2, \cdots, t)$, $x_j^{(1)} \leq x_i^{(2)}$, and for some j, say j_0, $x_{j_0}^{(1)} < x_{j_0}^{(2)}$, then we employ the "natural order" $Q_1 < Q_2$.

If two levels cannot be ordered in this fashion, they will be called independent. If, for two sets of level combinations, every member of one is independent of every member of the other, the sets are said to be "independent".

If $S = \{Q_1, Q_2, \cdots, Q_s\}$ is a set of level combinations, then its associated probability estimate, $\hat{P}$, is given by

$$\hat{P} = \frac{f_1 + \cdots + f_s}{n_1 + \cdots + n_s}$$

where n_i denotes the number of tests at Q_i and f_i is the number of responses observed in these n_i tests, for $i = 1, 2, \cdots, s$.

The complete order is found in the following manner.

1. The n distinct combinations are written so that all natural order requirements are satisfied, say

$$Q_1 < Q_2,\ Q_3 < Q_4 < Q_5,\ \cdots < Q_n$$

where these required natural orderings are shown by "<" and commas separate independent level combinations. All order requirements are not necessarily indicated here; for example, it may be that $Q_1 < Q_4$.

2. This order is now scanned for adjacent independent sets. The simplest way to do this is to consider sets separated by commas as we have done above. Thus, for example, we know Q_2, Q_3 are independent as indicated in the above sequence, but it may also be that $\{Q_1, Q_2\}$ is independent of $\{Q_3, Q_4, Q_5\}$.

3. If, for two adjacent independent sets S_1, S_2, the associated probability estimates $\hat{P}_1$ and $\hat{P}_2$ are such that $\hat{P}_2 < \hat{P}_1$, then the sets are interchanged. The new order satisfies all natural order requirements since the interchanged sets are independent.

4. This procedure is continued until all adjacent independent sets have non-decreasing associated estimates and a complete order has been derived.

5. Monotone estimates are now found by applying the method of reversals under this complete order.

<u>Example:</u> Consider a bivariate experiment $(t = 2)$ and suppose sensitivity tests are conducted at sixteen level combinations, $R_{ij} = R(x_1^{(i)}, x_2^{(j)})$ $(i = 1, 2, 3, 4;\ j = 1, 2, 3, 4)$, with the proportions of response at these points as follows:

	$x_1^{(1)}$	$x_1^{(2)}$	$x_1^{(3)}$	$x_1^{(4)}$
$x_2^{(4)}$	1/4	1/3	1/2	1/3
$x_2^{(3)}$	1/3	1/4	1/5	1/4
$x_2^{(2)}$	1/5	1/8	1/6	1/4
$x_2^{(1)}$	1/3	1/4	1/8	1/6

An order for R_{ij} that satisfies the requirements is given by

$R_{11}(1/3)\ R_{12}(1/5)\ R_{13}(1/3)\ R_{14}(1/4)\ |\ R_{21}(1/4)\ R_{22}(1/8)\ R_{23}(1/4)\ R_{24}(1/3)\ |$
$R_{31}(1/8)\ R_{32}(1/6)\ R_{33}(1/5)\ R_{34}(1/2)\ |\ R_{41}(1/6)\ R_{42}(1/4)\ R_{43}(1/4)\ R_{44}(1/3)$,

where adjacent independent levels are indicated by the vertical lines. For convenience the response fractions are given in parentheses after each level. We observe that the adjacent independent sets $\{R_{13},R_{14}\}$ and $\{R_{21},R_{22}\}$ have associated probabilities 2/7 and 2/12 and should be interchanged. Similarly, the following pairs of independent sets,

$\{R_{23},R_{24}\}$ and $\{R_{31},R_{32}\}$,
$\{R_{33},R_{34}\}$ and $\{R_{41}\}$,

may next be interchanged. This gives the order

$R_{11}\ R_{12}\ |\ R_{21}\ R_{22}\ |\ R_{13}\ R_{14}\ |\ R_{31}\ R_{32}\ |\ R_{23}\ R_{24}\ |\ R_{41}\ |\ R_{33}\ R_{34}\ |\ R_{42}\ R_{43}\ R_{44}$

and we note the following pairs should be reversed:

$\{R_{13},R_{14}\}$ and $\{R_{31},R_{32}\}$,
$\{R_{23},R_{24}\}$ and $\{R_{41}\}$,
$\{R_{34}\}$ and $\{R_{42},R_{43}\}$,

giving the order

$R_{11}\ R_{12}\ |\ R_{21}\ R_{22}\ |\ R_{31}\ R_{32}\ |\ R_{13}\ R_{14}\ |\ R_{41}\ |\ R_{23}\ R_{24}\ |\ R_{33}\ |\ R_{42}\ R_{43}\ |\ R_{34}\ R_{44}$.

There remain at this stage the reorderings

$$\{R_{13}, R_{14}\} \text{ and } \{R_{41}\} ,$$
$$\{R_{24}\} \text{ and } \{R_{33}, R_{42}, R_{43}\} ,$$

which result in the order

$R_{11}\ R_{12}\ R_{21}\ R_{22}\ R_{31}\ R_{32}\ R_{41}\ R_{13}\ R_{14}\ R_{23}\ R_{33}\ R_{42}\ R_{43}\ R_{24}\ R_{34}\ R_{44}$;
this satisfies all our conditions and is therefore final.

Now the method of reversals can be used as follows:

1/3,	1/5,	1/4,	1/8,	1/8,	1/6,	1/6,	1/3,	1/4,	1/4,	1/5,	1/4,	1/4,	1/3,	1/2, 1/3
2/8		2/12		1/8	1/6	1/6	2/7		2/9		1/4	1/4	1/3	2/5
	4/20			1/8	1/6	1/6		4/16			1/4	1/4	1/3	2/5
		5/28			1/6	1/6		4/16			1/4	1/4	1/3	2/5
				6/34		1/6		4/16			1/4	1/4	1/3	2/5
					7/40			4/16			1/4	1/4	1/3	2/5

and our final estimates are

$$\hat{P}_{11} = \hat{P}_{12} = \hat{P}_{21} = \hat{P}_{22} = \hat{P}_{31} = \hat{P}_{32} = \hat{P}_{41} = 7/40$$
$$\hat{P}_{13} = \hat{P}_{14} = \hat{P}_{23} = \hat{P}_{33} = \hat{P}_{42} = \hat{P}_{43} = 1/4$$
$$\hat{P}_{24} = 1/3$$
$$\hat{P}_{34} = \hat{P}_{44} = 2/5^{*} .$$

*In the multivariate case it is natural to define a plateau as the largest closed connected subset of the stress domain over which the response surface estimate is constant. Thus in the above example there are four plateaus.

IV-B7. ANALYSIS OF MULTIVARIATE SENSITIVITY DESIGN WITH NON-INTERACTING STIMULI

a. Introduction

For notational simplicity we consider explicitly the case of two stimulus variables. The analysis for more variables is conducted in an analogous manner. The design of multivariate experiments is discussed in Section III-B9.

Let the level combinations of the two stimulus variables be given by the grid of points $(x_i, y_j)(i=1,2,\ldots n_1;\ j=1,2,\ldots n_2)$. Then the probability of a nonresponse at the level combination (x_i, y_j) is binomially distributed with parameter $Q(x_i, y_j) = q_{x_i} \cdot q_{y_j}$, where q_{x_i} and q_{y_j} are the independent probabilities of nonresponses for the individual stimuli x and y, respectively.

Let N_i be the total number of nonresponses observed in ℓ tests at each of the stimulus levels given by $(x_i, y_j)(j=1,2,\ldots n_2)$. Then N_i has a Poisson generalized binomial distribution (see Ref. 26) such that $E(N_i) = \ell q_{x_i} \cdot \sum_{j=1}^{n_2} q_{y_j}$. Similarly, the number of nonresponses, M_j, in ℓ tests at the level combinations $(x_i, y_j)(i=1,2,\ldots n_1)$ is such that $E(M_j) = \ell q_{y_j} \sum_{i=1}^{n_1} q_{x_i}$; the total number of nonresponses, T, for all tests is such that $E(T) = \ell \sum_{i=1}^{n_1} q_{x_i} \sum_{j=1}^{n_2} q_{y_j}$. Eliminating $\sum_{i=1}^{n_1} q_{x_i}$ and $\sum_{j=1}^{n_2} q_{y_j}$ we have

$$Q(x_i,y_i) = q_{x_i} \cdot q_{y_j} = \frac{E(N_i)E(M_j)}{\ell \cdot E(T)} .$$

In the following sections this expression will be employed to determine estimates for Q and to test for interaction between x and y. (Interaction is discussed in Section III-B9.)

b. Monotone Estimates

Let $r_{ij}(i=1,2,\ldots n_1;\ j=1,2,\ldots n_2)$ be the observed number of nonresponses in ℓ tests at the level (x_i, y_j). An estimate, $\hat{Q}(x_i, y_j)$, of the probability of a response at this level can be found by using the following procedure. We first compute

$$N_i = \sum_{j=1}^{n_2} r_{ij}, \quad M_j = \sum_{i=1}^{n_1} r_{ij}, \quad T = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} r_{ij} .$$

Next consider the set $\{N_i\}(i=1,2,\ldots n_1)$ where, as before, N_i is the total number of nonresponses observed in ℓ tests at each of the stimulus levels given by $(x_i, y_j)(j=1,2,\ldots n_2)$. If the sequence $N_1, N_2, \ldots N_{n_1}$ is monotone decreasing, then $\hat{N}_i = N_i$. If not, the method of reversals is used (see Section IV-B6), assuming all N_i have equal weight. That is, if $N_i < N_{i+1}$, then both N_i and N_{i+1} are replaced by $\frac{N_i+N_{i+1}}{2}$. If $\frac{N_i+N_{i+1}}{2} < N_{i+2}$ all are replaced by $\frac{N_i+N_{i+1}+N_{i+2}}{3}$, etc. Proceeding in this fashion one obtains the monotone nonicreasing estimates $\hat{N}_i$. The estimates $\hat{M}_j$ are determined in the same way. The $\hat{N}_i$, $\hat{M}_j$, and T can now be used as estimates of $E(N_i)$, $E(M_j)$, and $E(T)$, respectively, and we obtain finally

$$\hat{Q}(x_i, y_j) = \begin{cases} \frac{\hat{N}_i\hat{M}_j}{\ell T}, & \text{if } \frac{\hat{N}_i\hat{M}_j}{\ell T} \le 1 \\ 1, & \text{otherwise.} \end{cases}$$

c. Test for Interaction

The assumption of noninteraction can now be tested by a χ^2-test; if the quantity

$$\chi^2 = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \frac{\left[\frac{\hat{N}_i\hat{M}_j}{T} - r_{ij}\right]^2}{\frac{\hat{N}_i\hat{M}_j}{T}},$$

which has $(n_1-1)(n_2-1)$ degrees of freedom, should be sufficiently large, the null hypothesis of non-interaction would be rejected.

d. Example

The following true nonresponse probabilities were selected as the basis for a brief simulation:

True Values of $Q(x_i, y_j)$

.36	.32	.24	.20
.54	.48	.36	.30
.72	.64	.48	.40
.90	.80	.60	.50

For these values a table of random numbers was used to generate the following hypothetical data for $n_1 = n_2 = 4$ and $\ell = 2$:

Experimental Data (r_{ij})

	x_1	x_2	x_3	x_4
y_4	0	1	1	1
y_3	1	1	1	0
y_2	2	2	1	2
y_1	2	1	0	1

where the number of nonresponses are indicated. From this we find the $N_i, (i=1,2,3,4)$ are given, respectively, by 4, 7, 3, 3. Using the method of reversals one obtains the set of corresponding monotone estimates $\{\hat{N}_i\} = \{5.5, 5.5, 3, 3\}$. Similarly, $\{M_j\} = \{5, 5, 3, 4\}$ which gives $\{\hat{M}_j\} = \{5, 5, 3.5, 3.5\}$. Our estimates

$$\hat{Q}(x_i, y_j) = \frac{\hat{N}_i \hat{M}_j}{\ell T}$$

are now given in the following table ($\ell = 2$, $T = 17$).

Estimates $\hat{Q}(x_i, y_j)$

	x_1	x_2	x_3	x_4
y_4	.44	.44	.31	.31
y_3	.44	.44	.31	.31
y_2	.81	.81	.57	.57
y_1	.81	.81	.57	.57

A χ^2-test can now be used to test for noninteraction:

$$\chi^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} \frac{(2\hat{Q}(x_i, y_j) - r_{ij})^2}{2\hat{Q}(x_i, y_j)} \sim 4.6$$

From tables of χ^2 we find that for $\chi^2 = 4.6$, with 9 degrees of freedom, the assumption of noninteraction would not be rejected.

e. <u>Extension to More Variables</u>

1. Let the stimulus variables be $x^{(1)}, x^{(2)}, \ldots, x^{(t)}$. Perform ℓ tests at each of the $n_1 \times n_2 \times \ldots \times n_t$ points on the t-dimensional rectangular grid given by $(x_{i_1}^{(1)}, x_{i_2}^{(2)}, \ldots, x_{i_t}^{(t)})$ $(i_1=1,2,\ldots,n_1;\ i_2=1,2,\ldots,n_2;\ \ldots;\ i_t=1,2,\ldots,n_t)$.

2. Let $N_{i_s}^{(2)}$ be the number of nonresponses at the point $(x_{i_1}^{(1)} \ldots, x_{i_t}^{(t)})$ $(i_1=1,2,\ldots,n_1;\ \ldots, i_{j-1}=1,2,\ldots,n_{j-1};\ i_{j+1}=1,2,\ldots,n_{j+1}; \ldots, i_t=1,2,\ldots,n_t)$ [i.e., $x^{(j)}$=fixed], and let T be the total number of nonresponses. Then

$$Q(x_{i_1}^{(1)}, \ldots, x_{i_t}^{(t)}) = \begin{cases} \dfrac{\hat{N}_{i_1}^{(1)} \ldots \hat{N}_{i_t}^{(t)}}{\ell T^{t-1}}, & \text{if } \dfrac{\hat{N}_{i_1} \ldots \hat{N}_{i_t}}{\ell T^{t-1}} \le 1 \\ 1 & \text{, otherwise.} \end{cases}$$

As in the bivariate case $\{\hat{N}_{i_1}^{(1)}\}$ are the monotone estimates of $\{N_{i_1}\}$ determined from

the sequence $N_1^{(1)}$, $N_2^{(1)}$, ..., $N_{n_1}^{(1)}$ by the method of reversals, and similarly for $\hat{N}_{i_2}^{(2)}$, ..., $\hat{N}_{i_t}^{(t)}$. The total number of tests, N, required is given by

$$N = \ell \times n_1 \times n_2 \times \ldots, \times n_t .$$

IV-B8. PROBIT ANALYSIS

a. Introduction

This method of analysis was developed for use in connection with the probit design (Section III-B17) and is based on the assumption that the response function is cumulative normal. A transformation, either performed graphically or analytically, is used to change the observed fraction response to probits in terms of which the cumulative normal distribution becomes linear. This approach has been previously employed in biological applications (e.g., dosage mortality).

b. Graphical method

The observed percentage response at each stimulus level is transformed to a probit, i.e.g, to a standard normal deviate increased by 5. (This increase serves to simplify the computation by avoiding negative values.) The stimulus level is then plotted against a linear scale of probits. A regression line can then be fitted and the probit for any stimulus level can either be read directly from the graph or determined from the equation for the line. The latter should approximate the relationship between dosage and response which is given by

$$Y = 5 + \frac{1}{\sigma}(x - \mu).$$

c. Analytic method

This method is based on maximum likelihood estimation procedure and is applied iteratively until satisfactory estimates are obtained. A "provisional" probit versus stimulus level line is drawn; from it, expected probits, Y, for each stimulus level can be determined. A "working" probit, y, is now found for each stimulus level in one of the following ways:

1) Algebraic procedure: The equation

$$y = Y + \frac{p-P}{z},$$

is used, where Y is the expected probit, P and z the corresponding probability and ordinate, respectively (as read from the provisional line), and the observed response fraction is p.

2) Tables are used (Ref. 33) of the "maximum working probit"

$$y_{100} = Y + \frac{Q}{z}$$

and/or the "minimum working probit"

$$y_0 = Y - \frac{P}{z},$$

where $Q = 1 - P$ and z and Y are defined as before. Defining the "range" $R = y_{100} - y_0$, we find $R = \frac{1}{z}$ and then the following expressions are employed:

$$y = y_0 + pR$$

or

$$y = y_{100} - (1-p)R$$

where p is the observed response-fraction. More extensive tables which give the working probit y directly are also available (see Ref. 30).

When working probits have been found by any of these procedures then a linear regression line is determined using the weights $n \cdot w$ where

n = sample size at the given stimulus

and

$$w = \frac{z^2}{PQ} .$$

Now z, p, and Q can be determined from the provisional line but are in fact given exactly by

$$z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Y-5)^2}$$

$$P = 1-Q = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{Y-5} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

Tables of w, the weighting coefficient, can be found in Ref. 33. The resulting line is then used as a provisional line for the next iteration. This is continued until the parameters of the line remain essentially unchanged on successive iterations.

Transformations to other than probits for which tables are available (in Ref. 33) are the following:

Method	Transformation
Bliss-Fisher (Probit)	Transforms normal data to a straight line.
Logit	$f(p) = \frac{1}{2} \log_e \frac{p}{1-p}$
Log log.	$f(p) = \log_e\{-\log_e(1-p)\}$
Angular	$f(p) = \text{arc sin} \sqrt{p}$

One advantage of these "transformation" methods is that comparisons between various response populations can be more readily made in terms of the simpler transformed models.

Because of the extensive literature on the probit, logit, and other transformation methods of analysis we will not give any details here.

IV-B9. A LIKELIHOOD RATIO TEST FOR MONOTONICITY

a. Introduction

The distribution-free procedure given below is for testing the null hypothesis that the response function is monotone nondecreasing, a hypothesis which may be suspect if there are any reversals in the data. The test is based on the extent of the decrease in the sample likelihood when the sample is analyzed under the assumption of monotonicity.

b. Procedure

The sample response fractions $f_1/n_1, \cdots, f_k/n_k$ at the levels $\ell_1, \cdots, \ell_k$ are first analyzed by the method of reversals (Section IV-B6) to give estimates $\hat{p}_1, \cdots, \hat{p}_k$, with π_o plateaus. The maximized sample likelihood under the null hypothesis is then

$$L_o = \prod_{i=1}^{k} \hat{p}_i^{f_i} (1-\hat{p}_i)^{n_i-f_i}$$

In the absence of the null hypothesis of monotonicity, the usual binomial estimates $\hat{p}_i = f_i/n_i$ may be used, giving a sample likelihood

$$L = \prod_{i=1}^{k} \left[\frac{f_i^{f_i}(n_i-f_i)^{n_i-f_i}}{n_i^{f_i}} \right] .$$

Then $X^2 = -2\,\ell n(L_o/L)$ is approximately X^2 distributed with $d = k-\pi_o$ degrees of freedom.

c. Example

Stress Level	Response Fraction	Estimate
3	2/7 ┐	4/9
2	2/2 ┘	4/9
1	1/4	1/4

$k = 3, \quad x_o = 2, \quad d = 1,$

$$L_o/L = \frac{4^4 \cdot 5^5/9^9}{2^2 \cdot 5^5/7^7} = 4^3 \cdot 7^7/9^9 \quad ,$$

$X^2 \cong 3.99$, which corresponds to roughly 95.3% in the X^2 table for one degree of freedom.

d. Comments

A test for monotonicity would perhaps be the first order of business in analyzing a set of sensitivity data. If a distributional assumption is made, and if the set of data passes the test for monotonicity, the next step would be the goodness-of-fit test in the following section.

IV-B10. A LIKELIHOOD RATIO GOODNESS-OF-FIT TEST

a. Introduction

If a set of data passes the test for monotonicity (see Section IV-B9), and if a specific type of distribution is assumed, then the appropriateness of this assumption must be tested as the next step in analyzing the data. The test given below is based on an evaluation of the decrease in the sample likelihood when the sample is analyzed under the distributional assumption, rather than under the weaker assumption of monotonicity.

b. Test

The sample response fractions $f_1/n_1, \cdots, f_k/n_k$ at the levels $\ell_1, \cdots, \ell_k$ are analyzed by the method of reversals to give estimates $\hat{p}_1, \cdots, \hat{p}_k$, with π_1 plateaus. The maximized sample likelihood is then

$$L = \prod_{i=1}^{k} \hat{p}_i^{\,f_i} (1-\hat{p}_i)^{n_i-f_i} \quad .$$

The data are now analyzed under the distributional assumption, giving likelihood L_o based on π_o parameters (see the discussion on parameters given in Section IV-B5). Then

$$X^2 = -2 \ln(L_o/L)$$

is approximately X^2 distributed with $d = \pi_1-\pi_o$ degrees of freedom. The value corresponding to X^2 and $\max(d,1)$ degrees of freedom would be found in a X^2 table, and the distributional assumption would be rejected for values, say, over 95% or 99%.

c. Example: Test for Normality

Stress Level	Response Fraction	Response Probability Estimate from Method of Reversals	Response Probability Estimate from Maximum Likelihood Estimates $\hat{\mu}=2431.61$, $\hat{\sigma}=14.91$
2453	1/1	1.0	.924
2433	0/1	0.5	.537
2423	1/1	0.5	.282
2415	0/2	0.0	.133
		$\pi_1 = 3$	$\pi_o = 2$

Then

$$L = 1/4 \ ,$$

$$L_o = .0907 \ ,$$

$$X^2 = 2.03 \ .$$

For one degree of freedom, the corresponding X^2 value is about 84%, and the distributional assumption would not be rejected.

IV-B11. TEST FOR THE EXISTENCE OF AN UNKNOWN STIMULUS VARIABLE

a. Introduction

One of the possible causes of significant non-monotonicity in sensitivity data is the existence of an unknown (or perhaps known but uncontrolled) stimulus variable (see Section II-A4). In this section we give a distribution-free procedure to test for this possibility.

In Section V-B8 the distribution of the response fractions when there is a single fixed known stimulus variable and an uncontrolled variable is derived. This leads to the interesting result that the variance is smaller when such a stimulus is present. Thus the test given below should be used only when observed variance in response fraction for several sets of data at the same level of the known stimulus variables is smaller than that expected for a binomial variable.

b. Test

A χ^2-test can be used as follows:

1. Perform n sets of m tests each at a fixed level of the known stimulus variable. Those tests already completed which originally indicated non-monotonicity may be included in this analysis.
2. Suppose $f_i (i=1,2...n)$ responses occur; construct the following table

f_1	f_2	.....	f_n
$m-f_1$	$m-f_2$	.....	$m-f_n$

which may be considered a contingency table. The estimates of the expected values are $m\bar{p}$ for the first row and $m - m\bar{p}$ for the second, where

$$\bar{p} = \frac{\sum_{i=1}^{m} f_i}{n \cdot m}$$

3. Then using the classical χ^2-test we compute

$$\chi^2 = \sum_{i=1}^{n} \frac{(f_i - m\bar{p})^2}{m\bar{p}} + \sum_{i=1}^{n} \frac{[(m-f_i) - (m-n\bar{p})]^2}{m-m\bar{p}}$$

$$= \sum_{i=1}^{n} \frac{(f_i - m\bar{p})^2}{m\bar{p}(1-\bar{p})}$$

which has $n-1$ degress of freedom.

4. The null hypothesis that there are no hidden stimulus variables will be rejected if the probability of having a larger χ^2-value is less than some specified probability (.95, for example).

5. The choice of n and m should be such that

$$n > 5$$

$$m\bar{p} > 5$$

$$m(1-\bar{p}) > 5.$$

Since the nonmonotonicity was originally seen from tests at several levels of the known stimulus, the question arises as to which of these levels to choose for performing the test given above. The above formulas indicate that the best level to choose is that which corresponds to a response probability as close to .5 as possible. It is well known that this χ^2-test is weak in the sense that the null hypothesis is rejected only in extreme circumstances. This suggests that, if the computed χ^2-value corresponds to a probability as low as, say, .85, additional testing be conducted.

REFERENCES

1. Rothman, D. and Zimmerman, J. M. (1965). The design of complex sensitivity experiments. Proceedings of the Tenth Conference on the Design of Experiments in Army Research, Development and Testing, Washington, D. C.

2. Grant, R. L. and Van Dolah, R. W. (1964). Use of the up-and-down method with factorial designs. Proceedings of the Seventh Conference on the Design of Experiments in Army Research, Development and Testing, ARODR 62-2, 39-65.

3. Ref. 3 has been deleted.

4. Chernoff, H. (1962). Optimal design of experiments. Technical Report No. 82, Applied Mathematics and Statistics Laboratories, Stanford University.

5. Wetherill, G. B. (1963). Sequential estimation of quantal response curves. J. Roy. Statist. Soc. Ser. B, 25, 1-48.

6. Robbins, H. and Monro, S. (1951). A stochastic approximation method. Ann. Math. Statist., 22, 400-407.

7. Kesten, H. (1958). Accelerated stochastic approximation. Ann. Math. Statist., 29, 41-59.

8. Odell, P. L. (1962). Stochastic approximation and nonparametric interval estimation in sensitivity testing which involves quantal response data. Oklahoma State University, Ph.D. Thesis.

9. Cochran, W. G. and Davis, Miles. (1964). Stochastic approximation to the median effective dose in bioassay. Stochastic Models in Medicine and Biology. University of Wisconsin Press, Madison.

10. Bartlett, M. S. (1946). A modified probit technique for small probabilities. J. Roy. Statist. Soc. Suppl., 8, 113-117.

11. Hampton, L. D., Ayres, J. N., and Kabik, I. (1964). Estimation of high and low probability EED functioning levels. NOLTR 63-266, White Oak.

12. Dixon, W. J. and Mood, A. M. (1948). A method for obtaining and analyzing sensitivity data. J. Amer. Statist. Assoc., 43, 109-126.

13. Hodges, J. L., Jr., and Lehmann, E. L. (1956). Two approximations to the Robbins-Monro process. Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability. University of California Press, Vol. 1, 95-104.

14. Walsh, J. E. (1962). Handbook of nonparametric statistics. D. Van Nostrand Co., Inc., Princeton, N. J.

15. Derman, C. (1956). Stochastic approximation. Ann. Math. Statist., 27, 879-886.

16. Hewlett, P. S. and Plackett, R. L. (1952). Quantal responses to mixtures of poisons. J. Roy. Statist. Soc. Ser. B, 14, 141-163.

17. Langlie, H. J. (1962). A reliability test method for "one-shot" items. Aeronutronic Publication No. U-1792.

18. Marks, B. L. (1962). Some optimal sequential schemes for estimating the mean of a cumulative normal quantal response curve. J. Roy. Statist. Soc. Ser. B, 24, 393-400.

19. Dvoretzky, A. (1956). On stochastic approximation. Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability. University of California Press, Vol. 1, 39-55.

20. Eissner, R. M. (1953). A method of analyzing sensitivity data from "contaminated" populations. BRL Memo. Rep. No. 722.

21. Golub, A. and Grubbs, F. E. (1956). Analysis of sensitivity experiments when the levels of stimulus cannot be controlled. J. Amer. Statist. Assoc., 51, 257-265.

22. Dixon, W. J. and Massey, F. J. (1957). Introduction to statistical analysis. McGraw-Hill, New York, second edition, 318-327.

23. Brownlee, K. A., Hodges, J. L., Jr., and Rosenblatt, M. (1953). The up-and-down method with small samples. J. Amer. Statist. Assoc., 48, 262-277.

24. Derman, C. (1957). Non-parametric up-and-down experimentation. Ann. Math. Statist., 28, 795-798.

25. Dixon, W. J. (1965). The up-and-down method for small samples. To be published in the J. Amer. Statist. Assoc.

26. Cramér, H. (1946). Mathematical methods of statistics. Princeton University Press, Princeton.

27. Brunk, H. (1955). Maximum likelihood estimates of monotone parameters. Ann. Math. Statist., 26, 607-616.

28. Anderson, T. W., McCarthy, P. J., and Tukey, J. W. (1946). "Staircase" methods of sensitivity testing. NAVORD Report 65-46, 135 pp.

29. Aitchison, J. and Silvey, S. D. (1957). The generalisation of probit analysis to the case of multiple responses. Biometrika, 44, 131-140.

30. Finney, D. J. (1952). Probit analysis. Cambridge University Press, Cambridge, second edition.

31. Ling, R. C. and Hess, W. R. (1960). An interpretation of the sensitivity of explosives. FRL-TR-21, Feltman Research Laboratory, Picatinny Arsenal.

32. Hammer, C. (1955). Statistical methods in initiator evaluation. Franklin Institute Laboratories for Research and Development, Interim Report No. I-1804-1 (CONFIDENTIAL).

33. Fisher, R. A. and Yates, F. (1957). Statistical tables. Hafner Publishing Co., Inc., New York, 5th edition.

34. Ayer, M., Brunk, H. D., Ewing, G. M., Reid, W. T., and Silverman, E. (1955). An empirical distribution function for sampling with incomplete information. Ann. Math. Statist., 26, 641-647.

35. Van Eeden, C. (1958). Testing and estimating ordered parameters of probability distributions. Studentendrukkerij Pootpers N. V.

APPENDIX

V-A. COMPUTER PROGRAM LISTINGS

V-A1. INPUT DATA FOR "MLE": A FORTRAN PROGRAM TO COMPUTE MAXIMUM-LIKELIHOOD ESTIMATES OF THE PARAMETERS OF A CUMULATIVE NORMAL RESPONSE FUNCTION

a. First Card: Format (4I12)

1. Number ($2 \leq LVL \leq 200$) of distinct stimulus levels
2. Maximum number of iterations (generally 15)
3. 1 if MOS is to be used for first guesses.

   0 if initial guesses and bounds are to be input

   -1 if MOS and 2-subset methods are to be used for first guesses (preferred)
4. 0 if ordinary output desired

   1 if intermediate output desired

b. Second Card: Format (6E12.8)

(To be used only if the third entry on the first card is 0)

1. initial guess for μ
2. initial guess for σ
3. lower bound on μ
4. upper bound on μ
5. lower bound on σ
6. upper bound on σ

c. Third Card: Format (12A6)

Up to 72 characters to be printed as output title

d. Fourth Card: Format (6E12.8)

1. stimulus level ℓ_1
2. number of successes at ℓ_1
3. number of tests at ℓ_1 (total number of tests at all levels ≤ 3000)
4. stimulus level $\ell_2 > \ell_1$
5. number of successes at ℓ_2
6. number of tests at ℓ_2

e. <u>Fifth Card</u> (if necessary): Format (6E12.8)

1. stimulus level $\ell_3 > \ell_2$, etc.

V-A2. LISTINGS AND SAMPLE OUTPUTS

We give below the listings of MLE and MOR, followed by three examples of the output of MLE. MOR is given in subroutine form. The calling sequence required is given on the first page of the listing.

```
C     MAXIMUM LIKELIHOOD ESTIMATES OF MU AND SIGMA                      00310100
C        *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *  00310300
C     DEFINITION OF VARIABLES                                           00310400
C       SS(I,1)= STIMULUS LEVELS  (MAXIMUM = 200)                       00310500
C       SS(I,2)= NUMBER OF SUCCESSES (ONES) AT A GIVEN STIMULUS LEVEL   00310600
C       SS(I,3)= NUMBER OF TESTS AT A GIVEN STIMULUS LEVEL              00310700
C                (IF OPTION OF MOS NEG. IS USED, THIS VECTOR MUST       00310800
C                BE MONOTONICALLY INCREASING.)                          00310801
C       KLVL   = NUMBER OF DISTINCT STIMULUS LEVELS                     00310900
C       ITMAX  = MAXIMUM NUMBER OF ITERATIONS                           00310950
C       TEST   = POSITIVE VALUE CAUSES INTERMEDIATE (CHECKOUT) PRINTOUT.00311000
C       MOS    = LOGICAL VARIABLE. POS.=COMPUTE MU,SIG FROM MINIMUM     00311030
C                OVERLAPPING SUBSET, ZERO, USE INPUT GUESSES            00311040
C                NEGATIVE, USE MIXED METHOD OF ARRIVING AT INITIAL      00311045
C                GUESSES AND MAXIMA,MINIMA FOR MU,SIG.                  00311046
C       EMMY   = WHEN MOS IS ZERO, INITIAL GUESS AT MU                  00311050
C       SIGGY  = WHEN MOS IS ZERO, INITIAL GUESS AT SIG                 00311060
C       HOLL   = ONE CARD FOR CASE IDENTIFICATION                       00311100
C        *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *  00311200
      DIMENSION SS(200,3),T(3000),IND(3000),BUFR(6000),SUM(5),HOLL(12)  00311300
      COMMON T,BUFR,IND,SS                                              00311400
 5    READ INPUT TAPE 5,6, KLVL,ITMAX,MOS,NTEST                         00311450
 6    FORMAT (4I12)                                                     00311460
      IF (MOS) 9,7,9                                                    00311490
 7    READ INPUT TAPE 5,8, EMMY,SIGGY,EMLO,EMHI,SIGMIN,SIGMAX           00311500
 8    FORMAT (6E12.8)                                                   00311510
 9    READ INPUT TAPE 5,10, HOLL, ((SS(I,J),J=1,3),I=1,KLVL)            00311520
 10   FORMAT (12A6/(6E12.8))                                            00311550
C     INITIALIZATION                                                    00311555
      TEST = NTEST                                                      00311560
      EMZERO = EMMY                                                     00311565
      SGZERO = SIGGY                                                    00311570
      FLAGMU= 0.0                                                       00311575
      FLAGSG = 0.0                                                      00311580
      IF (KLVL-200) 15,15,12                                            00311600
 12   WRITE OUTPUT TAPE 6,13, KLVL                                      00311650
 13   FORMAT (1H05X55HPROGRAM DIMENSIONS EXCEEDED. KLVL MAXIMUM = 200. K00311700
     1LVL =I5)                                                          00311750
 14   CALL EXIT                                                         00311800
```

```
15    SAVE = 0.                                                          00311850
      WINS = 0.0                                                         00311860
      DO 16 I=1,KLVL                                                     00311900
      WINS = WINS + SS(I,2)                                              00311920
16    SAVE = SAVE + SS(I,3)                                              00311950
      ALPHA = ABSF(WINS/SAVE - 0.5)                                      00311960
      IF (SAVE-3000.) 20,20,17                                           00312000
17    WRITE OUTPUT TAPE 6,18, SAVE                                       00312050
18    FORMAT (1H05X77HPROGRAM DIMENSIONS EXCEEDED. MAXIMUM NO. OF TESTS  00312100
     1= 3000. NUMBER CALLED OUT =F9.0)                                   00312150
      GO TO 14                                                           00312200
20    NUMBR = 0                                                          00312250
      DO 30 I=1,KLVL                                                     00312300
      KSS = SS(I,3)-SS(I,2)                                              00312350
      IF (KSS) 21,30,25                                                  00312400
21    WRITE OUTPUT TAPE 6,22, I,KSS                                      00312450
22    FORMAT (1H05X30HCHECK INPUT OF SS VECTOR AT I=I5,17H AND BELOW. KS00312500
     1S =I5)                                                             00312550
      GO TO 14                                                           00312600
25    DO 27 II=1,KSS                                                     00312650
      NUMBR = NUMBR + 1                                                  00312700
27    T(NUMBR) = SS(I,1)                                                 00312750
30    CONTINUE                                                           00312800
      JJ = NUMBR                                                         00312850
      JJP1 = JJ+1                                                        00312900
      DO 40 I=1,KLVL                                                     00312950
      KSS  = SS(I,2)                                                     00313000
      IF (KSS) 32,40,35                                                  00313050
32    WRITE OUTPUT TAPE 6,22, I,KSS                                      00313100
      GO TO 14                                                           00313150
35    DO 37 II=1,KSS                                                     00313200
      NUMBR = NUMBR + 1                                                  00313250
37    T(NUMBR) = SS(I,1)                                                 00313300
40    CONTINUE                                                           00313350
      N = NUMBR                                                          00313400
      KK = N-JJ                                                          00313450
C     NOW SORT LOWER AND UPPER PARTS OF VECTOR SEPARATELY                00313600
      IF (JJ-1) 140,140,110                                              00313700
110   CALL SORT (JJ,T,IND,BUFR)                                          00313800
```

```
      DO 120 I=1,JJ                                                     00313900
      II = IND(I)                                                       00314000
 120  BUFR(I) = T(II)                                                   00314100
      DO 130 I=1,JJ                                                     00314200
 130  T(I) = BUFR(I)                                                    00314300
 140  IF (KK) 180,180,150                                               00314400
 150  CALL SORT (KK,T(JJP1),IND,BUFR)                                   00314500
      DO 160 I=1,KK                                                     00314600
      II = IND(I) + JJ                                                  00314800
      IJJ = I+JJ                                                        00314900
 160  BUFR(IJJ) = T(II)                                                 00315000
      DO 170 I=JJP1,N                                                   00315100
 170  T(I) = BUFR(I)                                                    00315200
 180  IF (TEST) 185,185,181                                             00315240
 181  WRITE OUTPUT TAPE 6,182, JJ, (T(I),I=1,JJ)                        00315250
 182  FORMAT (1H05X,3HJJ=I4,5X10HZEROS ARE 3E20.8/(5E20.8))             00315260
      WRITE OUTPUT TAPE 6,183, JJP1, (T(I),I=JJP1,N)                    00315270
 183  FORMAT (1H03X,5HJJP1=I4,5X10HONES ARE  3E20.8/(5E20.8))           00315280
C                                                                       00315281
C        *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *  00315282
C     TWO-SUBSET INITIAL GUESSES FOR MU AND SIGMA ARE COMPUTED          00315283
C        *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *  00315284
C                                                                       00315285
 185  IF (MOS) 1851,518,188                                             00315290
 1851 KLVLM = KLVL-1                                                    00315300
      IF (KLVLM-1) 1852,1852,1855                                       00315310
 1852 IF (SS(1,2)/SS(1,3) - SS(2,2)/SS(2,3)) 1853,1862,1862             00315320
 1862 WRITE OUTPUT TAPE 6,1863, HOLL                                    00315321
 1863 FORMAT (1H15X8HON CASE 12A6/1H05X62HNO SUBSETS WERE FOUND WHICH SA00315322
     1TISFIED THE PROGRAMMED CRITERIA./1H05X36HRESUBMIT CASE USING INITI00315323
     2AL GUESSES.)                                                      00315324
      GO TO 5                                                           00315325
 1853 ND = 1                                                            00315330
      NU = 2                                                            00315340
      R1 = SS(1,2)/SS(1,3)                                              00315350
      R2 = SS(2,2)/SS(2,3)                                              00315360
      SUM(3) = SS(1,3)                                                  00315365
      GO TO 1870                                                        00315370
 1855 R1 = SS(1,2)/SS(1,3)                                              00315380
```

```
      R2 = (WINS-SS(1,2))/(SAVE-SS(1,3))                                00315390
      SUM(1) = SS(1,2)                                                  00315400
      SUM(2) = SS(1,3)                                                  00315410
      SUM(3) = SUM(2)                                                   00315412
      FUN = ABSF(R1*(1.0-R1)*SUM(2) - R2*(1.0-R2)*(SAVE-SUM(2)))        00315420
      J  = 1                                                            00315430
      SUM(1) = SUM(1) + SS(I,2)                                         00315450
      SUM(2) = SUM(2) + SS(I,3)                                         00315455
      TR1= SUM(1)/SUM(2)                                                00315460
      TR2 = (WINS-SUM(1))/(SAVE-SUM(2))                                 00315465
      FUNNY = ABSF(TR1*(1.-TR1)*SUM(2) - TR2*(1.-TR2)*(SAVE-SUM(2)))    00315470
      IF (FUN-FUNNY) 1858,1858,1857                                     00315475
 1857 FUN = FUNNY                                                       00315480
      J  = I                                                            00315490
      R1 = TR1                                                          00315500
      R2 = TR2                                                          00315510
      SUM(3) = SUM(2)                                                   00315512
 1858 CONTINUE                                                          00315520
 1864 ND = J                                                            00315530
      NU = J+1                                                          00315540
 1870 EL1 = 0.0                                                         00315550
      EL2 = 0.0                                                         00315560
      DO 1871 I=1,ND                                                    00315570
 1871 EL1 = EL1 + SS(I,1)*SS(I,3)                                       00315580
      EL1 = EL1/SUM(3)                                                  00315590
      DO 1872 I=NU,KLVL                                                 00315595
 1872 EL2 = EL2 + SS(I,1)*SS(I,3)                                       00315600
      EL2 = EL2/(SAVE-SUM(3))                                           00315610
      CALL INVNRM (.0005,R1,T1)                                         00315650
      CALL INVNRM (.0005,R2,T2)                                         00315660
      SIG1 = (EL2-EL1)/(T2-T1)                                          00315670
      EMU1 = EL1 - T1*SIG1                                              00315671
      IF (KLVL-2) 1865,1865,1866                                        00315674
 1865 EMMY = EMU1                                                       00315675
      SIGGY = SIG1                                                      00315676
      EMLO = 0.9*EMMY                                                   00315677
      EMHI = 1.1*EMMY                                                   00315678
      SIGMIN = 0.9*SIGGY                                                00315679
```

```
      SIGMAX = 1.1*SIGGY
      GO TO 5102
 1866 CONTINUE
      IF (TEST) 1875,1875,1873
 1873 WRITE OUTPUT TAPE 6,1874, EMU1,SIG1,WINS,SAVE,EL1,EL2,T1,T2,R1,R2
 1874 FORMAT (31H1ESTIMATES FROM 2-SUBSET METHOD/1H05X5HEMU1=E13.6,5X5HS
     1IG1=E13.6,5X5HWINS=F5.0,5X5HSAVE=F5.0/1H 5X4HEL1=E13.6,5X4HEL2=E13
     2.6,5X3HT1=F6.3,5X3HT2=F6.3/1H 5X3HR1=E13.6,5X3HR2=E13.6)
 1875 CONTINUE
C        *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *
C     INITIAL GUESSES FOR MU, SIGMA COMPUTED FROM MINIMUM OVERLAPPING
C     SUBSET
C        *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *
 188  IF (JJ) 190,190,200
C     ALL RESPONSES ARE ONES.
 190  WRITE OUTPUT TAPE 6,195, T(1)
 195  FORMAT (1H-45HALL RESPONSES ARE SUCCESSES. LOWEST LEVEL IS E17.8)
      GO TO 5
C     ALL RESPONSES ARE ZEROS.
 200  IF (KK) 210,210,280
 210  WRITE OUTPUT TAPE 6,215, T(N)
 215  FORMAT (1H-45HALL RESPONSES ARE FAILURES. HIGHEST LEVEL IS E17.8)
      GO TO 5
 280  AU = (T(JJ)+T(JJP1))/2.
C     IS THERE AN OVERLAP.
      IF (T(JJP1)-T(JJ)) 300,281,290
C     NO OVERLAP, AND SCORES TIED.
 281  DO 282 I=JJP1,N
      IF (T(I)-T(JJP1)) 282,282,283
 282  KUP = I
C       (A NORMAL LOOP EXIT INDICATES ALL VALUES THE SAME IN UPPER END)
      DIFUP = 0.
      GO TO 284
C       (IF-EXIT FROM LOOP OCCURS WHEN A DIFFERENCE IS NOTED)
 283  DIFUP = T(I)-T(JJP1)
 284  DO 285 I=1,JJ
      JJBK = JJP1-I
      IF (T(JJBK)-T(JJ)) 286,285,285
 285  KDN = JJBK
      DIFDN = 0.
```

```
      GO TO 287                                                         00317750
  286 DIFDN = T(JJ)-T(JJBK)                                             00317755
  287 CONTINUE                                                          00317760
      ENS = KUP-JJ                                                      00317765
      ENT = KUP-KDN + 1                                                 00317770
      PP = ENS/ENT                                                      00317775
      ELNL = ENS*LOGEF(ENS) + (ENT-ENS)*LOGEF(ENT-ENS) - ENT*LOGEF(ENT) 00317780
      WRITE OUTPUT TAPE6,288,(T(JJ)),PP,ELNL                            00317785
  288 FORMAT (1H-41HNO OVERLAP, AND A TIE AT STIMULUS LEVEL =E17.8,4H P 00317790
     1=E17.8,7H LN L =E17.8)                                            00317791
      GO TO 5                                                           00317795
C     NO OVERLAP.                                                       00317800
  290 WRITE OUTPUT TAPE 6,291,AU                                        00317900
  291 FORMAT (1H-28HNO OVERLAP. ESTIMATE OF MU =E17.8)                  00318000
      GO TO 5                                                           00318100
C     FIND THE MINIMUM OVERLAPPING SUBSET                               00318200
  300 AUN = 2.0                                                         00318300
      JREF = JJ                                                         00318320
      KREF = JJP1                                                       00318330
      IF (N-2) 450,450,305                                              00318340
  305 IF (JJ-1) 310,310,315                                             00318350
  310 DOWN = 0.0                                                        00318360
      NDOWN = 0                                                         00318370
      GO TO 320                                                         00318380
  315 DOWN = 1.0                                                        00318390
      NDOWN = 1                                                         00318400
  320 IF (JJP1-N) 330,325,325                                           00318500
  325 UP = 0.0                                                          00318600
      NUP = 0                                                           00318700
      GO TO 335                                                         00318800
  330 UP = 1.0                                                          00318900
      NUP = 1                                                           00319000
  335 IF (DOWN) 336,336,340                                             00319100
  336 IF (UP) 450,450,365                                               00319200
  340 IF (XMAXOF(NUP,NDOWN)) 450,450,345                                00319300
  345 IF (T(JREF-1)-AU) 360,350,350                                     00319400
  350 NDOWN = 1                                                         00319500
      JREF = JREF-1                                                     00319600
      AU = (AU*AUN + T(JREF))/(AUN + 1.0)                               00319700
```

```
      AUN = AUN + 1.0                                                   00319800
      IF (JREF-1) 355,355,365                                           00319900
  355 DOWN = 0.0                                                        00320000
      NDOWN = 0                                                         00320100
      GO TO 365                                                         00320200
  360 NDOWN = 0                                                         00320300
      IF (NUP) 450,450,365                                              00320400
  365 IF (UP) 335,335,370                                               00320500
  370 IF (XMAXOF(NUP,NDOWN)) 450,450,375                                00320600
  375 IF (AU - T(KREF+1)) 390,380,380                                   00320700
  380 NUP = 1                                                           00320800
      KREF = KREF + 1                                                   00320900
      AU = (AU*AUN + T(KREF))/(AUN+1.0)                                 00321000
      AUN = AUN + 1.0                                                   00321100
      IF (KREF-N) 335,385,385                                           00321200
  385 UP = 0.0                                                          00321300
      NUP = 0                                                           00321400
      GO TO 335                                                         00321500
  390 NUP = 0                                                           00321600
      IF (NDOWN) 450,450,335                                            00321700
C     ALL POSSIBLE CANDIDATES HAVE BEEN EXHAUSTED.                      00322300
  450 SAVE = 0.                                                         00322400
      IF (TEST) 459,459,451                                             00322420
  451 WRITE OUTPUT TAPE 6,452, JREF,KREF, (T(I),I=JREF,KREF)            00322430
  452 FORMAT (1H05X53HINDICES AND MEMBERS OF THE MINIMUM OVERLAPPING SUB00322440
     1SET/1H05X5HJREF=I4,5X5HKREF=I4,5X4HMOS=3E20.8/(5E20.8))           00322450
  459 CONTINUE                                                          00322460
      DO 460 I=JREF,KREF                                                00322500
  460 SAVE = SAVE + (T(I)-AU)**2                                        00322600
      SAVE = SQRTF(SAVE/AUN)                                            00322700
      V = AUN/FLOATF(N)                                                 00322800
      IF (V-0.25) 470,470,480                                           00322900
  470 SIGGY = SAVE * 3.31779348 * SQRTF(V)                              00323000
      GO TO 510                                                         00323100
  480 EO = MAX1F (0.5-V,.04)                                            00323200
  500 SIGGY = SAVE * 0.82944837/SQRTF(EO)                               00323415
  510 EMMY = AU                                                         00323420
      F1 = FLOATF(KREF-JJ)/FLOATF(KREF-JREF+1)                          00323425
      KREFP1 = KREF+1                                                   00323430
```

```
      CNTUP  = 0.                                                       00323435
      COUNT  = 0.0                                                      00323442
      IF (KREFP1-N) 5111,5111,5115                                      00323445
 5111 FCSIG  = AU + 0.67449 * SIGGY                                     00323450
      DO 5113 II=KREFP1,N                                               00323455
      IF (T(II)-FCSIG) 5112,5112,5115                                   00323460
 5112 CNTUP  = CNTUP + 1.0                                              00323465
 5113 COUNT  = COUNT + 1.0                                              00323470
 5115 FCSIG  = AU - 0.67449 * SIGGY                                     00323475
      JREFM1 = JREF-1                                                   00323480
      IF (JREFM1) 5120,5120,5117                                        00323485
 5117 DO 5118 II=1,JREFM1                                               00323490
      III = JREF - II                                                   00323495
      IF (FCSIG-T(III)) 5118,5118,5120                                  00323500
 5118 COUNT  = COUNT + 1.0                                              00323505
 5120 IF (COUNT) 5121,5121,5122                                         00323510
 5121 F2 = 0.5                                                          00323515
      GO TO 5123                                                        00323520
 5122 F2 = CNTUP/COUNT                                                  00323525
 5123 EMMY  = AU -((F1+F2-1.0)/0.8)*SIGGY                               00323530
      EMOS  = EMMY                                                      00323531
      SMOS  = SIGGY                                                     00323532
      IF (MOS) 5201,5202,5202                                           00323533
 5201 EMMY  = ALPHA*EMU1 + (1.0-ALPHA)*EMMY                             00323534
      SIGGY  = ALPHA*SIG1 + (1.0-ALPHA)*SIGGY                           00323535
      GO TO 5203                                                        00323536
 5202 EMU1  = EMMY                                                      00323537
      SIG1  = SIGGY                                                     00323538
 5203 CONTINUE                                                          00323539
      IF (V-.25) 5205,5206,5206                                         00323540
 5205 SIGO  = (20.*SAVE)*SQRTF(3.1415927/AUN)                           00323541
      GO TO 5207                                                        00323542
 5206 SIGO  = SIGGY*SQRTF(3.1415927/AUN)/.05                            00323543
 5207 EMLO  = EMMY - SIGO                                               00323544
      EMHI  = EMMY + SIGO                                               00323545
      IF (V-.058) 5125,5125,5124                                        00323546
 5124 ZEBRA  = MAX1F(0.5-V,.04)                                         00323550
      GO TO 5126                                                        00323555
 5125 ZEBRA  = 7.68*V                                                   00323560
```

```
 5126 RANGER  = 1.0 - ABSF(F1+F2-1.0)                                   00323565
      SGX2RT  = 2.0*SIGGY * SQRTF(1.80/(FLOATF(N)*ZEBRA*RANGER))        00323570
     1  * 14.0                                                          00323571
      SIGMAX  = SIGGY + SGX2RT                                          00323575
      SIGMIN  = 0.1*SIGGY                                               00323600
      IF (MOS) 5101,5102,5102                                           00323655
 5101 EMLO  = MIN1F(EMU1,EMOS,EMLO)                                     00323660
      EMHI  = MAX1F(EMU1,EMOS,EMHI)                                     00323665
      SIGMIN  = MIN1F(SIG1,SMOS,SIGMIN)                                 00323670
      SIGMAX  = MAX1F(SIG1,SMOS,SIGMAX)                                 00323680
 5102 CONTINUE                                                          00323709
      EMZERO  = EMMY                                                    00323710
      SGZERO  = SIGGY                                                   00323711
      IF (IEST) 518,518,512                                             00323720
 512  WRITE OUTPUT TAPE 6,513, EMMY,SIGGY                               00323730
 513  FORMAT (1HO5X28HINITIAL GUESSES ARE    EMUL =E20.8,5X6HSIGL =E20.8)00323740
 518  CONTINUE                                                          00323750
C     THE VARIABLES  EMMY AND SIGGY NOW CONTAIN FIRST GUESSES AT        00323800
C       EMUL AND SIGL.                                                  00323900
      NTRIPS = 0                                                        00324000
 520  IF (NTRIPS - ITMAX) 525,561,561                                   00324100
 525  DO 530 I=1,5                                                      00324200
 530  SUM(I) = 0.                                                       00324300
      DO 540 I=1,JJ                                                     00324400
      IF (I-1) 534,534,532                                              00324420
 532  IF (T(I)-T(I-1)) 534,536,534                                      00324430
 534  TT = (T(I)-EMMY)/SIGGY                                            00324500
      CALL NORM (TT,Z,P)                                                00324600
      Q = 1.0-P                                                         00324700
      ZQ = Z/Q                                                          00324800
      TZQ = TT*ZQ                                                       00324900
      ZQZQ = ZQ*ZQ                                                      00325000
      TTZQ = TT*TZQ                                                     00325100
      TZQZQ = TT*ZQZQ                                                   00325200
 536  SUM(1) = SUM(1) + ZQ                                              00325300
      SUM(2) = SUM(2) + TZQ                                             00325400
      SUM(3) = SUM(3) + TZQ - ZQZQ                                      00325500
      SUM(4) = SUM(4) + TTZQ - TZQZQ - ZQ                               00325600
 540  SUM(5) = SUM(5) + TT*TTZQ - 2.0*TZQ - TT*TZQZQ                    00325700
```

```
      DO 550 I=JJP1,N                                                   00325800
      IF (I-JJP1) 544,544,542                                           00325820
542   IF (T(I)-T(I-1)) 544,546,544                                      00325830
544   TT = (T(I)-EMMY)/SIGGY                                            00325900
      CALL NORM (TT,Z,D)                                                00326000
      ZD = Z/D                                                          00326100
      TZD = TT*ZD                                                       00326200
      ZDZD = ZD*ZD                                                      00326300
      TTZD = TT*TZD                                                     00326400
      TZDZD = TT*ZDZD                                                   00326500
546   SUM(1) = SUM(1) - ZD                                              00326600
      SUM(2) = SUM(2) - TZD                                             00326700
      SUM(3) = SUM(3) - TZD - ZDZD                                      00326800
      SUM(4) = SUM(4) - TTZD - TZDZD + ZD                               00326900
550   SUM(5) = SUM(5) - TT*TTZD + 2.0*TZD - TT*TZDZD                    00327000
      SUM(1) = SUM(1)/SIGGY                                             00327100
      SUM(2) = SUM(2)/SIGGY                                             00327200
      SIGGY2 = SIGGY*SIGGY                                              00327300
      DO 555 I=3,5                                                      00327400
555   SUM(I) = SUM(I)/SIGGY2                                            00327500
      DET = SUM(3)*SUM(5) - SUM(4)**2                                   00327600
      DELMU = -SUM(1)*SUM(5)/DET + SUM(2)*SUM(4)/DET                    00327700
      DELSIG = SUM(1)*SUM(4)/DET - SUM(2)*SUM(3)/DET                    00327800
      EMMY = EMMY + DELMU                                               00327900
      IF (EMMY-EMHI) 5552,5551,5551                                     00327920
5551  EMMY = EMHI                                                       00327925
      FLAGMU = 1.0                                                      00327930
      GO TO 5558                                                        00327935
5552  IF (EMMY-EMLO) 5553,5553,5554                                     00327940
5553  EMMY = EMLO                                                       00327945
      FLAGMU = 1.0                                                      00327950
      GO TO 5558                                                        00327955
5554  SIGGY = SIGGY + DELSIG                                            00328000
      IF (SIGGY-SIGMIN) 5555,5555,5556                                  00328020
5555  SIGGY = SIGMIN                                                    00328025
      FLAGSG = 1.0                                                      00328030
      GO TO 5558                                                        00328035
5556  IF (SIGGY-SIGMAX) 5558,5558,5557                                  00328040
5557  SIGGY = SIGMAX                                                    00328045
```

```
      FLAGSG = 1.0                                                      00328050
 5558 CONTINUE                                                          00328060
      NTRIPS = NTRIPS + 1                                               00328100
      IF (TEST) 559,559,556                                             00328120
 556  WRITE OUTPUT TAPE 6,557, NTRIPS, SUM,DET,DELMU,DELSIG,EMMY,SIGGY  00328130
 557  FORMAT (1H05X50HVALUES OF INTERMEDIATE QUANTITIES AFTER ITERATION 00328140
     111/1H 5X7HSUM(I)=5E17.6/1H 5X4HDET=E13.6,3X6HDELMU=E13.6,3X7HDELS100328150
     2G=E13.6,3X5HEMMY=E13.6,3X6HSIGGY=E13.6)                           00328160
 559  CONTINUE                                                          00328170
      IF (ABSF(DELMU/EMMY) - 1.E-5) 560,560,520                         00328200
 560  IF (ABSF(DELSIG/SIGGY) - 1.E-5) 561,561,520                       00328300
 561  EMU = EMMY                                                        00328400
      SIG = SIGGY                                                       00328500
      IF (DET) 562,562,563                                              00328600
 562  MINMAX = 3                                                        00328650
      GO TO 570                                                         00328700
 563  IF (SUM(3)) 564,562,565                                           00328750
 564  MINMAX = 1                                                        00328800
      GO TO 570                                                         00328850
 565  MINMAX = 2                                                        00328900
 570  DO 580 I=3,5                                                      00329200
 580  SUM(I) = 0.0                                                      00329300
      ELNL = 0.0                                                        00329350
      DO 590 I=1,N                                                      00329400
      IF (I-1) 584,584,582                                              00329420
 582  IF (T(I)-T(I-1)) 584,588,584                                      00329430
 584  TT = (T(I)-EMMY)/SIGGY                                            00329500
      CALL NORM (TT,Z,P)                                                00329600
      PQ = P*(1.0-P)                                                    00329700
      IF (PQ-.000032) 5842,5842,5843                                    00329800
 5842 ABTT = ABSF(TT)                                                   00329820
      ZZDPQ = Z/((1./ABTT)-(1./ABTT**3)+(3./ABTT**5))                   00329830
      GO TO 5845                                                        00329840
 5843 ZZDPQ = Z*Z/PQ                                                    00329850
 5845 IF (I-JJ) 585,585,586                                             00329860
 585  EL = LOGEF(MAXIF(1.0-P,1.E-08))                                   00329870
      GO TO 588                                                         00329880
 586  EL = LOGEF(MAXIF(P,1.E-08))                                       00329890
 588  ELNL = ELNL + EL                                                  00329900
```

```
      SUM(3) = SUM(3) + ZZDPQ                                           00329950
      SUM(4) = SUM(4) + TT*ZZDPQ                                        00330000
590   SUM(5) = SUM(5) + TT*TT*ZZDPQ                                     00330100
      SIGGY2 = SIGGY*SIGGY                                              00330200
      DO 600 I=3,5                                                      00330300
600   SUM(I) = SUM(I)/SIGGY2                                            00330400
      DET = SUM(3)*SUM(5) - SUM(4)**2                                   00330500
      E11 = SUM(5)/DET                                                  00330600
      E22 = SUM(3)/DET                                                  00330700
      E12 = -SUM(4)/DET                                                 00330800
      IF (TEST) 605,605,601                                             00330820
601   WRITE OUTPUT TAPE 6,602, DET,(SUM(I),I=3,5),E11,E12,E22           00330830
602   FORMAT (1H05X56HVALUES OF INTERMEDIATE QUANTITIES USED IN COMPUTIN00330840
     1G SIGL/1H 5X4HDET=E13.6,3X,7HSUM(I)=3E16.6/1H 5X4HE11=E13.6,3X4HE100330850
     22=E13.6,3X4HE22=E13.6)                                            00330860
605   CONTINUE                                                          00330870
      IF (E11) 606,606,620                                              00330920
606   WRITE OUTPUT TAPE 6,607, HOLL                                     00330922
607   FORMAT (1H115X12A6)                                               00330924
610   WRITE OUTPUT TAPE 6,615, E11,SUM(3),SUM(4),SUM(5),DET,J,N         00330930
615   FORMAT (38HESTIMATE OF VARIANCE IS NEGATIVE. E11=E18.8/1H05X7HSUM(00330940
     13)=E18.8,5X7HSUM(4)=E18.8,5X7HSUM(5)=E18.8/1H010X,4HDET=E18.8,5X2H00330950
     2J=I1,5X2HN=I4)                                                    00330960
620   CONTINUE                                                          00330980
      WRITE OUTPUT TAPE 6,621,HOLL,EMU,SIG                              00331000
621   FORMAT (1H115X12A6/1H-5X31HMAXIMUM LIKELIHOOD ESTIMATES --/1H025X600331010
     1HMEAN =E17.8,5X7HSIGMA =E17.8)                                    00331020
      GO TO (6221,6223,6225),MINMAX                                     00331030
6221  WRITE OUTPUT TAPE 6,6222                                          00331040
6222  FORMAT (1H010X53H(THE SOLUTION GIVEN ABOVE IS NEAR A RELATIVE MAXI00331050
     1MUM))                                                             00331060
      GO TO 6227                                                        00331070
6223  WRITE OUTPUT TAPE 6,6224                                          00331080
6224  FORMAT (1H010X53H(THE SOLUTION GIVEN ABOVE IS NEAR A RELATIVE MINI00331090
     1MUM))                                                             00331100
      GO TO 6227                                                        00331105
6225  WRITE OUTPUT TAPE 6,6226                                          00331110
6226  FORMAT (1H010X51H(THE SOLUTION GIVEN ABOVE IS NEAR A SINGULAR POIN00331115
     1T))                                                               00331120
```

```
6227 CONTINUE                                                          00331125
     WRITE OUTPUT TAPE 6,623, ELNL,E11,E12,E12,E22,NTRIPS,ITMAX        00331130
623  FORMAT (1H-5X43HNATURAL LOGARITHM OF MAXIMIZED LIKELIHOOD =F8.2/1H00331135
     1-28X44HESTIMATED COVARIANCE MATRIX OF THE ESTIMATES/1H028X2E20.8/100331136
     2H028X2E20.8/1H-/1H-20X19HNO. OF ITERATIONS =I3,9X17HMAXIMUM ALLOWE00331137
     3D =I3)                                                           00331138
     WRITE OUTPUT TAPE 6,625, EMZERO,SGZERO, EMLO,EMHI,SIGMIN,SIGMAX   00331150
625  FORMAT (1H-5X36HINITIAL VALUES, MINIMA AND MAXIMA --/1H023X4HMEAN400331200
     16X5HSIGMA/1H018XE14.6,36XE14.6/1H010X2E15.6,20X2E15.6)           00331300
     IF (FLAGMU) 635,635,630                                           00331400
630  WRITE OUTPUT TAPE 6,631                                           00331500
631  FORMAT (1H05X21HLIMIT OF MU WAS USED.)                            00331600
     GO TO 5                                                           00331700
635  IF (FLAGSG) 5,5,640                                               00331800
640  WRITE OUTPUT TAPE 6,645                                           00331900
645  FORMAT (1H05X24HLIMIT OF SIGMA WAS USED.)                         00332000
     GO TO 5                                                           00332100
     END                                                               00332200
```

CARC COUNT CC561

```
      SUBROUTINE SORT (NN,A,IND,INT)
      DIMENSION  A(10), IND(10), INT(2,10)
      N = NN
      IF (N) 300,300,10
C          SAVE SENSE LIGHT STATUS
 10   ISL = 0
      IF (SENSE LIGHT 1) 20,30
 20   ISL = 1
C          INITIALIZE SUBSCRIPT ARRAY,
C          FIND AMIN LESS OR EQUAL TO SMALLEST A ELEMENT
 30   JMOD = 1
      AMIN = 0.
      DO 40 I=1,N
      INT (1,I) = I
 40   AMIN = MIN1F(AMIN,A(I))
C
C          INITIALIZE FOR NEW PASS
 1000 IU = INT(JMOD,1)
      IL = INT(JMOD,N)
      IJ = 1
      IK = N
      U = A(IU)
      AL = A(IL)
      X = AMIN
      INDEX = 1
      K = 0
      L = 1
      M = 1
      SENSE LIGHT 1
      JMOD = 3-JMOD
      KMOD = 3-JMOD
C
C          BEGIN PASS LOOP
      DO 200 I=1,N
      IF (X-U) 50,50,70
 50   IF (X-AL) 60,60,110
 60   IF (U-AL) 120,120,130
```

```
 70    IF (X-AL) 130,130,80                                          00003750
C          STORE AT OPPOSITE END                                    00003760
 80    IF (M) 95,95,90                                               00003800
 90    M = -1                                                        00003900
       L = INDEX                                                     00004000
       INDEX = N-K                                                   00004100
       GO TO 100                                                     00004200
 95    M = 1                                                         00004300
       K = N-INDEX                                                   00004400
       INDEX = L                                                     00004500
 100   IF (U-AL) 110,130,130                                         00004600
C          ELEMENT OUT OF SORT                                      00004700
 110   IF (SENSE LIGHT 1) 120,120                                    00004800
C          PICK UP FROM TOP                                         00004900
 120   INT(JMOD,INDEX) = IU                                          00005000
       IJ = IJ + 1                                                   00005100
       IU = INT(KMOD,IJ)                                             00005200
       X = U                                                         00005300
       U = A(IU)                                                     00005400
       GO TO 200                                                     00005500
C          OUT OF SORT, PICK UP FROM BOTTOM                         00005600
 130   IF (SENSE LIGHT 1) 140,140                                    00005700
 140   INT (JMOD,INDEX) = IL                                         00005800
       IK = IK-1                                                     00005900
       IL = INT(KMOD,IK)                                             00006000
       X = AL                                                        00006100
       AL = A(IL)                                                    00006200
 200   INDEX = INDEX+M                                               00006300
C                        *****                        *****          00006400
       IF (SENSE LIGHT 1) 250,1000                                   00006500
C          SORT COMPLETED                                           00006600
 250   DO 260 I=1,N                                                  00006700
 260   IND(I) = INT(JMOD,I)                                          00006800
       IF (ISL) 300,300,270                                          00006900
 270   SENSE LIGHT 1                                                 00007000
 300   RETURN                                                        00007100
       END                                                           00007200
                                                                     00007300
                                                                     00007400
```

```
      SUBROUTINE INVNRM (CRIT,PP,TRY)                                   00660100
      TRY = (2.*PP-1.)*1.2533141                                        00660200
      TRY = TRY + TRY**3/6.0                                            00660300
5     CALL NORM (TRY,Z,P)                                               00660400
      TRY = TRY - (P-PP)/Z                                              00660500
      IF (ABSF(P-PP)-CRIT) 10,10,5                                      00660600
10    RETURN                                                            00660700
      END                                                               00660800
```

```
      SUBROUTINE NORM (TEA,Z,P)                                         00000010
      T = ABSF(TEA)                                                     00000012
      IF (T- 9.) 1,1,101                                                00000013
101   Z = 1.0E-06                                                       00000014
      GO TO 11                                                          00000015
1     Z = EXPF(-T/2.*T)/2.50662827464                                   00000020
      IF (T-5.) 2,2,11                                                  00000021
11    IF (TEA) 12,12,13                                                 00000022
12    P = 1.0E-07                                                       00000023
      GO TO 6                                                           00000024
13    P = 0.9999999                                                     00000025
      GO TO 6                                                           00000026
2     ETA = 1./(T*.23164188827+1.)                                      00000030
3     P=Z*(((((-1.3302744296*ETA+1.8212559790)*ETA-1.7814779373)*ETA+.35600000040
     15637818)*ETA-.31938153C5)*ETA+1.                                  00000050
      IF (TEA) 4,6,6                                                    00000055
4     P = 1.0-P                                                         00000056
6     RETURN                                                            00000060
      END                                                               00000070
```

```
C       METHOD OF REVERSALS                                              00500000
C          *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *  00500100
        SUBROUTINE MOR (NSL,SL,SLSUC,SLTOT,P,KK,EPSI,CARMU)              00500200
C          *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *  00500300
C       DEFINITION OF VARIABLES                                          00500400
C         NSL      = NUMBER OF STRESS LEVELS                             00500500
C         SL       = VECTOR OF STRESS VALUES                             00500600
C         SLSUC    = NUMBER OF SUCCESSFUL TESTS AT EACH STRESS LEVEL     00500700
C         SLTOT    = TOTAL NUMBER OF TESTS AT EACH STRESS LEVEL          00500800
C         P        = PROBABILITY ASSOCIATED WITH EACH STRESS LEVEL       00500900
C         KK       = INDEX OF THE CURRENT TEST LEVEL WITHIN THE SL VECTOR 00501000
C                    (COMPUTED BY THE SUBR.)                             00501100
C         EPSI     = EPSILON VALUE USED IN CONJUNCTION WITH CARMU, SL TO 00501200
C                    FIND KK.                                            00501300
C         CARMU    = THE CURRENT STRESS LEVEL                            00501400
C          *   *   *   *   *   *   *   *   *   *   *   *   *   *   *   *  00501500
        DIMENSION SL(1024),SLSUC(1024),SLTOT(1024),P(1024)               00501600
        IF (NSL-1) 1,1,2                                                 00501700
 1      KK = 1                                                           00501800
        P(1) = SLSUC(1)/SLTOT(1)                                         00501900
        RETURN                                                           00502000
 2      KK = 0                                                           00502100
        NSLM1 = NSL-1                                                    00502200
        DO 10 I=1,NSL                                                    00502300
        IF (ABSF(CARMU-SL(I))-EPSI) 5,5,10                               00502400
 5      KK = I                                                           00502500
 10     P(I) = SLSUC(I)/SLTOT(I)                                         00502600
 11     NREV = 0                                                         00502700
        DO 20 I=2,NSL                                                    00502800
        IF (P(I)-P(I-1)) 12,20,20                                        00502900
C       A REVERSAL HAS BEEN FOUND                                        00503000
 12     I1 = I-1                                                         00503100
        NREV = NREV+1                                                    00503200
C       HOW FAR DOES THE REVERSAL EXTEND.                                00503300
        DO 13 J=I1,NSLM1                                                 00503400
        IF (P(J+1)-P(J)) 13,13,14                                        00503500
 13     I2 = J+1                                                         00503600
```

```
C       NORMAL EXIT MEANS THAT THE CURRENT REVERSAL EXTENDS TO THE END OF 00503700
C         THE VECTOR. IN EITHER EVENT, I1 STORES INDEX OF BEGINNING OF    00503800
C         THE REVERSAL, I2 STORE INDEX OF ITS END.                        00503900
 14     UP = C.                                                           00504000
        DOWN = C.                                                         00504100
C       THE P VALUES ARE RECOMPUTED OVER THE REGION OF THE REVERSAL.      00504200
        DO 15 J=I1,I2                                                     00504300
        UP = UP + SLSUC(J)                                                00504400
 15     DOWN = DOWN + SLTOT(J)                                            00504500
        PP = UP/DOWN                                                      00504600
        DO 16 J=I1,I2                                                     00504700
 16     P(J) = PP                                                         00504800
 20     CONTINUE                                                          00504900
C       THE SUBROUTINE RETURNS WITH THE COMPUTED P-VECTOR UPON COMPLETING 00505000
C         ONE PASS WITH NO REVERSALS.                                     00505100
        IF (NREV) 25,25,11                                                00505200
 25     RETURN                                                            00505300
        END                                                               00505400
```

CARC COUNT 00066

EXAMPLE I

MAXIMUM LIKELIHOOD ESTIMATES --

MEAN = 0.24999999E 01 SIGMA = 0.85640163E 00

(THE SOLUTION GIVEN ABOVE IS NEAR A RELATIVE MAXIMUM)

NATURAL LOGARITHM OF MAXIMIZED LIKELIHOOD = -3.94

ESTIMATED COVARIANCE MATRIX OF THE ESTIMATES

0.19519781E-00	-0.77882460E-08
-0.77882460E-08	0.31420967E-00

NO. OF ITERATIONS = 4 MAXIMUM ALLOWED = 15

INITIAL VALUES, MINIMA AND MAXIMA --

MEAN		SIGMA	
0.250000E 01		0.829448E 00	
-0.182912E 02	0.232912E 02	0.100000E-19	0.228622E 02

EXAMPLE II

MAXIMUM LIKELIHOOD ESTIMATES --

MEAN = 0.24316110E 04 SIGMA = 0.14907530E 02

(THE SOLUTION GIVEN ABOVE IS NEAR A RELATIVE MAXIMUM)

NATURAL LOGARITHM OF MAXIMIZED LIKELIHOOD = -2.40

ESTIMATED COVARIANCE MATRIX OF THE ESTIMATES

0.11220745E 03	0.46580564E 02
0.46580564E 02	0.14389422E 03

NO. OF ITERATIONS = 5 MAXIMUM ALLOWED = 15

INITIAL VALUES, MINIMA AND MAXIMA --

MEAN			SIGMA		
	0.242861E 04			0.123168E 02	
0.211988E 04		0.273735E 04	0.100000E-19		0.666661E 03

EXAMPLE III

MAXIMUM LIKELIHOOD ESTIMATES --

MEAN = -0.10537833E-00 SIGMA = 0.74843461E 00

(THE SOLUTION GIVEN ABOVE IS NEAR A RELATIVE MAXIMUM)

NATURAL LOGARITHM OF MAXIMIZED LIKELIHOOD = -4.36

ESTIMATED COVARIANCE MATRIX OF THE ESTIMATES

0.30569106E-00	0.18411386E-00
0.18411386E-00	0.19069899E-00

NO. OF ITERATIONS = 7 MAXIMUM ALLOWED = 15

INITIAL VALUES, MINIMA AND MAXIMA --

MEAN		SIGMA	
-0.484764E-00		0.479975E-00	
-C.130179E 02	0.120484E 02	0.100000E-19	0.685480E 01

APPENDIX

V-B. MATHEMATICAL DETAILS

V-B1. GROWTH OF INFORMATION FOR THE 50% INVERSE RESPONSE PROBLEM WHEN THE RESPONSE FUNCTION IS CUMULATIVE NORMAL

a. Introduction

To discuss the growth of information in sensitivity experiments, we require a characterization of the amount of information available before an entire group of tests is conducted (from previous studies, etc.). This prior information is expressed as that number of equivalent asymptotically optimal tests which would provide the same asymptotic information. Our approach is based on the use of asymptotic expressions to characterize this growth of information. Attention is limited to the case in which the response function is a normal cdf, and to simplify the calculations we assume that the sole aim of the tests is to estimate the median critical stress level (i.e., $\alpha = 50\%$). Our analysis is carried out without actually specifying the test levels to be used in each of the blocks, although it is known (see Ref. 4) that for this type of experiment any test sequence converging to the median is asymptotically optimal in terms of efficiency in estimating the median.

b. Efficiency and Growth of Information

Suppose we have a cumulative normal response function with (unknown) parameters μ and σ. Let $\hat{\mu}$ denote the maximum-likelihood estimate of μ. Consider a design with T tests whose goal is the estimation of μ. An asymptotic expression for the variance of $\hat{\mu}$ (as $T \to \infty$) is given by

$$\sigma^2_{\hat{\mu}}(T) \sim C_2\sigma^2/(C_0C_2-C_1^2) \tag{1}$$

where L_i = the level of the stimulus variable on the ith test,

$$
\begin{aligned}
t_i &= (L_i-\mu)/\sigma \quad , \\
Z_i &= \frac{1}{\sqrt{2\pi}} e^{-t_i^2/2} \quad , \\
p_i &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t_i} e^{-u^2/2}\, du \quad , \\
q_i &= 1-p_i \quad , \\
V_i &= Z_i^2/p_i q_i \quad , \\
C_j &= \sum_{i=1}^{T} V_i t_i^j
\end{aligned}
\tag{2}
$$

Since the goal of the experiment is the estimation of μ, it is not unreasonable to restrict attention to designs which are asymptotically symmetric with respect to μ. Then

$$C_1 \sim 0 \quad ,$$

$$\sigma^2_{\hat{\mu}}(T) \sim \sigma^2/C_0 \quad .$$

It has been shown (Ref. 4) that $\sigma^2_{\hat{\mu}}(T)$ is asymptotically minimized when $t_i = 0$, $i = 1, \cdots, T$; this minimum value is

$$\sigma^2_{\hat{\mu}}(T) \sim (\pi/2)\sigma^2/T \quad .$$

The asymptotic information after T tests, I_T, may be expressed by the reciprocal of the variance of $\hat{\mu}$, or

$$I_T \sim C_0/\sigma^2 = \sum_{i=1}^{T} Z_i^2/p_i q_i \sigma^2 \quad . \tag{3}$$

Thus the information contribution of a test at $t_i = t$ is given by

$$I(t) \sim e^{-t^2}/2\pi pq\sigma^2 \quad . \tag{4}$$

Since this is maximized at $t = 0$, where we have

$$I(0) \sim 2/\pi\sigma^2 \quad , \tag{5}$$

the efficiency, defined as the relative information of an individual test at stimulus level t, may be written as

$$E(t) \equiv I(t)/I(0) \sim e^{-t^2}/4pq \quad . \tag{6}$$

The function $E(t)$ is tabulated below for selected values:

$\lvert t \rvert$	0	.1	.2	.5	.75	1.0	1.5	2.0	3.0	4.0	5.0
$E(t)$	1	.9964	.9856	.9127	.8128	.6888	.4226	.2060	.0229	.00089	.000012

It may be noticed that the efficiency declines rapidly in the range $.75 < |t| < 2$. Tests for which $|t| > 3$ are very inefficient in the long run, although they may provide a large fractional increase in information early in the experiment.

In order to derive an expected value for $E(t)$, we express it in a more explicit form. It can be shown that the following expansions are convergent for all values of t:

$$p = 1/2 + t/\sqrt{2\pi} - t^3/6\sqrt{2\pi} + t^5/40\sqrt{2\pi}$$
$$- t^7/336\sqrt{2\pi} + t^9/3456\sqrt{2\pi} - t^{11}/42240\sqrt{2\pi} + t^{13}/599040\sqrt{2\pi} - \cdots , \qquad (7)$$

$$q = 1/2 - t/\sqrt{2\pi} + t^3/6\sqrt{2\pi} - t^5/40\sqrt{2\pi}$$
$$+ t^7/336\sqrt{2\pi} - t^9/3456\sqrt{2\pi} + t^{11}/42240\sqrt{2\pi} - t^{13}/599040\sqrt{2\pi} + \cdots . \qquad (8)$$

Therefore

$$pq = 1/4 - t^2/2\pi + t^4/6\pi - 7t^6/180\pi + t^8/140\pi$$
$$- 83t^{10}/75600\pi + 73t^{12}/498960\pi - 523t^{14}/30270240\pi + \cdots , \qquad (9)$$

and

$$E(t) \sim e^{-t^2}/(1-2t^2/\pi+2t^4/3\pi-7t^6/45\pi+t^8/35\pi$$
$$- 83t^{10}/18900\pi+73t^{12}/124740\pi-523t^{14}/7567560\pi + \cdots) . \qquad (10)$$

We have finally

$$e^{t^2}E(t) \sim 1 + 2t^2/\pi - 2t^4/3\pi + 4t^4/\pi^2 + 7t^6/45\pi - 8t^6/3\pi^2$$
$$+ 8t^6/\pi^3 - t^8/35\pi + 16t^8/15\pi^2 - 8t^8/\pi^3 + 16t^8/\pi^4$$
$$+ 83t^{10}/18900\pi - 304t^{10}/945\pi^2 + 68t^{10}/15\pi^3 - 64t^{10}/3\pi^4$$
$$+ 32t^{10}/\pi^5 - 73t^{12}/124740\pi + 1132t^{12}/14175\pi^2$$
$$- 356t^{12}/189\pi^3 + 704t^{12}/45\pi^4 - 160t^{12}/3\pi^5 + 64t^{12}/\pi^6$$
$$+ 523t^{14}/7567560\pi - 296t^{14}/17325\pi^2 + 599t^{14}/945\pi^3$$
$$- 7808t^{14}/945\pi^4 + 48t^{14}/\pi^5 - 128t^{14}/\pi^6 + 128t^{14}/\pi^7 + \cdots . \qquad (11)$$

In general, since μ and σ are unknown, we are uncertain as to just which value of t we are testing at; let this uncertainty be represented by a density $f(t)$ with mean M and variance U. Since we are trying to test at $x = \mu$ $(t = 0)$, and since $\hat{\mu}$ is unbiased and asymptotically normal, we have

$$M = 0 \tag{12}$$

$$U \equiv \sigma_t^2 = \sigma_{\hat{\mu}}^2(T)/\sigma^2 \quad . \tag{13}$$

Then the expected test efficiency is given by

$$\overline{E(t)} \equiv \int_{-\infty}^{\infty} E(t)f(t)dt$$

$$\sim \int_{-\infty}^{\infty} \frac{E(t)}{\sqrt{2\pi U}} e^{-t^2/2U} dt$$

$$= \frac{1}{\sqrt{2U+1}} \int_{-\infty}^{\infty} \frac{e^{t^2} E(t)}{\sqrt{2\pi v}} e^{-t^2/2v} dt \quad , \tag{14}$$

using the substitution $v = U/(2U+1)$.

Because of (11) this integral can be thought of as a sum (with coefficients given by (11)) of the even central moments, M_{2n}, of a normal distribution with variance v. We have

$$\begin{aligned} M_0 &= 1 \quad , \\ M_{2n} &= (2n-1)vM_{2n-2} \quad , \end{aligned} \tag{15}$$

from which it follows that $M_2 = v$, $M_4 = 3v^2$, $M_6 = 15v^3$, $M_8 = 105v^4$, $M_{10} = 945v^5$, $M_{12} = 10395v^6$, and $M_{14} = 135135v^7$. Then from (14) and (11) we have

$$\overline{E(t)} \sim \Big[1 + \frac{2}{\pi} v + \Big(\frac{12}{\pi^2} - \frac{2}{\pi}\Big) v^2 + \Big(\frac{7}{3\pi} - \frac{40}{\pi^2} + \frac{120}{\pi^3}\Big) v^3 + \Big(\frac{1680}{\pi^4} - \frac{840}{\pi^3} + \frac{112}{\pi^2} - \frac{3}{\pi}\Big) v^4$$

$$+ \Big(\frac{83}{20\pi} - \frac{304}{\pi^2} + \frac{4284}{\pi^3} - \frac{20160}{\pi^4} + \frac{30240}{\pi^5}\Big) v^5$$

$$+ \Big(\frac{665280}{\pi^6} - \frac{554400}{\pi^5} + \frac{162624}{\pi^4} - \frac{19580}{\pi^3} + \frac{12452}{15\pi^2} - \frac{73}{12\pi}\Big) v^6$$

$$+ \Big(\frac{523}{56\pi} - \frac{11544}{5\pi^2} + \frac{85657}{\pi^3} - \frac{1116544}{\pi^4} + \frac{6486480}{\pi^5} - \frac{17297280}{\pi^6}$$

$$+ \frac{17297280}{\pi^7}\Big) v^7 + \cdots \Big] / \sqrt{2U+1}$$

$$\cong [1 + .636620v + .579234v^2 + .560060v^3 + .548604v^4$$

$$+ .539890v^5 + .534223v^6 + .530582v^7] / \sqrt{2U+1} \quad . \tag{16}$$

This function is tabulated below for selected values:

U	0.0	0.2	0.5	1.0	1.5	2.0	3.0	4.0	6.0	8.0	10.0	15.0	20.0	40.0	100.0
$\overline{E(t)}$	1.000	.934	.853	.75	.68	.63	.55	.49	.42	.37	.34	.28	.25	.18	.11

For small values of U, the approximation given by the first two terms of the expansion,

$$\overline{E(t)} \approx 1 - .3634U \quad , \tag{17}$$

is sufficiently accurate. For large values of U we have $v \cong .5$ and numerical evaluation of equation (16) leads directly to

$$\overline{E(t)} \approx 1.132/\sqrt{U} \tag{18}$$

At this point we have only an asymptotic formula (16) for computing the expected efficiency of a new test, given $\sigma^2_{\hat{\mu}}$. But denoting by E_0 the prior information in terms of equivalent efficient tests, and by E_i the efficiency of the ith test, we have from our definition of the information after T tests that

$$I_T \sim I_{T-1} + 2E_T/\pi\sigma^2 \quad . \tag{19}$$

Then one can show by an elementary induction that

$$I_T \sim \frac{2}{\pi\sigma^2} \sum_{i=0}^{T} E_i \quad , \tag{20}$$

or

$$\sigma^2_{\hat{\mu}}(T) \sim (\pi/2)\sigma^2 / \sum_{i=0}^{T} E_i \quad . \tag{21}$$

Equations (13), (16), and (21) can be used to asymptotically describe the growth of information in sensitivity experiments.

Of these equations, only (13) is exact. Equation (21) is asymptotically valid as $\Sigma_{i=0}^{T} E_i$ goes to infinity, which will happen if and only if E_0 and/or T become arbitrarily large. Equation (16) holds asymptotically on the $(j+1)^{st}$ test as $\Sigma_{i=0}^{j} E_i$ goes to infinity, in which case U goes to zero. But note that (16) and (21) do not give unreasonable results even for large values of U and for small values of E_0 and T. Thus we shall attempt to draw tentative conclusions even in the latter cases.

The above asymptotic theory has been tested by means of a computer program for simulating sensitivity experiments. The value of $\sigma^2_{\hat{\mu}}$ given by this theory is much more realistic than the value $(\pi/2)\sigma^2/T$, but still sometimes conservative by a factor of three.

In the following example we see how slowly the individual test efficiencies increase in the course of a purely sequential sensitivity experiment. Note that we never have to specify the individual test levels in this line of reasoning.

<u>Example</u>: If $E_0 = .10$, then from (21), $\sigma^2_{\hat{\mu}}(0) \sim 15.71\sigma^2$. From (13), $U \sim 15.71$, and from (16), $\bar{E}_1 \sim .275$. Continuing in this manner, we have $\sigma^2_{\hat{\mu}}(1) \sim 4.19\sigma^2$,

$$U \sim 4.19, \quad \bar{E}_2 \sim .486,$$
$$U \sim 1.82, \quad \bar{E}_3 \sim .647,$$
$$U \sim 1.042, \quad \bar{E}_4 \sim .748,$$
$$U \sim .696, \quad \bar{E}_5 \sim .811,$$
$$U \sim .512, \quad \bar{E}_6 \sim .851,$$
$$U \sim .401, \quad \bar{E}_7 \sim .878,$$
$$U \sim .328, \quad \bar{E}_8 \sim .899,$$
$$U \sim .276, \quad \bar{E}_9 \sim .911, \text{ etc.}, \quad \sum_{i=0}^{9} \bar{E}_i = 6.606 \quad .$$

It should be noted that the above machinery permits the characterization of experiments in which the stress variable has an independent "setting" error, such as the projectile velocity in projectile penetration tests.* Let this setting error be normally distributed with mean 0 and variance σ^2_s. Then the only change in the above formulae is in the expression for U, which is now

$$U = (\sigma^2_{\hat{\mu}} + \sigma^2_s)/\sigma^2 \quad .$$

<u>Example</u>: Let $\sigma^2_s = 2\sigma^2$. Then the asymptotic efficiency of even a purely sequential design in this case is only 63%, since $U \sim 2$ (see table below Eq. (16)).

*See Section III-A3.

c. Block-Sequential Designs (See Sections III-A1 and III-B8)

To "stage" a "block-sequential" design, in which each block of tests is planned after all previous test results have been analyzed, is to assign sample sizes to each of a given number of blocks, given the total sample size T. The "optimum" staging of a block-sequential sensitivity experiment is that staging which produces the greatest expected gain in information. The asymptotic methodology derived above may be used to compute optimum stagings for block-sequential sensitivity experiments as a function of the amount of prior information. These results are relatively independent of E_0, which is fortunate because this parameter is in practice very difficult to evaluate. For example, if our prior density on μ is uniform in $[A,B]$, then

$$\sigma^2_{\hat{\mu}}(0) = (B-A)^2/12$$

But to compute

$$E_0 = (\pi/2)\sigma^2/\sigma^2_{\hat{\mu}}(0) \quad ,$$

we must know σ^2, and such information is almost always unavailable.

Optimal stagings given by the above theory are not completely rigorous even for large values of E_0 and/or T, since we compute expected information in the second stage as a function of expected information in the first stage, rather than in terms of the distribution of this information. But the results are all plausible and of practical value precisely because they are similar for $E_0 = .02$ and $E_0 = 2.00$, for example, in two-block experiments. Results of simulations of some two-block designs are reported in the Simulation Section.

V-B2. MAXIMUM LIKELIHOOD ESTIMATES OF PARTIALLY ORDERED PARAMETERS

a. Introduction

In Section IV-B6 two algorithms are described by which monotone estimates of either partially or completely ordered parameters can be found. A proof that the first algorithm (method of reversals) produces maximum-likelihood estimates of completely ordered parameters can be found in Ref. 34. Conditions which determine MMLE's (monotone maximum-likelihood estimates) have been derived by C. van Eeden (Ref. 35) for partially ordered parameters, but specific techniques for obtaining them are not given. It will be shown in this appendix that if the method of reversals is applied to an ordering determined by the second algorithm in Section IV-B6, the estimates are, in fact, MMLE's.

b. Analysis

Let $X_1, X_2, \cdots, X_k$ be k independent binomially distributed random variables each specified by the single parameter Q_i $(i = 1, 2, \cdots, k)$, respectively (k represents the number of grid points in a multidimensional grid design). Further, to each Q_i we associate n real numbers $(x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)})$, which specify a level combination for n stimulus variables. We then consider a function $Q(x_1, x_2, \cdots, x_n)$ which takes on the value Q_i when $x_j = x_j^{(i)}$, $j = 1, 2, \cdots, n$ and which is monotone nondecreasing in each of the separate x_j $(j = 1, 2, \cdots, n)$.

Let O be the set of all orderings of $\{Q_i\}$ consistent with the monotonicity constraint on Q. If $\{\hat{Q}_i\}$ denotes the set of MMLE's, then it is clear that $\{\hat{Q}_i\}$ can be written in some nondecreasing order, O_M, such that $O_M \in O$.

Given an ordering, O_1, the MMLE's corresponding to that order are determined by the method of reversals. The application of this algorithm results in a "partitioning", $P(O_1)$, whose sections must satisfy the following conditions:

(a) If P_1, a section of $P(O_1)$, is written in any manner as two ordered sets S_1 S_2, then $t_1 \geq t_2$ (for definitions of these S's and t's see Section IV-B6).

(b) If the ordered sets S_1 S_2 belong to different sections, then $t_1 < t_2$.

(When these conditions are satisfied the MMLE's corresponding to the order O_1 are such that the estimates are constant within sections.)

Let A be the set of all orderings which satisfy the conditions of the algorithm given in Section IV-B6. Then $A \subset O$ and we shall show that for any A_1, such that $A_1 \in A$, $P(A_1) = P(O_M)$.

Theorem 1: There is an $A_1 \in A$ such that $P(A_1) = P(O_M)$, where as before O_M is a MMLE ordering.

Proof: If $O_M \in A$, then $A_1 = O_M$ and the theorem is proved. If this is not the case, then in O_M there are two adjacent independent subsets S_1 and S_2 such that the corresponding estimates t_1, t_2 satisfy $t_1 > t_2$. Let O^* be the ordering with S_1 and S_2 interchanged. By property (a) of the method of reversals, S_1 and S_2 must be in the same section and $P(O^*) = P(O_M)$. If $O^* \notin A$, we continue interchanging independent subsets until finally $P(O_M) = P(A_1)$.

Lemma: If $P(O^*)$ has ℓ sections which we denote by $P_1, P_2, \cdots, P_\ell$ and if a set of elements S_1 is deleted from the set, $S = (P_1, \cdots, P_i)$ where

$t_1 \geq t$ and $i < \ell$, then the partitioning of $P_{i+1}, \cdots, P_\ell$ remains unchanged. The proof follows immediately from the properties of the method of reversals.

Theorem 2: Let A_1 and A_2 be any two different orderings in A, then $P(A_1) = P(A_2)$.

Proof: Our proof will be by induction. If $k = 1, 2$, the theorem follows immediately. Assume it is true for $1, 2, \cdots, k-1$. We use an indirect proof to complete the induction. Suppose for k elements there are two different partitions, $P(A_1)$, $P(A_2)$ with sections given by

$$P(A_1): \quad P_{11}, P_{12}, \cdots, P_{1\ell}$$

$$P(A_2): \quad P_{21}, P_{22}, \cdots, P_{2m}$$

where without loss of generality we let $\ell \leq m$. The proof for all $\ell = 1, 2, 3, \cdots$ depends on using the lemma; we will give details only for $\ell > 2$.

Assume the sections P_{11} and P_{21} have a set S_1 of common elements so that

$$P_{11} = S_1 \cup S_1'$$

$$P_{21} = S_1 \cup S_2' \quad .$$

Then, since P_{11} has less than k elements, it must have a unique partitioning. Then it follows that $t_1' \leq t_1$ and similarly $t_2' \leq t_2$. Delete set S_1; by our lemma the remaining sections are unchanged and the partitioning must be unique unless S_1 is the null set. If S_1 is the null set, we consider the set of common elements in $P_{11} \cup P_{12}$ and $P_{21} \cup P_{22}$. In the same manner, either our theorem is proved or $P_{11} \cup P_{12}$ and $P_{21} \cup P_{22}$ have no elements in common.

Continuing in this manner, we show that the first ℓ-1 partitions have no common elements and thus $P_{1\ell}$ must contain ℓ-1 partitions, which is a contradiction. The treatment of $\ell = 1, 2$ is similar. This completes the proof.

V-B3. MINIMUM OVERLAPPING SUBSET

a. Introduction

The minimum overlapping subset (MOS) of a sample of sensitivity data is a subset whose points are all on the "wrong" side of the average stimulus level for the subset, while the points of its complement are all on the "right" side of this mean. That is, the MOS consists of all the responses below its mean and all of the non-responses above it. In this appendix we relate the mean and standard deviation of this subset to the mean and standard deviation of the critical level density in a special case, and use this result to motivate the use of the MOS in more general cases. We conclude by constructing the MOS and proving that the constructed set has the desired properties.

b. The MOS of a Special Infinite Sample

Theorem 1: Suppose we have a population of sensitivity data for which the test stimulus density and the critical stimulus density are both the same normal density $f(x)$ with mean μ and standard deviation σ. Let M denote the subpopulation of test levels at which the test results are "unexpected" (i.e., those stimulus levels above μ at which there are no responses, and those stimulus levels below μ at which there are responses). Then

1. One quarter of the parent population is in M,
2. $\mu_M = \mu$, and
3. $\sigma_M^2 = \sigma^2(1-2/\pi)$.

Proof: Let F denote the cumulative normal distribution, where $F'=f$. Then the probability is F that a point below μ in the parent population is also in M, and the probability is $1-F$ that a point above μ in the parent population is also in M. Hence the fraction of the parent population in M is

$$\int_{-\infty}^{\mu} f(x)F(x)dx + \int_{\mu}^{\infty} f(x)[1-F(x)]dx = 1/4.$$

Then if $f(x)$ denotes the density of M, we have

$$g(x) = 4f\begin{cases} F & , \text{if } x \le \mu \\ 1-F & , \text{if } x > \mu \end{cases}$$

If we use the transformation $y = 2\mu - x$, so that $f(y) = f(x)$, $F(x) = 1-F(y)$, we find

$$\begin{aligned} \mu_M &\equiv \int_{-\infty}^{\infty} xg(x)dx \\ &= 4\left[\int_{-\infty}^{\mu} xf(x)F(x)dx + \int_{\mu}^{-\infty} (-1)(2\mu-y)f(y)F(y)dy\right] \\ &= 4\int_{\mu}^{-\infty} (-2\mu fF) \\ &= \mu. \end{aligned}$$

Finally, taking advantage of the symmetry of g, and using $f' = -(x-\mu)f/\sigma^2$, we find

$$\begin{aligned} \sigma_M^2 &\equiv \int_{-\infty}^{\infty} (x-\mu_M)^2 g \\ &= 2\int_{-\infty}^{\mu} 4(x-\mu)(-\sigma^2)f'F \\ &= -8\sigma^2\left[(x-\mu)fF\Big|_{-\infty}^{\mu} - \int_{-\infty}^{\mu} f(F+(x-\mu)f)\right] \\ &= 8\sigma^2\int_{-\infty}^{\mu} (fF-\sigma^2 f'f) \\ &= 8\sigma^2\left[1/8 - \sigma^2 f^2/2\Big|_{-\infty}^{\mu}\right] \\ &= \sigma^2(1-2/\pi)\ .^{*} \end{aligned}$$

*The recursion formula for the even central moments of g is actually

$$M_{2n+2} = (2n+1)\sigma^2 M_{2n} - (2/\pi)n!\sigma^{2n+2}$$

Suppose we now have a finite sample of sensitivity data. Let $\hat{\mu}$ and $\hat{\sigma}$ denote the estimates of μ and σ. If T denotes the MOS of the sample, then let $\bar{X}_T$ and S_T denote the mean and standard deviation of the stimulus levels in T. Then Theorem 1 makes it reasonable to use the approximations

$$\hat{\sigma} \cong WS_T/\sqrt{1-2/\pi}$$

$$\hat{\mu} \cong \bar{X}_T + U\hat{\sigma} ,$$

where U and W are correction factors which should compensate for departure from the ideal test stimulus density; this is the form of the estimators given in Section IV-B2. As might be expected, $W = 1$ when v (the fraction of the parent sample in T) is 1/4; and $U = 0$ when the data in and near the MOS are evenly divided between responses and nonresponses.

If all of the test levels are near μ, then σ_T will be extremely small compared to σ, and v will be almost .5; if the test levels are scattered very widely about μ, then σ_T will be large compared to σ, and v will be almost 0. This is the reasoning which leads to the correction factor

$$W = \begin{cases} \sqrt{v/.25} & \text{, for } v \le .25 \\ \sqrt{.25/(.5-v)} & \text{, for } v > .25 . \end{cases}$$

c. <u>The Construction of the MOS</u>

We now provide a method for constructing T by first introducing a series of associated sets, T_r, as follows:

Let $x_1 \le x_2 \le x_3 \ldots$ denote the stimulus levels corresponding to each and every individual response, and let $y_1 \ge y_2 \ge y_3 \ldots$ denote the stimulus levels corresponding to each and every nonresponse. Let $T_0 = \emptyset$ (the null set), $\bar{x}_0 = (x_1+y_1)/2$, and $i_0 = j_0 = 0$.

We now define the sets T_r recursively. Suppose we have arrived at $T_r = (x_1, x_2, \ldots x_{i_r}, y_1, y_2, \ldots y_{j_r})$, $r \geq 0$, at some point in our process. Let $\bar{x}_r$ denote the mean of T_r. Then we define

$$T_{r+1} = T_r \cup Z_r ,$$

where

$$Z_r = \begin{cases} (x_{i_r+1}) & , \text{ if } x_{i_r+1} \leq \bar{x}_r \text{ but } y_{j_r+1} < \bar{x}_r \\ (y_{j_r+1}) & , \text{ if } y_{j_r+1} \geq \bar{x}_r \text{ but } x_{i_r+1} > \bar{x}_r \\ (x_{i_r+1}, y_{j_r+1}) & , \text{ if } x_{i_r+1} \leq \bar{x}_r \leq y_{j_r+1} \\ \emptyset & , \text{ if } y_{j_r+1} < \bar{x}_r < x_{i_r+1} \end{cases}$$

Finally, when $Z_r = \emptyset$, then $T = T_r$.

Theorem 2: The subset T constructed above is the unique MOS.

Proof: Since $T = T_r$ when $Z_r = \emptyset$, the complement of T consists only of points on the "right" side of $\bar{x}_r = \bar{x}_T$.

If T did not consist entirely of points on the "wrong" side of $\bar{x}_T$, then it would follow that, for some K,

$$x_{i_K+1} \leq \bar{x}_K$$

and yet

$$x_{i_K+1} > \bar{x}_r$$

(i.e., a response entered at one stage of the process but later no longer satisfied the desired entry condition). But $\bar{x}_r$ is a weighted average of $\bar{x}_K$ and points to the right of x_{i_K+1}, and so we have a contradiction.

Thus T is an MOS, and we must show only its uniqueness. But any other MOS must also have the form

$$U = (x_1, x_2, \ldots x_{i_s}, y_1, y_2, \ldots y_{j_s}) .$$

There are essentially four simple cases to be treated here.

For example, if $i_s > i_r$ and $j_s > j_r$, we have, from the construction of T, $y_{j_s} < x_{i_s}$, and we cannot have both of these two points of U on the "wrong" side of $\bar{x}_U$.

If $i_s > i_r$ and $j_s \leq j_r$, then

$$\bar{x}_U \geq x_{i_s} \geq x_{i_s-1} \geq \cdots \geq x_{i_r+1} > \bar{x}_T ,$$

$$\bar{x}_U > y_{j_s+1} \geq \cdots \geq y_{j_r-1} \geq y_{j_r} \geq \bar{x}_T$$

But if $y_{j_s+1}, \ldots y_{j_r}$ are deleted, the mean cannot be made larger than $\bar{x}_T$, and then adding $x_{i_r+1}, \ldots, x_{i_s}$ cannot possibly drive the mean as high as x_{i_s}. Therefore $\bar{x}_U < x_{i_s}$, a contradiction. The remaining cases are treated in a similar manner.

V-B4. APPROXIMATE NUMBER OF TESTS IN ALEXANDER DESIGN FOR SMALL α

In open-ended designs it is of interest to have some approximate idea of how many tests will be necessary. An approximate upper bound for the average number of tests required by the Alexander design for small α (when the step size chosen is not very small relative to the uncertanity in the initial guess of x_α) can be found as follows:

Assume that a design has been completed and the lowest response was obtained at the level L_1. Then there will be sequences of levels below L_1 at which only non-responses have occurred. From the rules of the Alexander design the last of these sequences will be that which ends two levels below L_1 and such that N nonresponses have been observed in its associated zero-region. Here N can be found from the expression

$$(1-\alpha)^{N_o} = 1-P_o \qquad N = \begin{cases} N_o & , \ N_o \text{ an integer} \\ [N_o] + 1 & , \ \text{otherwise} \end{cases}$$

where P_o is some prescribed probability. Thus the first sequence can involve tests at not more than N levels. In the second sequence, the nonresponses from the tests in the first sequence can be counted if they are in the appropriate zero region. Thus, the zero region corresponding to a level $N/2$ steps below L_1 will contain at least N nonresponses. From the stopping rules the design will terminate when a sequence ends two levels below L_1. Thus, proceeding in this manner we see that at levels below L_1 there can be a total of

$$N + \frac{N}{2} + \frac{N}{3} + \ldots + \frac{2N}{N}$$

tests in all. (In evaluating this expression, $\frac{N}{i}$ must be replaced by $\left[\frac{N}{i}\right] + 1$, if it is not an integer.

In approximating the number of tests at levels above L_1 we use an analogous procedure. In this case we consider how many levels a sequence can climb before a response should occur (with probability P_o). In the first sequence, a response is expected before no more than N-levels have been used; in the second by the $\frac{N^{th}}{2}$ level, etc.

Thus combining these results we have that an expected upper bound, $\bar{N}$, on the average number of tests in the design is given by

$$\bar{N} \sim 2N(1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{N/2})$$

(Thus $\bar{N}$ is roughly given by $2N\log N$.) For example, for $\alpha = .05$, $P_o = .5$, we have $\bar{N} \sim 76$. The results of simulating this design under these condition (Appendix V-C) indicated an average sample size of 45, although a few designs required as many as 100 tests.

From the table given in Section III-B6, for $\alpha = .01$, $P_o = .5$, an N of 69 is required. Thus for a design under these conditions, $\bar{N} \sim 604$.

V-B5. DERIVATION OF DISCRIMINATION DESIGNS

Suppose sensitivity tests are being conducted on two materials (e.g., two brands of igniters), and suppose we wish to prove that the two populations are really not the same without making any assumptions about their response functions. Let L_1 denote the likelihood of the sample from the first material, L_2 denote the likelihood of the sample from the second material, L_3 denote the likelihood of the joint sample under the null hypothesis that both response functions are identical. Then

$$X^2 = -2 \ln \lambda, \quad \text{where} \quad \lambda = L_3/L_1L_2$$

is asymptotically χ^2-distributed, and a natural rejection test would be based on large values of X^2. Therefore, for discrimination purposes, we want to test at a stimulus level, S, such that $E[X^2]$ is maximized.

Let $p_1(S)$ and $p_3(S)$ denote the probability of a response from population 1 at stimulus S, and the response probability under the null hypothesis, respectively. Then a single response for population 1, say, changes the values of λ from λ_0 to $\lambda_0 p_3(S)/p_1(S)$, while a non-response changes λ_0 to $\lambda_0 q_3(S)/q_1(S)$, where $q = 1-p$. The desired stimulus level, S, can then be found by maximizing

$$T_1 = p_1(S)[-2\ln\{\lambda_0 p_3(S)/p_1(S)\}] + q_1(S)[-2\ln\{\lambda_0 q_3(S)/q_1(S)\}]$$

which is equivalent to maximizing

$$D_1 = p_1 \ln(p_1/p_3) + q_1 \ln(q_1/q_3) \ . \tag{1}$$

Suppose further that our two critical level distributions are normal with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively. Then

$$\frac{\partial p_1(S)}{\partial S} = Z_1(S) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(S-\mu_1)^2/2\sigma_1^2} , \tag{2}$$

and a similar expression may be given for $Z_3(S)$. If D is to be a maximum, then from (1) and (2) we have

$$\frac{\partial D}{\partial S} = -Z_1 \ln(p_3 q_1/q_3 p_1) - Z_3(p_1/p_3 - q_1/q_3) = 0 . \tag{3}$$

This equation can be solved approximately as follows:

Let

$$\Delta = \mu_1 - \mu_3 . \quad \epsilon = \mu_1 - S .$$

Then

$$Z_1 = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\epsilon^2/2\sigma_1^2} , \quad Z_3 = \frac{1}{\sqrt{2\pi}\sigma_3} e^{-(\Delta-\epsilon)^2/2\sigma_3^2}$$

$$Z_1/Z_3 = (\sigma_3/\sigma_1)\, e^{[(\Delta-\epsilon)^2/\sigma_3^2 - \epsilon^2/\sigma_1^2]/2}$$

Let $\Delta/\sigma_3 = \omega$, $\epsilon/\Delta = t$, $\sigma_3/\sigma_1 = r$. Then

$$Z_1/Z_3 = r e^{\omega^2[1-2t + t^2(1-r^2)]/2} . \tag{4}$$

Also, from expansions about μ_1 and μ_3 we have for sufficiently small values of ϵ/σ_1 and $(\Delta-\epsilon)/\sigma_3$, respectively.

$$p_1 \cong 1/2 - \epsilon/\sqrt{2\pi}\,\sigma_1 + \epsilon^3/6\sqrt{2\pi}\,\sigma_1^3 ,$$

$$p_3 \cong 1/2 + (\Delta-\epsilon)/\sqrt{2\pi}\,\sigma_3 - (\Delta-\epsilon)^3/6\sqrt{2\pi}\,\sigma_3^3 .$$

Therefore to the same order of approximation

$$p_1/p_3 - q_1/q_3 \cong -4\epsilon/\sqrt{2\pi}\,\sigma_1 - 4(\Delta-\epsilon)/\sqrt{2\pi}\,\sigma_3 + 2\epsilon^3/3\sqrt{2\pi}\,\sigma_1^3$$

$$+ 2(\Delta-\epsilon)^3/3\sqrt{2\pi}\,\sigma_3^3 - 16(\Delta-\epsilon)^3/(2\pi)^{3/2}\sigma_3^3 - 16\epsilon(\Delta-\epsilon)^2/(2\pi)^{3/2}\sigma_1\sigma_3^2 .$$

Using the standard approximation for $\ln[(1+x)/(1-x)]$, we find to the same order of approximation,

$$\ln(p_3q_1/q_3p_1) \cong 4\epsilon/\sqrt{2\pi}\,\sigma_1 + 4(\Delta-\epsilon)/\sqrt{2\pi}\,\sigma_3 - 2\epsilon^3/3\sqrt{2\pi}\,\sigma_1^3$$

$$+ 16\epsilon^3/3(2\pi)^{3/2}\sigma_1^3 - 2(\Delta-\epsilon)^3/3\sqrt{2\pi}\,\sigma_3^3 + 16(\Delta-\epsilon)^3/3(2\pi)^{3/2}\sigma_3^3 .$$

Now if $G \equiv p_1/p_3$, $H \equiv q_1/q_3$, then

$$R \equiv (\ln G - \ln H)/(G-H) \cong 1 + \frac{[32(\Delta-\epsilon)^3/\sigma_3^3 + 48\epsilon(\Delta-\epsilon)^2/\sigma_3^2\sigma_1 - 16\epsilon^3/\sigma_1^3]/3(2\pi)^{3/2}}{-4\epsilon/\sqrt{2\pi}\,\sigma_1 - 4(\Delta-\epsilon)/\sqrt{2\pi}\,\sigma_3}$$

$$= 1 - \frac{2\omega^2}{3\pi}\,\frac{2(1-t)^3 - t^3r^3 + 3tr(1-t)^2}{tr + 1 - t} . \tag{5}$$

Since R is strictly positive, its logarithm is defined and we may therefore use the elementary approximation $\log R \cong -1+R$. Then from (3), (4), and (5), we find

$$\ln r + \omega^2 \left[\frac{1-2t+t^2(1-r^2)}{2} - \frac{2}{3\pi}\,\frac{2(1-t)^3 - t^3r^3 + 3tr(1-t)^2}{tr+1-t}\right] = 0 \tag{6}$$

If $r \approx 1$, one iteration of the Newton method, starting at $t_o = .20801$, gives

$$t \cong .20801 + 2.75194\,(\ln r)/\omega^2 + .16474\,(1-r) \tag{7}$$

For all practical purposes the term on the right may be ignored, and (7) may be rewritten as

$$t_{opt} \cong .208 + 2.572\,(\ln r)/\omega^2 , \tag{8}$$

or

$$S_{opt} \cong .792\mu_1 + .208\mu_3 + 2.752\,[\sigma_3^{\,2}/(\mu_3-\mu_1)]\,\ln(\sigma_3/\sigma_1)\ . \tag{9}$$

This approximate result gives in closed form the next test level. However, if the experimenter has the time, he may use the estimates of μ and σ to compute estimates of p_1, p_2, and p_3 at various nearby stress levels to get a more precise optimization through (1). He may do this for both items, and select the one which gives a greater D value; this is not always the item with the smaller sample. However, testing the item with the smaller sample, as recommended in Section IIIB-11, Phase II, leads to a sequence in which testing is performed on alternating items, a procedure which may be advantageous for other reasons.

V-B6. THE INVERSE RESPONSE PROBLEM FOR THE CUMULATIVE NORMAL RESPONSE FUNCTION

Suppose we have a mean response function, $M(x)$, which is a normal cdf with parameters μ and σ, both unknown. Suppose we want to find that value $x_\alpha = \mu + k\sigma$ for which $M(x_\alpha) = \alpha$, where α is a response probability specified in advance.

If L_i denotes the ith stress level in an experiment of size N, and r_i denotes the corresponding response (1 or 0), then the sample likelihood is given by

$$L = \prod_{i=1}^{N} (p_i^{r_i} q_i^{1-r_i}),$$

where

$$t_i = (L_i - \mu)/\sigma,$$

$$z = (2\pi)^{-\frac{1}{2}} \exp(-t^2/2),$$

$$p_i = \int_{-\infty}^{t_i} z \, dt, \quad \text{and}$$

$$q_i = 1 - p_i .$$

The maximum likelihood estimates of μ and σ are the solution of the simultaneous equations

$$\frac{\partial \ln L}{\partial \mu} = 0, \qquad \frac{\partial \ln L}{\partial \sigma} = 0.$$

From the Cramér-Rao bounds, and the asymptotic efficiency and unbiasedness of the maximum likelihood estimators, it is known that the covariance matrix of these estimates is asymptotic to

$$Q = \begin{pmatrix} -E\left[\frac{\partial^2 \ln L}{\partial \mu^2}\right] & -E\left[\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\right] \\ -E\left[\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\right] & -E\left[\frac{\partial^2 \ln L}{\partial \sigma^2}\right] \end{pmatrix}^{-1}$$

It may be shown that

$$-E\left[\frac{\partial^2 \ln L}{\partial \mu^2}\right] = C_o/\sigma^2 ,$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\right] = C_1/\sigma^2 ,$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \sigma^2}\right] = C_2/\sigma^2 ,$$

where

$$C_j = \sum_i V_i t_i^{\,j} ,$$

$$V_i = z(t_i)^2/p_i q_i .$$

Then

$$Q = \sigma^2 \begin{pmatrix} C_2 & -C_1 \\ -C_1 & C_o \end{pmatrix} / (C_o C_2 - C_1^{\,2}) ,$$

and the variance of the estimate of x_α is asymptotic to

$$H(\hat{\mu} + k\hat{\sigma}) = (1, \ k)\ Q \begin{pmatrix} 1 \\ k \end{pmatrix}.$$

Thus

$$H(\hat{\mu} + k\hat{\sigma})/\sigma^2 = (C_2 - 2kC_1 + k^2 C_o)/(C_o C_2 - C_1^{\,2}) .$$

Our goal is to minimize H. This will be only an asymptotic solution, since H is only asymptotic to the variance of the estimate.

Suppose now that N_1 of the N tests are at a stimulus $\mu+t_1\sigma$ and that $N_2 = N-N_1$ tests are at $\mu+t_2\sigma$. (Any possible set of values for C_o, C_1, and C_2, including the optimum combination, would be achieved by some set of values for t_1, t_2, and N_1, if not for the integer constraint on N_1; but this constraint is asymptotically innocuous.) Then

$$H/\sigma^2 = \frac{N_1V_1t_1^2+N_2V_2t_2^2-2k(N_1V_1t_1+N_2V_2t_2)+k^2(N_1V_1+N_2V_2)}{(N_1V_1t_1^2+N_2V_2t_2^2)(N_1V_1+N_2V_2)-(N_1V_1t_1+N_2V_2t_2)^2}$$

$$= \frac{(t_1-k)^2}{N_2V_2(t_1-t_2)^2} + \frac{(t_2-k)^2}{N_1V_1(t_1-t_2)^2} .$$

The value of N_1 which minimizes H is found by differentiating this expression and is given by

$$N_1 = N/\left(1 + \left|\frac{t_1-k}{t_2-k}\right|\sqrt{\frac{V_1}{V_2}}\right) .$$

For this value of N_1 we have

$$\sqrt{NH}/\sigma = \frac{|t_1-k|/\sqrt{V_2} + |t_2-k|/\sqrt{V_1}}{|t_1-t_2|} \equiv f(t_1,t_2) .$$

In order to find the (asymptotically) optimum values of t_1 and t_2, there are two fundamental cases to analyze: $t_1 \geq t_2 \geq k$ and $t_1 \geq k \geq t_2$. Both analyses use local reasoning, but computer studies have verified that the solutions are global optimums.

Case I: $t_1 \geq t_2 \geq k$

$$\frac{\partial f}{\partial t_1} = \frac{k-t_2}{(t_1-t_2)^2}\left[\frac{1}{\sqrt{V_2}} + \frac{1}{\sqrt{V_1}} + \frac{t_1-t_2}{2}\,\frac{\partial \ln V_1/\partial t_1}{\sqrt{V_1}}\right],$$

$$\frac{\partial f}{\partial t_2} = \frac{t_1-k}{(t_1-t_2)^2}\left[\frac{1}{\sqrt{V_1}} + \frac{1}{\sqrt{V_2}} - \frac{t_1-t_2}{2}\,\frac{\partial \ln V_2/\partial t_2}{\sqrt{V_2}}\right].$$

These partial derivatives are both zero if

$$t_1 = -t_2 \equiv t,$$

$$t(\partial \ln V/\partial t) = -2 ,$$

in which case we find

$$t \cong 1.5750360 ,$$

$$p \cong .94238 ,$$

$$N_1/N \cong .5 + .787518/k ,$$

$$H = 1.6436\, k^2\sigma^2/N .$$

Thus $N(.5 + .787518/k)$ of the tests should be at $\mu + 1.5750360\sigma$, and $N(.5 - .787518/k)$ of the tests should be at $\mu - 1.5750360\sigma$. This solution corresponds to values of k for which $k \leq -1.5750360$. The solution in the case $k \geq t_1 \geq t_2$ is identical except that $k \geq 1.5750360$. Thus the above solution holds whenever $|k| \geq 1.5750360$.

Since we do not know μ and σ, the best we can do is to find designs with $\mu + t_1\sigma$ and $\mu + t_2\sigma$ as limit points of convergent subsequences. Such a design will then be asymptotic to an asymptotically optimal design, and will therefore still be asymptotically optimal.

The above design gives a variance of $\hat{x}_\alpha$ which is asymptotic to $1.6436k^2\sigma^2/N$, whereas the Langlie design and the Bruceton design, with interval width equal to the

standard deviation, are roughly only half as efficient, with variances asymptotic to $(2.5+3.2k^2)\sigma^2/N$ (Ref. 17, pp. 17, 18) and $(2.0+3.8k^2)\sigma^2/N$ (Ref. 28), respectively.

Case II: $t_1 \geq k \geq t_2$

$$\frac{\partial f}{\partial t_1} = \frac{k-t_2}{(t_1-t_2)^2}\left[\frac{1}{\sqrt{V_2}} - \frac{1}{\sqrt{V_1}} - \frac{t_1-t_2}{2}\,\frac{\partial \ln V_1/\partial t_1}{\sqrt{V_1}}\right]$$

$$\frac{\partial f}{\partial t_2} = \frac{t_1-k}{(t_1-t_2)^2}\left[-\frac{1}{\sqrt{V_1}} + \frac{1}{\sqrt{V_2}} + \frac{t_1-t_2}{2}\,\frac{\partial \ln V_2/\partial t_2}{\sqrt{V_2}}\right].$$

A tedious and uninteresting computation shows that the limits of both of these derivatives are zero if both t_1 and t_2 approach k. Thus the optimum solution in this case is to take all of the tests at $\mu + k\sigma$, which is the only solution for $|k| < 1.5750360$. As before, this design is impossible to realize and must be replaced by a sequential parametric stochastic approximation to $\mu + k\sigma$.

The variance of the estimate is asymptotic to $\sigma^2/V_k N$. For $k = 0$, which corresponds to an estimate of the distribution mean, we have $V_o = 2/\pi$ and

$$H(\hat{\mu}) \sim \left(\frac{\pi}{2}\right) \sigma^2/N .$$

But for the Langlie design, we have

$$H(\hat{\mu}) \sim 2.5\sigma^2/N ,$$

and for the Bruceton up-and-down design with spacing $.67\sigma$ (very favorable circumstances for this design), we have (Ref. 28)

$$H(\hat{\mu}) \sim 1.85\sigma^2/N .$$

V-B7. RESPONSE DISTRIBUTION AT A SPECIFIED COMBINATION OF STIMULUS VARIABLES IN THE ABSENCE OF INTERACTION

In the nonparametric, noninteraction, multivariate grid design (Section III-B9), say with stimulus variables $x, y, \ldots, w$, the distribution at any level combination is taken to be binomial with parameter

$$p = 1-(1-p_x)(1-p_y)\ldots(1-p_w),$$

where $p_x, p_y, \ldots, p_w$ are the probabilities of a response when only one stimulus is applied. This result can be shown as follows.

Because of the definition of noninteraction we can think of the stimuli as applied one at a time. The result will be the same as applying all stimuli simultaneously.

Let x be the first stimulus applied; then the number of responses, λ_x, is known to have a binomial distribution

$$\text{Prob}\ \{\lambda_x | n\} = \frac{n!}{\lambda_x!(n-\lambda_x)!}\, p_x^{\lambda_x}\, q_x^{n-\lambda_x}\ .$$

If exactly λ_x responses occur, then only the remaining $n-\lambda_x$ nonresponses can be affected by the second stimulus, y. The number of additional responses, λ_y, will again be binomially distributed:

$$\text{Prob}\ \{\lambda_y | n-\lambda_x\} = \frac{(n-\lambda_x)!}{\lambda_y!(n-\lambda_x-\lambda_y)!}\, p_y^{\lambda_y}\, q_y^{n-\lambda_x-\lambda_y}\ .$$

If only the two stimuli were present, then the total number of responses $\lambda_{xy} = \lambda_x + \lambda_y$ is given by

$$\text{Prob}\,\{\lambda_{xy}|n\} = \sum_{\lambda_x+\lambda_y=\lambda_{xy}} \text{Prob}\,\{\lambda_x|n\} \cdot \text{Prob}\,\{\lambda_y|n-\lambda_x\}$$

$$= \sum_{\lambda_x=0}^{n} \frac{n!}{\lambda_x!(n-\lambda_x)!} p_x^{\lambda_x} q_x^{n-\lambda_x} \frac{(n-\lambda_x)!}{(\lambda_{xy}-\lambda_x)!(n-\lambda_{xy})!} p_y^{\lambda_{xy}-\lambda_x} q_y^{n-\lambda_{xy}}$$

$$= \frac{n!}{(n-\lambda_{xy})!} q_x^{n} q_y^{n-\lambda_{xy}} \sum_{\lambda_x=0}^{n} \frac{1}{\lambda_x!(\lambda_{xy}-\lambda_x)!} \left(\frac{p_x}{q_x}\right)^{\lambda_x} p_y^{\lambda_{xy}-\lambda_x}$$

$$= \frac{n!}{\lambda_{xy}!(n-\lambda_{xy})!} q_x^{n} q_y^{n-\lambda_{xy}} \sum_{\lambda_x=0}^{n} \frac{\lambda_{xy}!}{\lambda_x!(\lambda_{xy}-\lambda_x)!} \left(\frac{p_x}{q_x}\right)^{\lambda_x} \cdot p_y^{\lambda_{xy}-\lambda_x}$$

$$= \frac{n!}{\lambda_{xy}!(n-\lambda_{xy})!} (q_x q_y)^{n-\lambda_{xy}} (p_x+q_x p_y)^{\lambda_{xy}}$$

and since $p_x + q_x p_y + q_x q_y = 1$, this expression has the form of a binomial distribution with parameter

$$p = p_x + q_x p_y = 1 - q_x q_y.$$

If we continue applying the stimuli consecutively, we find that at any fixed combination of stimulus levels a binomial distribution is generated with parameter p given by

$$p = 1 - q_x q_y \cdots q_w .$$

V-B8. DISTRIBUTION OF RESPONSES DUE TO AN UNKNOWN STIMULUS VARIABLE

a. Distribution of Responses

Assume that the unknown stimulus (y) may be at levels $L_1, L_2, \cdots, L_k$ and that the probabilities of being at these levels are $t_1, t_2, \cdots, t_k$, respectively. Then in n tests the number of tests, z_r, conducted at each level, L_r, will have the multinomial distribution

$$T(z_1, z_2, \cdots, z_k) = \frac{n!}{z_1!z_2!\cdots z_k!} t_1^{z_1} t_2^{z_2} \cdots t_k^{z_k} , \tag{1}$$

where

$$\sum_{r=1}^{k} t_r = 1 \quad \text{and} \quad \sum_{r=1}^{k} z_r = n \quad .$$

Let the probability of a response for y at level L_r be p_r; then the probability of s_r responses in z_r tests is given by

$$\frac{z_r!}{s_r!(z_r-s_r)!} p_r^{s_r} q_r^{z_r-s_r} \quad .$$

If z_r $(r = 1, 2, \cdots, k)$ is given, then the probability of a total of ν responses can be expressed by

$$P\{\nu|z_1, z_2, \cdots, z_r\} = \sum_{\substack{\Sigma s_r=\nu \\ s_r \le z_r}} \prod_{r=1}^{k} \frac{z_r!}{s_r!(z_r-s_r)!} p_r^{s_r} q_r^{z_r-s_r}$$

and the joint distribution of $z_1, z_2, \cdots, z_k$ and ν is

$$T(z_1, z_2, \cdots, z_r) \sum_{\substack{\Sigma s_r=\nu \\ s_r \le z_r}} \prod_{r=1}^{k} \frac{z_r!}{s_r!(z_r-s_r)!} p_r^{s_r} q_r^{z_r-s_r} \quad .$$

We may now find the joint distribution, g, of $z_1, z_2, \cdots, z_k$ and λ, where λ is the number of responses when both stimuli are applied. Here the controlled stimulus is assumed to be at some fixed level for which the probability of failure is p_x. One obtains

$$g(\lambda, z_1, z_2, \cdots, z_k) = T(z_1, z_2, \cdots, z_k) \cdot$$

$$\sum_{\nu=0} \Big[\sum_{\substack{\Sigma s_r = \nu \\ s_r \leq z_r}} \prod_{r=1}^{k} \frac{z_r!}{s_r!(z_r-s_r)!} p_r^{s_r} q_r^{z_r-s_r} \Big] \frac{(n-\nu)!}{(\lambda-\nu)!(n-\lambda)!} p_x^{\lambda-\nu} q_x^{n-\lambda} \cdot$$

Eliminating $z_1, z_2, \cdots, z_k$ by summation, we have the marginal distribution for λ:

$$f(\lambda) = \sum_{\sum_{r=1}^{k} z_r = n} g(\lambda, z_1, z_2, \cdots, z_k) \cdot \tag{2}$$

We now show that the mean and variance of $f(\lambda)$ are given by

$$\begin{aligned} E_f(\lambda) &= np_x + nq_x \Sigma t_r p_r , \\ D_f^2(\lambda) &= np_x q_x + nq_x(q_x - p_x)\Sigma t_r p_r - nq_x^2 \Sigma t_r p_r^2 \cdot \end{aligned} \tag{3}$$

After a series of n tests, let the number of tests at each level, L_r, be given by z_r. Then the number of responses, λ, has a Poisson generalized binomial distribution,* $\phi(\lambda | z_1, z_2, \cdots, z_k)$, with mean and variance given by

*See, for example, Ref. 26.

$$E_{\phi}(\lambda) = \sum_{1}^{k} z_r(p_x+q_x p_r)$$

$$D_{\phi}^2(\lambda) = \sum_{1}^{k} z_r(p_x+q_x p_r)(q_x q_r) \quad . \tag{4}$$

The probability of any set $z_1, z_2, \cdots, z_k$ is given by

$$T(z_1, z_2, \cdots, z_k) = \frac{n!}{z_1! z_2! \cdots z_k!} t_1^{z_1} t_2^{z_2} \cdots t_k^{z_k} , \tag{5}$$

where

$$\sum_{1}^{k} t_r = 1 \quad \text{and} \quad \sum_{1}^{k} z_r = n \quad .$$

Then the distribution (2) can be written

$$f(\lambda) = \sum_{\sum_1^k z_r = n} T(z_1, z_2, \cdots, z_k) \cdot \phi(\lambda | z_1, \cdots, z_k) \quad . \tag{6}$$

The mean and variance are defined by

$$E_f(\lambda) = \sum_{\lambda=0}^{n} \lambda \, f(\lambda)$$

and

$$D_f^2(\lambda) = \sum_{\lambda=0}^{n} \lambda^2 \, f(\lambda) \quad . \tag{7}$$

From (6) and (7) we find

$$\begin{aligned} E_f(\lambda) &= \sum_{\lambda=0}^{n} \lambda \cdot \sum_{\Sigma z_r = n} T(z_1, z_2, \cdots, z_k) \phi(\lambda | z_1, \cdots, z_k) \\ &= \sum_{\Sigma z_r = n} T(z_1, z_2, \cdots, z_k) \sum_{\lambda=0}^{n} \lambda \, \phi(\lambda | z_1, \cdots, z_k) \quad . \end{aligned} \tag{8}$$

From (4)

$$\sum_{\lambda=0}^{n} \lambda\, \phi(\lambda|z_1, \cdots, z_k) = \sum_{r=1}^{k} z_r(p_x+q_xp_r) \quad .$$

Substituting in (8) we have

$$E_f(\lambda) = \sum_{\Sigma z_r=n} T(z_1, \cdots, z_k) \sum_{r=1}^{k} z_r(p_x+q_xp_r)$$

$$= \sum_{r=1}^{k} (p_x+q_xp_r) \sum_{\Sigma z_r=n} z_r\, T(z_1, \cdots, z_k)$$

$$= \sum_{r=1}^{k} (p_x+q_xp_r) \sum_{\Sigma z_r=n} z_r \cdot \frac{n!}{z_1!z_2!\cdots z_k!} t_1^{z_1} \cdots t_k^{z_k}$$

$$= n \sum_{r=1}^{k} (p_x+q_xp_r)t_r \sum_{\Sigma z_r=n} \frac{(n-1)!}{z_1!\cdots(z_r-1)!z_k!} t_1^{z_1} \cdots t_r^{z_r-1} \cdots t_k^{z_k} \quad .$$

By inspection we see that the inner summand is a multinomial distribution and therefore the sum is 1. Thus

$$E_f(\lambda) = n \sum_{r=1}^{k} t_r(p_x+q_xp_r)$$

$$= np_x + nq_x \sum_{r=1}^{k} t_rp_r \quad .$$

In the same manner, one may show that

$$D_f^2(\lambda) = \sum_{r=1}^{k} nt_r q_x q_r (p_x + p_r q_x)$$

$$= np_x q_x + nq_x(q_x - p_x) \sum t_r p_r - nq_x^2 \sum t_r p_r^2 \quad .$$

b. Comparison of Variances

Consider an experiment in which s responses occur in n tests at a single fixed level of a known stimulus, x. Then either of the following assumptions can be made:

H_o: The experiment was univariate with only x present.

H_1: An additional unknown stimulus, y, as well as known stimulus, x, was present.

Under H_o, an estimate of the mean λ can be determined from the associated binomial distribution by means of $\hat{\lambda} = s$. Then the binomial parameter p_1 is estimated by

$$\hat{p}_1 = \hat{\lambda}/n \tag{9}$$

and the variance σ_o^2 by

$$\hat{\sigma}_o^2 = n\,\hat{p}_1(1-\hat{p}_1) \quad . \tag{10}$$

For the distribution (2), which is implied by H_1, the mean λ and variance σ^2 are given by (3). Then an estimate of the parameter p_x can be given implicitly by

$$\hat{\lambda} = n\hat{p}_x + n\hat{q}_x\, \Sigma t_r \hat{p}_r \tag{11}$$

and an estimate of the variance σ_1^2 will then be given directly by

$$\hat{\sigma}_1^2 = n\hat{p}_x\hat{q}_x + n\hat{q}_x(\hat{q}_x - \hat{p}_x)\, \Sigma t_r \hat{p}_r - n\hat{q}_x^2\, \Sigma t_r \hat{p}_r^2 \quad . \tag{12}$$

Thus from (9) and (11) the parameters of the two distributions satisfy the relationship

$$\hat{p}_1 = \hat{p}_x + \hat{q}_x \, \Sigma t_r p_r \tag{13}$$

Now (10) can be written in the form

$$\begin{aligned} \sigma_o^2 &= n(\hat{p}_x+\hat{q}_x\Sigma t_r\hat{p}_r)(1-\hat{p}_x-\hat{q}_x\Sigma t_r\hat{p}_r) \\ &= n\hat{p}_x\hat{q}_x + n\hat{q}_x(\hat{q}_x-\hat{p}_x)\Sigma t_r\hat{p}_r - n\hat{q}_x^2(\Sigma t_r\hat{p}_r)^2 \end{aligned} \tag{14}$$

and we can now directly compare our two variances.

From (12) and (14) we have

$$\begin{aligned} \sigma_o^2 - \sigma_1^2 &= n\hat{q}_x^2 \, \Sigma t_r\hat{p}_r^2 - n\hat{q}_x^2(\Sigma t_r\hat{p}_r)^2 \\ &= n\hat{q}_x^2 \, \Sigma t_r(\hat{p}_r-\Sigma t_r\hat{p}_r)^2 > 0 \quad . \end{aligned}$$

We thus have the interesting result that the presence of an unknown stimulus results in a smaller variance.

V-B9. MODELS FOR MULTIVARIATE EXPERIMENTS WITH INTERACTING STIMULI

Introduction

Problems of general interest in univariate sensitivity experiments are the determination of response functions and the inverse response problem. The analogous problems when there are several stimulus variables are the determination of response surfaces and the determination of equal-probability contour surfaces, respectively. In the former case, if we have n stimulus or stress variables $x_1, x_2, \cdots, x_n$ then we seek an $(n+1)$-dimensional response surface

$$z = z(x_1, x_2, \cdots, x_n) \tag{1}$$

where z is the probability of a response. In the latter situation we wish to find a contour function, $f_\alpha(x_1, x_2, \cdots, x_n)$ such that for a given α, Prob{fraction response on $f_\alpha(x_1, x_2, \cdots, x_n) = 0\} = \alpha$.

By imposing conditions on (1) we will determine both response surfaces and contour surfaces. To simplify the notation the case of two stress variables, x and y, will be considered; the methods presented can be extended to any number of variables.

The response surface (1) will, at first, be assumed to satisfy the condition that for any fixed x, say x_0, or fixed y, say y_0, all the conditional distributions

$$z = z(x, y_0)$$
$$z = z(x_0, y) \tag{2}$$

are of the same form, but with different parameters. This condition will later be relaxed to hold only for $x_0 = 0$ and $y_0 = 0$.

Normal Conditional Distributions

If the distributions in (2) are assumed to be cumulative normal with parameters $\mu(y_0)$, $\sigma(y_0)$ and $\nu(x_0)$, $\tau(x_0)$, respectively, then the response surface (1) can be written in two equivalent ways:

$$z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x_0 - \mu(y_0)}{\sigma(y_0)}} e^{-\frac{1}{2}t^2}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{y_0 - \nu(x_0)}{\tau(x_0)}} e^{-\frac{1}{2}t^2}\, dt \quad . \tag{3}$$

It follows from (3) that

$$\frac{x_0-\mu(y_0)}{\sigma(y_0)} = \frac{y_0-\nu(x_0)}{\tau(x_0)} \tag{4}$$

for any x_0 and y_0. If we consider the identity (4) for the four sets of (x_0, y_0) values $(0,0)$, $(x,0)$, $(0,y)$, and (x,y) then by algebraic elimination from the four resulting equations it can be shown that

$$\frac{x}{\frac{1}{\tau(x)} - \frac{1}{\tau(0)}} = \frac{y}{\frac{1}{\sigma(y)} - \frac{1}{\sigma(0)}} \quad . \tag{5}$$

Since (5) must hold for any values of x and y it follows that

$$\frac{y}{\frac{1}{\sigma(y)} - \frac{1}{\sigma(0)}} \equiv \frac{1}{B} \equiv \frac{x}{\frac{1}{\tau(x)} - \frac{1}{\tau(0)}}$$

for some constant B. Letting $\sigma(0) = \sigma$, $\tau(0) = \tau$ we have therefore

$$\begin{aligned} \sigma(y) &= \frac{\sigma}{B\sigma y+1} \\ \tau(x) &= \frac{\tau}{B\tau x+1} \quad . \end{aligned} \tag{6}$$

Let $\mu(0) = \mu$ and $\nu(0) = \nu$, then from (4) and (6) we obtain the additional relationships

$$\begin{aligned} \mu(y) &= \frac{\nu-y}{\tau}\,\frac{\sigma}{B\sigma y+1} \\ \nu(x) &= \frac{\mu-x}{\sigma}\,\frac{\tau}{B\sigma x+1} \quad . \end{aligned} \tag{7}$$

Our response surface (3) can now be written in the form

$$z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Bxy+\frac{x}{\sigma}+\frac{y}{\tau}-\frac{\nu}{\tau}} e^{-\frac{1}{2}t^2}\, dt \quad . \tag{8}$$

The normal surface defined by (8) belongs to a class, C, of more general response surfaces satisfying (2) which are of the form

$$\begin{aligned} z &= z(u) \\ u &= Axy + bx + cy + d \end{aligned} \tag{9}$$

where A is an interaction parameter and b, c, and d depend on the parameters of the assumed underlying distribution. For this class of surfaces it is clear that the equal probability contours are all hyperbolas. In particular for normal conditional distributions we have from (8) that the p-contour is the hyperbola

$$Bxy + \frac{x}{\sigma} + \frac{y}{\tau} - \frac{\nu}{\tau} - K_p = 0 \quad ,$$

where K_p is the standard normal deviate exceeded with probability 1-p.

In deriving (8) it is assumed that (4) is satisfied for all x_0 and y_0. In particular, if $x_0 = y_0 = 0$ it must follow that $\frac{\mu}{\sigma} = \frac{\nu}{\tau}$. This condition is not as restrictive as it seems. If for $x = 0$ and $y = 0$ our cumulative normal distributions are known and have parameters ν_1, τ_1 and μ_1, σ_1 respectively but $\frac{\mu_1}{\sigma_1} \neq \frac{\nu_1}{\tau_1}$, we can consider the translated variable $y' = y + d$. Then the cumulative distribution of y' is normal with parameters $\nu_1 + d$, τ_1. Let $d = \frac{\mu_1}{\sigma_1}\tau_1 - \nu_1$, then the variables x and y' have distributions which satisfy the condition. If our distributions are unknown but there is reason to believe that $\frac{\mu}{\sigma} \neq \frac{\nu}{\tau}$ then the translation parameter, d, must also be estimated.

Other Conditional Distributions

It has been shown that if the conditional distributions satisfying (2) are cumulative normal, then the response surface belongs to the class of surfaces given by

(9). We shall now verify that several other conditional distributions satisfying (2) have response surfaces belonging to the same class.

Uniform Conditional Distributions

Consider the response surface defined in the first octant by

$$\begin{aligned} z &= Dxy + ax + by \qquad & 0 \le z \le 1 \\ z &= 1 & \text{otherwise} \end{aligned} \tag{10}$$

then in the region, R, where $0 \le z \le 1$, (10) belongs to C. If we let $x = x_0$, a constant, then

$$z(x_0, y) = y(Bx_0+b) + ax_0$$

which has a uniform distribution in R. Since we obtain a similar result when y is fixed it follows that in R (10) satisfies (2) with uniform conditional distributions.

If we let the uniform distributions corresponding to $x = 0$ and $y = 0$ for z in R be given by

$$\begin{aligned} z(0,y) &= m_2 y \\ z(x,0) &= m_1 x \end{aligned}$$

then we can express a and b in terms of the distribution parameters m_1 and m_2. Thus our p-contour becomes

$$p = Dxy + m_1 x + m_2 y$$

where D is an interaction parameter.

Exponential Conditional Distributions

Next we consider the member of C given by

$$z = 1 - e^{-(Exy+ax+by)} \tag{11}$$

It is easily seen that the conditional distributions satisfying (2) are exponential. As in the previous cases E will be an interaction parameter and we shall express a and b in terms of the parameters of the conditional distributions $z(x,0)$ and $z(0,y)$. Since these are known to be exponential they can be written in the form

$$z(x,0) = 1 - e^{-\beta x}$$

$$z(0,y) = 1 - e^{-\gamma y}$$

with parameters β and γ, respectively. Our hyperbolic p-contour then is given by

$$Exy + \beta x + \gamma y = -\ln(1-p) .$$

Weibull Conditional Distributions
with Common Shape Parameter

If we consider the response surface belonging to C of the form

$$z = 1 - e^{-(Fxy+ax+by)^{\lambda}}$$

we are led to Weibull conditional distributions satisfying (2) which have a common shape parameter, λ. In this case again the equal probability contours are hyperbolic.

No-Interaction Conditions

It would be desirable to determine a value for the interaction parameter, B, under each of the previously made distributional assumptions which would reduce the problem to a model without interactions. This can only be done for the exponential and the uniform distributions by setting $C = 0$ and $D = -m_1 m_2$, respectively. For normal conditional distributions letting $B = 0$ results in $\sigma(y) = \sigma$ and $\tau(x) = \tau$. Thus the interaction is of a simple nature in that only the means of the two distributions are affected by each other's stimulus levels.

More General Response Surfaces

In all of the cases considered the response surface was derived under the assumption that all the conditional distributions for any fixed x or y were of a given form. Now, we will relax the condition to require that only two conditional distributions (those for $x = 0$ and $y = 0$) be of the same given form. Our response surfaces will now belong to a more general class. The following model apparently describes the most general surface of this class

$$z = z(u)$$

$$u = ax + by + c + Axy + xy\,\phi(x,y)$$

where $\phi(x,0)$ and $\phi(0,y)$ are finite. The identity of the conditional distributions at $x = 0$, $y = 0$ is assured in this model. In particular, consider the response surface

$$z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sigma} + \frac{y}{\tau} - \frac{\nu}{\tau} + Bxy + xy\phi(x,y)} e^{-\frac{1}{2}t^2}\, dt \quad .$$

For $x = 0$ and $y = 0$ we obtain the conditional distributions

$$z(0,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{y}{\tau} - \frac{\nu}{\tau}} e^{-\frac{1}{2}t^2}\, dt$$

$$z(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sigma} - \frac{\mu}{\sigma}} e^{-\frac{1}{2}t^2}\, dt$$

which are both of the same form, i.e., cumulative normal, but if $x = x_0 \neq 0$ or $y = y_0 \neq 0$, the distributions are not necessarily cumulative normal.

By means of the function $\phi(x,y)$ in this more general model, <u>several</u> interaction parameters can be introduced.

BIBLIOGRAPHY

GENERAL

Aitchison, J. and Silvey, S. D. (1957). The generalisation of probit analysis to the case of multiple responses. Biometrika, 44, 131-140.

Anderson, T. W., McCarthy, P. J., and Tukey, J. W. (1946). "Staircase" methods of sensitivity testing. Bureau of Ordnance, Washington, Navord Report 65-46, 135 pp.

Anscombe, F. J. (1948). The validity of comparative experiments. J. Roy. Statist. Soc., 111, 181-211.

Anscombe, F. J. (1949). Note on a problem in probit analysis. Ann. Appl. Biol., 36, 203-205.

Anscombe, F. J. (1956). On estimating binomial response relations. Biometrika, 43, 461-464.

Armitage, P. and Allen, I. (1950). Methods of estimating the LD50 in quantal response data. J. of Hygiene, 48, 298-322.

Bartlett, M. S. (1937). Some examples of statistical methods of research in agriculture and applied biology. J. Roy. Statist. Soc., 137-183.

Bartlett, M. S. (1946). A modified probit technique for small probabilities. J. Roy. Statist. Soc. Suppl., 8, 113-117.

Berkson, J. (1944). Application of the logistic function of bio-assay. J. Amer. Statist. Assoc., 39, 357-365.

Berkson, J. (1946). Approximation of chi-square by "probits" and by "logits". J. Amer. Statist. Assoc., 41, 70-74.

Berkson, J. (1949). Minimum χ^2 and maximum likelihood solution in terms of a linear transform with particular reference to bio-assay. J. Amer. Statist. Assoc., 44, 273-278.

Berkson, J. (1951). Why I prefer logits to probits. Biometrics, 7, 327-339.

Berkson, J. (1951). Relative precision of minimal chi-square and maximum likelihood estimates. Second Berkeley Symposium on Math. Stat. Probability, Vol. I, 471-479.

Berkson, J. (1953). A statistically precise and relatively simple method of estimating the bio-assay with quantal response, based on the logistic function. J. Amer. Statist. Assoc., 48, 565-599.

Berkson, J. (1955). Estimate of the integrated normal curve by minimum normit chi-square with particular reference to bio-assay. J. Amer. Statist. Assoc., 50, 529-549.

Berkson, J. (1955). Maximum likelihood and minimum χ^2 estimates of the logistic function. J. Amer. Statist. Assoc., 50, 130-162.

Berkson, J. (1957). Tables for the maximum likelihood estimate of the logistic function. Biometrics, 13, 28-34.

Berkson, J. (1957). Tables for use in estimating the normal distribution function by normit analysis. Biometrika, 44, 411-435.

Black, A. (1950). Weighted probits and their use. Biometrika, 37, 158-167.

Bliss, C. I. (1934). The method of probits. Science, 79, 38-39.

Bliss, C. I. (1934). The method of probits - a correction. Science, 79, 409-410.

Bross, I. (1950). Estimates of the LD_{50}: a critique. Biometrics, 6, 413-423.

Brownlee, K. A., Hodges, J. L., Jr., and Rosenblatt, M. (1953). The up-and-down method with small samples. J. Amer. Statist. Assoc., 48, 262-277.

Chernoff, H. (1962). Optimal design of experiments. Technical Report No. 82, Applied Mathematics and Statistics Laboratories, Stanford University.

Churchman, C. W. and Epstein, B. (1944). On the statistics of sensitivity data. Ann. Math. Statist., 15, 90-97.

Churchman, C. W. and Epstein, B. (1946). Tests of increased severity. J. Amer. Statist. Assoc., 41, 567-590.

Cochran, W. G. (1938). Appendix to a paper by Tattersfield, F. and Martin, J. T. Ann. Appl. Biol., 25, 426-429.

Cochran, W. G. (1942). The χ^2 correction for continuity. Iowa St. Coll. J. Sci., 16, 421-436.

Cochran, W. G. and Davis, M. (1960). Sequential experiments for estimating the median lethal dose. Memo., Department of Statistics, Harvard University.

Cornfield, J. and Mantel, N. (1950). Some new aspects of the application of maximum likelihood to the calculation of the dosage response curve. J. Amer. Statist. Assoc., 45, 181-210.

Crow, E. L., Davis, F. A., and Maxfield, M. W. (1960). Sensitivity testing. Chap. 4 of Statistical Manual, New York, Dover, 91-97. (This manual was first released within the Department of Defense as NAVORD Report 3369-NOTS 948).

De Beer, E. J. (1945). The calculation of the biological assay results by graphic methods. The all-or-none type response. J. Pharmacol., 85, 1-13.

Derman, C. (1947). Non-parametric up-and-down experimentation. Ann. Math. Statist., 28, 795-798.

Dixon, W. J. and Massey, F. J., Jr. (1951). An introduction to statistical analysis. McGraw-Hill, New York, 279-287.

Dixon, W. J. and Mood, A. M. (1948). A method for obtaining and analyzing sensitivity data. J. Amer. Statist. Assoc., 43, 109-126.

Duncan, D. B. and Rhodes, R. C. (1952). Multiple regression with quantal response. Ann. Math. Statist., 23, 300.

van Eeden, C. (1957). Maximum likelihood estimation of partially or completely ordered parameters I, II. Indagationes Mathematicae, 19, 128-136, 201-211.

Eissner, R. M. (1958). A method of analyzing sensitivity data from "contaminated" populations. Proceedings of the Stat. Techniques in Missile Evaluating Symposium, Boyd Harshbarger, Editor, Bassett Printing Corp., Blacksburg, Va., 199-221. also, Ballistic Research Laboratories Memorandum Rept. No. 722, Aberdeen (1953).

Ferguson, G. A. (1942). Item selection by the constant process. Psychometrika, 7, 19-29.

Fieller, E. C. (1944). A fundamental formula in the statistics of biological assay, and some applications. Quart. J. Pharm., 17, 117-123.

Finney, D. J. (1944). The application of the probit method to toxicity test data adjusted for mortality in the controls. Ann. Appl. Biol., 31, 68-74.

Finney, D. J. (1944). Mathematics of biological assay. Nature, Lond., 153-284.

Finney, D. J. (1949). On a method of estimating frequencies. Biometrika, 36, 233-234.

Finney, D. J. (1949). The adjustment for a natural response rate in probit analysis. Ann. Appl. Biol., 36, 187-195.

Finney, D. J. and Stevens, W. L. (1948). A table for the calculation of working probits and weights in probit analysis. Biometrika, 35, 191-201.

Finney, D. J. (1952). Probit analysis. Cambridge Univ. Press, London, 2nd edition.

Fisher, R. A. (1935). Appendix to Bliss, C. I.: The case of zero survivors. Ann. Appl. Biol., 22, 164-165.

Fisher, R. A. (1942). The design of experiments. Edinburgh: Oliver and Boyd, 3rd edition.

Fisher, R. A. and Yates, F. (1949). Statistical tables for biological, agricultural and medical research. Hafner, New York, 3rd edition.

Friedman, M. (1947). Planning an experiment for estimating the mean and standard deviation of a normal distribution from observations on the cumulative distribution. Chap. 11 of Statistical Research Group, Columbia Univ., Selected Techniques of Statistical Analysis (edited by C. Eisenhart, M. W. Hastay, and W. A. Wallis). McGraw-Hill, New York, 339-352.

Garwood, F. (1941). The application of maximum likelihood to dosage-mortality curves. Biometrika, 32, 46-58.

Golub, A. and Grubbs, F. E. (1956). Analysis of sensitivity experiments when the levels of stimulus cannot be controlled. J. Amer. Statist. Assoc., 51, 257-265.

Grant, R. L. and Van Dolah, R. W. (1964). Use of the up-and-down method with factorial designs. Proceedings of the Seventh Conference on the Design of Experiments in Army Research, Development, and Testing. ARODR 62-2, 39-65.

Grizzle, J. E. (1961). A new method of testing hypotheses and estimating parameters for the logistic model. Biometrics, 17, 372-385.

Gurian, J. M. (1956). Note on fitting the multi-hit survival curve. Biometrics, 12, 123-126.

Guttman L. and Guttman, R. (1959). An illustration of the use of stochastic approximation. Biometrics, 15, 551-559.

Hampton, L. D., Ayres, J. N., and Kabik, I. (1964). Estimation of high and low probability EED functioning levels. NOLTR 63-266, White Oak, 8 pp.

Harris, E. K. (1959). Confidence limits for the LD_{50} using the moving average-angle method. Biometrics, 15, 424-432.

Healy, M. J. R. (1950). The planning of probit assays. Biometrics, 6, 424-431.

Hodges, J. L., Jr. (1958). Fitting the logistic by maximum likelihood. Biometrics, 14, 453-461.

Irwin, J. O. and Cheeseman, E. A. (1939). On an approximate method of determining the median effective dose and its error in the case of quantal response. J. Hyg. Camb., 39, 574-580.

Irwin, J. O. and Cheeseman, E. A. (1939). On the maximum likelihood method of determining dosage-response-curves and approximations to the median-effective dose, in cases of quantal response. J. Roy. Statist. Soc. Suppl., 6, 174-185.

Karber, G. (1931). Beitrag zur kollektiven behandlung pharmakologischer reihenversuche. Arch. Exp. Path. Pharmak., 162, 480-487.

Kimball, A. W. (1953). The fitting of multi-hit survival curves. Biometrics, 9, 201-211.

Langlie, H. J. (1962). A reliability test method for "one-shot" items. Aeronutronic Publication No. U-1792.

Lawley, D. N. (1943). On problems connected with item selection and test construction. Proc. Roy. Soc. Edinburgh. Sect. A, 61, 273-287.

Lawley, D. N. (1944). The factorial analysis of multiple item tests. Proc. Roy. Soc. Edinburgh. Sect. A, 62, 74-82.

Marks, B. L. (1962). Some optimal sequential schemes for estimating the mean of a cumulative normal quantal response curve. J. Roy. Statist. Soc. Ser. B, 24, 393-400.

Miller, L. C. and Tainter, M. L. (1944). Estimation of ED 50 and its error by means of logarithmic-probit graph paper. Proc. Soc. Exp. Biol., N. Y., 57, 261-264.

Odell, P. L. and Franck, W. E., Jr. (1963). A method for obtaining nonparametric confidence intervals in up-and-down testing. NAVWEPS Report 7988, U. S. Naval Weapons Evaluation Facility, Albuquerque, New Mexico.

Reed, L. J. and Muench, H. (1938). A simple method of estimating fifty per cent endpoints. Amer. J. Hyg., 27, 493-497.

Rothman, D. and Zimmerman, J. M. (1965). The design of complex sensitivity experiments. Proceedings of the Tenth Conference on the Design of Experiments in Army Research, Development and Testing, Washington.

Sampford, M. R. (1952). The estimation of response-time distributions, part I: Biometrics, 8, 13-32; part II: Biometrics, 8, 307-369.

Sampford, M. R. (1954). The estimation of response-time distributions, III: Truncation and survival. Biometrics, 10, 531-561.

Silverstone, H. (1957). Estimating the logistic curve. J. Amer. Statist. Assoc., 52, 567-577.

Spearman, C. (1908). The method of "right and wrong cases" ("constant stimuli") without Gauss's formulae. British J. of Psych., 2, 227-242.

Thompson, W. R. (1947). Use of moving averages and interpolation to estimate median-effective dose. Bact. Rev., 11, 115-145.

Thomson, G. H. (1914). The accuracy of the $\xi(Y)$ process. British J. of Psych., 7, 44-55.

Tocher, K. D. (1949). A note on the analysis of grouped probit data. Biometrika, 36, 9-17.

Worcester, J. and Wilson, E. B. (1943). A table determining L.D. 50 or the fifty per cent end point. Proc. Nat. Acad. Sci., 29, 207-212.

Wright, S. (1926). A frequency curve adapted to variation in percentage occurrence. J. Amer. Statist. Assoc., 21, 162-178.

...(1944) Statistical analysis for a new procedure in sensitivity experiments. Statistical Research Group, Princeton, No. 40 (Applied Mathematics Panel Report No. 101.1R), 58 pp.

STOCHASTIC APPROXIMATION

Block, H. D. (1957). On stochastic approximation. Office of Naval Research Report, ASTIA No. 123792, 16 pp.

Blum, J. R. (1954). Multidimensional stochastic approximation procedures. Ann. Math. Statist., 25, 737-744.

Blum, J. R. (1954). Approximation methods which converge with probability one. Ann. Math. Statist., 25, 382-386.

Burkholder, D. L. (1956). On a class of stochastic approximation processes. Ann. Math. Statist., 27, 1044-1059.

Chung, K. L. (1954). On a stochastic approximation method. Ann. Math. Statist., 25, 463-483.

Derman, C. (1956). Stochastic approximation. Ann. Math. Statist., 27, 879-886.

Dvoretzky, A. (1956). On stochastic approximation. Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Univ. of Calif. Press, Vol. 1, 39-55.

Gardner, L. A. (1960). Stochastic approximation and minimax problems. MIT-L, TR-219.

Hodges, J. L. Jr., and Lehmann, E. L. (1956). Two approximations to the Robbins-Monro process. Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Univ. of Calif. Press, Vol. 1, 95-104.

Kallianpur, G. (1954). A note on the Robbins-Monro stochastic approximation method. Ann. Math. Statist., 25, 386-388.

Kesten, H. (1958). Accelerated stochastic approximation. Ann. Math. Statist., 29, 41-59.

Robbins, H. and Monro, S. (1951). A stochastic approximation method. Ann. Math. Statist., 22, 400-407.

Odell, P. L. (1962). Stochastic approximation and non-parametric interval estimation in sensitivity testing which involves quantal response data. Oklahoma State University Ph.D. Thesis.

Sacks, J. (1958). Asymptotic distribution of stochastic approximation procedures. Ann. Math. Statist., 29, 373-405.

Schmetterer, L. (1961). Stochastic approximation. Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Vol. I, 587-609.

Wolfowitz, J. (1956). On stochastic approximation methods. Ann. Math. Statist., 27, 1151-1156.

SIMULATIONS OF SENSITIVITY EXPERIMENTS

Cramer, E. M. (1962). A comparison of three methods of fitting the normal ogive. Psychometrika, 27, 183-192.

Cramer, E. M. (1964). Some comparisons of methods of fitting the dosage response curve for small samples. J. Amer. Statist. Assoc., 59, 779-793.

Davis, M. (1965). Sequential experiments in bioassay. Presented at ASA-Biom. Soc.--IMS meeting, Florida State Univ., Tallahassee, 30 April 1965, 46 pp.

Gayle, J. B. (1963). Computer simulation study of Bruceton and probit methods of sensitivity testing. MTP-P&VE-M-63-2, NASA, Marshall Space Flight Center, Huntsville.

Hampton, L. D. (1951). Bruceton test results with non-normal populations. NAVORD Report 2129, White Oak, 5 pp.

Martin, J. W. and Saunders, J. (1963). Bruceton tests: Results of a computer study on small sample accuracy. Presented at the International Conference on Sensitivity and Hazards of Explosives, London, 1-3 October 1963, 21 pp.

Odell, P. L. (1961). An empirical study of three stochastic approximation techniques applicable to sensitivity testing. NAVWEPS Report 7837, U. S. Naval Weapons Evaluation Facility, Albuquerque, 44 pp.

Votaw, D. F., Jr. (1948). Effects of non-normality on "staircase" methods of sensitivity testing. Statistical Research Group, Princeton.

Wetherill, G. B. (1963). Sequential estimation of quantal response curves. J. Roy. Statist. Soc. Ser. B, 25, 1-48.

ENGINEERING

Baber, B. B., Blackstone, W. R., Burton, R. A., Ku, P. M., and Russell, J. A. (1964). Lubrication research and test method development for aerospace propulsion systems. Technical Documentary Report No. APL-TDR-64-50, AFSC, Wright-Patterson Air Force Base, 135 pp.

Baber, B. B., Blackstone, W. R., and Ku, P. M. (1964). Lubrication research and test method development for aerospace propulsion systems, progress report No. 4. Report No. RS-413, Southwest Research Institutue, San Antonio, 30 pp (not releasable to OTS).

Baber, B. B. and Chang, F. (1963). LOX-lubricant impact sensitivity results for ASD Cooperative Test Program No. 3. Report No. RS-369, Southwest Research Institute, San Antonio, 60 pp.

Bulfinch, A. (1959). Characteristics of various methods of collecting data in tests of increased severity. Feltman Research and Engineering Laboratories Technical Report 2627, Picatinny Arsenal, Dover, 24 pp.

Bulfinch, A. (1963). Evaluation of various laboratory methods for determining reliability. Quality Assurance Directorate GS 6-106, Picatinny Arsenal, Dover, pp. 17-31.

Clancey, V. J. (1963). Tests applied to industrial explosives, Part 1. Summarized descriptions of tests. Royal Armament Research and Development Establishment Memorandum (X) 18/63, pp. 1-3.

Clear, A. J. (1961). Standard laboratory procedures for sensitivity brisance, and stability of explosives. Technical Report FRL-TR-25, Feltman Research Laboratory, Picatinny Arsenal, Dover, 43 pp.

Combs, L. P. (1959). Rating of the A7 and A6 Redstone engines by gas pulse iniated combustion instability, summary report. Research Memo 528/91, Rocketdyne, Canoga Park.

Culling, H. P. (1953). Statistical methods appropriate for evaluation of fuze explosive-train safety and reliability. NAVORD Report 2101, White Oak, 69 pp.

Dorough, G. D. and Gray, D. T. (1962). The SUSAN test for evaluating the sensitivity of explosives to crushing impact (U). NWL Report No. 1805. Dahlgren (Confidential).

Grant, R. L. and Van Dolah, R. W. (1961). Statistical analysis of gallery variables affecting the probability of ignition by explosives. Bureau of Mines Report of Inv. 5760, 27 pp.

Hammer, C. (1955). Statistical methods in initiator evaluation. Franklin Institute Laboratories for Research and Development, Interim Report No. I-1804-1 (Confidential).

Hampton, L. D. and Ayres, J. N. (1961). Characterization of Squib MK 1 Mod 0: determination of the statistical model. NAVWEPS Report 7347, White Oak, 25 pp.

Hartvigsen, D. E. and Vanderbeck, J. P. (1955). Sensitivity tests for fuzes. NAVORD Report 3496, China Lake.

Harvey, J. R. (1954). The "up-and-down" method for conduct and analysis of fuze sphere tests. NOTS Technical Memorandum 1787, China Lake.

Keefe, R. L. (1961). Establishing a valid test method for obtaining the zero initiation level. 7-2-15, Allegheny Ballistic Laboratory, Rocket Center.

Keefe, R. L. (1961). Investigation of test methods for determining zero iniation levels. 7-2-15, Allegheny Ballistic Laboratory, Rocket Center.

Key, C. F. and Riehl, W. A. (1963). Compatibility of materials with liquid oxygen. MTP-P&VE-M-63-14, George C. Marshall Space Flight Center, Huntsville, 94 pp (for internal use only).

Levy, P. W. (1958). The impact testing of explosives. Nature, 182, 37-39.

Ling, R. C. and Hess, W. R. (1960). An interpretation of the sensitivity of explosives. TR-21, Feltman Research Laboratories, Picatinny Arsenal, Dover, 10 pp.

Naval Ordnance Laboratory (1951). Sensitivity test of primers and detonators... NAVORD OD 5823, White Oak, 23 pp.

Naval Ordnance Laboratory (1956). The development of impact sensitivity tests at the Explosives Research Laboratory, Bruceton, Pennsylvania, during the years 1941-1945. NAVORD Rept. 4236, White Oak (Confidential).

Pennington, O. K. (1961). Standard laboratory procedures, part 1. Stability, sensitivity, physical properties, and analyses for high explosives. NAVWEPS Report 7764 (NOTS TP 2742), China Lake, pp 11-13.

Quattlebaum, C. A. (1961). A method of improving the reproducibility and confidence in the zero initiation level. 7-2-15, Allegheny Ballistic Laboratory, Rocket Center.

Rothman, D. (1959). Sequential design of multi-level pulse testing of rocket engines when each test series must terminate at the first level which gives instability. Research Report 59-28, Rocketdyne, Canoga Park, 11 pp.

Van Dolah, R. W., Hanna, N. E., and Grant, R. L. (1963). Effect of natural gas concentration on ignitibility of gallery atmospheres. Bureau of Mines Report of Inv. 6340, 15 pp.

NASA-Langley, 1966 CR-62026

NASA CR-62026 – Vol. II
R-6152

THE DESIGN AND ANALYSIS OF SENSITIVITY EXPERIMENTS

VOLUME II

By D. Rothman, M. J. Alexander,
and J. M. Zimmerman

May 1965

Distribution of this report is provided in the interest of information exchange. Responsibility for the contents resides in the author or organization that prepared it.

Prepared under Contract No. NAS 8-11061 by
ROCKETDYNE
Canoga Park, Calif.

for George C. Marshall Space Flight Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – Price $5.00

APPENDIX

V-C. SIMULATION PROGRAM DESCRIPTION AND RESULTS

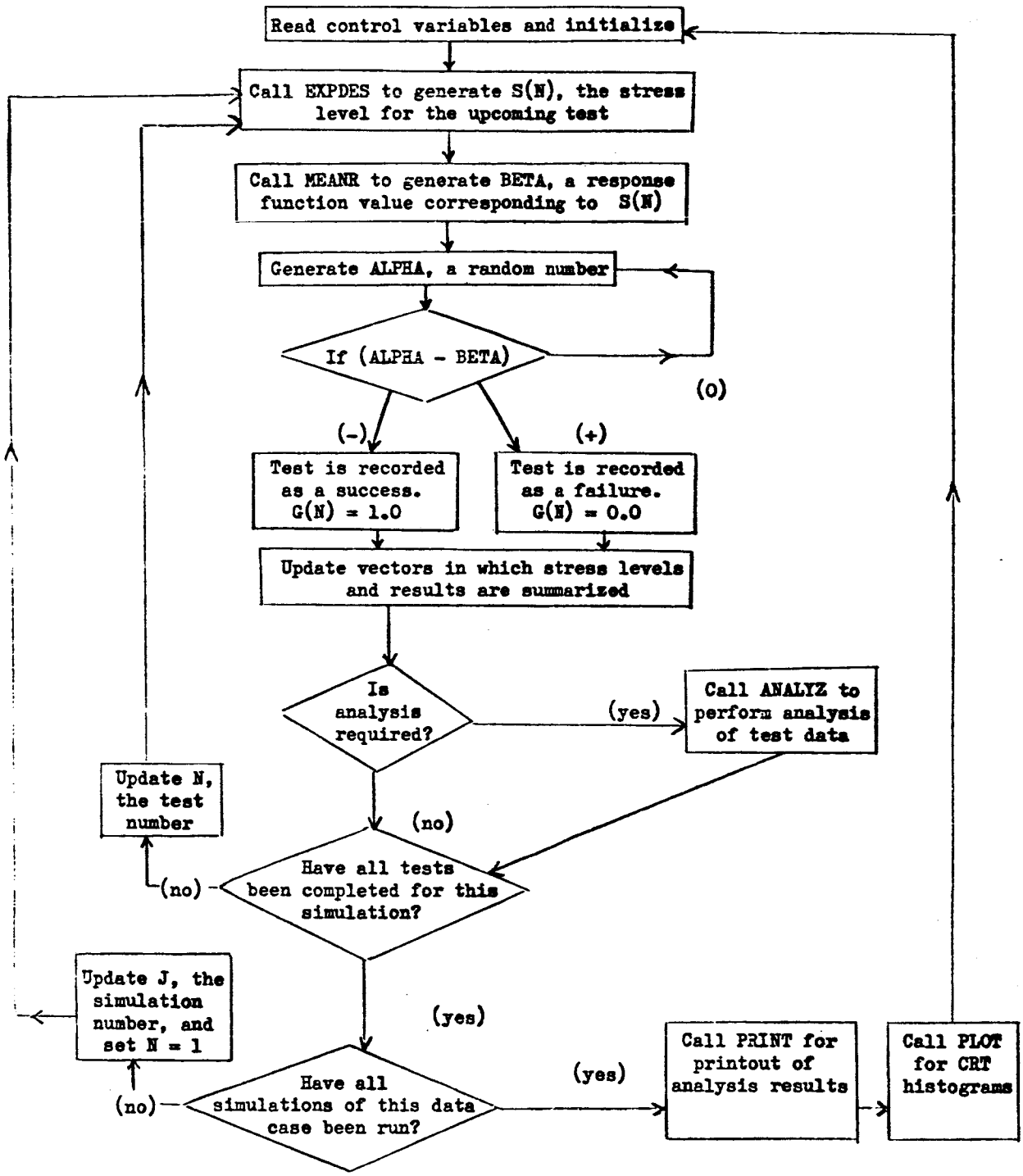
V-C SIMULATION PROGRAM DESCRIPTION AND RESULTS

a. Introduction

Simulation of the design, test, and analysis phases of numerous types of sensitivity experiments has been performed by means of a general purpose, FØRTRAN II computer program, coded for the IBM 7094. To date it has been used to simulate the Bartlett, run-down, 20-ZIL, block sequential, Alexander, and Rothman designs. Response functions which have been used in conjunction with these designs include the cumulative normal and cumulative uniform, those of Odell (Ref. 8), and several special functions. Maximum-likelihood estimation, predicated on the normal response function, and the method of reversals are the analysis techniques which have received the most extensive evaluation.

A modular concept has been employed in coding the program, which consists of a relatively small main program and a large number of specialized subroutines. These routines may be categorized roughly as design, response function, analysis, or utility subroutines. In the flow diagram of the main program (Fig. 1), five SUBROUTINE Call statements are indicated. Three of these subroutines (EXPDES, MEANR, and ANALYZ) are essentially "buffer" routines whose function is to call the specific routine required to implement, respectively, the particular experimental design, response function, or analysis specified by means of input control variables. The remaining two routines, PRINT and PLOT, are utility routines which handle program output. Utilization of a modular structure has made it possible to provide a wide range of design-response function combinations and to facilitate program growth (addition, deletion, or modification of subroutines to implement different types of designs, etc.).

Fig. 1. Flow Diagram of Sensitivity Experiment Simulation Main Program



b. General Program Capabilities

A maximum of 100 simulations, each consisting of up to 1024 tests, may be processed as a single data case (i.e., one specific combination of design, response function, and analysis). Results over the desired number of simulations are cumulatively analyzed at test numbers which are represented by 2^k, for any specified values of k greater than one and less than or equal to 10. Alternatively, analysis may be started when some other specified minimum number of tests have been completed. Results from a maximum of nine analyses may be stored for printout at the completion of the data case. The present program is limited to situations in which no more than two response function quantities are to be estimated.

c. Experimental Design Simulation

Sensitivity experimental design methods which at present may be simulated with the program are the Bartlett, run-down, 20-ZIL, block sequential, Alexander, and Rothman designs. Each of these is implemented by a separate subroutine called by subroutine EXPDES. A feature provided in each of these routines is the randomization of the first test level about an initial input mean level using a standard derivation which is furnished as program input; this device corresponds to the "first guess" made in actual testing. Subsequent test levels are furnished by the design routine on the basis of the programmed design criteria.

d. Simulation of Testing

Sensitivity testing is simulated by the program in the following manner. Given a stress level furnished by the design routine, the corresponding response value, BETA, is computed by the appropriate response function routine. Then BETA is tested against ALPHA, a random number (uniformly distributed in [0,1]) generated within the program, yielding either a response or a nonresponse on the given test.

Nine response function are available in the sensitivity experiment simulation program, each implemented by a separate routine called for through subroutine MEANR. These response functions are as follows:

1. cumulative normal; $\mu = 0$, $\sigma = 1$
2. cumulative uniform; x, $0 \le x \le 1$
3. The five functions given by Odell (Ref. 8)

   (a) $x^{\frac{1}{4}}$, $0 \le x \le 1$

   (b) $\begin{cases} 4x^2, & 0 \le x \le \frac{1}{4} \\ 1-4(1-x)^2/3, & \frac{1}{4} \le x \le 1 \end{cases}$

   (c) $\begin{cases} 2x^2, & 0 \le x \le \frac{1}{2} \\ 1-2(1-x)^2, & \frac{1}{2} \le x \le 1 \end{cases}$

   (d) $\begin{cases} 4x^2/3, & 0 \le x \le \frac{3}{4} \\ 1-4(1-x)^2, & \frac{3}{4} \le x \le 1 \end{cases}$

   (e) x^4, $0 \le x \le 1$
4. Two special function whose property of interest is that, at zero stress level ($M(0) = .5$ in both cases), the derivatives of these function are zero and infinity, respectively:

   (a) $.5 + x^5$, $-.87056 \le x \le .87056$

   (b) $\begin{cases} .5, & x = 0 \\ .5 + \dfrac{1}{2+|x|^{-.2}}, & x > 0 \\ .5 - \dfrac{1}{2+|x|^{-.2}}, & x < 0 \end{cases}$

e. Analysis Procedures

At the time of each analysis the results of J simulations, each consisting of 2^k tests ($2 \leq k \leq 10$), are available in core storage. For each simulation, estimates of the desired parameters based on these 2^k tests are computed. Various characteristics of the response function can be estimated, e.g., the stimulus levels corresponding to the 1% and 0.1% response levels, or the mean and standard derivation of the distribution.

From these estimates the program then computes, over the J simulations, the mean, median, standard error, and maximum and minimum values for each parameter estimate. These are then printed out in the analysis summary along with the true value of each parameter. When there are two parameters, their sample covariance matrix is also provided. Several examples of this output are given in the following section.

Maximum-likelihood estimation predicated on the cumulative normal response function (Section IV-B1) and the method of reversals (Section IV-B6) are the two types of analysis which have been used most extensively within the sensitivity experiment simulation program. The analysis method used and the parameters estimated for each type of design are shown in Table 3.

Maximum-likelihood estimation is implemented within the simulation program by several coordinated subroutines. An independent program version of these subroutines, "MLE", has been formulated for general usage. Contained within MLE are the minimum overlapping subset (Section IV-B2) and two-subset (Section IV-B3) methods of obtaining initial guesses, prior to entering the iterative phase of the maximum-likelihood estimation procedure. A listing of this program, together with sample data input sheets may be found in Appendix V-A.

The method of reversals is implemented within the simulation program by subroutine, "MOR". A listing of subroutine MOR is included in Appendix V-A. Application of subroutine MOR outside the simulation program may be accomplished simply by coding a small main program to furnish data to this routine and print its results.

Design	Analysis Type	Parameter 1	Parameter 2
Block Sequential Two Blocks: (m_1, m_2)	Maximum Likelihood Estimates Predicated on Normal cdf Response	mean (μ)	Standard Dev. (σ)
Run down (200)	MLE (")	$x_{.01}(\mu-2.326\sigma)$	$x_{.001}(\mu-3.090\sigma)$
20 ZIL (3 replications)	MLE (") or Level below lowest response	$x_{.01}(\mu-2.326\sigma)$ or Safe Firing Level (level below lowest response)	$x_{.001}(\mu-3.090\sigma)$ or $x_{.01222}(u-2.25\sigma)$
Bartlett	MLE (")	$x_{.01}(\mu-2.326\sigma)$	$x_{.001}(\mu-3.090\sigma)$
Alexander	Method of Reversals (MOR) with linear interpolation	x_α	Required Sample Size
Rothman	Method of Reversals (MOR) using centers of approximate 50% confidence intervals	x_α	

Table 3. Summary of Simulations

f. Results of Simulations

The computer output summary pages for the simulations are ordered primarily according to the sensitivity design being evaluated, as in the above table. Each set of results for a design is ordered in turn according to response functions, as in the table of these functions given above.

The sample sizes are not given in the summary pages if the particular simulated experiments were actually terminated.

The 20ZIL design replicated three times generally takes about 100-110 tests. Thus if sample size 64 is given, probably none of the simulated experiments were actually finished, whereas if sample size 128 is given, almost all of the simulations were actually finished at a smaller sample size.

In the following tables the sample variances of the estimators over each set of simulations are summarized and compared with the asymptotic or approximate minimum for all inverse response problems in which the (known or unknown) response function is cumulative normal. Simulations for other response functions would be evaluated in the same way, using the methods of Sections III-A4, 5.

Block Sequential Design for 50%

Cumulative Normal Inverse Response Problem

Design (size of first block size of second block)	Prior information	Asymptotic minimum ($\pi c^2/2N$)	"Growth of information" minimum	Actual variance in simulations
(1, 2)	2	.52	.351	.45
	.02	.52	2.0	8.8
(2, 1)	2	.52	.353	.42
	.02	.52	2.2	7.5
(1, 4)	2	.314	.255	.41
	.02	.314	1.1	4.9
(2, 3)	2	.314	.254	.34
	.02	.314	1.0	3.3
(3, 2)	2	.314	.255	.29
	.02	.314	1.1	3.2
(8, 24)	2	.049	.051	.080
	.02	.049	.091	.28

Rundown Design (N = 200),

20ZIL Design (N = 125, roughly), and

Bartlett Design for 1% and .1% Cumulative Normal Inverse Response Problem

Design		Prior σ	Asymptotic minimum for 1% problem ($8.89\sigma^2/N$)	Actual variance of 1% estimator	Asymptotic minimum for .1% problem ($15.69\sigma^2/N$)	Actual variance of 1% estimator in simulations
Rundown $\Delta = .5$		2	.044	.29	.078	.63
Rundown $\Delta = 1.0$		2	.044	.27	.078	.57
20ZIL $\Delta = 1.0$		.2	.071	.22	.13	.57
20ZIL $\Delta = .5$		.2	.071	.18	.13	.43
20ZIL $\Delta = 1.0$		1	.071	.19	.13	.49
20ZIL $\Delta = .5$		1	.071	.14	.13	.37
Bartlett $\Delta = .5$	N					
	32	.2	.28	.62	.49	1.4
	64		.14	.25	.25	.63
	128		.069	.11	.12	.27
	256		.035	.070	.061	.18
	512		.017	.036	.031	.086
	1024		.0087	.025	.015	.050
Bartlett $\Delta = 1.0$	32	.2	.28	.64	.49	1.5
	64		.14	.23	.25	.58
	128		.069	.13	.12	.31
	256		.035	.074	.061	.17
	512		.017	.059	.031	.13
	1024		.0087	.065	.015	.097

Alexander Design and Rothman Design for Nonparametric 50%, 5%, and 1% Inverse Response Problems When Response Function is Actually Cumulative Normal

Design	Prior σ	Fixed or median required* sample size	Asymptotic minimum for 50% problem ($\pi\sigma^2/2N$)	Actual variance of 50% estimator in simulations
Alexander (.5) $\Delta = .1$	1	17	.092	.14**
Rothman (.5)	1	16	.098	.15
Alexander (.5) $\Delta = 1$	10	14	.11	.39
Rothman (.5)	10	16	.098	.38
			Asymptotic minimum for 5% problem ($4.46\sigma^2/N$)	Actual variance of 5% estimator in simulations
Alexander (.05) $\Delta = .025$	1	62	.072	.10
Rothman (.05)	1	64	.070	.092
Alexander (.05) $\Delta = .25$	10	65	.069	.23
Rothman (.05)	10	64	.070	.14
			Asymptotic minimum for 1% problem ($13.91\sigma^2/N$)	Actual variance of 1% estimator in simulations
Rothman (.01)	1	256	.054	.17***
Rothman (.01)	10	256	.054	.072

*It is believed that $\overline{(\text{estimator error})^2(\text{required sample size})}$ is better approximated by $\overline{(\text{estimator error})^2}(\text{median required sample size})$ than by $\overline{(\text{est. err.})^2} \cdot \overline{(\text{req. sam. size})}$.

**Of course, the estimators would be better if the experimentor made use of the hypothesis of normality.

Footnotes (continued)

***The term $+1.167w(\alpha-\alpha_r)$ in the Rothman design is responsible for this anomaly. For very large sample sizes the coefficient should be changed to $1/M'(x_\alpha)$; the quantity $M'(x_\alpha)$ should be estimated from a "local" curve-fit of the data. At any rate, since $M'(x_\alpha) = .02668$ in this case, the simulations with the "head start" $(w = 1)$ eventually had too small step-sizes to "reach" $x_{.01}$; almost all of the variance of .17 in the table was due to a bias of .39.

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=2.0 (1, 2)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	-1.0142E-01	2.9541E-01
SAMPLE MEAN OF THE ESTIMATES	-1.3469E-01	5.7756E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.7388E-01	6.9419E-01
SMALLEST ESTIMATE	-1.7871E 00	2.3082E-01
LARGEST ESTIMATE	1.6810E 00	2.7150E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.5411E-01	6.4810E-02
6.4810E-02	4.8190E-01

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=2.0 (2, 1)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	-1.3252E-01	3.5162E-01
SAMPLE MEAN OF THE ESTIMATES	-1.2823E-01	5.5302E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.4896E-01	6.0659E-01
SMALLEST ESTIMATE	-1.9403E 00	2.2882E-01
LARGEST ESTIMATE	1.1892E 00	2.4508E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.2115E-01	6.2861E-02
6.2861E-02	3.6795E-01

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=2.0 (1, 4)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	8.0146E-02	2.0349E-01
SAMPLE MEAN OF THE ESTIMATES	3.9667E-02	3.6544E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.3834E-01	7.5707E-01
SMALLEST ESTIMATE	-1.2838E 00	2.9208E-02
LARGEST ESTIMATE	1.8346E 00	2.3989E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.0748E-01	2.3833E-02
2.3833E-02	5.7315E-01

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=2.0 (2, 3)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	-6.3785E-02	2.2554E-01
SAMPLE MEAN OF THE ESTIMATES	-8.8046E-02	3.8494E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	5.8515E-01	7.4994E-01
SMALLEST ESTIMATE	-1.6604E 00	4.6022E-02
LARGEST ESTIMATE	1.2860E 00	1.8619E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

3.4240E-01	1.7342E-02
1.7342E-02	5.6242E-01

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=2.0 (3, 2)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	5.5183E-03	2.7508E-01
SAMPLE MEAN OF THE ESTIMATES	-1.5634E-02	4.7732E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	5.4048E-01	7.3391E-01
SMALLEST ESTIMATE	-1.5388E 00	7.5464E-02
LARGEST ESTIMATE	1.3397E 00	2.4990E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

2.9212E-01	2.5364E-02
2.5364E-02	5.3862E-01

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=2.0 (8,24)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES OF MU, AND SIGMA**
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	1.3653E-02	4.1355E-01
SAMPLE MEAN OF THE ESTIMATES	8.3913E-03	5.1041E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.8364E-01	5.6652E-01
SMALLEST ESTIMATE	-1.2369E 00	7.4490E-02
LARGEST ESTIMATE	8.9744E-01	1.4063E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

8.0449E-02	-9.9455E-03
-9.9455E-03	3.2095E-01

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=.02 (1, 2)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	-1.8608E-02	7.7904E 00
SAMPLE MEAN OF THE ESTIMATES	2.9206E-02	5.8556E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.9595E 00	5.5003E 00
SMALLEST ESTIMATE	-7.0723E 00	2.3082E 00
LARGEST ESTIMATE	6.7387E 00	7.7904E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

8.7585E 00	7.7429E-01
7.7429E-01	3.0253E 01

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=.02 (2 1)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	1.6197E-01	3.5162E 00
SAMPLE MEAN OF THE ESTIMATES	-6.5448E-02	4.9572E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.7405E 00	4.3439E 00
SMALLEST ESTIMATE	-7.5544E 00	2.2882E 00
LARGEST ESTIMATE	7.5892E 00	6.9094E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

7.5101E 00	-2.4775E-01
-2.4775E-01	1.8870E 01

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=.02 (1 4)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	5.9929E-02	2.0349E 00
SAMPLE MEAN OF THE ESTIMATES	1.1383E-01	3.7871E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.2168E 00	3.6111E 00
SMALLEST ESTIMATE	-8.2297E 00	2.9208E-01
LARGEST ESTIMATE	5.8450E 00	6.4421E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.9140E 00	5.3138E-01
5.3138E-01	1.3040E 01

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=.02 (2 3)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	-2.5403E-01	2.0085E 00
SAMPLE MEAN OF THE ESTIMATES	-6.8436E-02	2.4242E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.8112E 00	2.0666E 00
SMALLEST ESTIMATE	-4.6486E 00	4.6022E-01
LARGEST ESTIMATE	5.3874E 00	5.3423E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

3.2805E 00	2.1677E-01
2.1677E-01	4.2708E 00

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=.02 (3 2)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	4.4466E-01	2.7508E 00
SAMPLE MEAN OF THE ESTIMATES	3.6883E-01	2.6726E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.7951E 00	2.2935E 00
SMALLEST ESTIMATE	-4.7385E 00	7.5464E-01
LARGEST ESTIMATE	4.2112E 00	5.1000E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

3.2224E 00	1.4389E 00
1.4389E 00	5.2603E 00

EXPERIMENTAL DESIGN-- BLOCK SEQ., BLK. MEMBERS EQUISPACED IN PROB., PRIOR INFO.=.02 (8 24)
ANALYTICAL PROCEDURE-- **MAXIMUM LIKELIHOOD ESTIMATES** OF MU AND SIGMA
RESPONSE FUNCTION-- CUMULATIVE NORMAL RESPONSE FUNCTION, MU=0, SIG=1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 MU	PARAMETER 2 SIG
TRUE VALUE	0.	1.0000E 00
SAMPLE MEDIAN OF THE ESTIMATES	1.2676E-01	5.0076E-01
SAMPLE MEAN OF THE ESTIMATES	9.5602E-02	5.3982E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	5.2648E-01	5.9928E-01
SMALLEST ESTIMATE	-1.2411E 00	5.7217E-02
LARGEST ESTIMATE	2.6065E 00	2.0410E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

2.7718E-01	-5.7557E-02
-5.7557E-02	3.5914E-01

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 2.0 AND STEP SIZE = 0.5
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MOS + MLE)
RESPONSE FUNCTION-- CUMULATIVE NORMAL
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN OF THE ESTIMATES	-2.1379E 00	-2.8223E 00
SAMPLE MEAN OF THE ESTIMATES	-2.1437E 00	-2.8173E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	5.4279E-01	7.9287E-01
SMALLEST ESTIMATE	-3.7407E 00	-5.0457E 00
LARGEST ESTIMATE	-5.8688E-01	-5.8688E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

2.9462E-01	4.2622E-01
4.2622E-01	6.2864E-01

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 2.0 AND STEP SIZE = 1.0
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MOS + MLE)
RESPONSE FUNCTION-- CUMULATIVE NORMAL
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN OF THE ESTIMATES	-2.1236E 00	-2.7772E 00
SAMPLE MEAN OF THE ESTIMATES	-2.2096E 00	-2.8960E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	5.2083E-01	7.5751E-01
SMALLEST ESTIMATE	-4.7539E 00	-6.3449E 00
LARGEST ESTIMATE	-1.0625E 00	-1.2264E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

2.7127E-01	3.9035E-01
3.9035E-01	5.7381E-01

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.25
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MOS + MLE)
RESPONSE FUNCTION-- CUMULATIVE UNIFORM
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-03	10.0000E-04
SAMPLE MEDIAN OF THE ESTIMATES	-1.7359E-01	-3.8119E-01
SAMPLE MEAN OF THE ESTIMATES	-1.9709E-01	-4.0731E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.8012E-01	4.8695E-01
SMALLEST ESTIMATE	-8.7902E-01	-1.3013E 00
LARGEST ESTIMATE	2.7511E-01	2.7511E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

7.8465E-02	1.3432E-01
1.3432E-01	2.3712E-01

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.5
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MOS + MLE)
RESPONSE FUNCTION-- CUMULATIVE UNIFORM
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-03	10.0000E-04
SAMPLE MEDIAN OF THE ESTIMATES	-1.9876E-01	-4.0716E-01
SAMPLE MEAN OF THE ESTIMATES	-2.1597E-01	-4.2932E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.8412E-01	4.9113E-01
SMALLEST ESTIMATE	-7.4586E-01	-1.1440E 00
LARGEST ESTIMATE	3.2843E-01	2.4600E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

8.0724E-02	1.3709E-01
1.3709E-01	2.4121E-01

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.25
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MOS + MLE)
RESPONSE FUNCTION-- CDELL, Y=X**0.25
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-09	1.0000E-12
SAMPLE MEDIAN OF THE ESTIMATES	-1.6457E-01	-2.4217E-01
SAMPLE MEAN OF THE ESTIMATES	-1.8686E-01	-2.7584E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.3841E-01	3.3919E-01
SMALLEST ESTIMATE	-7.5359E-01	-1.0339E 00
LARGEST ESTIMATE	6.3499E-03	-1.2210E-02

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

5.6841E-02	8.0388E-02
8.0388E-02	1.1505E-01

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.5
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MCS + MLE)
RESPONSE FUNCTION-- CDELL, Y=X**0.25
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-09	1.0000E-12
SAMPLE MEDIAN OF THE ESTIMATES	-1.9885E-01	-2.8528E-01
SAMPLE MEAN OF THE ESTIMATES	-2.6906E-01	-3.7659E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	3.5316E-01	4.7605E-01
SMALLEST ESTIMATE	-1.2689E 00	-1.6939E 00
LARGEST ESTIMATE	2.5637E-02	-2.1074E-02

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.2472E-01	1.6684E-01
1.6684E-01	2.2662E-01

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.25
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MCS + MLE)
RESPONSE FUNCTION-- GDELL, Y=4.*X**2 IF X BELOW OR =.25, OR Y=1.-4.*(1.-X)**2/3.
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	5.0000E-02	1.5800E-02
SAMPLE MEDIAN OF THE ESTIMATES	-1.0659E-02	-1.2711E-01
SAMPLE MEAN OF THE ESTIMATES	-2.6917E-02	-1.5473E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.1747E-01	2.1145E-01
SMALLEST ESTIMATE	-3.4195E-01	-5.6597E-01
LARGEST ESTIMATE	1.5131E-01	7.1113E-02

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.3798E-02	2.3939E-02
2.3939E-02	4.4713E-02

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.5
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MCS + MLE)
RESPONSE FUNCTION-- DDLLL, Y=4.*X**2 IF X BELOW OR =.25, OR Y=1.-4.*(1.-X)**2/3.
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	5.0000E-02	1.5800E-02
SAMPLE MEDIAN OF THE ESTIMATES	-2.5654E-02	-1.5746E-01
SAMPLE MEAN OF THE ESTIMATES	-5.2804E-02	-1.9180E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.4880E-01	2.5383E-01
SMALLEST ESTIMATE	-5.0090E-01	-7.9547E-01
LARGEST ESTIMATE	1.2741E-01	6.2382E-02

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

2.2142E-02	3.6772E-02
3.6772E-02	6.4431E-02

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.25
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MCS + MLE)
RESPONSE FUNCTION-- UDELL, Y=2.*X**2 IF X BELOW CR =.5, CR Y=1.-2.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	7.0700E-02	2.2360E-02
SAMPLE MEDIAN OF THE ESTIMATES	3.0608E-02	-1.1098E-01
SAMPLE MEAN OF THE ESTIMATES	2.6078E-02	-1.2297E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	9.4195E-02	1.8776E-01
SMALLEST ESTIMATE	-2.4373E-01	-5.1754E-01
LARGEST ESTIMATE	2.0405E-01	1.1757E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

8.8727E-03	1.6154E-02
1.6154E-02	3.5254E-02

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.5
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MCS + MLE)
RESPONSE FUNCTION-- CASE1, Y=2.*X**2 IF X BELOW OR =.5, OR Y=1.-2.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	7.0700E-02	2.2360E-02
SAMPLE MEDIAN OF THE ESTIMATES	5.2521E-02	-9.7107E-02
SAMPLE MEAN OF THE ESTIMATES	4.0284E-02	-1.0796E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.0124E-01	1.8613E-01
SMALLEST ESTIMATE	-2.0970E-01	-4.5241E-01
LARGEST ESTIMATE	3.4565E-01	2.7450E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.0250E-02	1.6714E-02
1.6714E-02	3.4644E-02

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.25
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MOS + MLE)
RESPONSE FUNCTION-- LDELL, Y=(4.*X**2)/3. IF X BELOW OR =.75, OR Y=1.-4.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	8.6600E-02	2.7400E-02
SAMPLE MEDIAN OF THE ESTIMATES	9.0867E-02	-5.7469E-02
SAMPLE MEAN OF THE ESTIMATES	9.0454E-02	-6.8152E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.1544E-01	1.8602E-01
SMALLEST ESTIMATE	-1.9177E-01	-4.5597E-01
LARGEST ESTIMATE	4.5663E-01	4.5663E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.3328E-02	1.7745E-02
1.7745E-02	3.4604E-02

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.5
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MCS + MLE)
RESPONSE FUNCTION-- CDELL, Y=(4.*X**2)/3. IF X BELOW OR =.75, OR Y=1.-4.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	8.6600E-02	2.7400E-02
SAMPLE MEDIAN OF THE ESTIMATES	6.6436E-02	-9.7078E-02
SAMPLE MEAN OF THE ESTIMATES	6.7982E-02	-9.2661E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.2385E-01	2.0113E-01
SMALLEST ESTIMATE	-2.4438E-01	-5.2555E-01
LARGEST ESTIMATE	4.4146E-01	4.0336E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.5338E-02	2.1312E-02
2.1312E-02	4.0452E-02

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.25
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MOS + MLE)
RESPONSE FUNCTION-- ODELL, Y=X**4
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	3.1620E-01	1.7800E-01
SAMPLE MEDIAN OF THE ESTIMATES	4.0097E-01	2.6943E-01
SAMPLE MEAN OF THE ESTIMATES	4.0320E-01	2.6814E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.2085E-01	1.4701E-01
SMALLEST ESTIMATE	2.0583E-01	-2.4785E-02
LARGEST ESTIMATE	5.5625E-01	4.9247E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.4605E-02	1.7529E-02
1.7529E-02	2.1612E-02

EXPERIMENTAL DESIGN-- RUNDOWN, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.5
ANALYTICAL PROCEDURE-- 1 AND 0.1 PERCENT RESPONSE LEVELS (MU, SIGMA FROM MCS + MLE)
RESPONSE FUNCTION-- ODELL, Y=X**4
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	3.1620E-01	1.7800E-01
SAMPLE MEDIAN OF THE ESTIMATES	4.0148E-01	2.7874E-01
SAMPLE MEAN OF THE ESTIMATES	4.1829E-01	2.9271E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.5024E-01	1.9214E-01
SMALLEST ESTIMATE	1.2797E-01	-1.4711E-01
LARGEST ESTIMATE	6.8839E-01	6.3249E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

2.2573E-02	2.8534E-02
2.8534E-02	3.6919E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 1.0
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CUMULATIVE NORMAL
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		-2.2500E 00
SAMPLE MEDIAN OF THE ESTIMATES	-2.8398E 00	-2.2400E 00
SAMPLE MEAN OF THE ESTIMATES	-2.6561E 00	-2.2537E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.9272E-01	4.3731E-01
SMALLEST ESTIMATE	-3.9125E 00	-3.1903E 00
LARGEST ESTIMATE	-1.0768E 00	-1.0768E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.7986E-01	1.9422E-01
1.9422E-01	1.9124E-01

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2314E 00	-2.9043E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.2774E 00	-2.8918E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	4.7248E-01	7.5428E-01
SMALLEST ESTIMATE	-3.7508E 00	-5.1133E 00
LARGEST ESTIMATE	-9.9610E-01	-9.9610E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.2324E-01	3.2415E-01
3.2415E-01	5.6894E-01

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 0.5
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CUMULATIVE NORMAL
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		-2.2500E 00
SAMPLE MEDIAN OF THE ESTIMATES	-2.3065E 00	-2.1609E 00
SAMPLE MEAN OF THE ESTIMATES	-2.2928E 00	-2.1786E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	3.5981E-01	3.9162E-01
SMALLEST ESTIMATE	-3.1727E 00	-3.4883E 00
LARGEST ESTIMATE	-1.4242E 00	-1.2631E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.2946E-01	1.0455E-01
1.0455E-01	1.5337E-01

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 0.5
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 128 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2774E 00	-2.9418E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.2500E 00	-2.9334E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	4.2180E-01	6.5485E-01
SMALLEST ESTIMATE	-3.2204E 00	-4.5483E 00
LARGEST ESTIMATE	-1.2964E 00	-1.6324E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.7792E-01	2.6815E-01
2.6815E-01	4.2883E-01

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 1.0
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CUMULATIVE NORMAL
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		-2.2500E 00
SAMPLE MEDIAN OF THE ESTIMATES	-2.5491E 00	-2.2360E 00
SAMPLE MEAN OF THE ESTIMATES	-2.5870E 00	-2.2184E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.7986E-01	4.4186E-01
SMALLEST ESTIMATE	-4.0128E 00	-3.4467E 00
LARGEST ESTIMATE	-9.3428E-01	-9.3428E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.6221E-01	1.8570E-01
1.8570E-01	1.9524E-01

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2494E 00	-2.7874E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3092E 00	-2.9058E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	4.3910E-01	6.9887E-01
SMALLEST ESTIMATE	-3.5558E 00	-5.5942E 00
LARGEST ESTIMATE	-1.1818E 00	-1.1818E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.9281E-01	2.6535E-01
2.6535E-01	4.8843E-01

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 1.0 AND STEP SIZE = 0.5
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CUMULATIVE NORMAL
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		-2.2500E 00
SAMPLE MEDIAN OF THE ESTIMATES	-2.3254E 00	-2.1597E 00
SAMPLE MEAN OF THE ESTIMATES	-2.3279E 00	-2.1822E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	4.0525E-01	3.6622E-01
SMALLEST ESTIMATE	-3.3228E 00	-3.2736E 00
LARGEST ESTIMATE	-1.4192E 00	-1.3522E 00

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.6423E-01	1.1672E-01
1.1672E-01	1.3412E-01

RESPØNSE FUNCTION-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 128 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2308E 00	-2.9177E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.2630E 00	-2.9510E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.7461E-01	6.1128E-01
SMALLEST ESTIMATE	-3.4032E 00	-4.8747E 00
LARGEST ESTIMATE	-1.4085E 00	-1.6824E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.4033E-01	2.2112E-01
2.2112E-01	3.7366E-01

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CUMULATIVE UNIFORM
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		1.2220E-02
SAMPLE MEDIAN OF THE ESTIMATES	-9.8562E-02	-6.2920E-02
SAMPLE MEAN OF THE ESTIMATES	-8.3491E-02	-6.2632E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.1466E-01	1.0301E-01
SMALLEST ESTIMATE	-1.8588E-01	-2.2585E-01
LARGEST ESTIMATE	1.0916E-01	1.0269E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.3147E-02	9.6287E-03
9.6287E-03	1.0612E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE UNIFØRM
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-03	10.0000E-04
SAMPLE MEDIAN ØF THE ESTIMATES	-5.7321E-02	-2.0966E-01
SAMPLE MEAN ØF THE ESTIMATES	-6.8027E-02	-2.2087E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.0531E-01	2.5144E-01
SMALLEST ESTIMATE	-2.3720E-01	-5.6583E-01
LARGEST ESTIMATE	4.1032E-02	2.4946E-02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.1090E-02	2.4518E-02
2.4518E-02	6.3224E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CUMULATIVE UNIFORM
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		1.2220E-02
SAMPLE MEDIAN OF THE ESTIMATES	-2.4313E-01	-1.5212E-01
SAMPLE MEAN OF THE ESTIMATES	-2.0301E-01	-1.4465E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.5040E-01	1.9498E-01
SMALLEST ESTIMATE	-3.8760E-01	-4.0386E-01
LARGEST ESTIMATE	1.3082E-01	1.3082E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

6.2700E-02	4.5655E-02
4.5655E-02	3.8017E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE UNIFØRM
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-03	10.0000E-04
SAMPLE MEDIAN ØF THE ESTIMATES	-1.3842E-01	-2.7804E-01
SAMPLE MEAN ØF THE ESTIMATES	-1.3996E-01	-2.6197E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.9601E-01	3.2004E-01
SMALLEST ESTIMATE	-3.8806E-01	-6.6320E-01
LARGEST ESTIMATE	1.3532E-01	1.2001E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.8420E-02	5.7163E-02
5.7163E-02	1.0243E-01

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CUMULATIVE UNIFORM
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		1.2220E-02
SAMPLE MEDIAN OF THE ESTIMATES	-7.6834E-02	-5.3905E-02
SAMPLE MEAN OF THE ESTIMATES	-7.4014E-02	-5.5689E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.0788E-01	9.8341E-02
SMALLEST ESTIMATE	-1.8844E-01	-2.0139E-01
LARGEST ESTIMATE	1.7922E-01	1.2615E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.1638E-02	8.1311E-03
8.1311E-03	9.6710E-03

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE UNIFØRM
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-03	10.0000E-04
SAMPLE MEDIAN ØF THE ESTIMATES	-7.5709E-02	-2.1790E-01
SAMPLE MEAN ØF THE ESTIMATES	-7.1049E-02	-2.1202E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.0912E-01	2.4762E-01
SMALLEST ESTIMATE	-2.4993E-01	-5.3409E-01
LARGEST ESTIMATE	9.9586E-02	1.7687E-02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.1906E-02	2.5033E-02
2.5033E-02	6.1316E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CUMULATIVE UNIFORM
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		1.2220E-02
SAMPLE MEDIAN OF THE ESTIMATES	-1.5809E-01	-1.1126E-01
SAMPLE MEAN OF THE ESTIMATES	-1.5986E-01	-1.2293E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.1454E-01	1.8159E-01
SMALLEST ESTIMATE	-3.9333E-01	-3.7189E-01
LARGEST ESTIMATE	1.2123E-01	1.2123E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.6027E-02	3.5679E-02
3.5679E-02	3.2975E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE UNIFØRM
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-03	10.0000E-04
SAMPLE MEDIAN ØF THE ESTIMATES	-1.1374E-01	-2.1789E-01
SAMPLE MEAN ØF THE ESTIMATES	-1.2211E-01	-2.2514E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.7819E-01	2.7856E-01
SMALLEST ESTIMATE	-4.1916E-01	-7.1540E-01
LARGEST ESTIMATE	1.1811E-01	8.4889E-02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.1753E-02	4.4154E-02
4.4154E-02	7.7598E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=X**0.25
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		2.2300E-08
SAMPLE MEDIAN OF THE ESTIMATES	-1.0202E-01	-1.0655E-01
SAMPLE MEAN OF THE ESTIMATES	-9.7964E-02	-1.0331E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.1472E-01	1.2455E-01
SMALLEST ESTIMATE	-1.9901E-01	-2.7667E-01
LARGEST ESTIMATE	-1.3045E-03	2.9138E-02

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.3160E-02	1.3978E-02
1.3978E-02	1.5513E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=X**0.25
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-09	1.0000E-12
SAMPLE MEDIAN ØF THE ESTIMATES	-1.2856E-01	-1.5418E-01
SAMPLE MEAN ØF THE ESTIMATES	-1.1807E-01	-1.7673E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.3894E-01	2.1513E-01
SMALLEST ESTIMATE	-3.0611E-01	-5.0225E-01
LARGEST ESTIMATE	1.2095E-02	-3.2115E-04

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.9305E-02	2.8752E-02
2.8752E-02	4.6280E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- COELL, Y=X**0.25
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		2.2300E-08
SAMPLE MEDIAN OF THE ESTIMATES	-2.7076E-01	-2.7076E-01
SAMPLE MEAN OF THE ESTIMATES	-2.3037E-01	-2.3683E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.6174E-01	2.7187E-01
SMALLEST ESTIMATE	-3.9932E-01	-4.9914E-01
LARGEST ESTIMATE	-1.6423E-04	3.7862E-02

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

6.8509E-02	7.0858E-02
7.0858E-02	7.3912E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=X**0.25
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-09	1.0000E-12
SAMPLE MEDIAN ØF THE ESTIMATES	-2.8935E-01	-2.9286E-01
SAMPLE MEAN ØF THE ESTIMATES	-2.6012E-01	-2.9566E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.8926E-01	3.4008E-01
SMALLEST ESTIMATE	-5.5673E-01	-8.6914E-01
LARGEST ESTIMATE	1.2446E-03	-2.0034E-04

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

8.3669E-02	9.5612E-02
9.5612E-02	1.1566E-01

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=X**0.25
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		2.2300E-08
SAMPLE MEDIAN OF THE ESTIMATES	-1.0965E-01	-1.0505E-01
SAMPLE MEAN OF THE ESTIMATES	-1.0275E-01	-1.0257E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.2329E-01	1.2620E-01
SMALLEST ESTIMATE	-1.9991E-01	-2.6820E-01
LARGEST ESTIMATE	1.9628E-01	1.9628E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.5200E-02	1.5174E-02
1.5174E-02	1.5928E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=X**0.25
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-09	1.0000E-12
SAMPLE MEDIAN ØF THE ESTIMATES	-1.1092E-01	-1.2977E-01
SAMPLE MEAN ØF THE ESTIMATES	-1.0919E-01	-1.5835E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.3402E-01	2.0232E-01
SMALLEST ESTIMATE	-2.7285E-01	-4.5201E-01
LARGEST ESTIMATE	1.9997E-01	1.9997E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.7960E-02	2.5991E-02
2.5991E-02	4.0933E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=X**0.25
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		2.2300E-08
SAMPLE MEDIAN OF THE ESTIMATES	-2.1279E-01	-1.9890E-01
SAMPLE MEAN OF THE ESTIMATES	-2.0385E-01	-1.9972E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.3237E-01	2.3030E-01
SMALLEST ESTIMATE	-3.9813E-01	-4.9013E-01
LARGEST ESTIMATE	-6.7677E-03	9.9562E-03

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

5.3996E-02	5.3109E-02
5.3109E-02	5.3039E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=X**0.25
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	10.0000E-09	1.0000E-12
SAMPLE MEDIAN ØF THE ESTIMATES	-1.7353E-01	-2.3208E-01
SAMPLE MEAN ØF THE ESTIMATES	-2.0465E-01	-2.2656E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.3770E-01	2.6157E-01
SMALLEST ESTIMATE	-4.0481E-01	-6.0883E-01
LARGEST ESTIMATE	1.9231E-02	-3.6632E-03

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

5.6501E-02	6.0272E-02
6.0272E-02	6.8420E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=4.*X**2 IF X BELOW OR =.25, OR Y=1.-4.*(1.-X)**2/3.
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		5.5300E-02
SAMPLE MEDIAN OF THE ESTIMATES	-1.5415E-02	2.6033E-02
SAMPLE MEAN OF THE ESTIMATES	-1.3013E-02	2.7362E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.0366E-01	7.3886E-02
SMALLEST ESTIMATE	-1.7359E-01	-1.7891E-01
LARGEST ESTIMATE	2.2639E-01	2.2639E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.0745E-02	5.8529E-03
5.8529E-03	5.4592E-03

E PERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
AN LYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=4.*X**2 IF X BELØW ØR =.25, ØR Y=1.-4.*(1.-X)**2/3.
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	5.0000E-02	1.5800E-02
SAMPLE MEDIAN ØF THE ESTIMATES	1.5149E-02	-7.3570E-02
SAMPLE MEAN ØF THE ESTIMATES	1.0082E-02	-8.1275E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	7.5999E-02	1.4332E-01
SMALLEST ESTIMATE	-2.0205E-01	-3.8437E-01
LARGEST ESTIMATE	2.0273E-01	2.0273E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

5.7759E-03	9.5149E-03
9.5149E-03	2.0542E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- UDELL, Y=4.*X**2 IF X BELOW CR =.25, OR Y=1.-4.*(1.-X)**2/3.
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		5.5300E-02
SAMPLE MEDIAN OF THE ESTIMATES	-5.5132E-02	-5.2150E-02
SAMPLE MEAN OF THE ESTIMATES	-7.8020E-02	-6.7641E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.8022E-01	1.6413E-01
SMALLEST ESTIMATE	-3.6622E-01	-3.4437E-01
LARGEST ESTIMATE	1.4725E-01	1.4725E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

3.2478E-02	2.8430E-02
2.8430E-02	2.6939E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=4.*X**2 IF X BELØW ØR =.25, ØR Y=1.-4.*(1.-X)**2/3.
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	5.0000E-02	1.5800E-02
SAMPLE MEDIAN ØF THE ESTIMATES	-5.6532E-02	-6.5086E-02
SAMPLE MEAN ØF THE ESTIMATES	-6.7979E-02	-9.6534E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.5131E-01	1.6900E-01
SMALLEST ESTIMATE	-3.4367E-01	-4.0703E-01
LARGEST ESTIMATE	1.4972E-01	1.4972E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.2895E-02	2.3169E-02
2.3169E-02	2.8563E-02

EXPERIMENTAL DESIGN-- 20-21L, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- LDELL, Y=4.*X**2 IF X BELOW CR =.25, OR Y=1.-4.*(1.-X)**2/3.
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		5.5300E-02
SAMPLE MEDIAN OF THE ESTIMATES	-1.0848E-02	1.9079E-02
SAMPLE MEAN OF THE ESTIMATES	-9.5247E-03	2.0334E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	9.5105E-02	7.0361E-02
SMALLEST ESTIMATE	-1.5247E-01	-1.2156E-01
LARGEST ESTIMATE	1.4650E-01	1.4650E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

9.0449E-03	5.3899E-03
5.3899E-03	4.9506E-03

EX .RIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØ.SE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=4.*X**2 IF X BELØW ØR =.25, ØR Y=1.-4.*(1.-X)**2/3.
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	5.0000E-02	1.5800E-02
SAMPLE MEDIAN ØF THE ESTIMATES	1.3885E-02	-5.7147E-02
SAMPLE MEAN ØF THE ESTIMATES	2.1600E-02	-6.2540E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	7.4861E-02	1.3152E-01
SMALLEST ESTIMATE	-1.2330E-01	-3.5548E-01
LARGEST ESTIMATE	2.1220E-01	1.8716E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

5.6041E-03	7.9883E-03
7.9883E-03	1.7298E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- COELL, Y=4.*X**2 IF X BELOW OR =.25, OR Y=1.-4.*(1.-X)**2/3.
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		5.5300E-02
SAMPLE MEDIAN OF THE ESTIMATES	-1.2171E-01	-9.5111E-02
SAMPLE MEAN OF THE ESTIMATES	-1.2570E-01	-9.7034E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.2653E-01	1.9673E-01
SMALLEST ESTIMATE	-3.4421E-01	-3.4421E-01
LARGEST ESTIMATE	1.6347E-01	1.6347E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

5.1318E-02	4.1836E-02
4.1836E-02	3.8701E-02

EX RIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=4.*X**2 IF X BELØW ØR =.25, ØR Y=1.-4.*(1.-X)**2/3.
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	5.0000E-02	1.5800E-02
SAMPLE MEDIAN ØF THE ESTIMATES	-8.4270E-02	-1.1425E-01
SAMPLE MEAN ØF THE ESTIMATES	-8.8373E-02	-1.2049E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.8765E-01	2.0228E-01
SMALLEST ESTIMATE	-3.6814E-01	-5.0317E-01
LARGEST ESTIMATE	1.4568E-01	1.4199E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.5213E-02	3.5296E-02
3.5296E-02	4.0915E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- LOELL, Y=2.*X**2 IF X BELOW CR =.5, OR Y=1.-2.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		7.8200E-02
SAMPLE MEDIAN OF THE ESTIMATES	8.6091E-03	6.3932E-02
SAMPLE MEAN OF THE ESTIMATES	7.1195E-03	6.7356E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.1596E-01	8.1059E-02
SMALLEST ESTIMATE	-1.6928E-01	-1.4993E-01
LARGEST ESTIMATE	2.5725E-01	2.5725E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.3448E-02	6.2291E-03
6.2291E-03	6.5705E-03

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=2.*X**2 IF X BELØW ØR =.5, ØR Y=1.-2.*(1.-X)**2
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	7.0700E-02	2.2360E-02
SAMPLE MEDIAN ØF THE ESTIMATES	4.5465E-02	-6.6581E-02
SAMPLE MEAN ØF THE ESTIMATES	5.1121E-02	-6.9653E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	7.5763E-02	1.4606E-01
SMALLEST ESTIMATE	-1.5354E-01	-4.2258E-01
LARGEST ESTIMATE	2.4369E-01	2.2324E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

5.7400E-03	9.0134E-03
9.0134E-03	2.1333E-02

EXPERIMENTAL DESIGN-- 20-ZIL; INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=2.*X**2 IF X BELOW CR =.5, OR Y=1.-2.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		7.8200E-02
SAMPLE MEDIAN OF THE ESTIMATES	5.7058E-03	1.4986E-02
SAMPLE MEAN OF THE ESTIMATES	-4.0714E-02	-1.7789E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.8543E-01	1.5872E-01
SMALLEST ESTIMATE	-3.4296E-01	-2.9176E-01
LARGEST ESTIMATE	2.4320E-01	2.4320E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

3.4384E-02	2.6842E-02
2.6842E-02	2.5193E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=2.*X**2 IF X BELØW ØR =.5, ØR Y=1.-2.*(1.-X)**2
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	7.0700E-02	2.2360E-02
SAMPLE MEDIAN ØF THE ESTIMATES	-3.6444E-03	-2.1666E-02
SAMPLE MEAN ØF THE ESTIMATES	-2.9565E-02	-5.7377E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.7464E-01	1.8528E-01
SMALLEST ESTIMATE	-3.5752E-01	-4.4016E-01
LARGEST ESTIMATE	2.2651E-01	2.2651E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.0498E-02	2.9778E-02
2.9778E-02	3.4328E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=2.*X**2 IF X BELOW OR =.5, OR Y=1.-2.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		7.8200E-02
SAMPLE MEDIAN OF THE ESTIMATES	1.2528E-02	6.1058E-02
SAMPLE MEAN OF THE ESTIMATES	1.7274E-02	6.1189E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.0226E-01	7.7529E-02
SMALLEST ESTIMATE	-1.6609E-01	-1.1513E-01
LARGEST ESTIMATE	2.3261E-01	2.2564E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.0457E-02	5.3242E-03
5.3242E-03	6.0108E-03

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=2.*X**2 IF X BELØW ØR =.5, ØR Y=1.-2.*(1.-X)**2
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	7.0700E-02	2.2360E-02
SAMPLE MEDIAN ØF THE ESTIMATES	5.3318E-02	-5.7139E-02
SAMPLE MEAN ØF THE ESTIMATES	4.9286E-02	-6.4568E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.5789E-02	1.5836E-01
SMALLEST ESTIMATE	-1.7205E-01	-4.2971E-01
LARGEST ESTIMATE	2.4426E-01	2.4047E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

7.3598E-03	1.1428E-02
1.1428E-02	2.5077E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=2.*X**2 IF X BELOW CR =.5, OR Y=1.-2.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		7.8200E-02
SAMPLE MEDIAN OF THE ESTIMATES	-5.2487E-02	-3.7218E-02
SAMPLE MEAN OF THE ESTIMATES	-5.8366E-02	-4.0646E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.9150E-01	1.7240E-01
SMALLEST ESTIMATE	-3.3104E-01	-3.1148E-01
LARGEST ESTIMATE	3.0798E-01	3.0798E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

3.6673E-02	3.1347E-02
3.1347E-02	2.9723E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=2.*X**2 IF X BELØW ØR =.5, ØR Y=1.-2.*(1.-X)**2
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	7.0700E-02	2.2360E-02
SAMPLE MEDIAN ØF THE ESTIMATES	-6.5938E-02	-1.2877E-01
SAMPLE MEAN ØF THE ESTIMATES	-6.0408E-02	-1.0346E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.7628E-01	1.8781E-01
SMALLEST ESTIMATE	-2.6579E-01	-3.7162E-01
LARGEST ESTIMATE	2.2437E-01	2.2437E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.1075E-02	2.9801E-02
2.9801E-02	3.5271E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDLLL, Y=(4.*X**2)/3. IF X BELOW OR =.75, OR Y=1.-4.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		9.5700E-02
SAMPLE MEDIAN OF THE ESTIMATES	2.6696E-02	8.4157E-02
SAMPLE MEAN OF THE ESTIMATES	2.6510E-02	8.1460E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.1536E-01	8.0217E-02
SMALLEST ESTIMATE	-1.6357E-01	-1.2509E-01
LARGEST ESTIMATE	2.9554E-01	2.9554E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.3308E-02	6.0388E-03
6.0388E-03	6.4347E-03

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=(4.*X**2)/3. IF X BELØW ØR =.75, ØR Y=1.-4.*(1.-X)**2
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	8.6600E-02	2.7400E-02
SAMPLE MEDIAN ØF THE ESTIMATES	5.4878E-02	-9.4535E-02
SAMPLE MEAN ØF THE ESTIMATES	5.7949E-02	-1.0136E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.9942E-02	1.8446E-01
SMALLEST ESTIMATE	-1.2684E-01	-3.7349E-01
LARGEST ESTIMATE	2.8697E-01	2.7321E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

8.0895E-03	1.4122E-02
1.4122E-02	3.4027E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=(4.*X**2)/3. IF X BELOW OR =.75, OR Y=1.-4.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		9.5700E-02
SAMPLE MEDIAN OF THE ESTIMATES	-1.0037E-01	3.0474E-02
SAMPLE MEAN OF THE ESTIMATES	-6.6228E-02	1.1452E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.2577E-01	1.5294E-01
SMALLEST ESTIMATE	-3.2252E-01	-3.0727E-01
LARGEST ESTIMATE	2.4806E-01	2.4806E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

5.0974E-02	2.8175E-02
2.8175E-02	2.3391E-02

EX .RIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=(4.*X**2)/3. IF X BELØW ØR =.75, ØR Y=1.-4.*(1.-X)**2
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	8.6600E-02	2.7400E-02
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2117E-02	-1.3207E-01
SAMPLE MEAN ØF THE ESTIMATES	-2.0352E-02	-8.6594E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.6972E-01	2.0128E-01
SMALLEST ESTIMATE	-2.7919E-01	-4.9883E-01
LARGEST ESTIMATE	2.5483E-01	2.5483E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.8805E-02	2.9990E-02
2.9990E-02	4.0514E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=(4.*X**2)/3. IF X BELØW ØR =.75, ØR Y=1.-4.*(1.-X)**2
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 128 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	8.6600E-02	2.7400E-02
SAMPLE MEDIAN ØF THE ESTIMATES	6.5333E-02	-7.4044E-02
SAMPLE MEAN ØF THE ESTIMATES	6.7317E-02	-6.6616E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	7.7187E-02	1.5182E-01
SMALLEST ESTIMATE	-1.1947E-01	-3.4182E-01
LARGEST ESTIMATE	2.6260E-01	2.0426E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

5.9578E-03	9.0200E-03
9.0200E-03	2.3049E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=(4.*X**2)/3. IF X BELOW OR =.75, OR Y=1.-4.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		9.5700E-02
SAMPLE MEDIAN OF THE ESTIMATES	-9.1361E-02	-3.6049E-02
SAMPLE MEAN OF THE ESTIMATES	-7.2966E-02	-3.3547E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.1346E-01	1.7850E-01
SMALLEST ESTIMATE	-3.2628E-01	-3.2628E-01
LARGEST ESTIMATE	2.4268E-01	2.4268E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.5564E-02	3.3887E-02
3.3887E-02	3.1861E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=(4.*X**2)/3. IF X BELOW OR =.75, OR Y=1.-4.*(1.-X)**2
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		9.5700E-02
SAMPLE MEDIAN OF THE ESTIMATES	4.5390E-02	1.0196E-01
SAMPLE MEAN OF THE ESTIMATES	4.0159E-02	9.1260E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.0429E-01	7.8730E-02
SMALLEST ESTIMATE	-1.4524E-01	-1.5643E-01
LARGEST ESTIMATE	2.3813E-01	2.3813E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.0877E-02	4.9805E-03
4.9805E-03	6.1984E-03

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=(4.*X**2)/3. IF X BELØW ØR =.75, ØR Y=1.-4.*(1.-X)**2
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	8.6600E-02	2.7400E-02
SAMPLE MEDIAN ØF THE ESTIMATES	-5.1123E-02	-1.1756E-01
SAMPLE MEAN ØF THE ESTIMATES	-4.6081E-02	-1.0868E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.8185E-01	2.1333E-01
SMALLEST ESTIMATE	-2.9630E-01	-6.2951E-01
LARGEST ESTIMATE	3.0374E-01	3.0374E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.3069E-02	3.3966E-02
3.3966E-02	4.5512E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=X**4
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		3.3200E-01
SAMPLE MEDIAN OF THE ESTIMATES	2.6084E-01	3.4701E-01
SAMPLE MEAN OF THE ESTIMATES	2.5328E-01	3.4568E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.3414E-01	9.0331E-02
SMALLEST ESTIMATE	-1.2929E-02	1.3757E-01
LARGEST ESTIMATE	5.6653E-01	5.9900E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.7992E-02	5.7390E-03
5.7390E-03	8.1596E-03

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=X**4
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	3.1620E-01	1.7800E-01
SAMPLE MEDIAN ØF THE ESTIMATES	3.2870E-01	1.8460E-01
SAMPLE MEAN ØF THE ESTIMATES	3.3535E-01	1.8962E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	9.3976E-02	1.4489E-01
SMALLEST ESTIMATE	1.2634E-01	-2.1323E-01
LARGEST ESTIMATE	5.7393E-01	5.7393E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

8.8315E-03	1.2512E-02
1.2512E-02	2.0994E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- ODELL, Y=X**4
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		3.3200E-01
SAMPLE MEDIAN OF THE ESTIMATES	1.6612E-01	2.4885E-01
SAMPLE MEAN OF THE ESTIMATES	2.1620E-01	2.5554E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.1091E-01	1.6900E-01
SMALLEST ESTIMATE	-2.0877E-01	-9.4620E-02
LARGEST ESTIMATE	5.5208E-01	5.5208E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.4482E-02	3.2153E-02
3.2153E-02	2.8560E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.1 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=X**4
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	3.1620E-01	1.7800E-01
SAMPLE MEDIAN ØF THE ESTIMATES	2.4305E-01	1.7725E-01
SAMPLE MEAN ØF THE ESTIMATES	2.5213E-01	1.9667E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.6323E-01	1.8786E-01
SMALLEST ESTIMATE	-1.0903E-01	-1.9250E-01
LARGEST ESTIMATE	5.7717E-01	5.7717E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.6645E-02	2.3484E-02
2.3484E-02	3.5293E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=X**4
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		3.3200E-01
SAMPLE MEDIAN OF THE ESTIMATES	2.5672E-01	3.4182E-01
SAMPLE MEAN OF THE ESTIMATES	2.5323E-01	3.3524E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.3530E-01	9.9137E-02
SMALLEST ESTIMATE	-3.6046E-02	5.7300E-02
LARGEST ESTIMATE	5.8289E-01	5.8289E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.8306E-02	7.5017E-03
7.5017E-03	9.8280E-03

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=X**4
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	3.1620E-01	1.7800E-01
SAMPLE MEDIAN ØF THE ESTIMATES	3.3410E-01	1.8305E-01
SAMPLE MEAN ØF THE ESTIMATES	3.2005E-01	1.8967E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.1843E-01	1.8977E-01
SMALLEST ESTIMATE	-3.4551E-02	-4.5688E-01
LARGEST ESTIMATE	6.3647E-01	6.3647E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.4025E-02	2.0670E-02
2.0670E-02	3.6014E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PROCEDURE-- SAFE FIRING LEVEL AND 1.222 PERCENT RESPONSE LEVEL
RESPONSE FUNCTION-- CDELL, Y=X**4
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 SAFE FIRING LEVEL	PARAMETER 2 1.222 PERCENT RESPONSE LEVEL
TRUE VALUE		3.3200E-01
SAMPLE MEDIAN OF THE ESTIMATES	2.0985E-01	2.2927E-01
SAMPLE MEAN OF THE ESTIMATES	1.7758E-01	2.1851E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.2244E-01	1.7212E-01
SMALLEST ESTIMATE	-2.1435E-01	-1.4581E-01
LARGEST ESTIMATE	5.6435E-01	5.6435E-01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

4.9481E-02	3.3148E-02
3.3148E-02	2.9626E-02

EXPERIMENTAL DESIGN-- 20-ZIL, INITIAL TEST LEVEL SIGMA = 0.4 AND STEP SIZE = 0.4
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- ØDELL, Y=X**4
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	3.1620E-01	1.7800E-01
SAMPLE MEDIAN ØF THE ESTIMATES	2.3020E-01	1.7555E-01
SAMPLE MEAN ØF THE ESTIMATES	2.3036E-01	1.7524E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.6135E-01	1.7799E-01
SMALLEST ESTIMATE	-5.3656E-02	-4.3447E-01
LARGEST ESTIMATE	6.9087E-01	6.9087E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.6035E-02	2.0650E-02
2.0650E-02	3.1682E-02

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 0.5
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 32 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-1.9243E 00	-2.4222E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.1027E 00	-2.7044E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	7.8882E-01	1.1809E 00
SMALLEST ESTIMATE	-6.7011E 00	-9.8602E 00
LARGEST ESTIMATE	-9.9548E-01	-1.3633E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

6.2223E-01	9.1910E-01
9.1910E-01	1.3945E 00

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 0.5
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.1630E 00	-2.8415E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.2324E 00	-2.9258E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	5.0439E-01	7.9252E-01
SMALLEST ESTIMATE	-4.0235E 00	-6.3508E 00
LARGEST ESTIMATE	-1.1568E 00	-1.4446E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.5441E-01	3.9299E-01
3.9299E-01	6.2809E-01

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 0.5
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 128 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2840E 00	-3.0375E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.2946E 00	-3.0136E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.2796E-01	5.2162E-01
SMALLEST ESTIMATE	-3.2488E 00	-4.5351E 00
LARGEST ESTIMATE	-1.6233E 00	-1.9793E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.0756E-01	1.6513E-01
1.6513E-01	2.7209E-01

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 0.5
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 256 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.3342E 00	-3.0401E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3334E 00	-3.0730E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.6366E-01	4.2550E-01
SMALLEST ESTIMATE	-3.3245E 00	-4.6346E 00
LARGEST ESTIMATE	-1.8210E 00	-2.2285E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

6.9518E-02	1.0626E-01
1.0626E-01	1.8105E-01

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 0.5
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 512 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.3264E 00	-3.0685E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3370E 00	-3.0766E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.9095E-01	2.9311E-01
SMALLEST ESTIMATE	-2.9500E 00	-4.0023E 00
LARGEST ESTIMATE	-1.8934E 00	-2.3454E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.6463E-02	5.0349E-02
5.0349E-02	8.5912E-02

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 0.5
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 1024 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.3383E 00	-3.0725E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3372E 00	-3.0761E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.5807E-01	2.2400E-01
SMALLEST ESTIMATE	-2.8955E 00	-3.8017E 00
LARGEST ESTIMATE	-1.9748E 00	-2.6160E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.4985E-02	3.0018E-02
3.0018E-02	5.0177E-02

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 32 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.0789E 00	-2.6246E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.2649E 00	-2.8553E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.0225E-01	1.2184E 00
SMALLEST ESTIMATE	-5.6511E 00	-8.3684E 00
LARGEST ESTIMATE	-7.6672E-01	-9.8687E-01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

6.4361E-01	9.3302E-01
9.3302E-01	1.4845E 00

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2893E 00	-2.9866E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3298E 00	-3.0191E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	4.7525E-01	7.6278E-01
SMALLEST ESTIMATE	-4.1128E 00	-6.0954E 00
LARGEST ESTIMATE	-1.5177E 00	-1.8761E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.2587E-01	3.4123E-01
3.4123E-01	5.8183E-01

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 128 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2844E 00	-2.9546E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3486E 00	-3.0739E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.6121E-01	5.5941E-01
SMALLEST ESTIMATE	-3.3241E 00	-4.7140E 00
LARGEST ESTIMATE	-1.5900E 00	-1.9490E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.3047E-01	1.8931E-01
1.8931E-01	3.1293E-01

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 256 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2946E 00	-3.0347E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3389E 00	-3.0712E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.7247E-01	4.1005E-01
SMALLEST ESTIMATE	-3.0517E 00	-4.0472E 00
LARGEST ESTIMATE	-1.6749E 00	-2.1682E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

7.4238E-02	1.0250E-01
1.0250E-01	1.6814E-01

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 512 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2888E 00	-3.0317E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3299E 00	-3.0615E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.4386E-01	3.6326E-01
SMALLEST ESTIMATE	-2.9315E 00	-3.9268E 00
LARGEST ESTIMATE	-1.5830E 00	-2.1562E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

5.9467E-02	8.0655E-02
8.0655E-02	1.3196E-01

EXPERIMENTAL DESIGN-- BARTLETT, INITIAL TEST LEVEL SIGMA = 0.2 AND STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- 1 AND 0.1 PERCENT RESPØNSE LEVELS (MU, SIGMA FRØM MØS + MLE)
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 1024 **

	PARAMETER 1 1 PERCENT LEVEL	PARAMETER 2 0.1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00	-3.0900E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.3151E 00	-3.0600E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3464E 00	-3.0743E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.5564E-01	3.1166E-01
SMALLEST ESTIMATE	-3.8529E 00	-3.8529E 00
LARGEST ESTIMATE	-1.8829E 00	-2.4068E 00

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

6.5350E-02	6.5998E-02
6.5998E-02	9.7129E-02

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIOR SIGMA = 1.0, STEP SIZE = 0.1
ANALYTICAL PROCEDURE-- METHOD OF REVERSALS WITH LINEAR INTERPOLATION BETWEEN PLATEAUS
RESPONSE FUNCTION-- CUMULATIVE NORMAL, MU = 0, SIGMA = 1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	0.	1.4000E 01
SAMPLE MEDIAN OF THE ESTIMATES	-4.4842E-02	1.7000E 01
SAMPLE MEAN OF THE ESTIMATES	-4.3226E-02	1.9370E 01
SAMPLE STANDARD ERROR OF THE ESTIMATES	3.7748E-01	1.6410E 01
SMALLEST ESTIMATE	-9.1514E-01	4.0000E 00
LARGEST ESTIMATE	8.1740E-01	1.1900E 02

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.4249E-01	-6.1055E-01
-6.1055E-01	2.6929E 02

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIØR SIGMA = 10.0, STEP SIZE = 1.0
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0, SIGMA = 1
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	0.	1.4000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	1.8964E-01	1.4000E 01
SAMPLE MEAN ØF THE ESTIMATES	9.9830E-02	1.5360E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	6.2283E-01	7.4606E 00
SMALLEST ESTIMATE	-1.4709E 00	4.0000E 00
LARGEST ESTIMATE	1.6747E 00	4.3000E 01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.8791E-01	-3.5244E-01
-3.5244E-01	5.5660E 01

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIOR SIGMA = 0.2, STEP SIZE = 0.02
ANALYTICAL PROCEDURE-- METHOD OF REVERSALS WITH LINEAR INTERPOLATION BETWEEN PLATEAUS
RESPONSE FUNCTION-- CUMULATIVE UNIFORM
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	5.0000E-01	1.4000E 01
SAMPLE MEDIAN OF THE ESTIMATES	4.9388E-01	1.0500E 01
SAMPLE MEAN OF THE ESTIMATES	5.0995E-01	1.4500E 01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.2266E-01	1.0157E 01
SMALLEST ESTIMATE	2.6568E-01	4.0000E 00
LARGEST ESTIMATE	7.7824E-01	5.2000E 01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.5095E-02	-9.8665E-02
-9.8665E-02	1.0316E 02

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIØR SIGMA = 2.0, STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- CUMULATIVE UNIFØRM
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	5.0000E-01	1.4000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	4.8951E-01	1.4000E 01
SAMPLE MEAN ØF THE ESTIMATES	5.1276E-01	1.5660E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.7790E-01	8.3594E 00
SMALLEST ESTIMATE	7.6591E-02	4.0000E 00
LARGEST ESTIMATE	9.2079E-01	5.6000E 01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.1649E-02	-5.6473E-02
-5.6473E-02	6.5880E 01

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIOR SIGMA = 0.2, STEP SIZE = 0.02
ANALYTICAL PROCEDURE-- METHOD OF REVERSALS WITH LINEAR INTERPOLATION BETWEEN PLATEAUS
RESPONSE FUNCTION-- DOELL (Y=X**0.25)
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	6.2500E-02	1.4000E 01
SAMPLE MEDIAN OF THE ESTIMATES	8.0075E-02	1.5000E 01
SAMPLE MEAN OF THE ESTIMATES	9.8032E-02	1.8010E 01
SAMPLE STANDARD ERROR OF THE ESTIMATES	8.3783E-02	1.2683E 01
SMALLEST ESTIMATE	-2.6302E-03	4.0000E 00
LARGEST ESTIMATE	3.9375E-01	7.4000E 01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

7.0197E-03	1.2891E-01
1.2891E-01	1.6087E 02

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIØR SIGMA = 2.0, STEP SIZE = 0.2
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=X**0.25)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	6.2500E-02	1.4000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	1.0309E-01	1.2500E 01
SAMPLE MEAN ØF THE ESTIMATES	1.2676E-01	1.4950E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.3758E-01	8.2771E 00
SMALLEST ESTIMATE	-7.5215E-02	4.0000E 00
LARGEST ESTIMATE	5.1305E-01	5.5000E 01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.8929E-02	1.5009E-01
1.5009E-01	6.8510E 01

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIØR SIGMA = 0.2, STEP SIZE = 0.02
ANALYTICAL PROCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- PDELL (Y=2.*X**2, Y=1.-2.*(1.-X)**2)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	5.0000E-01	1.4000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	4.9536E-01	1.5500E 01
SAMPLE MEAN ØF THE ESTIMATES	4.9698E-01	1.5590E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.4281E-02	1.6151E 01
SMALLEST ESTIMATE	2.6221E-01	4.0000E 00
LARGEST ESTIMATE	7.5901E-01	8.0000E 01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

7.1033E-03	-1.0770E-02
-1.0770E-02	2.6087E 02

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIOR SIGMA = 2.0, STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- METHOD OF REVERSALS WITH LINEAR INTERPOLATION BETWEEN PLATEAUS
RESPONSE FUNCTION-- DOELL (Y=2.*X**2, Y=1.-2.*(1.-X)**2)
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	5.0000E-01	1.4000E 01
SAMPLE MEDIAN OF THE ESTIMATES	4.9327E-01	1.4000E 01
SAMPLE MEAN OF THE ESTIMATES	4.9751E-01	1.5370E 01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.1045E-01	7.7389E 00
SMALLEST ESTIMATE	2.5014E-01	4.0000E 00
LARGEST ESTIMATE	7.2051E-01	5.2000E 01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.2200E-02	-1.9705E-01
-1.9705E-01	5.9890E 01

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIOR SIGMA = 0.2, STEP SIZE = 0.02
ANALYTICAL PROCEDURE-- METHOD OF REVERSALS WITH LINEAR INTERPOLATION BETWEEN PLATEAUS
RESPONSE FUNCTION-- DOELL (Y=X**4)
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	8.4090E-01	1.4000E 01
SAMPLE MEDIAN OF THE ESTIMATES	8.3383E-01	1.7000E 01
SAMPLE MEAN OF THE ESTIMATES	8.3136E-01	1.8460E 01
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.1412E-02	1.2108E 01
SMALLEST ESTIMATE	6.1847E-01	4.0000E 00
LARGEST ESTIMATE	9.4691E-01	5.4000E 01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

3.7714E-03	-4.1216E-02
-4.1216E-02	1.4660E 02

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIØR SIGMA = 2.0, STEP SIZE = 0.2
ANALYTICAL PROCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEALS
RESPØNSE FUNCTIØN-- ØDELL (Y=X**4)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	8.4090E-01	1.4000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	8.2448E-01	1.3000E 01
SAMPLE MEAN ØF THE ESTIMATES	8.2251E-01	1.4410E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.0685E-02	6.5123E 00
SMALLEST ESTIMATE	5.9755E-01	4.0000E 00
LARGEST ESTIMATE	9.9206E-01	3.6000E 01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

6.5100E-03	-2.4844E-02
-2.4844E-02	4.2410E 01

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIOR SIGMA = 0.1, STEP SIZE = 0.01
ANALYTICAL PROCEDURE-- METHOD OF REVERSALS WITH LINEAR INTERPOLATION BETWEEN PLATEALS
RESPONSE FUNCTION-- PATHOLOGICAL -- M-PRIME(C) = C
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	0.	1.4000E 01
SAMPLE MEDIAN OF THE ESTIMATES	2.3349E-03	1.1000E 01
SAMPLE MEAN OF THE ESTIMATES	4.5819E-04	1.5370E 01
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.0660E-01	1.1980E 01
SMALLEST ESTIMATE	-2.7640E-01	4.0000E 00
LARGEST ESTIMATE	2.6191E-01	7.6000E 01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.1364E-02	2.0706E-01
2.0706E-01	1.4351E 02

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIØR SIGMA = 1.0, STEP SIZE = 0.1
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- PATHØLØGICAL -- M-PRIME(C) = C
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	0.	1.4000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	-2.3346E-01	1.3000E 01
SAMPLE MEAN ØF THE ESTIMATES	-1.5111E-01	1.7260E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	4.9759E-01	1.3920E 01
SMALLEST ESTIMATE	-8.2893E-01	4.0000E 00
LARGEST ESTIMATE	8.2315E-01	8.5000E 01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.4759E-01	-7.5726E-01
-7.5726E-01	1.9378E 02

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIOR SIGMA = 0.1, STEP SIZE = 0.01
ANALYTICAL PROCEDURE-- METHOD OF REVERSALS WITH LINEAR INTERPOLATION BETWEEN PLATEAUS
RESPONSE FUNCTION-- PATHOLOGICAL -- M-PRIME(C) = INFINITY
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	0.	1.4000E 01
SAMPLE MEDIAN OF THE ESTIMATES	-1.0199E-04	1.5000E 01
SAMPLE MEAN OF THE ESTIMATES	-2.9642E-03	1.5970E 01
SAMPLE STANDARD ERROR OF THE ESTIMATES	5.8924E-02	9.9403E 00
SMALLEST ESTIMATE	-1.8857E-01	4.0000E 00
LARGEST ESTIMATE	2.9347E-01	4.5000E 01

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

3.4721E-03	3.4800E-02
3.4800E-02	9.8810E 01

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 50 PERCENT LEVEL, PRIØR SIGMA = 1.0, STEP SIZE = 0.1
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEALS
RESPØNSE FUNCTIØN-- PATHØLØGICAL -- M-PRIME(0) = INFINITY
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 50 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	0.	1.4000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	7.1756E-03	1.4500E 01
SAMPLE MEAN ØF THE ESTIMATES	1.8954E-02	1.6300E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.1634E-01	9.3306E 00
SMALLEST ESTIMATE	-9.3729E-01	4.0000E 00
LARGEST ESTIMATE	2.1521E 00	4.8000E 01

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.0007E-01	-2.0168E-02
-2.0168E-02	8.7066E 01

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL, PRIOR SIGMA = 1.0, STEP SIZE = 0.025
ANALYTICAL PROCEDURE-- METHOD OF REVERSALS WITH LINEAR INTERPOLATION BETWEEN PLATEAUS
RESPONSE FUNCTION-- CUMULATIVE NORMAL, MU = 0, SIGMA = 1
NUMBER OF SIMULATIONS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	-1.6450E 00	4.8000E 01
SAMPLE MEDIAN OF THE ESTIMATES	-1.7263E 00	6.2000E 01
SAMPLE MEAN OF THE ESTIMATES	-1.7423E 00	7.1170E 01
SAMPLE STANDARD ERROR OF THE ESTIMATES	3.2098E-01	4.0859E 01
SMALLEST ESTIMATE	-2.7355E 00	3.2000E 01
LARGEST ESTIMATE	-1.0176E 00	2.4000E 02

SAMPLE COVARIANCE MATRIX OF THE ESTIMATES

1.0303E-01	-1.8351E 00
-1.8351E 00	1.6727E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL, PRIØR SIGMA = 1C.C, STEP SIZE = 0.25
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = C, SIGMA = 1
NUMBER ØF SIMULATIØNS-- 1CC

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	-1.6450E OC	4.ECCOE C1
SAMPLE MEDIAN ØF THE ESTIMATES	-1.6433E CC	6.5CCOE C1
SAMPLE MEAN ØF THE ESTIMATES	-1.7287E OC	6.8330E C1
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	4.7662E-C1	3.1253E C1
SMALLEST ESTIMATE	-2.8643E OC	3.3CCOE C1
LARGEST ESTIMATE	-8.5327E-01	1.68COE C2

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.2717E-01	-1.6456E CC
-1.6456E CC	5.7673E C2

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL, PRIØR SIGMA = 0.2, STEP SIZE = 0.005
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- CUMULATIVE UNIFØRM
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	5.0000E-02	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	5.2398E-02	7.2000E 01
SAMPLE MEAN ØF THE ESTIMATES	6.2714E-02	8.0640E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.8576E-02	5.1049E 01
SMALLEST ESTIMATE	1.1295E-03	3.3000E 01
LARGEST ESTIMATE	1.7050E-01	1.9400E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.4801E-03	3.3576E-01
3.3576E-01	2.6060E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL,PRIØR SIGMA = 2.0, STEP SIZE = 0.05
ANALYTICAL PROCEDURE-- METHOD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- CUMULATIVE UNIFØRM
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	5.0000E-02	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	6.1966E-02	6.7500E 01
SAMPLE MEAN ØF THE ESTIMATES	7.0780E-02	7.1670E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	5.8368E-02	3.8357E 01
SMALLEST ESTIMATE	-2.0554E-02	3.3000E 01
LARGEST ESTIMATE	2.6854E-01	1.5800E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

3.4069E-03	5.8181E-01
5.8181E-01	1.4712E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL, PRIØR SIGMA = 0.2, STEP SIZE = 0.005
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=X**0.25)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	6.2500E-06	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	-1.3219E-03	6.4000E 01
SAMPLE MEAN ØF THE ESTIMATES	-9.5233E-04	6.7590E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.1548E-03	3.3294E 01
SMALLEST ESTIMATE	-4.0366E-03	3.3000E 01
LARGEST ESTIMATE	6.0799E-03	1.2800E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

4.6429E-06	-1.9901E-02
-1.9901E-02	1.1085E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL,PRIØR SIGMA = 2.0, STEP SIZE = 0.05
ANALYTICAL PROCEDURE-- METHOD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPONSE FUNCTIØN-- ØDELL (Y=X**0.25)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	6.2500E-06	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	-1.8546E-02	6.5500E 01
SAMPLE MEAN ØF THE ESTIMATES	-1.8972E-02	6.6220E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.3529E-02	3.2168E 01
SMALLEST ESTIMATE	-4.4821E-02	3.3000E 01
LARGEST ESTIMATE	1.5274E-02	1.3800E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

5.5361E-04	-4.5815E-01
-4.5815E-01	1.0348E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL, PRIØR SIGMA = 0.2, STEP SIZE = 0.005
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=4.*X**2, Y=1.-4.*(1.-X)**2/3.)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	1.1180E-01	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	1.2075E-01	7.4500E 01
SAMPLE MEAN ØF THE ESTIMATES	1.1998E-01	7.9180E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.7615E-02	4.7537E 01
SMALLEST ESTIMATE	2.6331E-02	3.3000E 01
LARGEST ESTIMATE	2.3390E-01	1.7100E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.4149E-03	2.7891E-01
2.7891E-01	2.2598E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL,PRIØR SIGMA = 2.0, STEP SIZE = 0.05
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=4.*X**2, Y=1.-4.*(1.-X)**2/3.)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	1.1180E-01	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	1.2471E-01	7.0000E 01
SAMPLE MEAN ØF THE ESTIMATES	1.2478E-01	7.1870E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	5.3773E-02	3.5790E 01
SMALLEST ESTIMATE	-2.5577E-03	3.5000E 01
LARGEST ESTIMATE	2.5384E-01	1.3500E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

2.8916E-03	3.9877E-01
3.9877E-01	1.2809E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL, PRIØR SIGMA = 0.2, STEP SIZE = 0.005
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=2.*X**2, Y=1.-2.*(1.-X)**2)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	1.5810E-01	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	1.5695E-01	7.6500E 01
SAMPLE MEAN ØF THE ESTIMATES	1.5968E-01	8.1480E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	4.3805E-02	5.2004E 01
SMALLEST ESTIMATE	5.1437E-02	3.3000E 01
LARGEST ESTIMATE	2.8098E-01	2.5400E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

1.9189E-03	-3.1956E-02
-3.1956E-02	2.7044E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL, PRIØR SIGMA = 2.0, STEP SIZE = 0.05
ANALYTICAL PROCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=2.*X**2, Y=1.-2.*(1.-X)**2)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	1.5810E-01	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	1.7498E-01	6.7360E 01
SAMPLE MEAN ØF THE ESTIMATES	1.6776E-01	7.1661E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	7.8781E-02	3.7531E 01
SMALLEST ESTIMATE	5.8243E-03	3.4600E 01
LARGEST ESTIMATE	3.5885E-01	1.4500E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

6.1986E-03	-1.0918E-01
-1.0918E-01	1.4086E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL, PRIØR SIGMA = 0.2, STEP SIZE = 0.005
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=(4.*X**2)/3., Y=1.-4.*(1.-X)**2)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	1.9360E-01	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	2.1004E-01	7.6000E 01
SAMPLE MEAN ØF THE ESTIMATES	2.1676E-01	8.0700E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	6.5707E-02	5.1794E 01
SMALLEST ESTIMATE	9.1480E-02	3.3000E 01
LARGEST ESTIMATE	3.9203E-01	1.9800E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

4.3174E-03	7.5579E-01
7.5579E-01	2.6826E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL,PRIØR SIGMA = 2.0, STEP SIZE = 0.05
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=(4.*X**2)/3., Y=1.-4.*(1.-X)**2)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	1.9360E-01	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	2.1106E-01	7.0000E 01
SAMPLE MEAN ØF THE ESTIMATES	2.1447E-01	7.3970E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.7919E-02	3.9526E 01
SMALLEST ESTIMATE	3.4647E-02	3.3000E 01
LARGEST ESTIMATE	4.1328E-01	1.5500E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

7.7298E-03	9.2555E-01
9.2555E-01	1.5623E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL, PRIØR SIGMA = 0.2, STEP SIZE = 0.005
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=X**4)
NUMBER ØF SIMULATIØNS-- 100

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	4.7290E-01	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	4.8501E-01	6.5000E 01
SAMPLE MEAN ØF THE ESTIMATES	4.8216E-01	6.8720E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.4761E-02	3.7672E 01
SMALLEST ESTIMATE	1.5760E-01	3.3000E 01
LARGEST ESTIMATE	6.4038E-01	1.5700E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

7.1844E-03	5.6313E-01
5.6313E-01	1.4192E 03

EXPERIMENTAL DESIGN-- ALEXANDER DESIGN, 5 PERCENT LEVEL,PRIØR SIGMA = 2.0, STEP SIZE = 0.05
ANALYTICAL PRØCEDURE-- METHØD ØF REVERSALS WITH LINEAR INTERPØLATIØN BETWEEN PLATEAUS
RESPØNSE FUNCTIØN-- ØDELL (Y=X**4)
NUMBER ØF SIMULATIØNS-- 10C

	PARAMETER 1 5 PERCENT LEVEL	PARAMETER 2 REQUIRED SAMPLE SIZE
TRUE VALUE	4.7290E-01	4.3000E 01
SAMPLE MEDIAN ØF THE ESTIMATES	5.0999E-01	6.7000E 01
SAMPLE MEAN ØF THE ESTIMATES	4.9691E-01	7.3420E 01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.8257E-02	3.9813E 01
SMALLEST ESTIMATE	1.5419E-01	3.3000E 01
LARGEST ESTIMATE	7.0752E-01	1.4400E 02

SAMPLE CØVARIANCE MATRIX ØF THE ESTIMATES

7.7893E-03	2.5234E-01
2.5234E-01	1.5851E 03

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 1.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 16 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	0.
SAMPLE MEDIAN ØF THE ESTIMATES	-1.2114E-02
SAMPLE MEAN ØF THE ESTIMATES	-1.0096E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.9171E-01
SMALLEST ESTIMATE	-9.3626E-01
LARGEST ESTIMATE	8.7347E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 50 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 1.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- CUMULATIVE NORMAL, MU = C.0, SIGMA = 1.C
NUMBER OF SIMULATIONS-- 1CC

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	0.
SAMPLE MEDIAN OF THE ESTIMATES	2.CCC9E-02
SAMPLE MEAN OF THE ESTIMATES	4.4656E-03
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.C222E-01
SMALLEST ESTIMATE	-4.3547E-01
LARGEST ESTIMATE	4.8312E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 50 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 10.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- CUMULATIVE NORMAL, MU = 0.0, SIGMA = 1.0
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 16 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	0.
SAMPLE MEDIAN OF THE ESTIMATES	-2.2252E-02
SAMPLE MEAN OF THE ESTIMATES	-4.0600E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.1396E-01
SMALLEST ESTIMATE	-3.3546E 00
LARGEST ESTIMATE	1.2563E 00

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 10.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

PARAMETER 1
50 PERCENT LEVEL

TRUE VALUE	0.
SAMPLE MEDIAN ØF THE ESTIMATES	1.2231E-02
SAMPLE MEAN ØF THE ESTIMATES	1.0394E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.4748E-01
SMALLEST ESTIMATE	-6.4273E-01
LARGEST ESTIMATE	6.8552E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 0.2
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE UNIFØRM
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	5.0000E-01
SAMPLE MEDIAN ØF THE ESTIMATES	5.0900E-01
SAMPLE MEAN ØF THE ESTIMATES	5.0323E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.1149E-02
SMALLEST ESTIMATE	2.8353E-01
LARGEST ESTIMATE	7.1345E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 2.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE UNIFØRM
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	5.0000E-01
SAMPLE MEDIAN ØF THE ESTIMATES	5.2176E-01
SAMPLE MEAN ØF THE ESTIMATES	5.0663E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.5681E-02
SMALLEST ESTIMATE	3.0145E-01
LARGEST ESTIMATE	7.3062E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 0.2
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- ØDELL (Y=X**0.25)
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	6.2500E-02
SAMPLE MEDIAN ØF THE ESTIMATES	6.9155E-02
SAMPLE MEAN ØF THE ESTIMATES	7.6820E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	5.0743E-02
SMALLEST ESTIMATE	-4.9804E-03
LARGEST ESTIMATE	3.0106E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 2.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- ØDELL (Y=X**0.25)
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	6.2500E-02
SAMPLE MEDIAN ØF THE ESTIMATES	8.8293E-02
SAMPLE MEAN ØF THE ESTIMATES	9.1363E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	5.3909E-02
SMALLEST ESTIMATE	9.7077E-03
LARGEST ESTIMATE	2.1034E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 0.2
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- ØDELL (Y=2.*X**2, Y=1.-2.*(1.-X)**2)
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	5.0000E-01
SAMPLE MEDIAN ØF THE ESTIMATES	5.0422E-01
SAMPLE MEAN ØF THE ESTIMATES	4.9875E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	5.4521E-02
SMALLEST ESTIMATE	3.5668E-01
LARGEST ESTIMATE	6.9366E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 2.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- ØDELL (Y=2.*X**2, Y=1.-2.*(1.-X)**2)
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	5.0000E-01
SAMPLE MEDIAN ØF THE ESTIMATES	5.0068E-01
SAMPLE MEAN ØF THE ESTIMATES	4.9558E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	5.0670E-02
SMALLEST ESTIMATE	3.6864E-01
LARGEST ESTIMATE	6.1754E-01

'TI

SE

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 0.2
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- ØDELL (Y=X**4)
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	8.4090E-01
SAMPLE MEDIAN ØF THE ESTIMATES	8.4028E-01
SAMPLE MEAN ØF THE ESTIMATES	8.3881E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.6704E-02
SMALLEST ESTIMATE	7.2400E-01
LARGEST ESTIMATE	9.1556E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 2.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- ØDELL (Y=X**4)
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	8.4090E-01
SAMPLE MEDIAN ØF THE ESTIMATES	8.3035E-01
SAMPLE MEAN ØF THE ESTIMATES	8.3469E-01
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.3328E-02
SMALLEST ESTIMATE	7.4689E-01
LARGEST ESTIMATE	9.1157E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 50 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 0.1
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- PATHOLOGICAL -- M-PRIME(0) = 0
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	0.
SAMPLE MEDIAN OF THE ESTIMATES	-5.7477E-02
SAMPLE MEAN OF THE ESTIMATES	-2.3923E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.7149E-01
SMALLEST ESTIMATE	-5.9144E-01
LARGEST ESTIMATE	5.3700E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 50 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 1.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- PATHØLØGICAL -- M-PRIME(0) = 0
NUMBER ØF SIMULATIØNS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	0.
SAMPLE MEDIAN ØF THE ESTIMATES	3.5768E-02
SAMPLE MEAN ØF THE ESTIMATES	3.5088E-02
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	4.2902E-01
SMALLEST ESTIMATE	-7.1680E-01
LARGEST ESTIMATE	7.1948E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 50 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 0.1
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- PATHOLOGICAL -- M-PRIME(0) = INFINITY
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	0.
SAMPLE MEDIAN OF THE ESTIMATES	-8.5230E-05
SAMPLE MEAN OF THE ESTIMATES	-9.2400E-04
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.5041E-02
SMALLEST ESTIMATE	-8.4948E-02
LARGEST ESTIMATE	6.4428E-02

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 50 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 1.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- PATHOLOGICAL -- M-PRIME(0) = INFINITY
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 50 PERCENT LEVEL
TRUE VALUE	0.
SAMPLE MEDIAN OF THE ESTIMATES	-7.6184E-04
SAMPLE MEAN OF THE ESTIMATES	5.9660E-04
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.0195E-02
SMALLEST ESTIMATE	-4.8659E-02
LARGEST ESTIMATE	1.5897E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 1.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- CUMULATIVE NORMAL, MU = 0.0, SIGMA = 1.0
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	-1.6450E 00
SAMPLE MEDIAN OF THE ESTIMATES	-1.4673E 00
SAMPLE MEAN OF THE ESTIMATES	-1.4696E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	3.0332E-01
SMALLEST ESTIMATE	-2.0468E 00
LARGEST ESTIMATE	-6.1747E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 10.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- CUMULATIVE NORMAL, MU = 0.0, SIGMA = 1.0
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	-1.6450E 00
SAMPLE MEDIAN OF THE ESTIMATES	-1.6368E 00
SAMPLE MEAN OF THE ESTIMATES	-1.6822E 00
SAMPLE STANDARD ERROR OF THE ESTIMATES	3.7836E-01
SMALLEST ESTIMATE	-2.8514E 00
LARGEST ESTIMATE	-9.5035E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 0.2
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- CUMULATIVE UNIFORM
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	5.0000E-02
SAMPLE MEDIAN OF THE ESTIMATES	7.9465E-02
SAMPLE MEAN OF THE ESTIMATES	7.9324E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	5.1422E-02
SMALLEST ESTIMATE	9.8974E-03
LARGEST ESTIMATE	2.2789E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 2.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- CUMULATIVE UNIFORM
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	5.0000E-02
SAMPLE MEDIAN OF THE ESTIMATES	7.4429E-02
SAMPLE MEAN OF THE ESTIMATES	7.6312E-02
SAMPLE STANDARD ERROR OF THE ESTIMATES	4.3357E-02
SMALLEST ESTIMATE	1.1762E-02
LARGEST ESTIMATE	1.8464E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 0.2
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- ODELL (Y=X**0.25)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	6.2500E-06
SAMPLE MEDIAN OF THE ESTIMATES	1.1620E-03
SAMPLE MEAN OF THE ESTIMATES	1.5820E-03
SAMPLE STANDARD ERROR OF THE ESTIMATES	2.5373E-03
SMALLEST ESTIMATE	-2.2653E-03
LARGEST ESTIMATE	9.9565E-03

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 2.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- ODELL (Y=X**0.25)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	6.2500E-06
SAMPLE MEDIAN OF THE ESTIMATES	4.7063E-03
SAMPLE MEAN OF THE ESTIMATES	7.4135E-03
SAMPLE STANDARD ERROR OF THE ESTIMATES	1.0762E-02
SMALLEST ESTIMATE	-2.9127E-03
LARGEST ESTIMATE	3.8213E-02

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 0.2
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- ODELL (Y=4.*X**2, Y=1.-4.*(1.-X)**2/3.)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	1.1180E-01
SAMPLE MEDIAN OF THE ESTIMATES	1.3013E-01
SAMPLE MEAN OF THE ESTIMATES	1.2634E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	3.5687E-02
SMALLEST ESTIMATE	5.1060E-02
LARGEST ESTIMATE	2.3197E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 2.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPCNSE FUNCTION-- CDELL (Y=4.*X**2, Y=1.-4.*(1.-X)**2/3.)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	1.1180E-01
SAMPLE MEDIAN OF THE ESTIMATES	1.2525E-01
SAMPLE MEAN OF THE ESTIMATES	1.2644E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	3.9416E-02
SMALLEST ESTIMATE	4.6010E-02
LARGEST ESTIMATE	2.0832E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 0.2
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- ODELL (Y=2.*X**2, Y=1.-2.*(1.-X)**2)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	1.5810E-01
SAMPLE MEDIAN OF THE ESTIMATES	1.8919E-01
SAMPLE MEAN OF THE ESTIMATES	1.8867E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	5.2985E-02
SMALLEST ESTIMATE	8.0508E-02
LARGEST ESTIMATE	2.9000E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 2.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- ODELL (Y=2.*X**2, Y=1.-2.*(1.-X)**2)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	1.5810E-01
SAMPLE MEDIAN OF THE ESTIMATES	1.7063E-01
SAMPLE MEAN OF THE ESTIMATES	1.7516E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.3150E-02
SMALLEST ESTIMATE	2.9658E-02
LARGEST ESTIMATE	3.8036E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 0.2
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- ODELL (Y=(4.*X**2)/3., Y=1.-4.*(1.-X)**2)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	1.9360E-01
SAMPLE MEDIAN OF THE ESTIMATES	2.5054E-01
SAMPLE MEAN OF THE ESTIMATES	2.5150E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	8.0566E-02
SMALLEST ESTIMATE	1.0600E-01
LARGEST ESTIMATE	3.6996E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 2.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- ODELL (Y=(4.*X**2)/3., Y=1.-4.*(1.-X)**2)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	1.9360E-01
SAMPLE MEDIAN OF THE ESTIMATES	2.0731E-01
SAMPLE MEAN OF THE ESTIMATES	2.0597E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	6.4695E-02
SMALLEST ESTIMATE	4.2110E-02
LARGEST ESTIMATE	3.8052E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 0.2
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- ODELL (Y=X**4)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	4.7290E-01
SAMPLE MEDIAN OF THE ESTIMATES	5.2549E-01
SAMPLE MEAN OF THE ESTIMATES	5.2512E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	8.2015E-02
SMALLEST ESTIMATE	3.6369E-01
LARGEST ESTIMATE	6.6377E-01

EXPERIMENTAL DESIGN-- ROTHMAN DESIGN, 5 PERCENT INVERSE RESPONSE PROBLEM, PRIOR SIGMA = 2.0
ANALYTICAL PROCEDURE-- CENTER OF APPROXIMATE 50 PERCENT CONFIDENCE INTERVAL
RESPONSE FUNCTION-- ODELL (Y=X**4)
NUMBER OF SIMULATIONS-- 100

** SAMPLE SIZE 64 **

	PARAMETER 1 5 PERCENT LEVEL
TRUE VALUE	4.7290E-01
SAMPLE MEDIAN OF THE ESTIMATES	4.7713E-01
SAMPLE MEAN OF THE ESTIMATES	4.8087E-01
SAMPLE STANDARD ERROR OF THE ESTIMATES	8.3267E-02
SMALLEST ESTIMATE	1.8147E-01
LARGEST ESTIMATE	6.6410E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 1 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 1.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 25

** SAMPLE SIZE 32 **

	PARAMETER 1 1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-1.2358E 00
SAMPLE MEAN ØF THE ESTIMATES	-1.2740E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.1131E 00
SMALLEST ESTIMATE	-2.0325E 00
LARGEST ESTIMATE	-6.9609E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 1 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 1.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 25

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-1.5073E 00
SAMPLE MEAN ØF THE ESTIMATES	-1.5573E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	8.0099E-01
SMALLEST ESTIMATE	-2.0325E 00
LARGEST ESTIMATE	-1.2077E 00

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 1 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 1.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 25

** SAMPLE SIZE 128 **

	PARAMETER 1 1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-1.7777E 00
SAMPLE MEAN ØF THE ESTIMATES	-1.7291E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	6.2093E-01
SMALLEST ESTIMATE	-2.0875E 00
LARGEST ESTIMATE	-1.4286E 00

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 1 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 1.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 25

** SAMPLE SIZE 256 **

	PARAMETER 1 1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-1.9450E 00
SAMPLE MEAN ØF THE ESTIMATES	-1.9347E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	4.1832E-01
SMALLEST ESTIMATE	-2.1816E 00
LARGEST ESTIMATE	-1.5610E 00

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 1 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 10.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 25

** SAMPLE SIZE 32 **

	PARAMETER 1 1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-1.4052E 00
SAMPLE MEAN ØF THE ESTIMATES	-1.4222E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	1.0458E 00
SMALLEST ESTIMATE	-2.4802E 00
LARGEST ESTIMATE	-3.3540E-01

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 1 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 10.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 25

** SAMPLE SIZE 64 **

	PARAMETER 1 1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-1.8810E 00
SAMPLE MEAN ØF THE ESTIMATES	-1.8773E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	5.7018E-01
SMALLEST ESTIMATE	-2.4802E 00
LARGEST ESTIMATE	-1.2413E 00

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 1 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 10.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 25

** SAMPLE SIZE 128 **

	PARAMETER 1 1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.2691E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.1516E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	3.2643E-01
SMALLEST ESTIMATE	-2.6458E 00
LARGEST ESTIMATE	-1.6268E 00

EXPERIMENTAL DESIGN-- RØTHMAN DESIGN, 1 PERCENT INVERSE RESPØNSE PRØBLEM, PRIØR SIGMA = 10.0
ANALYTICAL PRØCEDURE-- CENTER ØF APPRØXIMATE 50 PERCENT CØNFIDENCE INTERVAL
RESPØNSE FUNCTIØN-- CUMULATIVE NØRMAL, MU = 0.0, SIGMA = 1.0
NUMBER ØF SIMULATIØNS-- 25

** SAMPLE SIZE 256 **

	PARAMETER 1 1 PERCENT LEVEL
TRUE VALUE	-2.3260E 00
SAMPLE MEDIAN ØF THE ESTIMATES	-2.4031E 00
SAMPLE MEAN ØF THE ESTIMATES	-2.3444E 00
SAMPLE STANDARD ERRØR ØF THE ESTIMATES	2.6780E-01
SMALLEST ESTIMATE	-2.7925E 00
LARGEST ESTIMATE	-1.9759E 00

NASA-Langley, 1966